A nonparametric technique is presented that is appropriate for comparing two social interaction matrices, either when both are obtained empirically, or when one is generated from some given theoretical position. Depending on whether the diagonals of the matrices are considered relevant to the analysis, the comparison can be carried out using some variation on a cross-product statistic. In all cases, the chosen index is referred to an exact (or approximate) permutation distribution, based on the random matching of the rows and columns of one matrix to those of a second. Several examples are provided along with formulas for the first two moments of the permutation distributions for the suggested indices.

The Analysis of Social Interaction Data

A Nonparametric Technique

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Almost by definition the social sciences are involved in the study of social interaction whether the behavior occurs, for example, in small groups (Hare et al., 1966), school classrooms (Flanders, 1970), between married couples (Weiss et al., 1973), or even among animals (Rajecki and Lake, 1972). Typically, a researcher interested in the area of social interaction relies on data collected formally or informally by a particular observational strategy and hopes that any patterning in the data will lead to a deeper understanding of the underlying social process. In general, two aspects of social interaction have received

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considerable attention in the literature. The first deals with communication networks where the direction and amount of interaction among the individual members in a group are of major concern. The basic data may be verbal communications among humans or physical contacts among animals, but in general they will consist of simple counts of the number of times each group member "communicates" with another. As a second area, the form of social interaction itself may be under study and some classification scheme devised for codifying n potential "states" of the interaction process. An observer uses a classification scheme[1] to record, on a time-sampled basis, the category of interaction that occurs, and a sequence of category numbers is generated. This sequence may then be summarized by an n × n matrix depicting the frequency of state changes.
matrix depicting the frequency of state changes. Some of the statistical problems associated with the analysis of such data were presented a generation ago by Bales (1951).

The type of observational data described above provides at least two methodological problems. First of all, the researcher may wish to determine whether the obtained data reflect some hypothesized structure. Second, if two such matrices are collected, say, at different points in time and/or under different conditions, do the matrices have similar empirical structures? Based on the work of Mantel (1967), Hubert and Schultz (1976), and others, this article discusses a general nonparametric technique that may be used to approach both of these questions. The methods to be described in the examples to follow are very different from the more traditional statistical procedures used by Hare and Bales (1963) or from those presented by Darwin (1959), and are based on the concept of object matching as reflected by the correspondence between two matrices. Although some of the examples employed below may also be approached via log linear models or network analysis, these procedures were not included in the present paper. Both of these techniques have an extensive literature and the interested reader can see Bishop and associates (1975) for the former, or the special issue of this journal (November 1978) for the latter.

TABLE 1
Average Interaction in 12 Laboratory Groups—Hare and Bales Data

		Person Receiving Interaction				
		1	2	3	4	5
Person Initiating Interaction	1	X	21	29	22	31
	2	25	X	24	20	33
	3	35	24	X	20	36
	4	23	19	19	X	21
	5	32	33	34	19	X

EXAMPLE 1

To provide a concrete illustration of the communication network problem, a data set collected by Hare and Bales (1963) will be employed. As a notational convenience, let S refer to a set of n objects, which in the first example will be people. Furthermore, suppose two n × n matrices are defined on S, denoted by $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$, that contain measures of relationship between pairs of objects from S. It is assumed, at least for now, that the main diagonals of both A and B are irrelevant and both could be set equal to 0. Here, the objects in S denote the placement of individuals seated at 3 sides of a rectangular table with the unused side facing a one-way mirror. From the observer's position looking out of the mirror, person 1 was seated at the left side, persons 2, 3, and 4 at the long side and person 5 at the right side. Experimentally, 12 different groups of 5 people each were given a topic to discuss and the direction of interaction recorded for a 40-minute session. Table 1 contains the average number of interactions (reported as integer numbers) among the 5 seating positions over the 12 groups. In our notation, the data of Table 1 constitute the matrix $\{a_{ij}\}$.

