分析学

$BV([0,1]) = C([0,1])'$ 测度视为对偶空间.

Ring, Algebra. Generated rings (algebras)

Ring $R \subset 2^{\Omega}$ i) $A \cup B$ ii) $A-B$ Algebra: R: ring, $\Omega \subset R$

σ-Ring $\mathcal{R} \subset 2^{\Omega}$ i) 可数并 ii) $A-B$

monotone class $M \subset 2^{\Omega}$ i) $A_n \uparrow \subset M \Rightarrow \bigcup_{n=1}^{\infty} A_n \in M$ ii) $A_n \downarrow \subset M \Rightarrow \bigcap_{n=1}^{\infty} A_n \in M$

monotone class generated by $\mathcal{F}$ $M(\mathcal{F})$: i) $\mathcal{F} \subset M(\mathcal{F})$ ii) $\mathcal{F} \subset M_1 \leftarrow$ monotone class $\Rightarrow M(\mathcal{F}) \subset M_1$

Thm Ring R. $M(R) = \Sigma(R)$ 由R生成的单调类 = 由R生成的σ-代数

先证明$M(R)$为环: 记 $K(A) = \{B \in 2^{\Omega} \mid A \cap B \quad A \cup B \quad A-B \quad B-A \in M(R)\}$

下证$K(A)$为单调类: 若 $\{A_n\}_{n=1}^{\infty} \subset K(A)$ $A_n \cup A \in M(R)$ $A_n \cap A \in M(R)$ $A_n - A \in M(R)$ $A - A_n \in M(R)$

$$\left(\bigcup_{n=1}^{\infty} A_n\right) \cup A = \bigcup_{n=1}^{\infty} (A_n \cup A) \in M(R)$$
$$\left(\bigcup_{n=1}^{\infty} A_n\right) \cap A = \bigcup_{n=1}^{\infty} (A_n \cap A) \in M(R)$$
$$\left(\bigcup_{n=1}^{\infty} A_n\right) - A = \bigcup_{n=1}^{\infty} (A_n - A) \in M(R)$$
$$A - \left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} (A - A_n) \in M(R)$$

$\Rightarrow \bigcup_{n=1}^{\infty} A_n \subset K(A)$

$$\left(\bigcap_{n=1}^{\infty} A_n\right) \cup A = \bigcap_{n=1}^{\infty} (A_n \cup A) \in M(R)$$
$$\left(\bigcap_{n=1}^{\infty} A_n\right) \cap A = \bigcap_{n=1}^{\infty} (A_n \cap A) \in M(R)$$
$$\left(\bigcap_{n=1}^{\infty} A_n\right) - A = \bigcap_{n=1}^{\infty} (A_n - A) \in M(R)$$
$$A - \left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} (A - A_n) \in M(R)$$

$\Rightarrow \bigcap_{n=1}^{\infty} A_n \in K(A)$

$\Rightarrow K(A)$为单调类

$\forall A \in R$. $R \subset K(A) \Rightarrow M(R) \subset K(A)$.

$\forall B \in M(R) \subset K(A)$. 有 $A \in K(B) \Rightarrow R \subset K(B) \Rightarrow M(R) \subset K(B) \Rightarrow \forall C, D \in M(R) \subset K(C) \subset K(D)$

$C \cup D \in M(R)$, $C \cap D \in M(R)$.

故$M(R)$为环

再证明$M(R)$为σ-环 $\forall \{A_n\}_n \subset M(R)$. 定义 $B_n = \bigcup_{j=1}^{n} A_j$ 则 $B_n \uparrow$, $\{B_n\} \subset M(R)$.

$\Rightarrow \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \in M(R)$

故 $M(R) \supset \Sigma(R)$. 而 $R \subset \Sigma(R)$ $M(R) \subset M(\Sigma(R)) = \Sigma(R)$. 故 $M(R) = \Sigma(R)$

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Measures on a ring

(X, R, μ) $\quad E_n \in R \quad \bigcup_n E_n \in R \qquad \mu(\bigcup_n E_n) = \sum_n \mu(E_n) \quad \mu(\emptyset) = 0$

(X, R_σ, μ) measure space

E is of finite measure : $\mu(E) < \infty$

E is of σ-finite measure : $\exists \{E_n\} \subset R_\sigma \quad \mu(E_n) < \infty \ (\forall n). \quad E = \bigcup_n E_n$

μ: finite $\qquad \forall E \in R_\sigma. \quad \mu(E) < \infty$

μ: σ-finite $\qquad \forall E \in R_\sigma. \quad E$ σ-finite

complete measure :

零测集的子集也在R_σ中 测度也为0

正测度分两类：全空间为∞ / 1 (有限)

Probability measure on a σ-algebra $\quad \mu \geq 0. \quad \mu(X) = 1.$

Property of positive measures (on a ring)

1. $E \subset F \Rightarrow \mu(E) \leq \mu(F) \qquad F = E \cup (F - E)$

$\mu(F - E) = \mu(F) - \mu(E)$

2. subadditivity $\quad \mu(\bigcup_n E_n) \leq \sum_n \mu(E_n)$

令 $F_1 = E_1, \ F_n = E_n - \bigcup_{j=1}^{n-1} E_j \quad (n \geq 2)$

则$\{F_n\}_n$ disjoint , $\bigcup_n F_n = \bigcup_n E_n \quad \mu(\bigcup_n E_n) = \mu(\bigcup_n F_n) = \sum_n \mu(F_n) \leq \sum_n \mu(E_n)$

3. If $\bigcup_{n=1}^{\infty} E_n \subset E$ and $\{E_n\}$ are mutually disjoint,

then $\sum_j \mu(E_n) \leq \mu(E)$

4. Continuity $\quad \{E_n\} \subset R, E_n \uparrow . \ \bigcup_n E_n \in R \Rightarrow \mu(\bigcup_n E_n) = \lim_n \mu(E_n)$

$\{F_n\} \subset R, \ F_n \downarrow \quad \bigcap F_n \in R, \ \mu(F_1) < \infty \Rightarrow \mu(\bigcap F_n) = \lim_n \mu(F_n)$

rmk: 若μ仅是σ-ring上的有限可加的，且有"4"中的任意一式成立，则也可得可列可加性，故μ是一个测度

例: 1. Dirac's measure $\quad \delta_{x_0}(E) = \begin{cases} 0 & x_0 \notin E \\ 1 & x_0 \in E \end{cases}$

2. $f: X \to [0, \infty]$ Define $\mu_f(E) = \sup\{\sum_{x \in F} f(x), F \subset E, F \text{ finite}\}$. 验证可列可加性!

2.1 $f(x_0) = 1, \ f(x) = 0 \ (x \neq x_0)$ 则 $\mu_f = \delta_{x_0}$

2.2 $f \equiv 1. \qquad \mu_f = \# \quad X = \mathbb{N}$.

3. Lebesgue's measure

$(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d), m_d)$ is a complete measure space

$(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), m_d)$ is not complete

3.1 If $f \geq 0$ $f \in L^1_{loc}(\mathbb{R})$ then

$\mu_f(E) := \int_E f\,dm$ $\quad \mu(\bigcup_{n=1}^{\infty} E_n) = \int_{\bigcup_{n=1}^{\infty} E_n} f\,dm = \sum_n \int_{E_n} f\,dm$

3.2 $f \in L^1(\mathbb{R})$. μ_f is a finite (signed) measure

$|\mu_f(E)| = |\int_E f\,dm| \leq \int_{\mathbb{R}} |f|\,dm < \infty.$

3.3 Lebesgue - Stieltjes measures on $\mathbb{R}$

$F: \mathbb{R} \to \mathbb{R}$ $\quad F \uparrow$ and left-continuous

$\mu_F([a,b)) = F(b) - F(a) \geq 0$ $\quad (\mu_F = dF)$

e.g. $H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$ $\quad \mu_H = \delta_0$

4. X: uncountable set

$\mathcal{A}_\sigma$: countable / co-countable $\quad \mu(E) = \begin{cases} 0, & E \text{ countable} \\ 1, & E \text{ co-countable} \end{cases}$.

5. X: infinite set $\quad \mu(E) = \begin{cases} 0, & E\text{有限} \\ \infty, & E\text{无限} \end{cases}$.

仅有限可加. 但不是可列可加的.

rmk $(X, \mathcal{A}_\sigma, \mu)$ $\quad \mathcal{N} = \{E \in \mathcal{A}_\sigma \mid \mu(E) = 0\}$

完备化

$\overline{\mathcal{A}_\sigma} = \{A \cup F \in 2^X \mid A \in \mathcal{A}_\sigma$. F为$\mathcal{N}$某元素的子集$\}$

$\bar{\mu}(A \cup F) := \mu(A)$ $\quad$ 则 $(X, \overline{\mathcal{A}_\sigma}, \bar{\mu})$ is complete

I.5 Outer measures $\mu^*: 2^X \to [0, \infty]$ is called an outermeasure on X

If 1° $\mu^*(\phi) = 0$. $\quad$ 2° $E \subset F \Rightarrow \mu^*(E) \leq \mu^*(F)$

3° $\mu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$ 次可列可加

Prop 5 $\mathcal{E} \subset 2^X$. $\rho: \mathcal{E} \to [0, \infty]$ s.t. $\phi, X \in \mathcal{E}$

For $\forall E \in 2^X$ define $\mu^*(E) = \inf\{\sum_{n=1}^{\infty} \rho(E_n) \mid \{E_n\} \subset \mathcal{E}, E \subset \bigcup_n E_n\}$

则 μ^* 是一个外测度.

Pf: $\mu^*(\bigcup E_n) \leq \sum_{n,j} \rho(E_{nj}) = \sum_{n=1}^{\infty}(\sum_{j=1}^{\infty} \rho(E_{nj})) \leq \sum_{n=1}^{\infty}(\mu^*(E_n) + 2^{-n}\varepsilon)$

$= \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon$ $\quad$ 令$\varepsilon \to 0$即得

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记号 $\mathcal{P}(X) = 2^X$

Thm (Carathéodory)

let $\mathcal{M}^* = \{A \in \mathcal{P}(X) \mid \mu^*(E) = \mu^*(E\cap A) + \mu^*(E\cap A^c)\ \forall E\}$,

Then $(X, \mathcal{M}^*, \mu^*)$ is a complete measure space i.e.

$\mathcal{M}^*$ is a σ-algebra $\mu^*|_{\mathcal{M}^*}$ is a complet measure.

Pf: σ-algebra: 1. $A\in\mathcal{M}^* \Rightarrow A^c\in\mathcal{M}^*$ ✓ 且 $X\in\mathcal{M}^*$ 显然

(先证有限并) 2. $A, B\in\mathcal{M}^*$, $A\cup B\in\mathcal{M}^*$ ✓

3. $\bigcup_{j=1}^\infty A_j$ 见下 取 $B_n=\bigcup_{j=1}^n A_j$. 再取 $C_n = B_n - B_{n-1}$ $\bigcup_{j=1}^\infty A_j = \bigcup_{n=1}^\infty C_n$

measure: 先证有限可加 1. $A, B\in\mathcal{M}^*$, $A\cap B=\emptyset$

$$\mu^*(A\cup B) = \mu^*((A\cup B)\cap A) + \mu^*((A\cup B)\cap A^c) = \mu^*(A)+\mu^*(B)$$

2. 可数可加 若 $\{A_j\}$ 两两不交. 定义 $B_n = \bigcup_{j=1}^n A_j$ $B=\bigcup_{j=1}^\infty A_j = \bigcup_{n=1}^\infty B_n$

$B_n \subset B$, $B_n^c \supset B^c$

$\forall n$. $\mu^*(E) = \mu^*(E\cap B_n) + \mu^*(E\cap B_n^c)$

$\geq \mu^*(\bigcup_{j=1}^n (E\cap A_j)) + \mu^*(E\cap B^c)$

$= \sum_{j=1}^n \mu^*(E\cap A_j) + \mu^*(E\cap B^c)$ 对 $n\to\infty$.

$\mu^*(E) \geq \sum_{j=1}^\infty \mu^*(E\cap A_j) + \mu^*(E\cap B^c) \geq \mu^*(E\cap B) + \mu^*(E\cap B^c)$

$\Rightarrow \mu^*(E) = \mu^*(E\cap B)+\mu^*(E\cap B^c) \Rightarrow B\in\mathcal{M}^*$.

且 取 $E=B$. 则 $\mu^*(\bigcup_{j=1}^\infty A_j) = \sum_{j=1}^\infty \mu^*(A_j)$ 可数可加

completeness $\forall$ $\mu^*(A)=0$. $\forall E$. $\mu^*(E) \leq \mu^*(E\cap A) + \mu^*(E\cap A^c)$

$\leq \mu^*(E\cap A^c) \leq \mu^*(E)$

$\Rightarrow$ 都取等 $\Rightarrow A\in\mathcal{M}^*$.

Hahn's extension Thm (from algebra to σ-algebra)

Assume $\mathcal{A}\subset\mathcal{P}(X)$ is an algebra and $\mu:\mathcal{A}\to[0,\infty]$ satisfies (也称之为预测度)

1° $\mu(\emptyset)=0$ 2° $\mu(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty\mu(A_n)$ $\forall\{A_n\}$ 不交 且 A_n, $\bigcup A_n$ 均在 $\mathcal{A}$ 中.

By Caratheodory (μ induced an outer measure μ^* on $\mathcal{P}(X)$) $(X,\mathcal{M}^*,\mu^*)$ is a measure space

且 1° $\mu^*|_{\mathcal{A}}=\mu$ 2° $\mathcal{A}\subset\mathcal{M}^*$ 3° $(X, \sigma(\mathcal{A}), \mu^*)$ is a measure space

由 $\sigma(\mathcal{A})\subset\mathcal{M}^*$

μ^* 对 $\mathcal{P}(X)$ 中所有元素均有定义 不仅仅定义在 $\mathcal{M}^*$ 上

4° 若 $(X, \sigma(\mathcal{A}), \bar{\mu})$ 是 $(\mathcal{A}, \mu)$ 的另一个扩张，则 $\bar{\mu} = \mu^*|_{\sigma(\mathcal{A})}$

记 $\mathcal{M}' = \{A \subset X \mid \mu^*(A) = \bar{\mu}(A),\ \mu^*(A \cap B) = \bar{\mu}(A \cap B)\ \forall B \in \mathcal{A}\}$，证明 $\mathcal{M}'$ 为单调类.

rmk 1. $(X, \sigma(\mathcal{A}), \mu^*|_{\sigma(\mathcal{A})})$ is not necessarily complete

2. Is $(X, \mathcal{M}^*, \mu^*)$ the unique complete measure? (若 μ 是 σ-有限的则 μ^* 唯一)

Total variation (of a signed measure)

$\mu: \mathcal{F} \to [-\infty, +\infty]$ ($\mathcal{F}$: σ-algebra)

For $\forall E \in \mathcal{F}$, define

$|\mu|(E) = \sup\{\sum_{j=1}^{n} |\mu(E_j)| \mid E_j \subset E,\ E_j$ 两两不交$\}$

先假设 $\sup_{E \in \mathcal{F}} |\mu(E)| < \infty$. 则 $\sum_{j=1}^{n} |\mu(E_j)| = \sum^+ \mu(E_j) - \sum^- \mu(E_j) = \mu(\bigcup^+ E_j) - \mu(\bigcup^- E_j)$

$\leq 2 \sup_{\mathcal{F}} |\mu(E)| < \infty$

Proof of Hahn's extension thm 首先，μ^* 是外测度（有可列次可加）By Prop 5.

1°: $\forall A \in \mathcal{A}$ $\mu^*(A) = \inf\{\sum_{j=1}^{\infty} \mu(A_j) \mid A_j \in \mathcal{A},\ A \subset \bigcup_{j=1}^{\infty} A_j\} = \inf S$

$\mu(A) \in S \Rightarrow \mu^*(A) \leq \mu(A)$

且 $\forall A \subset \bigcup_{j=1}^{\infty} A_j$，$A_j \in \mathcal{A}$. $\mu(A) \leq \sum_{j=1}^{\infty} \mu(A_j)$ 再取inf 得 $\mu(A) \leq \mu^*(A)$

故 $\mu(A) = \mu^*(A)$ $\forall A \in \mathcal{A}$

2°. $\forall A \in \mathcal{A}$ $\forall \varepsilon > 0$ $\exists E \subset \bigcup A_j$ …

对 $\mu^*(E) < \infty$. 见右

对 $\mu^*(E) = \infty$.

$\Rightarrow \mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ 也成立.

$\mu^*(E \cap A) + \mu^*(E \cap A^c)$

$\leq \mu^*(\bigcup_{j=1}^{\infty}(A_j \cap A)) + \mu^*(\bigcup_{j=1}^{\infty}(A_j \cap A^c))$

$\leq \sum_{j=1}^{\infty} \mu^*(A_j \cap A) + \sum_{j=1}^{\infty} \mu^*(A_j \cap A^c)$

$= \sum_{j=1}^{\infty} \mu(A_j \cap A) + \sum_{j=1}^{\infty} (A_j \cap A^c) = \sum_{j=1}^{\infty} \mu(A_j) \leq \mu^*(E) + \varepsilon$.

3° 由 $\mathcal{A} \subset \mathcal{M}^*$. $\mathcal{M}^*$ 为 σ-algebra 可得.

(未用到 $E \in \sigma(\mathcal{A})$ 这是rmk 2 的证明，不是4°的证明)

4° 首先，对 $\mu^*(E) < \infty$ 有 $\bar{\mu}(E) \leq \mu^*(E)$.

对 $E \subset \bigcup A_j$ $\{A_j\} \subset \mathcal{A}$ 由测度 $\bar{\mu}(E) \leq \sum \bar{\mu}(A_j) = \sum \mu(A_j)$ 取inf 得 $\bar{\mu}(E) \leq \mu^*(E)$

记 $A = \bigcup A_j$. 由测度 $\bar{\mu}(A) = \lim \bar{\mu}(\bigcup_{j=1}^{n} A_j) = \lim \mu(\bigcup_{j=1}^{n} A_j) = \mu^*(A)$

若 $\mu^*(E) < \infty$ 可取 $\mu^*(A) \leq \mu^*(E) + \varepsilon$ 故 $\mu^*(A - E) < \varepsilon$.

$\mu^*(E) \leq \mu^*(A) = \bar{\mu}(A) = \bar{\mu}(E) + \bar{\mu}(A - E) \leq \bar{\mu}(E) + \mu^*(A - E) < \bar{\mu}(E) + \varepsilon \Rightarrow \mu^*(E) \leq \bar{\mu}(E)$

故 $\bar{\mu}(E) = \mu^*(E)$ (对 $\forall \mu^*(E) < \infty$)

若 μ^* 是 σ-有限的，有 $X = \bigcup B_j$ 且 $\mu(B_j) < \infty$ $\forall j$. 不妨两两不交.（否则取 $B_n' = \bigcup_{j=1}^{n} B_j - \bigcup_{j<n} B_j$）

$\bar{\mu}(E) = \sum \bar{\mu}(E \cap B_j) = \sum \mu^*(E \cap B_j) = \mu^*(E)$

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Thm Assume $\mu: \mathcal{F} \to (-\infty, +\infty) = \mathbb{R}$ Then μ is bdd i.e. $\sup_{E \in \mathcal{F}} |\mu(E)| < \infty$

Pf: unbdd set E: $\sup_{\substack{F \subset E \\ F \in \mathcal{F}}} \mu(F) = +\infty$.

若 X is unbdd

$\exists E_1 \in \mathcal{F}$ s.t. $\mu(E_1) > 1$. 则 1. E_1 unbdd
2. E_1 bdd $\Rightarrow E_1^c$ unbdd

2-1 $\exists \overset{\text{unbdd}}{F_1} \subset E_1^c$ s.t. $\mu(F_1) > 1$ 令 F_1 为新的 E_1. $\to \exists E_2 \subset E_1$. $\mu(E_2) > 2$. 继续讨论

2-2 $\forall$ unbdd $F \subset E_1^c$, $\mu(F_1) \le 1$

若可找到 $E_1 \supset E_2 \supset \cdots \supset E_n \supset E_{n+1} \supset \cdots$ 且 $\mu(E_n) \ge n$ 则 $G = \bigcap_{n=1}^{\infty} E_n \in \mathcal{F}$ $\mu(G) = +\infty$ 矛盾

否则. $\exists N$ s.t. $\forall$ unbdd $F \subset E_{N-1} - E_N$ (且 $E_{N-1} - E_N$ unbdd) $\mu(F) \le N$. $\exists F_1 \subset E_{N-1} - E_N$. $\mu(F_1) > N$. $\Rightarrow F_1$ bdd $\Rightarrow E_{N-1} - E_N - F_1^c$ unbdd

$\exists A_1 \subset E_{N-1} - E_N - F_1^c$. $\mu(A_1) > 1$. $F_2 := F_1 \cup A_1$

$\mu(F_2) > N + 1 > N \Rightarrow F_2$ bdd

$E_{N-1} - E_N - F_2^c$ unbdd

$\Rightarrow A_2$,

$\Rightarrow \Rightarrow A_n \Rightarrow \cdots$

$\mu(F_1 \bigcup_{n=1}^{\infty} A_n) = +\infty$

$\in \mathcal{F}$

矛盾

Jordan's decomposition $\mu: \mathcal{F} \to \mathbb{R}$ is a signed measure.

Then $|\mu|: \mathcal{F} \to \mathbb{R}$ is a non-negative measure as well as μ^+, μ^-.

$$\mu^+ = \frac{1}{2}(|\mu| + \mu) \quad \mu^- = \frac{1}{2}(|\mu| - \mu) \quad \sup_{\mathcal{F}} \mu(E) < \infty$$

Pf: 1° let $E, F \in \mathcal{F}$ disjoint. $\forall A_j \subset E \cup F$, $j = 1, 2, \dots, n$ disjoint

$$\sum |\mu(A_j)| \le \sum |\mu(A_j \cap E)| + \sum |\mu(A_j \cap F)| \le |\mu|(E) + |\mu|(F)$$

$$\Rightarrow |\mu|(E \cup F) \le |\mu|(E) + |\mu|(F)$$

$\forall \varepsilon > 0$. $\exists E_j \subset E$ $F_j \subset F$

$$|\mu|(E) - \varepsilon < \sum |\mu(E_j)|$$

$$|\mu|(F) - \varepsilon < \sum |\mu(F_j)|$$

$$|\mu|(E) + |\mu|(F) - 2\varepsilon < \sum |\mu(E_j)| + \sum |\mu(F_j)| \le |\mu|(E \cup F)$$

$$\overset{\varepsilon \to 0}{\Rightarrow} |\mu|(E) + |\mu|(F) \le |\mu|(E \cup F) \Rightarrow |\mu|(E \cup F) = |\mu|(E) + |\mu|(F)$$

2° disjoint $E_n \in \mathcal{F}$ $n = 1, 2, 3, \dots$

$$\forall N \quad |\mu|\left(\bigcup_{n=1}^{\infty} E_n\right) \ge |\mu|\left(\bigcup_{n=1}^{N} E_n\right) = \sum_{n=1}^{N} |\mu|(E_n) \to \sum_{n=1}^{\infty} |\mu|(E_n) \text{ as } N \to \infty.$$

$\forall A_k \subset \bigcup E_n$ $k = 1, 2, \dots, K$ disjoint

$$|\mu|\left(\bigcup_{n=1}^{\infty} E_n\right) \leftarrow \sum_{k=1}^{K} |\mu(A_k)| = \sum_{k=1}^{K} \left|\sum_{n=1}^{\infty} \mu(A_k \cap E_n)\right| \le \sum_{k=1}^{K} \sum_{n=1}^{\infty} |\mu(A_k \cap E_n)| \le \sum_{n=1}^{\infty} |\mu|(E_n)$$

$$\Rightarrow |\mu|\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} |\mu|(E_n)$$

$\because$ $\mu^+(E) = \sup_{F\subset E} \mu(F)$ $\quad$ $\forall F\subset E$ $2\mu(F) = \mu(F)+\mu(E) - \mu(E-F)$

$\leq \mu(E) + |\mu(F)| + |\mu(E-F)|$

$\leq \mu(E) + |\mu|(E) = 2\mu^+(E).$

$\sup_{F\subset E} \mu(F) \leq \mu^+(E)$

$\forall \varepsilon>0.\ \exists E_j \subset E$ $j=1,2,\cdots,n$ disjoint $|\mu|(E) - \varepsilon < \sum_{j=1}^{n} |\mu(E_j)|$

$2\mu^+(E) - \varepsilon = \mu(E) + |\mu|(E) - \varepsilon < \sum_{j=1}^{n} |\mu(E_j)| + \mu(E) - \varepsilon$

$= \Sigma^+ \mu(E_j) + \Sigma^-(E_j) + \mu(\cup^+ E_j) - \mu(\cup^- E_j) = 2\mu(\cup^+ E_j) \leq 2\sup_{F\subset E} \mu(F)$

Hahn's decomposition $(X, \mathcal{F}, \mu)$

Let $\mu : \mathcal{F} \to (-\infty, +\infty]$ be (an extended) signed measure

There exists $P_0 \in \mathcal{F}$ s.t. $\mu(E) \geq 0$, $\forall E \subset P_0$ and

$\mu(E) \leq 0$ $\forall E \subset P_0^c := N_0$

i.e. $\exists$ Positive set P_0 and Negative set N_0 s.t. $X = P_0 \cup N_0$, $P_0 \cap N_0 = \phi$

Moreover, if $X = P_1 \cup N_1$ is another decomposition $P_0 \triangle P_1$ is μ-null set

$N \notin \mathcal{F}$, $\mu(F \cap N) \neq 0$.