Although it may be possible to identify structure in the underlying communication network by inspecting the matrix a_{ij}, the literature also hypothesizes that certain underlying structures should be present. Ideally, a methodology should be available to test the data directly against these structures. For instance,

TABLE 2
Pattern of Interaction Expected According to Steinzor

		Person Receiving Interaction				
		1	2	3	4	5
Person Initating Interaction	1	X	0	1	1	1
	2	0	X	0	1	1
	3	1	0	X	0	1
	4	1	1	0	X	0
	5	1	1	1	0	X

Steinzor (1950) hypothesized that group members will talk more to persons who sit opposite them than to persons sitting next to them. Therefore, the experimentally defined seating arrangement used by Hare and Bales suggests that certain entries in the data matrix will be relatively large, and conversely, certain entries will be relatively small. In particular, the "structure" matrix, i.e., the second matrix $\{b_{ij}\}$ given in Table 2, identifies adjacencies by 0's and nonadjacencies by 1's, and according to Steinzor the 0's should be reflected by low values in the $\{a_{ij}\}$ matrix of Table 1, and conversely, the 1's, by high values.

The correspondence between the two matrices $\{a_{ij}\}$ and $\{b_{ij}\}$ will be measured by a raw cross-product statistic of the form[2]

$$\Gamma_1 = \sum_{i \neq j} a_{ij} b_{ij}.$$

Based on the information given in Tables 1 and 2, Γ_1 is defined as the sum of entries in the data matrix $\{a_{ij}\}$ that identify communication among nonadjacent seating positions. In other words, if Steinzor's conjecture were reflected in the matrix $\{a_{ij}\}$, Γ_1 should be relatively large.

To formalize the notion of "relative," the effect of relabeling the rows and, simultaneously, the columns of the matrix $\{a_{ij}\}$ against a fixed matrix $\{b_{ij}\}$ is considered (or equivalently, relabeling the rows and columns of the matrix $\{b_{ij}\}$). Following Mantel (1967), the "null" hypothesis assumes that the rows and columns of $\{a_{ij}\}$ are labeled at random and thus the exact distribution of Γ_1 can be obtained from all such n! = 5! = 120 (considered equally likely) possible reorganizations of the matrix $\{a_{ij}\}$. If the observed value of Γ_1 were sufficiently extreme with respect to this permutation distribution, the hypothesis of a random relabeling could be rejected. Moreover, due to the construction of the structure matrix $\{b_{ij}\}$, this rejection in turn would provide support for the Steinzor hypothesis, i.e., that the sum of adjacent interactions is lower than what would be expected to occur by a chance relabeling of the seating positions. (This philosophy of confirmatory testing parallels the work of Hildebrand et al., 1977, in a related contingency table context.)

Carrying out the procedure introduced above and relying on the complete distribution of Γ_1 over the 120 possible permutations as given in Table 3, the probability of obtaining an index as large as or larger than the observed Γ_1 value of 347 is 2/120 = .017. Thus there appears to be strong support for Steinzor's conjecture in the data of Table 1. In fact, as is clear from Table 3, there are no other reorganizations of the matrix $\{a_{ij}\}$ that would lead to a larger value of Γ_1 when compared to the same structure matrix $\{b_{ij}\}$.

EXAMPLE 2

As a second, slightly more complicated illustration, we consider a classroom observation procedure due to Flanders (1965) defined by ten categories of verbal behavior. Seven of the categories deal with teacher talk, two with student talk, and the tenth covers pauses, silence, or periods of confusion. Based on this system, an observer records the category numbers for behaviors exhibited in the classroom over a sequential set of fixed time periods. The resulting number sequences are then used to construct an $n \times n$ transition matrix, where the off-diagonal terms indicate the frequency of state changes and those on the main diagonal indicate steady state behavior.

TABLE 3
Complete Permutation Distribution of Γ_1 for the Hare and Bales Data

Γ_1	Frequency	Γ_1	Frequency
275	2	315	2
281	2	316	2
288	4	317	2
293	2	318	2
294	2	319	2
297	2	320	2
298	2	321	2
299	2	322	2
300	8	323	4
301	2	324	4
302	8	325	6
303	2	326	2
304	4	329	4
305	6	330	4
306	4	331	4
307	4	332	2
309	2	333	2
311	4	341	2
312	2	327	2
313	2		
314	2		

Since the diagonal and off-diagonal elements in an $n \times n$ transition matrix are assumed to provide rather distinct sources of information (see the discussion of this point in Honigman, 1967), any comparison of two such matrices could be carried out in one

TABLE 4
Interaction Matrix for Indirect Mathematics Teachers: N = 7, Rates per 1000 Tallies

		To Category									
		1	2	3	4	5	6	7	8	9	10
From Category	1	.1	-	.1	.3	1.1	-	-	-	.1	.3
	2	-	1.6	2.7	2.8	5.4	.5	.2	.1	1.3	2.3
	3	.2	2.6	29.3	15.8	20.6	.8	.3	2.5	5.0	3.9
	4	.3	.5	.7	27.7	6.7	1.7	.3	74.1	4.6	8.5
	5	.8	1.3	2.1	47.0	383.0	8.8	1.7	.9	12.0	8.6
	6	-	-	.2	1.8	5.3	9.9	.3	6.1	1.9	8.5
	7	-	.1	.1	1.2	1.6	.6	2.5	.5	.6	2.3
	8	.3	7.3	29.0	17.9	23.0	3.5	1.1	20.3	3.0	1.9
	9	.3	2.9	16.1	3.6	8.4	1.8	.9	.1	25.0	2.1
	10	.2	.5	.6	7.2	11.1	6.3	2.2	2.6	7.9	7.8

SOURCE: Flanders, 1965: 76.

of three ways—comparing the diagonal and off-diagonal entries separately or together.[3] Irrespective of the particular alternative chosen, however, our interest is in determining whether the dynamics of one classroom differ from that of another. For example, Tables 4 and 5 were obtained from classes conducted by mathematics teachers characterized as "direct" and "indirect" by Flanders (1965). The actual cell entries in Table 4 are rates per thousand tallies obtained from seven classrooms conducted by "indirect" teachers; similarly, Table 5 contains rates for nine classrooms conducted by "direct" teachers. On a priori grounds, one could expect that the pattern of change from one category of verbal behavior to another may vary for such teachers, and possibly, the pattern of steady state behaviors may be different as well.

TABLE 5
Interaction Matrix for Direct Mathematics Teachers: N = 9, Rate per 1000 Tallies

		To Category									
		1	2	3	4	5	6	7	8	9	10
	1	.1	-	-	.1	.4	.1	.1	-	.2	.2
	2	.1	.9	.8	1.7	2.7	1.1	.1	.2	.9	2.2
	3	.1	.5	5.3	4.7	8.2	-	1.2	.4	1.0	2.8
	4	.1	.3	.1	6.8	3.4	2.6	.8	67.7	5.7	7.8
From Category	5	.4	1.3	.7	34.4	312.9	18.1	7.2	2.8	15.9	14.6
	6	-	.2	-	4.6	9.6	27.1	3.4	17.8	6.2	17.1
	7	-	.2	.1	3.4	7.5	4.0	18.8	1.2	3.2	8.1
	8	-	4.2	10.7	25.0	30.3	11.6	4.2	36.0	3.5	4.6
	9	.2	1.8	8.1	5.7	20.5	7.1	3.9	.4	14.9	4.8
	10	.1	1.3	.4	9.0	12.9	13.0	7.6	3.1	14.2	66.4

SOURCE: Flanders, 1965: 76.