注 $\mu^+(E) = \mu(P_0 \cap E)$ $\quad$ $\mu^-(E) = \mu^-(N_0 \cap E)$ $\quad$ $|\mu| = \mu^+ + \mu^-$

Pf: Let $\mathcal{N} = \{N \in \mathcal{F} : N \text{ is negative set}\}$

and consider $\inf_{N\in\mathcal{N}} \mu(N) > -\infty$

Choose $N_n \in \mathcal{N}$, $n = 1, 2, 3, \cdots$ s.t.

$\lim_n \mu(N_n) = \inf \mu(N)$

and let $N_0 = \bigcup_n N_n$

1° N_0 is μ-negative $\quad \forall N \subset N_0$

$\mu(N) = \sum_{n=1}^{\infty} \mu(N \cap (N_n - N_{n-1})) \leq 0.$ since $N_n - N_{n-1}$ is μ-negative

2° $\mu(N_0) = \inf_{N\in\mathcal{N}} \mu(N) = \lim_n \mu(N_n)$

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L_{n-1} is negative

$\mu(N_0) = \lim_n \mu(\bigcup_{j=1}^{n} N_j) = \lim_n \mu(N_n \cup L_{n-1}) = \lim_n (\mu(N_n) + \mu(L_{n-1} - N_n))$

$(L_n = \bigcup_{j=1}^{n} N_j)$ $\qquad \le \lim_n \mu(N_n) = \inf_{N \in \mathcal{N}} \mu(N) \le \mu(N_0)$

3°. $P_0 = N_0^c$ is μ-positive:

Assume $\exists A_0 \subset P_0$ s.t. $\mu(A_0) < 0$. (A_0 is not necessary μ-negative)

Let $\mathcal{P} = \{E \in \mathcal{F} \cap A_0 \mid \mu(E) \le \mu(A_0) < 0\}$.

定义 $\mathcal{P}$ 上偏序关系 "$\le$" $E_1 \le E_2 \iff E_1 \subset E_2$ 且 $\mu(E_1) \le \mu(E_2)$ (iff 二者都取等时取等)

则 $\mathcal{P}$ 中最小元 is μ-negative $\Rightarrow M \in \mathcal{N}$ 又 $M \subset A_0 \subset P_0 = N_0^c$, $M \cup N_0 \in \mathcal{N}$, $\mu(M \cup N_0) < \mu(N_0)$ 矛盾!

若 M 为最小元且非 μ-negative. 则 $\exists M_0 \subset M$ s.t. $\mu(M_0) > 0$.

$\Rightarrow M - M_0 \in \mathcal{P}$, $\mu(M - M_0) = \mu(M) - \mu(M_0) < \mu(M)$ $\Rightarrow M - M_0 < M$ 与 M 为最小元矛盾

$\mathcal{P}$ 中任意全序集有下界:

$\mathcal{P}_0$ 为这样的全序集, 则可找出一列 $\{A_n\} \subset \mathcal{P}_0$ $\mu(A_n) \downarrow \inf \mu(E)$

$\mu(A_n) \ge \mu(A_{n+1})$ 由全序, 必有 $A_n \supset A_{n+1}$

取 $B_0 = \bigcap_{n=1}^{\infty} A_n$ 则 $\mu(B_0) = \lim_n \mu(A_n) = \inf \mu(E)$

若 $B_0 \in \mathcal{P}_0$, 则 B_0 为 $\mathcal{P}_0$ 下界:

若 $B_0 \notin \mathcal{P}_0$

10.9 Def mutually singular

$\mu_1 \perp \mu_2$ $\quad \exists$ decomposition of X $\quad A_1 \cup A_2 = X$ $\quad A_1 \cap A_2 = \emptyset$

s.t. $\forall E \in \mathcal{F}$ $\quad \mu_1(A_2 \cap E) = 0$ $\quad \mu_2(A_1 \cap E) = 0$.

rmk: 1° (uniqueness)

(1) If $\mu = \mu_1 - \mu_2$, $\quad \mu_1, \mu_2 \ge 0$

then $\mu^+ \le \mu_1$, $\mu^- \le \mu_2$

$\because \mu^+(E) = \mu(P_0 \cap E) = \mu_1(P_0 \cap E) - \mu_2(P_0 \cap E) \le \mu_1(P_0 \cap E) \le \mu_1(E)$

(2) If $\mu = \mu_1 - \mu_2$ and $\mu_1 \perp \mu_2$ then $\mu^+ = \mu_1$, $\mu^- = \mu_2$ ✓

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Lebesgue decomposition

$f = f_{AC} + f_{Jump} + f_{singular}$ $\qquad \nu = \nu_a + \nu_s$

μ, ν 是 $(X, \mathcal{F})$ 上两个测度. 称 ν 为 μ-连续的. 若 $\lim\limits_{\mu(E)\to 0} \nu(E) = 0$.

($\Rightarrow$ 若 $\mu(E_0)=0$ 则 $\nu(E_0)=0$). 记为 $\nu \ll \mu$

1° If $\nu \perp \mu$ $\nu \ll \mu$ then $\nu = 0$.

2° $\begin{cases} \nu \perp \mu \Rightarrow \nu^+, \nu^-, |\nu| \perp \mu \\ \nu \ll \mu \Leftrightarrow |\nu| \ll \mu \end{cases}$

3° $\mu, \mu^+, \mu^- \ll |\mu|$

lemma $\nu \ll \mu \Leftrightarrow$ 若 $\forall E_0$. $\mu(E_0) = 0$ 则 $\nu(E_0) = 0$.

"$\Rightarrow$" ✓

"$\Leftarrow$" 否则. $\exists \delta_0 > 0$. $\exists \{E_n\}_n$ s.t. $\varlimsup\limits_{\mu(E_n)\to 0} \nu(E_n) > \delta_0$.

不妨 $\mu(E_n) < \frac{1}{2^n}$. $\quad \nu(E_n) > \delta_0$.

取 $E = \limsup\limits_{n\to\infty} E_n = \bigcap\limits_{k=1}^{\infty} \bigcup\limits_{n\geq k} E_n$

则 $\mu(E) \overset{(\forall k)}{\leq} \mu\left(\bigcup\limits_{n\geq k} E_n\right) = \frac{1}{2^{k-1}}$ $\qquad$ 故 $\mu(E) = 0$

$\Rightarrow \forall F \subset E$, $\mu(F)=0 \Rightarrow \nu(F)=0$. ~~$\Rightarrow \nu(E)=0$~~ 而 $|\nu|(E) = \lim\limits_{k\to\infty} |\nu|\left(\bigcup\limits_{n\geq k} E_n\right) \geq \delta_0$ 矛盾!

Thm Let μ and ν be two measures on $(X, \mathcal{F})$ and $|\nu|(X) < \infty$.
Then there exists a unique decomposition of ν

$\nu = \nu_a + \nu_s$ $\quad$ s.t. 1° $\nu_a \perp \nu_s$ $\quad$ 2° $\nu_a \ll \mu$, $\nu_s \perp \mu$.

Pf: 不妨 $\mu, \nu \geq 0$

令 $\mathcal{N}_\mu = \{E \in \mathcal{F} \mid \mu(E) = 0\}$ $\qquad C_0 := \sup\limits_{E\in\mathcal{N}_\mu} \nu(E)$ $(< \infty)$

则 $\exists \{E_n\}_n \subset \mathcal{N}_\mu$ s.t. $\lim\limits_{n\to\infty} \nu(E_n) = C_0$

$M_n = \bigcup\limits_{j=1}^{n} E_j \in \mathcal{N}_\mu$ $\qquad E_0 := \bigcup\limits_{n=1}^{\infty} M_n = \bigcup\limits_{n=1}^{\infty} E_n$

则 $\mu(E_0) = 0$ 故 $E_0 \in \mathcal{N}_\mu$ $\qquad \nu(E_0) = C_0$

Define for $\forall E \in \mathcal{F}$ $\quad \nu_s(E) := \nu(E_0 \cap E)$ $\qquad$ Then $\nu_a \perp \nu_s$, $\nu_s \perp \mu$

$\nu_a(E) := \nu(E_0^c \cap E)$

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$\forall E \in N_\mu,\ \nu_a(E) = 0. \Rightarrow \nu_a \ll \mu.$

$\in N_\mu$ $\quad$ N_μ 中最大

(对 $E \in N_\mu$. $0 \le \nu_a(E) = \nu(E \cap E_0^c) = \nu((E \cup E_0) - E_0) = \nu(E \cup E_0) - \nu(E_0) \le 0.$)

Regular measures on topological space (X, τ) $\mathcal{B} = \sigma(\tau)$: Borel σ-algebra

$(X, \mathcal{A}, \mu)$ additive "measure" 称 μ "regular". 若

$\forall E \in \mathcal{A}.\ \forall \varepsilon > 0$ $\exists F_\varepsilon \in \mathcal{A}$ s.t. $\overline{F_\varepsilon} \subset E.$

$G_\varepsilon \in \mathcal{A}, E \subset \mathring{G}_\varepsilon$ $\quad |\mu|(G_\varepsilon - F_\varepsilon) < \varepsilon.$

$|\mu|$

Thm (Alexandroff) bdd, regular, finite additive "measure" 在紧空间 (X, τ) 中 algebra $\mathcal{A}$ 是可数可加的.

紧性用到了?

Pf: 1° $|\mu|$ is countably additive: $\{E_n\} \subset \mathcal{A}$, 不交 $\quad E := \bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$

$$|\mu|\Big(\bigcup_{n=1}^{\infty} E_n\Big) \ge |\mu|\Big(\bigcup_{n=1}^{N} E_n\Big) = \sum_{n=1}^{N} |\mu|(E_n) \to \sum_{n=1}^{\infty} |\mu|(E_n)$$

故 $|\mu|\Big(\bigcup_{n=1}^{\infty} E_n\Big) \ge \sum_{n=1}^{\infty} |\mu|(E_n).$ (有限可加性, 见 Jordan 分解的证明第一部分)

反过来. $\forall \varepsilon > 0.$ $\exists F, G \in \mathcal{A}$ $\quad F \subset \overline{F} \subset E \subset \mathring{G} \subset G$ s.t. $|\mu|(G - F) < \varepsilon$

$\exists \{F_n\}\{G_n\} \subset \mathcal{A}$ $\quad F_n \subset \overline{F_n} \subset E_n \subset \mathring{G}_n \subset G_n$ s.t. $|\mu|(G_n - F_n) < \frac{\varepsilon}{2^n}$

$$\sum_{n=1}^{\infty} |\mu|(E_n) \ge \sum_{n=1}^{\infty} |\mu|(G_n) - \varepsilon \ge \sum_{n=1}^{N} |\mu|(G_n) - \varepsilon$$

$\Big(\overline{F} \subset \bigcup_{n=1}^{\infty} \mathring{G}_n$. 由 X 紧性, $\exists N$ s.t. $\overline{F} \subset \bigcup_{n=1}^{N} \mathring{G}_n \Rightarrow F \subset \bigcup_{n=1}^{N} G_n\Big) \ge |\mu|(F) - \varepsilon \ge |\mu|(E) - 2\varepsilon$

全部加 故 $\sum_{n=1}^{\infty} |\mu|(E_n) \ge |\mu|(E)$ $\quad$ (再由有限次可加性. 注意可数次可加性没有!)

即 $|\mu|\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} |\mu|(E_n)$

2° $\mu\Big(\bigcup_n E_n\Big) = \sum_n \mu(E_n)$

$$\Big|\mu\Big(E - \bigcup_{j=1}^{n} E_j\Big)\Big| \le \sum_{j=n+1}^{\infty} |\mu|(E_j) \to 0 \text{ as } n \to \infty \quad (\text{因}\ 收敛)$$

$$= \Big|\mu(E) - \sum_{j=1}^{n} \mu(E_j)\Big| \quad \Rightarrow \mu(E) = \sum_{j=1}^{\infty} \mu(E_j)$$

即证到 μ 可数可加.