To be more specific about the three alternatives for comparing the two matrices in Tables 4 and 5, we can define Γ_1, Γ_2, and Γ_3 as follows:

$$\Gamma_1 = \sum_{i \neq j} a_{ij}\, b_{ij};$$

$$\Gamma_2 = \sum_{i} a_{ii}\, b_{ii};$$

$$\Gamma_3 = \sum_{i,j} a_{ij}\, b_{ij}.$$

Obviously, Γ_1 is the same index used in the Hare-Bales example and requires off-diagonal entries only. The index Γ_2, on the other hand, considers *only* the diagonals, and in fact, could be treated without the extended matrix notation as a simple raw cross-product statistic. Finally, Γ_3 uses a sum over all pairs of row and column values, including the diagonals, and thus could be defined merely as the sum of Γ_1 and Γ_2. In short, either Γ_1 or Γ_2 may be evaluated separately or together as the sum $\Gamma_3 = \Gamma_1 + \Gamma_2$.

Although complete permutation distributions would be desirable, the computational burden this implies is enormous. Thus, Edgington's (1969) strategy for constructing an approximate permutation distribution will be adopted. In particular, if we are interested in, say, an upper one-tailed test of Γ (i.e., for Γ_1, Γ_2, or Γ_3), a random sample of size M, e.g., M = 999, is drawn with replacement from the complete permutation distribution.[4] The significance level of Γ is then defined as that proportion of M + 1 Γ values, i.e., the sample of 999 and the single observed statistic, that are as large as or larger than the observed statistic (a lower-tail significance test could be carried out in an analogous fashion). Such an approximate test is very accurate for a sample size as large as 999, and one loses surprisingly little over the unpleasant alternative of enumerating the complete distribution. For instance, Edgington (1969) shows that with probability .99, an obtained statistic judged significant at the .05 level by using the complete permutation distribution will be given a probability no larger than .066 by using an approximate distribution based on M = 999. For a detailed discussion, the reader is referred to Edgington (1969), Hope (1968), and Cliff and Ord (1973).

Based on a sample size of 999, Table 6 presents the cumulative frequency distributions for the three Γ statistics. Since the observed values, $\Gamma_1 = 10{,}236.06$, $\Gamma_2 = 122{,}403.92$ and $\Gamma_3 = 132{,}639.98$, all have upper one-tailed significance levels less than, say, .06, the two matrices apparently do reflect similar patterns. This tends to disconfirm our original conjecture that the classes taught by the "direct" and "indirect" teachers differed with respect to their verbal behavior.

EXAMPLE 3

In this last example, the structure matrix will be derived from a single variable X and the comparison method will be reinter-

TABLE 6
Approximate Permutation Cumulative Distributions for Γ_1, Γ_2, Γ_3 Based on the Flanders Data: M = 999

Cumulative Frequency	Cumulative Proportion	Γ_1	Γ_2	Γ_3
1	.001	661.56	1,669.33	2,647.18
5	.005	763.78	2,281.21	3.313.29
10	.010	854.05	2,661.08	3,740.56
50	.050	1,125.27	3,742.09	5,272.71
100	.100	1,308.96	4,649.16	6,467.80
200	.200	1,567.42	7,609.52	9,923.05
300	.300	1,782.44	10,141.08	12,171.80
400	.400	2,036.63	11,607.42	14,152.40
500	.501	2,262.21	13,419.54	16,491.02
600	.601	2,549.68	16,211.10	18,855.95
700	.701	2,854.73	18,492.39	21,339.30
800	.801	3,267.23	25,305.89	28,217.40
900	.901	3,915.02	121,071.23	122,615.87
950	.951	4,636.32	122,499.95	125,599.96
990	.991	7,248.58	123,489.08	127,254.37
995	.996	8,962.18	123,679.72	131,224.93
999	1.000	9,956.61	124,203.45	133,565.67

preted as a strategy for assessing the relationship between the data provided by $\{a_{ij}\}$ and n observations on X, say $X_1, \ldots X_n$. To illustrate this latter use of a Γ statistic, assume that a matrix $\{a_{ij}\}$ is again available on n individuals, but instead of a second matrix, suppose some outside variable is measured on the n individuals. The task is now to relate this information to the notion of "proximity" as defined by $\{a_{ij}\}$.

TABLE 7
$\{a_{ij}\}$ Matrix: A 1 Represents a Pair of Individuals Who Interact Socially

Person	Person								
	1	2	3	4	5	6	7	8	9
1	X	1	1	1	1	0	0	0	0
2	1	X	1	1	0	0	0	0	0
3	1	1	X	1	1	0	0	0	0
4	1	1	1	X	1	0	0	0	0
5	1	0	1	1	X	0	1	0	0
6	0	0	0	0	0	X	1	1	1
7	0	0	0	0	1	1	X	1	1
8	0	0	0	0	0	1	1	X	1
9	0	0	0	0	0	1	1	1	X

For a concrete example, Table 7 presents some well-known data on nine men who work in a "factory" (see Winsborough et al., 1963). Specifically, a 1 in Table 7 corresponds to a pair of individuals who played games together, and a 0 to the lack of this type of social interaction. The single outside variable is productivity of these workers and is given in Table 8. Thus the inference task is one of relating productivity and social interaction, or, stated in another way, do individuals who are similar in output also interact socially and conversely?

To place this comparison task in our context, a second $n \times n$ matrix $\{b_{ij}\}$ containing the absolute values of the differences between the output values for each pair of individuals (see Table 9) is defined. A correspondence between the entries in $\{b_{ij}\}$ and $\{a_{ij}\}$ as measured by a small value for Γ_1 would now suggest that men who interact socially are similar in productivity as well. Using the approximate permutation distribution in Table 10 based on M = 999, the observed value of $\Gamma_1 = 4100$ provides a

TABLE 8
Productivity Values for the Nine Individuals of Table 7

Individual	Output
1	724
2	860
3	823
4	757
5	804
6	822
7	651
8	710
9	416

lower tail significance level of .113. Thus, an association between productivity and social interaction is not confirmed, at least with respect to the rather lenient significance level of .10.

The procedure for comparing an outside variable to data on social interaction can be generalized substantially, and therefore, the work of Winsborough and associates (1963), who approached this same type of inference task with what is called the contiguity ratio, can be extended (Cliff and Ord, 1973). For instance, the matrix $\{a_{ij}\}$ does not have to contain 0—1 values, and also, more than a single outside variable could be considered at one time. Some distance function, such as Mahalanobis D^2, would be obtained between pairs of vectors where the vectors themselves contain variables collected on the n subjects. The more general distance measures merely take the place of the previous absolute differences on a single variable.

As an alternative application of these same principles, it is also possible to extend several analysis strategies proposed in the literature that are based on naturally occurring geometric models.

TABLE 9
Matrix $\{b_{ij}\}$ Derived by Taking the Absolute Values of the Differences in Productivity Scores Reported in Table 8

	1	2	3	4	5	6	7	8	9
1	X	136	99	33	80	98	73	14	308
2	136	X	37	103	56	38	209	150	444
3	99	37	X	66	19	1	172	113	407
4	33	103	66	X	47	65	106	47	341
5	80	56	19	47	X	18	153	94	388
6	98	38	1	65	18	X	171	112	406
7	73	209	172	106	153	171	X	59	235
8	14	150	113	47	94	112	59	X	294
9	308	444	407	341	388	406	235	294	X

Most of the appropriate references relate to the organization of a group of objects, e.g., people, census tracts, and so on, derived from some notion of geographic or spatial contiguity. Within this context, one of the major analysis tasks concerns the association between spatial contiguity and some other variable or variables measured on these same objects. For instance, in social psychology, Campbell and associates (1966) developed an index of seating aggregation and an associated significance-testing strategy for determining whether the observed black-white seating adjacencies within a classroom might be considered random. The geometric model in this case is defined by the occupied seats within a classroom, or, more specifically, by the spatial location of the students in a two-dimensional plane. The outside variable of interest is dichotomous, i.e., black or white, and the inference task is one of determining whether the spatial positioning of blacks and whites indicates aggregation, e.g., whether blacks sit with blacks and whites sit with whites. For a further discussion, see Freeman (1978).