Cor 紧上 有限可加的假测度 可以延拓成 $\sigma(\mathcal{A})$ 上 的 σ-regular measure (By Hahn's extension)

Chap II Measurable functions and Integral

§ II.1 $(X, \mathcal{F})$ measurable space

$f: X \to \mathbb{R}$ [or $\mathbb{R}^* = [-\infty, +\infty]$)

measurable functions $f^{-1}((\alpha, +\infty]) \in \mathcal{F}$ $\forall \alpha \in \mathbb{R}$
$= X(f>\alpha)$

$\Leftrightarrow f^{-1}([\alpha, +\infty]) \in \mathcal{F}^{\forall\alpha} \Leftrightarrow f^{-1}([-\infty, \alpha]) \in \mathcal{F}^{\forall\alpha} \Leftrightarrow f^{-1}(G) \in \mathcal{F}$ $\forall G$ open

$\Leftrightarrow f^{-1}(B) \in \mathcal{F}$ $\forall B \in \mathcal{B}(\mathbb{R})$

$f^{-1}: \mathcal{B}(\bar{\mathbb{R}}) \to \mathcal{F}$, $(\mathcal{B}(\bar{\mathbb{R}}), \mathcal{F})$-measurable in the following sense $(\forall B \in \mathcal{M})$

$(X, \mathcal{F})$, $(Y, \mathcal{M})$ mapping $f: X \to Y$ is $(\mathcal{F}, \mathcal{M})$-measurable if $f^{-1}(B) \in \mathcal{F}$

rmk 1 · $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ $f: \mathbb{R}^n \to \mathbb{R}$ if f is continuous then f is $(\mathcal{B}(\mathbb{R}^n), \mathcal{B}(\mathbb{R}))$-measurable

· If $f: \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{L}(\mathbb{R}^n)$-measurable, $g: \mathbb{R} \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R})$-measurable then $g \circ f(x)$ is $\mathcal{L}(\mathbb{R}^n)$-measurable

rmk 2 complex value functions

rmk 3 $\forall f: X \to (Y, \mathcal{M})$ define $\mathcal{F}_f := \{ f^{-1}(B) \in 2^X \mid B \in \mathcal{M} \}$
then f is $(\mathcal{F}_f, \mathcal{M})$-measurable

Measurable function space $MF(X, \mathcal{F})$ is suitable for limit operation

1. $f, g: X \to \mathbb{R}$ measurable $\Rightarrow$ $\max\{f(x), g(x)\}$ $\min\{f(x), g(x)\}$ $\in MF(X)$
2. $f_n \in MF(X)$ $n \in \mathbb{N}$ $\sup f_n, \inf f_n \in MF(X)$
3. $\varlimsup_n f_n$, $\varliminf_n f_n \in MF(X)$
   · If $\lim_{n\to\infty} f_n$ exists for $\forall x \in X$ then $\lim_n f_n \in MF(X)$.
4. $\chi_E(x) = \begin{cases} 1 & , x \in E \\ 0 & , x \notin E \end{cases} \in MF(X) \Leftrightarrow E \in \mathcal{F}$
   $\sum_{j=1}^{J} c_j \chi_{E_j}(x) \in MF(X) \Leftrightarrow E_j \in \mathcal{F}$ $(\forall j)$

thm 1. 非负函数总可写成非负不减简单函数序列的极限 $f_n(x) = \begin{cases} \frac{j-1}{2^n} & , f(x) \in [\frac{j-1}{2^n}, \frac{j}{2^n}) \quad 1 \le j \le n2^n \\ n & , f(x) \ge n \end{cases}$

2. 可测函数总可写成可测函数列的极限

cor 1. $f: X \to \mathbb{R}$ 可测 $\Leftrightarrow$ f 可写成简单函数的极限

2. $f, g \in MF(X)$ $\Rightarrow$ $f+g \in MF(X)$.

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$(X, \mathfrak{F}, \mu)$ measure space (complete) μ-a.e. $MF(X)/\mu\text{-a.e.}$

If $\lim_n f_n = f$ μ-a.e. 则 f在等价类下唯一

依测度收敛 $\lim_n |\mu|(X(|f_n - f| > \delta)) = 0$. $\forall \delta > 0$. $f_n \xrightarrow{\mu} f$

几乎一致收敛 almost uniform convergence $\forall \varepsilon > 0\ \exists E_\varepsilon$ $\mu(E_\varepsilon) < \varepsilon$. $f_n \to f$ uniformly in E_ε^c

$f_n \xrightarrow{a.u.} f$ (a.u. $\Rightarrow$ a.e.)

Thm (Riesz) $f_n \xrightarrow{\mu} f \Rightarrow \exists$ subseq $\{f_{n_k}\}_k$ s.t. $f_{n_k} \xrightarrow{a.e.} f$

Thm' (Riesz) $\{f_n\}$ is a Cauchy seq. in measure Then $\exists$ subseq. $\{f_{n_k}\}$ and f s.t. $f_{n_k} \xrightarrow{a.u.} f$

$\mu(|f_{n_k} - f_{n_{k+1}}| > \frac{1}{2^k}) < \frac{1}{2^k}$ $f := f_{n_1} + \sum_{k=1}^{\infty}(f_{n_{k+1}} - f_{n_k})$ 存在 a.e.有限

rmk Define $|f|_\rho := \inf_{\delta > 0} \arctan(\delta + |\mu|(X(|f| > \delta)))$ 则 $f_{n_k} \xrightarrow{\mu} f$, $f_n \xrightarrow{\mu} f$

Then $\rho(f, g) := |f - g|_\rho$ is a metric on $MF(X)$ and

$f_n \xrightarrow{\mu} f \iff f_n \xrightarrow{\rho} f$; $(MF(X), \rho)$ is complete 可由本书附

Thm (Egoroff) 若 $|\mu|(X) < \infty$ 则 $f_n \xrightarrow{a.e.} f \Rightarrow f_n \xrightarrow{a.u.} f$

10.23 简单函数 $s = \sum_{j=1}^{n} c_j \chi_{E_j}$ Define $\int_X s\, d\mu = \sum_{j=1}^{n} c_j \mu(E_j)$ 且 $X \subset \bigcup_{j=1}^{n} E_j$ 若 $\mu(E_j) = \infty$ 则 $c_j = 0$

$f \in MF(X, \mathfrak{F})$ 称f可积 若 $\exists$ $\{s_n\}_{n \geq 1}^{\infty}$ simple s.t. (1) $s_n \xrightarrow{\mu} f$ (as $n \to \infty$)

(2) $\int_X |s_m - s_n|\, d|\mu| \to 0$ $(m, n \to \infty)$

and define it's integral $\int_X f\, d\mu = \lim_n \int_X s_n\, d\mu \in \mathbb{R}$

well-defined: ①极限存在 ②极限唯一

Pf: ①由(2)即得. ②: 若有两列这样的简单函数列 $\{s_n\}_n$ $\{t_n\}_n$

∴ let $P_n(E) = \int_E |s_n - t_n|\, d|\mu|$ Then $P(E) := \lim_{n \to \infty} P_n(E)$ exists:

$$|P_n(E) - P_m(E)| \le \int_E |s_n - s_m|\, d|\mu| + \int_E |t_n - t_m|\, d|\mu|$$

$\to 0$ as $m, n \to \infty$.

故 (1) $\lim_{n \to \infty} P_n(E) = P(E)$ 在$E \in \mathfrak{F}$ 时收敛是一致的

(2) $\lim_{|\mu|(E) \to 0} P_n(E) = 0$ (简单函数的绝对连续性) $\Rightarrow \lim_{|\mu|(E) \to 0} P(E) = 0$.

$|s_n - t_n|$ 仍为简单函数. $\exists N$. $A_N = \operatorname{supp}|s_N - t_N|$ $\mu(A_N) < \infty$.

s.t. $|P_N(A_N^c) - P(A_N^c)| < \varepsilon$.

$= |P(A_N^c)|$

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Moreover. $s_n - t_n \xrightarrow{\mu} 0$ by Riesz's Thm.

$\forall \varepsilon > 0.\ \exists N_1 > N.\ B (= B_{N_1})$ with $|\mu|(B^c) < \varepsilon.$ (3)

$$|s_{N_1} - t_{N_1}| < \frac{\varepsilon}{1+|\mu|(A_N)} \quad \text{in } B$$

记 $A = A_N$ $\quad P(X) = P(A^c) + P(A) \leq \varepsilon + \underbrace{P(A \cap B)}_{\leq P_{N_1}(A\cap B)+\varepsilon} + \underbrace{P(A \cap B^c)}_{\leq \varepsilon}$ (by (3),(2), 取 N_1 足大满足)

(by (1). 一致收敛性. 取 N_1 足大, 满足)

$$P_{N_1}(A \cap B) = \int_{A\cap B} |s_{N_1} - t_{N_1}|\, d|\mu| < \frac{\varepsilon}{1+|\mu|(A)} |\mu|(A \cap B) < \varepsilon.$$

$\leq 4\varepsilon$

故 $P(X) = 0. \Rightarrow$ 极限唯一

用 $L(X)$ 记 X 上可积函数之集.

1. 若 $f \in L(X)$. 则 $|f| \in L(X)$ 且 $\left|\int_X f\, d\mu\right| \leq \int_X |f|\, d\mu.$

$$s_n \xrightarrow{\mu} f \Rightarrow |s_n| \xrightarrow{\mu} |f| \quad \text{且} \int \big||s_n| - |s_m|\big| \leq \int |s_n - s_m| \to 0 \text{ as } n, m \to \infty.$$

2. $f \geq 0$ and $\mu \geq 0 \Rightarrow \int_X f\, d\mu \geq 0.$ and $\int_X f\, d\mu = 0$ iff $f = 0$

$$\int_{A_N} |s_N - f| < \varepsilon \qquad s_n \uparrow f \qquad \int f = \underbrace{\int f - s_N}_{> -\varepsilon} + \underbrace{\int s_N}_{\geq 0}$$

证 (Chebyshev ineq) $\quad f \geq 0.\ \mu \geq 0$

"$\Rightarrow$" $0 = \int_E f\, d\mu \geq \int_{\{f \geq \delta\}} f\, d\mu \geq \delta\, \mu\{f \geq \delta\} \Rightarrow \mu\{f \geq \delta\} = 0 \quad \forall \delta > 0 \Rightarrow f = 0$ a.e.

3. $f \in L(X, d\mu)$ Then $\mu_f(E) = \int_E f\, d\mu \quad \forall E \in \mathcal{F}$ is a finite measure on $(X, \mathcal{F})$

with $|\mu_f|(E) = \int_E |f|\, d|\mu|$

证 ① 有限可加 $\int_{E_1 \cup E_2} f\, d\mu = \int_{E_1} f\, d\mu + \int_{E_2} f\, d\mu$

$$\chi_{E_1 \cup E_2} = \chi_{E_1} + \chi_{E_2}$$

② 绝对连续性. $|\mu|(E) \to 0 \quad \int_E |f|\, d|\mu| \to 0.$

③ 可数可加 $\quad E_n. \quad \left|\mu\left(\bigcup_{j=1}^{\infty} E_j\right)\right| < \infty.$ 由 ①.② 可得

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14) 简单函数s $|\mu_s|(E)=\int_E |s|\, d|\mu|$ $f\in L(X)$ ？定在干嘛？

$f\in L(X)$ $\int_E |s_n|\, d|\mu| \to \int_E |f|\, d|\mu|$

15) If $|f|\le g\in L(X)$, $f\in MF(X,\mathcal{F})$. then $f\in L(X)$.

$f_n \uparrow |f|$ 为如前定义的简单函数逼近 $|f|$ 的函数列

$\lim_{N\to\infty} |\mu|(g>N)=0$. $\int |f_m-f_n|\, d|\mu| = \int_{|f|<N} |f_m-f_n|\, d|\mu| + \int_{|f|>N} |f_m-f_n|\, d|\mu|$

$\le \int_{|f|<N} |f_m-f_n|\, d|\mu| + 2\int_{g>N} g\, d|\mu| \to 0$

$\Rightarrow |f|\in L(X)$

$f=f^+-f^-$ $|f^+|\le|f|$ $|f^-|\le|f|$. $\Rightarrow f^+, f^-\in L(X) \Rightarrow f\in L(X)$

10.25 Convergence in $L(X,d\mu) := \{f\in MF(X,\mathcal{F}) \mid |\int_X f\, d\mu|<\infty\}$ equipped with $\|\cdot\|_{L^1}$

$f_n \xrightarrow{L^1} f \Rightarrow f_n \xrightarrow{\mu} f$ $\|f\|_{L^1}=\int_X |f|\, d|\mu|$

Thm1 $f_n \xrightarrow{L^1} f \Leftrightarrow$
① $f_n \xrightarrow{\mu} f$ (or $\{f_n\}_n$ is μ-Cauchy)
② $\lim_{\mu(E)\to 0} \int_E |f_n|\, d|\mu|=0$ 且关于 n 是一致的. 即 $\forall\varepsilon>0, \exists\delta>0$ $\forall \mu(E)<\delta$ $\forall n$. $\int_E |f_n|\, d\mu|<\varepsilon$
③ $\forall\varepsilon>0$. $\exists E_\varepsilon$ s.t. $\mu(E_\varepsilon)<\infty$ 且 $\int_{E_\varepsilon^c} |f_n|\, d|\mu|<\varepsilon$ $\forall n$

Cor1 $L^1(X,d\mu)$ is a Banach space

Pf of cor1 $\forall \{f_n\}$ is L^1-Cauchy $\Rightarrow \{f_n\}$ is μ-Cauchy.

$\exists f$ s.t. $f_n \xrightarrow{\mu} f$

$\int_E |f_n|\, d|\mu| \le \int_E |f_n-f_N| + \int_E |f_N|$

Pf of Thm1 "$\Rightarrow$" 显然,

"$\Leftarrow$". $\int_X |f_n-f|\, d|\mu| = \int_{E_\varepsilon} |f_n-f| + \underbrace{\int_{E_\varepsilon^c} |f_n-f|}_{③\ \mu(E_\varepsilon)<\infty\ <\varepsilon}$

$\le \underbrace{\int_{E_\varepsilon(|f_n-f|>\delta)} |f_n-f|}_{②\ \mu(E_\varepsilon(|f_n-f|>\delta))<\delta_2 \ <2\varepsilon_2} + \underbrace{\int_{E_\varepsilon(|f_n-f|\le\delta)} |f_n-f|}_{\le \delta\, |\mu|(E_\varepsilon)} + \varepsilon.$

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Thm (Lebesgue's Dominated Convergence)

1. $f_n \xrightarrow{\mu} f \qquad |f_n| \le g \in L^1$ Then $f_n \xrightarrow{L^1} f \quad n \to \infty$.