TABLE 10
Approximate Cumulative Frequency Distribution of Γ_1 for Comparing Tables 7 and 9

Cumulative Frequency	Γ_1
1	2770
5	2996
10	3078
50	3884
100	4074
200	4272
300	4410
400	4518
500	4620
600	4724
700	4864
800	4992
900	5124
950	5224
990	5374
995	5430
999	5534

The confirmatory approach developed above includes the approach of Campbell and associates (1966) as a special case. The matrix $\{a_{ij}\}$ would contain the measures of spatial distance obtained from the observed seating pattern. Although very general measures of distance could be used, Campbell and associates (1966) consider a simple index defined (in our notation) as follows:

$$a_{ij} = \begin{cases} 1 & \text{if persons i and j are seated adjacently within a single row;} \\ 0 & \text{otherwise.} \end{cases}$$

The second matrix b_{ij} would be obtained from the outside variable of race:

$$b_{ij} = \begin{cases} 1 & \text{if persons i and j are both black or both white;} \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, the cross-product statistic Γ_1 is the number of same-race adjacencies observed in the given seating pattern and evidence for aggregation is indicated by a large value.

The concepts presented in this final example are appropriate in a number of situations of interest in social psychology that have been developed in slightly different directions. Suppose the objects are now census tracts and the task is to relate an outside variable to contiguity of the tracts defined geographically (see Cliff and Ord, 1971; Geary, 1954; Royaltey et al., 1975). The same analysis procedure is appropriate with geographical distance defining $\{a_{ij}\}$ and the measure $\{b_{ij}\}$ defined by some possibly more complicated function of a vector of variables available for each object. Also, the analysis strategy for observations connected through some general network structure, e.g., influence, can be approached in the same way and related to outside data, e.g., to socioeconomic status. (See Laumann et al., 1974; Burt, 1977a, 1977b).

EXACT MOMENTS FOR THE Γ STATISTICS

Although the significance testing of a Γ statistic was approached in the previous sections by either an exact or an approximate permutation test, several other alternatives could have been followed. For instance, it may be possible in some cases to use a normal approximation for the complete permutation distribution, e.g., in using Γ_2, see Wald and Wolfowitz, 1944; and for Γ_1 see Abe, 1969, or to generate a bound on a significance level based on a Chebyshev- or Cantelli-type inequality. Since these options require the exact mean and variance of the various statistics over all n! equally likely reorderings of the matrix a_{ij} against the matrix b_{ij} (or the converse), the first two moments for the three indices are given in Appendix A. It should be pointed out that these formulas are special cases of a set of expressions derived by Graves and Whinston (1970a, 1970b) for a very different purpose.

SUMMARY

This article provides a nonparametric approach to the analysis of social interaction data based on the hypothesis of a random relabeling of the rows (and simultaneously, the columns) of one matrix with respect to another. At present, these comparison problems when they arise in the literature are usually handled in a purely descriptive manner. As an example, the work of Burt (1976, 1977a, 1977b) deals explicitly with multiple matrices (or networks) of social distance among "actors," even though each of the matrices, defined by different sociometric questions, is analyzed separately. Here, an attempt is made to define structurally equivalent individuals in each of the various networks, based on some hierarchical cluster analysis. Burt's work could be easily augmented by the procedures discussed in this article, e.g., a formal test could be made of a hypothesis that individuals who are identified as structurally equivalent in one network are also "close" in a second. The first matrix $A = \{a_{ij}\}$ could specify the equivalence hypothesis by defining $a_{ij} = 1$ if i and j are assumed structurally equivalent and 0 otherwise; B could contain proximity information among the same actors but from a data source distinct from that used to generate A. More generally, the original proximity matrices could be compared directly without the imposition of an intermediate clustering solution.

In a similar way, the concerns of Laumann and others (Laumann et al., 1974; Laumann et al., 1977) regarding the use of significance tests in the comparisons of networks can be answered. The approach to matrix comparisons that has been presented will provide a legitimate inference base for the between-network correlations that may eventually be used in constructing path analytic models. The integrity of the matrices in the row-column permutation strategy is preserved and the degree of freedom issue of concern Proctor (1979) can be avoided entirely. In fact, the possible utility of our matrix comparison approach has been raised briefly in the blockmodel context by Arabie et al. (1978) based on some earlier work by the authors (Hubert and Baker, 1978).