(Pf: 由 $g \in L^1$ 得满足 Thm1 中的②③ 故由 Thm1, $f_n \xrightarrow{L^1} f$ □)

2. $f_n \xrightarrow{a.e.} f$ and $|f_n| \le g \in L^1(X) \Rightarrow f_n \xrightarrow{L^1} f$

$|f| \le g \Rightarrow f \in L^1(X)$

Pf: $\displaystyle\int_X |f_n - f| = \int_{E_\varepsilon} |f_n - f| + \underbrace{\int_{E_\varepsilon^c} |f_n - f|}_{\le 2\int_{E_\varepsilon^c} g}$

由 Egoroff $f_n \rightrightarrows f$ in F_ε^c, $\mu(F_\varepsilon) < \varepsilon$, 取 $\mu(E_\varepsilon) < \infty$. $\int_{E_\varepsilon^c} g < \varepsilon$.

$< \mu(E_\varepsilon)\,\delta(n) \qquad < 2\varepsilon$.

Thm 1' (Vitali Convergence Thm)

若 $f_n \xrightarrow{a.e.} f$ 则 $f_n \xrightarrow{L^1} f \iff \begin{cases} ② \\ ③ \end{cases}$

Pf "⇒" ✓

"⇐" 已有 L^1 完备. 只需证 $\{f_n\}$ 为 Cauchy

$$\int |f_m - f_n| \le \int_{E_\varepsilon} |f_m - f_n| + \underbrace{\int_{E_\varepsilon^c} |f_m|}_{<\varepsilon} + \underbrace{\int_{E_\varepsilon^c} |f_n|}_{<\varepsilon}$$

③ $\exists \mu(E_\varepsilon) < \infty$

$$\le \int_{E_\varepsilon} |f_m - f| + \int_{E_\varepsilon} |f_n - f| + 2\varepsilon.$$

~~Egoroff~~ F_ε.

$$\le \underbrace{\int_{E_\varepsilon \cap F_\varepsilon} |f_m - f|}_{<2\varepsilon \text{ by } ②} + \underbrace{\int_{E_\varepsilon \cap F_\varepsilon^c} |f_m - f|}_{<\delta(m)\mu(E_\varepsilon)} + \int_{E_\varepsilon \cap F_\varepsilon} |f_n - f| + \int_{E_\varepsilon \cap F_\varepsilon^c} |f_n - f| + 2\varepsilon$$

□

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Thm (Levi's Lemma) $\mu \geq 0$

$0 \leq f_n \uparrow f \Rightarrow \lim_{n\to\infty} \int_X f_n \, d\mu = \int_X \lim_n f_n \, d\mu$

Fatou's lemma $f_n \geq 0$ $\mu \geq 0$

$\int_X \liminf_n f_n \, d\mu \leq \liminf_n \int_X f_n \, d\mu$

rmk for $f \geq 0$, $\mu \geq 0$. $\mu_f(E) := \int_E f \, d\mu$ is an extended measure on $MF(\mathcal{F}, X)$

◦ Lebesgue-Radon-Nikodym Thm

σ-finite $\mu \geq 0$ ν : finite (signed)

Then (1) $\nu = \nu_a + \nu_s$, $\nu_a \perp \nu_s$

(2) $\nu_a \ll \mu$ $\nu_s \perp \mu$

(3) $\exists 1$ $f \in L^1(X, d\mu)$ s.t. $d\nu_a = f \, d\mu$ $\nu_a(E) = \int_E f \, d\mu$

Radon-Nikodym : $\nu \ll \mu \Rightarrow \exists 1$ $f \in L^1(d\mu)$ s.t. $\nu(E) = \int_E f \, d\mu$.

Pf: 1° μ有限, 故 $\nu(X) \leq M(X) < \infty$ ~~不妨设~~ $\nu \geq 0$. $\mathcal{F} := \{ g \in MF(X) \mid \int_E g \, d\mu \leq \nu(E)\ \forall E \in \mathcal{F}, g \geq 0 \}$

$\alpha := \sup_{g \in \mathcal{F}} \int_X g \, d\mu \leq \nu(X) < \infty$.

$\exists \{g_n\} \subset \mathcal{F}$ $\int_X g_n \, d\mu \to \alpha$.

$f_n(x) := \max_{1 \leq k \leq n} \{ g_k(x) \} \in \mathcal{F}$: 若 $g_1, g_2 \in \mathcal{F}$

$\int_E \max\{g_1(x), g_2(x)\} \, d\mu$

$= \int_{E(g_1 > g_2)} g_1 \, d\mu + \int_{E(g_1 \leq g_2)} g_2 \, d\mu$

$\leq \nu(E(g_1 > g_2)) + \nu(E(g_1 \leq g_2)) = \nu(E)$.

即 $\max\{g_1(x), g_2(x)\} \in \mathcal{F}$

$f_n \overset{\text{逐点}}{\uparrow} f_0$

$\alpha \geq \int_X f_n \, d\mu \geq \int_X g_n \, d\mu \to \alpha$.

$\Rightarrow \lim_n \int_X f_n \, d\mu = \alpha \Rightarrow \int_X f_0 \, d\mu = \alpha$.

现定义 $\nu_a(E) := \int_E f_0 \, d\mu$. 只需证 $0 \leq \nu_s := \nu - \nu_a \perp \mu$.

(Lemma μ_1 μ_2 为两个有限正测度 则 $\mu_1 \perp \mu_2$ 或 $\exists E_0 \in \mathcal{F}$. $\varepsilon_0 > 0$ s.t. $\mu_2(E_0) > 0$ 且 $(\mu_1 - \varepsilon_0 \mu_2)(E_1) \geq 0$ $(\forall E_1 \subset E_0)$ 未证)

若非如此 $\exists E_0$. $\mu(E_0) > 0$. $(\nu - \nu_a) - \varepsilon_0 \mu$ 在 E_0 上恒正.

考虑 $g_0 = f_0 + \varepsilon_0 \chi_{E_0}$. 则 $\int_X g_0 \, d\mu = \int_X f_0 \, d\mu + \varepsilon_0 \mu(E_0) > \alpha$. $g_0 \in \mathcal{F}$

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$$\int_E g_0 \, d\mu = \int_E f_0 + \varepsilon_0 \chi_{E_0} \, d\mu \le \nu_a(E) + \varepsilon_0 \mu(E \cap E_0)$$

$$= \nu_a(E \setminus E_0) + \nu_a(E \cap E_0) + \varepsilon_0 \mu(E \cap E_0) \le \nu(E \setminus E_0) + \nu(E \cap E_0) = \nu(E)$$

$g_0 \in \mathcal{F}$ 矛盾

附此 lemma 考虑 $\nu_n = \mu_1 - \frac{1}{n}\mu_2$

每个 ν_n 有 Hahn 分解 $X = P_n \cup N_n$.

$P = \bigcup P_n$ $\quad N = \bigcap N_n$ $\quad \Rightarrow \forall n, \nu_n(N) \le 0$

$0 \le \mu_1(N) \le \frac{1}{n}\mu_2(N) \to 0$ $\quad \Rightarrow \mu_1(N) = 0$.

若 $\mu_2(P) = 0$. 则 $\mu_1 \perp \mu_2$

若 $\mu_2(P) > 0$. $\exists n_0$ s.t. $\mu_2(P_{n_0}) > 0$. $\quad \nu_{n_0}$ 在 P_{n_0} 上 正. 且 $\mu_2(P_{n_0}) > 0$.

已
证.

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2° μ: σ-finite $\quad X = \bigcup_n E_n \quad E_n \uparrow \quad \mu(E_n) < \infty$. $\quad \forall E \subset E_n$

令 $\begin{cases} \mu_n = \mu|_{E_n} \\ \nu_n = \nu|_{E_n} \end{cases}$ $\quad \exists f_n \in L(d\mu_n)$ s.t. $\nu_n(E) = \int_E f_n \, d\mu_n$

(在 E_n 外为 0) $\quad \forall E \subset E_n$, 有 ν

由于 $\mu_{n+1}|_{E_n} = \mu_n$ $\quad f_{n+1}|_{E_n} = f_n$ $\quad \nu_{n+1}(E) = \int_E f_{n+1} \, d\mu = \int_E f_n \, d\mu = \nu_n(E)$

$\Rightarrow \tilde{f}_n \uparrow$

令 $f(x) = \lim_{n \to \infty} f_n(x)$

$$\nu(E) = \nu(E \cap (\bigcup_n E_n)) = \lim_{n \to \infty} \nu(E \cap E_n) = \lim_n \int_{E \cap E_n} f_n \, d\mu = \lim_n \int_X \tilde{f}_n \chi_{E \cap E_n} \, d\mu$$

$$(\text{by Levi's lemma}) = \int_E f \, d\mu$$

Radon-Nikodym 导数 $\quad d\nu = f \, d\mu \quad \nu(E) = \int_E f \, d\mu$

称 f 为 ν 关于 μ 的 R-N 导数, 记为 $f = \frac{d\nu}{d\mu}$

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then $fg \in L(d\mu)$ and

1. $g \in L(d\nu)$ $\quad \int_X g\, d\nu = \int_X gf\, d\mu$ $\quad$ 积分变换

Pf: 1° $g = \chi_E$ $\quad : \nu(E) = \int_E f\, d\mu$ $\quad$ g: simple functions

2° $g \in L(d\nu)$ $\exists g_n$ simple $\quad |g_n| \le |g|$ $\quad g_n \to g$

$$\left| \int_X g_n\, d\nu - \int_X g\, d\nu \right| \le \int_X |g_n - g|\, d|\nu| \to 0 \quad \text{as } n \to \infty$$

$$\overset{\shortparallel}{\int_X g_n f\, d\mu} \xrightarrow{\text{lem}} \int_X g f\, d\mu \qquad \Rightarrow \int_X g\, d\nu = \int_X g f\, d\mu$$

2. $\nu \ll \mu$ $\quad \mu \ll \lambda$ $\quad$ 则$\exists f, g$. $d\nu = f\, d\mu$ $\quad d\mu = g\, d\lambda$

$$\Rightarrow \frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} = fg$$

3. ν complex $\quad \nu \ll |\nu|$ $\quad \Rightarrow \exists f \in L(d|\nu|)$ $\quad$ s.t. $\nu = f\, d|\nu|$ $\quad$ Then $|f| = 1$ $|\nu|$-a.e.

$$\nu(E) = \int_E f\, d|\nu| \qquad |\nu|(E) = \int_E |f|\, d|\nu| = \int_E 1\, d|\nu|$$

4. $x_0 \in X$ $\quad \delta_{x_0}(E) = \begin{cases} 1, & x_0 \in E \\ 0, & x_0 \notin E \end{cases}$

$f: X \to \mathbb{R}/\mathbb{C}$ $\qquad \int_X f\, d\delta_{x_0} = \int_X f\, \delta_{x_0}(dx) = f(x_0)$

$\mu_N = \sum_{n=1}^{N} c_n \delta_{x_n}$ $\quad c_n > 0.$ $\quad \sum c_n = 1$ $\quad$ 则μ_n为概率测度 $\int_X f\, d\mu_N$ 为f的期望

5. $X = \mathbb{N}$. $\quad \mu = \# = \sum_{n=1}^{\infty} \delta_{\{n\}}$

$\ell^1 = L(d\#)$ $\quad f: \mathbb{N} \to \mathbb{R}$ $\quad a_n = f(n)$ $\qquad \int_X f\, d\mu = \sum_{n=1}^{\infty} a_n$ $\quad$ $f \in L(d\#)$ ⇔ 绝对收敛

6. $(\Omega, \mathcal{F}, P)$ 概率空间.

r.v. $\xi: \Omega \to \mathbb{R}$ $\qquad (\mathcal{F}, \mathcal{B})$-measurable

概率分布 $\mu_\xi(B) := P(\xi^{-1} B)$ $\qquad B \in \mathcal{B}$

$$E\xi = \int_\Omega \xi(\omega)\, P(d\omega) = \int_{\mathbb{R}} t\, P_\xi(t)\, dt$$

$(X, \mathcal{F})$ $\quad M(X,\mathcal{F}) :=$ measures on $(X,\mathcal{F})$ with finite total variation

(= real-valued measures) linear space

Define $\|\mu\| := |\mu|(X) < \infty$ which is a norm on $M(X)$

Then $(M(X), \|\cdot\|)$ is a Banach Space.