We hope that this study will stimulate a greater concern with these formal inference possibilities in the study of social interaction, and more specifically in network analysis. It is recognized that the method is obviously not a complete solution to all the comparison tasks that a researcher may wish to perform. For instance, there is no option at present to partial out the effect of one network on a second and no nonnull distributional theory has been proposed for our discriptive cross-product measure. In general, log linear solution, when appropriate, may be the most obvious way to proceed given the extensive information that can be extracted from such an analysis. Nevertheless, the generality of the permutation argument is impressive and can be used even when the original proximity data bear no discernible relationship to the frequency counts that are typically required for the use of a log linear model.

Appendix A

EXACT MOMENTS OF THE Γ STATISTIC UNDER PERMUTATION

Γ_1:

$$E(\Gamma_1) = \{1/(n(N-1))\} \sum_{i,j} a'_{ij} \sum_{i,j} b'_{ij};$$

$$V(\Gamma_1) = -\{1/(n(n-1))\}^2 B_1 + \{1/(n(n-1))\} \{B_2 + B_3\}$$
$$+ \{1/(n(n-1)(n-2))\} \{B_4 + 2B_5 + B_6\}$$
$$+ \{1/(n(n-1)(n-2)(n-3))\} B_7,$$

where

$$a'_{ij} = \begin{cases} a_{ij} & \text{if } i \neq j \\ 0 & \text{if } i = j, \end{cases}$$

$$b'_{ij} = \begin{cases} b_{ij} & \text{if } i \neq j \\ 0 & \text{if } i = j, \end{cases}$$

and

$$B_1 = \left(\sum_{i,j} a'_{ij}\right)^2 \left(\sum_{i,j} b'_{ij}\right)^2;$$

$$B_2 = \left(\sum_{i,j} a'^2_{ij}\right) \left(\sum_{i,j} b'^2_{ij}\right);$$

$$B_3 = \sum_i \left(\sum_j a'_{ij} a'_{ji}\right) \sum_i \left(\sum_j b'_{ij} b'_{ji}\right);$$

$$B_4 = \sum_i \left[\left(\sum_j a'_{ij}\right)^2 - \sum_j a'^2_{ij}\right] \sum_i \left[\left(\sum_j b'_{ij}\right)^2 - \sum_j b'^2_{ij}\right];$$

$$B_5 = \sum_i \left[\left(\sum_j a'_{ij}\right)\left(\sum_j a'_{ji}\right) - \sum_j a'_{ij}\, a'_{ji}\right] \times$$

$$\sum_i \left[\left(\sum_j b'_{ij}\right)\left(\sum_j b'_{ji}\right) - \sum_j b'_{ij}\, b'_{ji}\right];$$

$$B_6 = \sum_i \left[\left(\sum_j a'_{ji}\right)^2 - \sum_j a'^2_{ji}\right] \sum_i \left[\left(\sum_j b'_{ji}\right)^2 - \sum_j b'^2_{ji}\right];$$

$$B_7 = \left[\left(\sum_{i,j} a'_{ij}\right)^2 - \sum_j \left(\sum_i a'_{ij}\right)^2\right.$$

$$-2 \sum_i \left(\sum_j a'_{ij}\right)\left(\sum_j a'_{ji}\right) - \left(\sum_i \sum_j a'_{ij}\right)^2$$

$$\left. + \sum_i \left(\sum_j a'_{ij}\, a'_{ji}\right) + \sum_{i,j} a'^2_{ij}\right] \times$$

$$\left[\left(\sum_{i,j} b'_{ij}\right)^2 - \sum_j \left(\sum_i b'_{ij}\right)^2\right.$$

$$-2 \sum_i \left(\sum_j b'_{ij}\right)\left(\sum_j b'_{ji}\right) - \sum_i \left(\sum_j b'_{ij}\right)^2$$

$$\left. + \sum_i \left(\sum_j b'_{ij}\, b'_{ji}\right) + \sum_{i,j} b'^2_{ij}\right]$$