· Vitali - Hahn - Sales Thm

$(X, \mathcal{F}, \mu)$ Define an equivalent relation on $\mathcal{F}$

$$E \sim F : |\mu|(E \Delta F) = 0$$

$\mathcal{F}_\mu = \mathcal{F}/\sim \; = \mathcal{F}/N$ For $E, F \in \mathcal{F}$ define $\rho(E,F) = \arctan(|\mu|(E \Delta F))$

是 $\mathcal{F}_\mu$ 上的一个 metric：

$$|\mu|(E\Delta F) = \int_X |\chi_E - \chi_F| \, d|\mu| \le \int_X |\chi_E - \chi_G| \, d|\mu| + \int_X |\chi_F - \chi_G| \, d|\mu| = |\mu|(E\Delta G) + |\mu|(F\Delta G)$$

$$\rho(\chi_E, \chi_F) = \inf_{\alpha>0} \arctan(\alpha + |\mu|(|\chi_E - \chi_F| > \alpha)) = \arctan |\mu|(E\Delta F) = \rho(E,F)$$

Prop 1 $(\mathcal{F}_\mu, \rho)$ is complete

$E_n \in \mathcal{F}_\mu$ is Cauchy $\Leftrightarrow$ $\chi_{E_n} \in MF(X)$ is Cauchy

$\Rightarrow \chi_{E_n} \xrightarrow{\mu}$ some $\chi = \chi_E$ $\quad E \in \mathcal{F}_\mu$.

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rmks on finitely additive "measure" 有限可加"测度"

1. $FM(X, \mathcal{F}) :=$ finitely additive measure with finite variation

Define $\|\nu\| := |\nu|(X) < \infty$ Then $(FM(X,\mathcal{F}), \|\cdot\|)$ is Banach

2. $\mathcal{F}_\mu$ If $\nu \ll \mu$. than ν is well-defined on μ

即 $\lim_{\mu(E)\to 0} \nu(E) = 0$.

Moreover, $\nu : (\mathcal{F}_\mu, \rho) \to \mathbb{R}$ is continous if $E_n \xrightarrow{\rho} E$ $(|\mu|(E_n \Delta E) \to 0)$

$\Rightarrow \nu(E_n) \to \nu(E)$

$\nu(E_n) - \nu(E) = \nu(E_n - E) - \nu(E - E_n) \to 0$ since $\mu(E_n \Delta E) \to 0$.

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Vitali-Hahn-Saks Thm

$(X,\mathcal{F},\mu)$: measure space $\{\nu_n\}_{n=1}^{\infty} \subset FM(X,\mathcal{F})$ s.t. $\nu_n \ll \mu$ s.t. $\lim_n \nu_n(E)$ exists for all $E\in\mathcal{F}$

Then $\lim_{|\mu|(E)\to 0} \nu_n(E)=0$ is uniformly in $n\in\mathbb{N}$

Pf: $\forall \varepsilon>0.\ \exists N\ \forall m,n\geq N$ $\{E\in\mathcal{F}_\mu : |\nu_m(E)-\nu_n(E)|\leq\varepsilon\}$ which is close in $(\mathcal{F}_\mu,\rho)$ Ex: ✓

$=: \mathcal{E}_{m,n}(\varepsilon)$; $\mathcal{E}_N(\varepsilon)=\bigcap_{m,n\geq N}\mathcal{E}_{m,n}(\varepsilon)$: close

$\Rightarrow \mathcal{F}_\mu = \bigcup_{N=1}^{\infty}\mathcal{E}_N(\varepsilon)$ (第3 由 limit exists 得)

By Baire's Category Thm, $\exists N_0$ s.t. $\mathcal{E}_{N_0}(\varepsilon)$ has nonempty interior

$\exists\delta>0$. a point $A\in\mathcal{F}_\mu$ s.t. $\{E\in\mathcal{F}_\mu : \rho(E,A)<\arctan\delta\}\subset\mathcal{E}_{N_0}(\varepsilon)$

$\Leftrightarrow \forall E\in\mathcal{F}$ $|\mu|(E\Delta A)<\delta \Rightarrow |\nu_m(E)-\nu_n(E)|<\varepsilon\ \ \forall m,n\geq N_0$

$|\nu_j|(E)<\varepsilon$ $j=1,2,\dots,N-1$ provided $|\mu|(E)<\delta$ 取 δ 足小

$|\mu|((E\cup A)\Delta A)\leq|\mu|(E-A)\leq|\mu|(E)<\delta.$

$|\mu|((A-E)\Delta A)=|\mu|(E\cap A)<\delta$

$\Rightarrow |\nu_n(E)-\nu_{N_0}(E)|<2\varepsilon$

$|\nu_n(E)|\leq|\nu_{N_0}(E)|+|\nu_n(E)-\nu_{N_0}(E)|<3\varepsilon$ $\forall n\geq N_0$ $n\in\mathbb{N}$

Cor 1 $\nu_n\in FM(X,\mathcal{F})$ $\nu_n\ll\mu$ $|\mu|(X)<\infty$.

$\lim_{n\to\infty}\nu_n(E)$ exists $(:=\nu(E))$ $\forall E\in\mathcal{F}$

Then ν is a measure, countable additive

ν measure $\Leftrightarrow$ $E_k\searrow\phi \Rightarrow \lim_k\nu(E_k)=0$

$|\mu|(E_k)\to 0.$

$\lim_k\nu(E_k)=\lim_k\lim_n\nu_n(E_k)=\lim_n\lim_k\nu_n(E_k)=0$

$\left(F_k\downarrow F \Leftrightarrow (F_k-F)\downarrow\phi\right)$
$\nu(F_k-F)=\nu(F_k)-\nu(F)$

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一致弱可加（一致绝对连续的特例）

Cor 2 (Nikodym) $\mu_n \in M(X, \mathcal{F})$ is real/complex-valued $n \in \mathbb{N}$. 有限变差.
s.t. $\lim_n \mu_n(E)$ exists for $\forall E \in \mathcal{F}$
$=: \mu_\infty(E)$ Then μ_∞ is a measure
and for any $E_k \downarrow \phi$ $\lim_{k\to\infty} \mu_n(E_k) = 0$ is uniformly in $n \in \mathbb{N}$

Pf: Consider $\mu(E) = \sum_{n=1}^{\infty} \frac{|\mu_n|(E)}{2^n(1+|\mu_n|(X))} < \infty$ Then μ is a measure and $\mu_n \ll \mu$

Chapter III L^p-spaces

$(X, \mathcal{F}, \mu)$ $\mu \geq 0$ $p \in [1, \infty]$

$L^p(X, \mathcal{F}, \mu) = \{ f \in MF(X) : |f|^p \in L(X) \}$ $1 \leq p < \infty$.

$\|f\|_{L^p} = \left(\int_X |f(x)|^p \mu(dx) \right)^{\frac{1}{p}}$

$L^\infty(X, \mathcal{F}, \mu) =$ ess bdd functions $\|f\|_{L^\infty} := \inf\{ M \geq 0 : |f(x)| \leq M \ \ \mu\text{-a.e.} \}$
$= \inf_{\mu(E_0)=0} \sup_{x \in E_0^c} |f(x)|$
$= \inf\{ M \geq 0 : \mu(|f| > M) = 0 \}$

Property $\frac{1}{p} + \frac{1}{p'} = 1$

1. Hölder ineq
2. Minkowski ineq
3. L^p is Banach

3.1 $f_n \xrightarrow{L^p} f$ $(1 \leq p < \infty)$ $\Leftrightarrow$ $\begin{cases} f_n \xrightarrow{\mu} f \\ \forall \varepsilon > 0 \ \exists \delta \ \forall E \ \mu(E) < \delta \ \int_E |f_n|^p d\mu < \varepsilon^p \ \forall n. \\ \forall \varepsilon > 0, \exists A_\varepsilon \in \mathcal{F}, \mu(A_\varepsilon) < \infty \text{ s.t. } \int_{A_\varepsilon^c} |f_n|^p d\mu < \varepsilon^p \ \forall n. \end{cases}$

3.2 L^∞ $f_n \xrightarrow{L^\infty} f$
$\Leftrightarrow \exists \mu$-zero set E_0 s.t. $f_n \rightrightarrows f$ in E_0^c

4. {simple functions} is dense in L^p, $p < \infty$

General simple function $S(x) = \sum_{j=1}^{J} c_j \chi_{E_j}$ $\bigcup_{j=1}^{J} E_j = X$ 不要求 $\mu(E_j) = \infty \Rightarrow c_j = 0$

$\hookrightarrow$ is dense in L^∞

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§Ⅲ.2 Dual of L^p-spaces

Thm (Riesz) $p\in[1,\infty)$ $(L^p)' \sim L^{p'}$ in the following sense

$\forall F\in(L^p)'$ $\exists!\, g_F\in L^{p'}$ s.t. $F(f)=\int_X f g_F\, d\mu \quad \forall f\in L^p$

Pf 1. $\mu(X)<\infty$ define $\nu_F(E)=F(\chi_E)$

Claim1: ν_F is a measure

$$\nu_F\Big(\bigcup_{j=1}^\infty E_j\Big)=F\Big(\chi_{\bigcup_{j=1}^\infty E_j}\Big) = F\Big(\sum_{j=1}^\infty E_j\Big) \underset{\uparrow}{=} \sum_{j=1}^\infty F(E_j)=\sum_{j=1}^\infty \nu(E_j)$$

为连续线性泛函；线性得有限可加

Claim 2 $\nu_F \ll \mu$ $\quad \mu(E)\to 0 \Rightarrow \chi_E\to 0 \Rightarrow F(\chi_E)\to 0$

by Radon-Nikodym $\exists g(=g_F)\in L^1(d\mu)$ s.t.

$$\big(F(\chi_E)=\big)\ \nu_F(E)=\int_E g\, d\mu=\int_X g\chi_E\, d\mu$$

$$F(s)=\int_X gs\, d\mu,\quad s:\ \text{simple function}$$

$f\in L^p$, $\exists s_n \xrightarrow{L^p} f$

$$F(f)\leftarrow F(s_n)=\int_X g s_n\, d\mu\ \Big(\to\int_X gf\, d\mu\Big)$$

$$F(f\chi_E)\leftarrow F(s_n\chi_E)=\int_E s_n g\, d\mu$$

By Vitali-Hahn-Saks

$\nu_n(F)=s_n\, d\nu_F \qquad \nu_n(E)=s_n\, d\nu_F=s_n g\, d\mu$

$$\lim_{\mu(E)\to 0}\int_E s_n g\, d\mu=0 \text{ uniformly in } n$$

$\Rightarrow s_n g \xrightarrow{L^1} fg\in L^1 \quad \forall f\in L^p$. $fg\in L^1 \Rightarrow g\in L^{p'}$

11.6 Folland Chapter 6

$g\in L^1$ 哪来的?

Reverse Hölder Ineq : μ: σ-finite, $p\in[1,\infty]$

If $M_g:=\sup\{\int_X fg\, d\mu : f \text{ simple functions},\ \|f\|_{L^p}=1\}<\infty$

then $g\in L^{p'}$ and $\|g\|_{L^{p'}}=M_g$

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a direct proof

Let $g_1(x) = |g(x)|^{\frac{1}{p}} \operatorname{sgn} g(x) \in L^p$

$\int |g(x)|^{1+\frac{1}{p}} d\mu = \int g(x) g_1(x) d\mu = F(g_1) \le \|F\| \cdot \|g_1\|_{L^p} = \|F\| \left(\int_X |g(x)| d\mu\right)^{\frac{1}{p}} = \|F\| \left(\int_X g(x) \operatorname{sgn} g(x) d\mu\right)^{\frac{1}{p}}$

$= \|F\| \cdot \|F(\operatorname{sgn} g)\|^{\frac{1}{p}} \le \|F\|^{1+\frac{1}{p}} \mu(X)^{\frac{1}{p^2}}$

$\int |g(x)|^{1+\frac{1}{p}+\frac{1}{p^2}+\cdots+\frac{1}{p^n}} \le \|F\|^{1+\frac{1}{p}+\cdots+\frac{1}{p^n}} \mu(X)^{\frac{1}{p^{n+1}}}$

by Fatou, $\int |g|^{p'} \le \|F\|^{p'} < \infty \Rightarrow g \in L^{p'}$ $\|g\|_{L^{p'}} \le \|F\|$

2. μ: σ-finite $X = \bigcup_n E_n$ $E_n \uparrow$ $\mu(E_n) < \infty$ $F \in (L^p(X))' \Rightarrow F \in (L^p(E_n))'$

$\Rightarrow \exists g_n \in L^{p'}$ 记 $\tilde{g}_n = \begin{cases} g_n(x), & x \in E_n \\ 0, & x \notin E_n \end{cases}$ $g_{n+1}|_{E_n} = g_n$

$\{\tilde{g}_n\} \uparrow$ some g where $g|_{E_n} = g_n = \lim_{k\to\infty} g_k(x)$

3. μ: general case

3.1 $\forall E \in \mathcal{F}$ which is σ-finite, $\exists g_E$ s.t. $F(f) = \int_X g_E f \, d\mu$ $\forall f \in L^p(E)$

Moreover, if $E \subset E_1$ then $g_{E_1}|_E = g_E$ $\|g_E\|_{L^{p'}} \le \|g_{E_1}\|_{L^{p'}} \le \|F\|$

Denote $M = \sup_{E:\sigma\text{-finite}} \|g_E\|_{L^{p'}} \le \|F\|$ choose $\{E_n\}$ σ-finite $\|g_{E_n}\| \to M$ as $n \to \infty$

令 $E_\infty = \bigcup_{n=1}^{\infty} E_n$ which is still σ-finite

and $\|g_{E_\infty}\|_{L^{p'}} = M$

$M \le \|F\|$.

下证 g_{E_∞} 为 F 在 X 上的一个表示

1° 若 A 是个包含 E_∞ 的 σ-有限集. 则 $g_A = g_{E_\infty}$ μ-a.e.

则 $g_A|_{E_\infty} = g_{E_\infty}$. $\int_X |g_A|^p \le M = \int_X |g_{E_\infty}|^p \Rightarrow \int_{A \setminus E_\infty} |g_A|^p = 0$

$\Rightarrow g_A|_{A\setminus E_\infty} = 0$ μ-a.e. $\Rightarrow g_A = g_{E_\infty}$ μ-a.e.

2° Consider $\forall f \in L^p(X, d\mu)$ Note that $A_f := E_\infty \cup \{f \ne 0\}$ is σ-finite.

$\{f \ne 0\} = \bigcup_{n=1}^{\infty} \{|f| > \frac{1}{n}\}$

$\Rightarrow \int_X g_{E_\infty} f \, d\mu = \int_X g_{A_f} f = F(f)$ $\mu\{|f| > \frac{1}{n}\} < (n\|f\|_{L^p})^p < \infty$.

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Ex 引 L^∞ 说明 $\|F\|_{\ast} \to \|g\|_{L^\infty}$

Thm 1' ($p=1$) μ: σ-finite $\quad (L^1(d\mu))' = L^\infty(d\mu)$

$\mu\{|g|>M\} \cdot M^n \le \int_X |g|^n \le \|F\|^n \mu(X)$ $\qquad \mu\{|g|>M\} \le \left(\frac{\|F\|}{M}\right)^n \mu(X) \quad \forall n \in \mathbb{N}.$

若 $M > \|F\|$ 则 $\mu\{|g|>M\} = 0$

$\Rightarrow \|g\|_{L^\infty} \le \|F\|$

σ-有限时 再讨论

§ III.3 Convergence in L^p

1. (Strong) Convergence and Compactness

$f_n \xrightarrow{L^p} f$: $\|f_n - f\|_{L^p} \to 0$ as $n \to \infty$

Thm 2 $p \in [1,\infty)$ If $f_n, f \in L^p$ s.t. 1° $f_n \xrightarrow{\mu} f$ / $f_n \xrightarrow{\mu\text{-a.e.}} f$

2° $\|f_n\|_{L^p} \to \|f\|_{L^p}$ $n \to \infty$.

then $f_n \xrightarrow{L^p} f$ $n \to \infty$.

Pf ($p=1$) (1) $\int_X \min\{|f_n|, |f|\} d\mu \to \int_X |f| d\mu$ 由 $f_n \xrightarrow{\mu\text{-a.e.}} f$ + Lebesgue dominated convergence

(2) $|f| + |f_n| = \min\{|f|, |f_n|\} + \max\{|f|, |f_n|\}$

$\Rightarrow$ by 2° & 1. $\int_X \max\{|f|, |f_n|\} d\mu \to \int_X |f| d\mu.$

(3) $\int_X \big||f| - |f_n|\big| d\mu = \int_X \max\{|f|, |f_n|\} - \min\{|f|, |f_n|\} d\mu \to 0$ as $n \to \infty$.

$|f_n| \xrightarrow{L^1} |f|$

$|f_n| \le \big||f_n| - |f|\big| + |f|.$

$\mu(E) \to 0$ 时. $\int_E |f| \to 0$ $\qquad \int_E |f_n| \le \big\||f_n| - |f|\big\|_{L^1} + \int_E |f|$

$\forall \varepsilon > 0.$ $\exists \mu(A) < \infty$ $\qquad \int_{A^c} |f_n| \le \underbrace{\big\||f_n| - |f|\big\|_{L^1}}_{<\frac{\varepsilon}{2}} + \underbrace{\int_{A^c} |f|}_{<\frac{\varepsilon}{2}} < \varepsilon.$

$\forall n > N$

上方两半

Campus

2. Weak Convergence

$(B, \|\cdot\|)$ normed space $\quad B' = \{$bdd / continuous linear functional on $B\}$

$x_n \in B$ is weakly Cauchy : $\forall f \in B'$, $f(x_n)$ convergence

x_n converges weakly to x_0 : $\lim_{n\to\infty} f(x_n) = f(x_0)$ $\forall f \in B'$ $\quad x_n \rightharpoonup x$

weakly complete : 弱 Cauchy 列必有弱极限

weak-* : B' $\quad f_n \in B'$ $\quad f_0 \in B'$ $\quad f_n \xrightarrow{*} f_0$ $n\to\infty$: $f_n(x) \to f_0(x)$ $\forall x \in B$

$A \subset B$ 弱列紧(闭集) : A 中任意序列有弱收敛子列

rmk 1 Strong convergence $\Rightarrow$ weak convergence

The converse is (in general) not true $\quad$ 例 1° l^2 中 $\{e_n\}_k$

2° $L^2([-\pi,\pi])$ 中 $\{\frac{1}{\sqrt{\pi}}\sin(kx)\}_k$

3° $L^2(0,1)$ 中 $\{n^{\frac{1}{2}}\chi_{(0,\frac{1}{n})}\}_n$

11.13 用 x 表示 L^1 中的一个元素

- $x_n \rightharpoonup x_0$ in $L^1(\Omega, \mathcal{F}, \mu)$ iff $\int_E x_n(w)\mu(dw) \to \int_E x_0(w)\mu(dw)$ $\forall E \in \mathcal{F}$

即 $\forall y \in L^\infty$ $\int_\Omega x_n y \to \int_\Omega x_0 y$

· 若 $\{x_n\}$ 有界, 则强收敛可推出弱收敛

- $x_n \xrightarrow{\mu} x_0 \Rightarrow x_n \rightharpoonup x_0$: $\forall \mu(E)<\infty$ $\left|\int_E (x_n - x_0)d\mu\right| \le \int_{E\cap[|x_n-x_0|>\delta]} |x_n-x_0|d\mu + \int_{E\cap[|x_n-x_0|\le\delta]} |x_n-x_0|\,d\mu$

$\le 2M\cdot\mu[|x_n-x_0|>\delta] + \delta\,\mu(E)$

- $C([0,1])$ 中弱收敛 $\Leftrightarrow$ 点态收敛且有界 : "$\Rightarrow$" : $\{x_n\}$ 在 $C([0,1])$ 上有界 by Resonance Thm

$C([0,1])' = BV_0([0,1])$ $\quad$ 逐点收敛: 取 $\delta_{t_0} \in C([0,1])'$

"$\Leftarrow$" by LDCT

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- $C([0,1])$ is not weakly complete e.g. $X_n(t) = t^n$ $\lim_{n\to\infty} X_n(t) = \begin{cases} 0 & [0,1) \\ 1 & 1 \end{cases} =: y_0$ $\bar{\in} C([0,1])$

by LDT $\lim_{n\to\infty} \int_0^1 x_n(t)\, dF(t)$ exists and $= \int_0^1 y_0\, dF(t)$ $\forall F \in BV$

- Thm $L^1(X, \mathcal{F}, \mu)$ is weakly complete (μ: σ-finite)

pf: Let $\{f_n\}_{n=1}^{\infty} \subset L^1$ be a weak Cauchy seq

$\forall g \in L^{\infty}(X)$, $\lim_{n\to\infty} \int_X f_n g\, d\mu$ exists

$\forall E \in \mathcal{F}$. By taking $g = \chi_E$ we find

$\lim_{n\to\infty} \int_E f_n\, d\mu$ exists, $=: \nu(E)$

which is a measure and $\nu \ll \mu$ (by Vitali-Hahn-Saks)

by Radon-Nikodym $\Rightarrow$ $\exists f_0 \in L(X, \mathcal{F}, \mu)$ s.t. $d\nu = f_0\, d\mu$

i.e. $\lim_{n\to\infty} \int_E f_n\, d\mu = \int_E f_0\, d\mu$ □

- Cor For $f_n \in L^1$ 下列等价

$(L^1)^* = L^{\infty}$ 需 μ σ-finite

(1) $\{f_n\}_{n=1}^{\infty}$ is weakly Cauchy

(2) f_n converges weakly to some $f_0 \in L^1$

(3) $\|f_n\|_{L^1} \le C$ & $\lim_n \int_E f_n\, d\mu$ exists for $\forall E \in \mathcal{F}$

rmk: In Thm above. μ σ-finite is not necessary

- Lemma $(X, \mathcal{F}, \mu)$ $G \in L^p(X, \mathcal{F}, \mu)$ separable subset. Then $\exists X_1 \in \mathcal{F}$

sub σ-algebra $\mathcal{F}_1 \subset \mathcal{F}$ s.t. $(X_1, \mathcal{F}_1, \mu_1 = \mu|_{X_1})$ is a σ-finite measure space

and $L^p(X_1, \mathcal{F}_1, \mu_1)$ is separable $G \subset L^p(X_1, \mathcal{F}_1, \mu_1)$ $g|_{X - X_1} = 0$, $\forall g \in G$

$G = \{f_n\}_{n=1}^{\infty}$ $p \ge 1$

$S_k^{(n)} \xrightarrow{L^1} f_n$ $k \to \infty$ $S_k^{(n)}(x) = \sum_{j=1}^{J_k} C_{k,j}^{(n)} \chi_{E_{k,j}^{(n)}}$ $\{E_{k,j}^{(n)}\}$, $X_1 = \bigcup E_{k,j}^{(n)}$

$\mathcal{F}_1 = \sigma(\{E_{k,j}^{(n)}\})$

- weak compactness

$A \subset X$ (Banach space) is weakly (sequentially) compact if any seq. in A has a weak convergence subseq.

Campus e.g. $L^1([0,1], dx)$ $f_n = n\chi_{(0,\frac{1}{n})}$ [illegible]

Thm on weak compactness of $L^1(X,\mathcal{F},\mu)$

A bdd seq $\{f_n\}_n$ in L^1 is weakly seq. compact iff

$\{f_n d\mu\}_{n=1}^{\infty}$ 一致可列可加 (U.C.A.). 即 if $E_k \downarrow \emptyset$ then $\lim_{k\to\infty}\int_{E_k} f_n \, d\mu = 0$ uniformly in n

(名称由来 $\nu_n(\bigcup_{k=1}^{K} E_k) \to \nu_n(\bigcup_{k=1}^{\infty} E_k)$ 是一致的)

Pf:

11.15 "⇒" $f_{n_k} \rightharpoonup f_0$ in L^1 $\quad \int_E f_n \, d\mu \to \int_E f_0 \, d\mu \quad \forall E \in \mathcal{F}$.

By Vitali-Hahn-Saks. 说明 $\int_E f_n \, d\mu$ $\nu_n \ll \mu$. $\lim_{\mu(E)\to 0}\int_E f_n \, d\mu = 0$ 关于n一致.

可验证~~满足~~等价于 U.C.A.

"⇐" 对 $\mathcal{A} = \{E_k\}_k$ $E_k \downarrow \emptyset$. 可由对角化, 找到 $\{f_n\}_n$ 的一列子列 $\{f_{n_j}\}$

s.t. $\lim_{j\to\infty}\int_{E_k} f_{n_j} \, d\mu = 0$ exists.

Claim: $\forall E \in \sigma(\mathcal{A}) =: \mathcal{F}_1$, $\lim_{j\to\infty}\int_E f_{n_j} \, d\mu$ exists.

denote $\lim_{j\to\infty}\int_E f_n \, d\mu =: \nu(E)$ $\quad$ by VHS, $\exists f_0 \in L^1(X_1,\mathcal{F}_1,\mu_1)$ s.t. $f_n \rightharpoonup f_0$ in $L(X_1)$

$\forall g \in L^\infty(X)$. $F_g(f) = \int_X fg \, d\mu$. $\quad \forall f \in L^1(X_1)$ $\quad$ 要说明 $f_0 \in L^1(X,\mathcal{F},\mu)$

Cor 1 若 K 弱列紧 in L^1 则 $\{f \, d\mu : f \in K\}$ is U.A.C.

反过来. 若 $\mu(X)<\infty$. $\lim_{\mu(E)\to 0}\int_E f \, d\mu = 0$ uniformly in K. then U.A.C. ⇒ 弱紧.