Γ_2:

If $d_{ij} = a_{ii}\, b_{jj}$, $1 \leqslant i,\ j \leqslant n$,

then,

$$E(\Gamma_2) = \{1/n\} \sum_{i,j} d_{ij};$$

$$V(\Gamma_2) = \{1/(n^2(n-1)\}\ A_1 +$$

$$\{-1/n(n-1))\}\ \{A_2 + A_3\} +$$

$$\{1/(n-1)\}\ A_4,$$

where

$$A_1 = \Big(\sum_{i,j} d_{ij}\Big)^2;$$

$$A_2 = \sum_i \Big(\sum_j d_{ij}\Big)^2;$$

$$A_3 = \sum_j \Big(\sum_i d_{ij}\Big)^2;$$

$$A_4 = \sum_{i,j} d_{ij}^2.$$

Γ_3:

Since $\Gamma_3 = \Gamma_1 + \Gamma_2$,

$$E(\Gamma_3) = E(\Gamma_1) + E(\Gamma_2),$$

$$V(\Gamma_3) = V(\Gamma_1) + V(\Gamma_2) + 2\,\mathrm{Cov}(\Gamma_1, \Gamma_2).$$

Thus, only the covariance term needs a new formula:

$$\begin{aligned}\mathrm{Cov}(\Gamma_1, \Gamma_2) = {} & \left\{-1/(n^2(n-1))\right\} C_1 + \\ & \left\{1/(n(n-1))\right\} C_2 + \\ & \left\{1/(n(n-1)(n-2))\right\} C_3,\end{aligned}$$

where

$$C_1 = \Big(\sum_{i,j} d_{ij}\Big)\Big(\sum_{i,j} a'_{ij}\Big)\Big(\sum_{i,j} b'_{ij}\Big);$$

$$C_2 = \sum_{i,j} d_{ij} \sum_{j'} a'_{ij'} \sum_{j'} b'_{jj'} + \sum_{j'} a'_{j'i} \sum_{j'} b'_{j'j};$$

$$\begin{aligned}C_3 = \sum_{i,j} d_{ij} & \left\{\sum_{i'j'} a'_{i'j'} - \sum_{j'} a'_{ij'} - \sum_{i'} a'_{i'i}\right\} \times \\ & \left\{\sum_{i'j'} b'_{i'j'} - \sum_{j'} b'_{ij'} - \sum_{i'} b'_{i'i}\right\}.\end{aligned}$$

Before these moments are actually used by a researcher in an approximate significance test, or for that matter, before using an exact or approximate permutation distribution, it may be desirable to "normalize" the diagonal and off-diagonal elements differentially. The concern, however, appears to be more substantive than statistical, since all of the various procedures can be used under any appropriate transformation of the matrix entries.

NOTES

1. A variety of such classification schemes, particularly appropriate in evaluating classroom interaction, are summarized periodically in the series *Mirrors for Behavior* (Simon and Boyer, 1976 and later).

2. In our context, the cross-product statistic is equivalent to the use of several other indices of correspondence, e.g.,

$$\sum_{i \neq j} (a_{ij} - b_{ij})^2$$

or the Pearson product-moment correlation between the entries in $\{a_{ij}\}$ and $\{b_{ij}\}$. This latter statistic could be used as a final descriptive measure for indexing the extent to which two matrices are similar. It should be noted that our use of the term "cross-product" does not infer any use related to the odds-ratio that is common in contingency table analysis.

3. The strategy of treating diagonal and off-diagonal entries separately is similar to the approach used by Goodman (1972) in fitting social mobility tables.

4. The Quadratic Assignment computer program (Baker et al., 1977) for carrying out this type of analysis can be obtained from User Services, Madison Academic Computing Center, 1210 W. Dayton St., University of Wisconsin—Madison, Madison, WI 53706 at a nominal cost.

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