Cor 2 若 $K \subset L^1$ 弱紧. 则 $|K| := \{|f|, f \in K\}$ 也在 L^1 中紧

Cor 3. $f_n \xrightarrow{\|\cdot\|} f_0$ in L^1 iff $\begin{cases} f_n \rightharpoonup f_0 \\ f_n \xrightarrow{\mu} f_0 \end{cases}$ $\forall E \in \mathcal{F}$ 且 $\mu(E) < \infty$.

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Cor 4 If $(X, \mathcal{F}, \mu)$ is an atomic measure, $\mu(\{x\})>0$. $\forall x \in X$.
then weak and strong convergence are equivalent (L^1)

Thm. $p \in (1,\infty)$
1. $L^p(X)$ 弱完备
2. $K \subset L^p(X)$ 弱列紧 iff K is bdd ✓

Thm In a reflexive Banach Space B. $K \subset B$ is weakly seq. compact. iff K is bdd.
自证 By Banach Alaoglu

11.20 Chapter 4 Riesz representation thm

$(\mathbb{R}^N, \mathcal{B}^N)$ $\mu: \mathcal{B}^N \to \mathbb{R}_{\geq 0}$ measure s.t. $\mu(E)<\infty$ if E bdd.
Borel measure (记这样的测度的集合为 $M^+(\mathbb{R}^N)$)

regular $\begin{cases} \mu(E) = \inf\{\mu(G): E \subset G, G\text{开}\} \\ \quad = \sup\{\mu(K): K \subset E \quad K\text{紧}\} \\ \cancel{|\mu(K)| < \infty \quad \forall K\text{紧}} \end{cases}$

Radon measure: 紧集上有限. Borel集外正则. 开集内正则 的Borel测度.

$C_0(\mathbb{R}^N)$ = closure of $C_c(\mathbb{R}^N)$ in $\|\cdot\|_{C_0}$

Thm (Riesz) l 为 $C_c(\mathbb{R}^N)$ 上正线性泛函. 则存在一个 σ-algebra $\mathcal{M} \supset \mathcal{B}(\mathbb{R}^N)$ 及
唯一一个~~完备~~测度 $\mu \in M^+(\mathbb{R}^N)$ s.t. $l(f) = \int_{\mathbb{R}^N} f(x)\, d\mu(x)$ (*)
~~并且 μ 是正则的.~~

Pf: (唯一性) 若 μ_1, μ_2 为两个满足条件的测度. 由正则性. 只需证 $\forall$ 紧集 K. $\mu_1(K) = \mu_2(K)$.
$\forall \varepsilon>0$. 由外正则性. $\exists$ 开 $G_\varepsilon \supset K$ 且 $\mu_1(G_\varepsilon) < \mu_1(K) + \varepsilon$. 由 Uryshon 分离定理. $\exists f: \mathbb{R}^N \to [0,1]$
$f|_K = 1$
$f|_{G_\varepsilon^c} = 0$.
$\mu_2(K) = \int \chi_K \, d\mu_2 \leq \int f \, d\mu_2 = \int f \, d\mu_1 \leq \mu_1(G_\varepsilon) < \mu_1(K) + \varepsilon$.
故 $\mu_2(K) \leq \mu_1(K)$ ($\forall K$ 紧). 同理有 $\mu_1(K) \leq \mu_2(K)$ $\mu_1 = \mu_2 \Rightarrow$ 结果成立.

记: $K \prec f \prec U$
若 K 紧. U 开. $f \in C_c(\mathbb{R}^N)$. $0 \leq f \leq 1$ $f|_K = 1$ $f|_{U^c} = 0$.

(存在性). $\forall$ 开 $G \subset \mathbb{R}^N$. 定义 $\mu(G) = \sup\{l(f) : f \prec G\}$
$\forall E \subset \mathbb{R}^N$. 定义 $\mu^*(E) = \inf\{\mu(G): E \subset G, G\text{开}\}$ 则 $\mu^*(G) = \mu(G)$ 对G开.

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Claim 1: μ^* 是外测度. $\mu^*(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mu^*(A_n)$

只需证 ~~An~~ A_n 均为开集的情况.

$\forall f \prec G := \bigcup_{n=1}^{\infty} G_n$. ~~$\forall K$ 紧. K~~ $\forall K := \mathrm{supp} f$ 紧. $\exists n_0$. $K \subset \bigcup_{n=1}^{n_0} G_n$.

令 $(g_n)_{n=1}^{n_0}$ 为 $(K, \bigcup_{n=1}^{n_0} G_n)$ 上的单位分解: $g_n \prec G_n$, $\sum_{n=1}^{n_0} g_n = 1$ in K.

故 $f(x) = \sum_{n=1}^{n_0} f g_n$

$l(f) = \sum_{n=1}^{n_0} l(f g_n) \le \sum_{n=1}^{n_0} \mu(G_n) \le \sum_{n=1}^{\infty} \mu(G_n)$

$f g_n \prec G_n$

对 $l(f)$ 取 sup 得

~~由~~ 故 $\mu^*(\bigcup_{n=1}^{\infty} G_n) \le \sum_{n=1}^{\infty} \mu(G_n)$

Claim 2 对 $\mathbb{R}^N$ 中紧集 K. $\mu^*(K) = \inf\{l(f) : K \prec f\} < \infty$

"$\le$": $\forall f_0 \succ K$. 对 $\lambda \in (0,1)$ 考虑 $G_\lambda = \{f_0(x) > \lambda\}$. G_λ 开. $K \subset G_\lambda$.

$\mu^*(K) \le \mu(G_\lambda) = \sup\{l(f) : f \prec G\} \le l(\frac{f_0}{\lambda}) = \frac{1}{\lambda} l(f_0)$. 令 $\lambda \to 1$.

(外测度单调性). $f < 1 < \frac{f_0(x)}{\lambda}$ $(\forall x \in G_\lambda)$

$\mu^*(K) \le l(f_0)$. 对 f_0 取 inf

$\mu^*(K) \le \inf\{l(f) : K \prec f\}$

"$\ge$": $\forall K \subset G$. G 开. $\exists f \in C_c(\mathbb{R}^N)$ $K \prec f \prec G$ (by Uryshon)

$\Rightarrow l(f) \le \mu(G) \Rightarrow \inf\{l(f) : K \prec f\} \le \inf\{\mu(G) : K \subset G\}$
$= \mu^*(K)$

· 定义 $\mu_*(E) = \sup\{\mu^*(K) : K \subset E.\ K 紧\}$ $\forall E \subset \mathbb{R}^N$

$\mathcal{M}_F := \{E \in \mathcal{P}(\mathbb{R}^N) \mid \mu^*(E) = \mu_*(E) < \infty\}$.

$\mathcal{M} := \{E \in \mathcal{P}(\mathbb{R}^N) \mid E \cap K \in \mathcal{M}_F,\ \forall K 紧\}$.

Claim. $(\mathbb{R}^N, \mathcal{M}, \mu^*)$ 是一个包含 $\mathcal{B}(\mathbb{R}^N)$ 的测度.

1. $\forall K$ 紧. $K \in \mathcal{M}_F$. $K \in \mathcal{M}$.

2. $\forall G$ 开. $G \in \mathcal{M}$.

只需证 $\mu^*(G) = \mu_*(G)$

"$\le$" absolutely

"$\ge$": $\forall \lambda < \mu^*(G)$. $\exists$ $f_\lambda \prec G$ s.t. $\lambda < l(f_\lambda) \le \mu^*(G)$.

$K_\lambda = \mathrm{supp} f_\lambda$ 紧, $\forall$ 开 $U \supset K_\lambda$. $f_\lambda \prec U$ 故 $l(f_\lambda) \le \mu(U) \Rightarrow \lambda < l(f_\lambda) \le \mu^*(K_\lambda)$.

令 $\lambda \to \mu^*(G)$, 得 $\mu_*(G) \le \mu^*(G)$

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3. 紧集有有限可加性 $\forall K_1 \cap K_2 = \phi$ $\mu^*(K_1 \cup K_2) = \mu^*(K_1) + \mu^*(K_2)$

: $\forall \varepsilon > 0$ $\exists$ $K_1 \cup K_2 \prec f_\varepsilon$ s.t. $\ell(f_\varepsilon) < \mu^*(K_1 \cup K_2) + \varepsilon$

$\forall$ $K_1 \prec f \prec K_2^c$ $K_2 \prec 1-f \prec K_1^c$

$\mu^*(K_1) + \mu^*(K_2) \le \ell(f f_\varepsilon) + \ell((1-f) f_\varepsilon) = \ell(f_\varepsilon) < \mu^*(K_1 \cup K_2) + \varepsilon$.

故 $\mu^*(K_1) + \mu^*(K_2) \le \mu^*(K_1 \cup K_2)$ 又. μ^* 为外测度. 故 $\mu^*(K_1) + \mu^*(K_2) = \mu^*(K_1 \cup K_2)$

4. $\mathcal{M}_F$ 中 μ^* 满足可列可加即不交 $E_n \in \mathcal{M}_F$. $n = 1, 2, 3, \cdots$. $\mu^*(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu^*(E_n)$

进一步若 $\mu^*(\bigcup_{n=1}^{\infty} E_n) < \infty$、则 $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}_F$.

: 1° 若 $\mu^*(\bigcup_{n=1}^{\infty} E_n) = \infty$ 则 $\sum_{n=1}^{\infty} \mu^*(E_n) \ge \mu^*(\bigcup_{n=1}^{\infty} E_n) = \infty$. 等式成立.

2° 若 $\mu^*(\bigcup_{n=1}^{\infty} E_n) < \infty$. $\forall \varepsilon > 0$, $\exists$ compact $K_n \subset E_n$ s.t. $\mu^*(K_n) > \mu_*(E_n) - \frac{\varepsilon}{2^n}$

$\mu^*(\bigcup_{n=1}^{\infty} E_n) \ge \mu^*(\bigcup_{n=1}^{\infty} K_n) = \sum_{n=1}^{\infty} \mu^*(K_n) > \sum_{n=1}^{\infty} \mu_*(E_n) - \varepsilon. = \sum_{n=1}^{\infty} \mu^*(E_n) - \varepsilon$.

故满足可数可加.

$\mu_*(\bigcup_{n=1}^{\infty} E_n) = \mu^*(\bigcup_{n=1}^{\infty} E_n)$: "$\le$": absolutely

"$\ge$".

5. $\forall E \in \mathcal{M}_F$. $\forall \varepsilon > 0$ $\exists$ compact $K_\varepsilon \subset E$. open $G \supset E$. $\mu(G_\varepsilon - K_\varepsilon) < \varepsilon$.

: $\forall \varepsilon > 0$. $\exists$ G_ε 开. $\mu^*(E) + \frac{\varepsilon}{2} > \mu(G_\varepsilon)$

$\exists$ K_ε 紧 $\mu_*(E) - \frac{\varepsilon}{2} < \mu^*(K_\varepsilon)$ 又 $\mu^*(E) = \mu_*(E) = \mu(E)$.

$\mu(G_\varepsilon - K_\varepsilon) = \mu(G_\varepsilon) - \mu(K_\varepsilon) < \varepsilon$. (by 4. 可加性保证)

6. $\forall E_1, E_2 \in \mathcal{M}_F$. $E_1 \setminus E_2$. $E_1 \cap E_2$. $E_1 \cup E_2 \in \mathcal{M}_F$.

:

7. $\mathcal{M}$ is a σ-algebra, continuity $\mathcal{B}(\mathbb{R}^n)$.

7-1 For $A \in \mathcal{M}$. $A^c \cap K = K - (K \cap A) \subset \mathcal{M}_F$

7-2. $(\bigcup_n A_n) \cap K = \bigcup_n (A_n \cap K)$

why $\bigcup A_n \in \mathcal{M}$?

why $\mu(\bigcup A_n \cap K) < \infty$?

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7.3. ~~open $G \in \mathcal{M}$.~~ close F : $F\cap K$ is compact $\Rightarrow F\in\mathcal{M}$. 故 $\mathcal{B}(\mathbb{R}^N)\subset\mathcal{M}$.

(5) regularity of μ on $\mathcal{M}$.

① $\mathcal{M}_F\subset\mathcal{M}$. follows from $K\in\mathcal{M}_F$ and 6.

② $\mathcal{M}_F$ is nothing but $\mathcal{M}$ s.t. $\mu(E)<\infty$.

$\forall\varepsilon>0$. $\exists$ $K_\varepsilon\subset E\subset G_\varepsilon$ s.t. $\mu(G_\varepsilon-K_\varepsilon)<\varepsilon$. But $E\cap K_\varepsilon\in\mathcal{M}_F$ $\exists$ compact C_ε s.t.
$\mu^*(E\cap K_\varepsilon)<\mu(C_\varepsilon)+\varepsilon$.

Since $E\subset(E\cap K_\varepsilon)\cup(G_\varepsilon-K_\varepsilon)$. $\mu^*(E)\le\mu(E\cap K_\varepsilon)+\mu(G_\varepsilon-K_\varepsilon)<\mu(C_\varepsilon)+2\varepsilon<\infty$

$\Rightarrow E\in\mathcal{M}_F$.

(6) $\mu^*|_{\mathcal{M}}$ is a measure $\mu^*(\bigcup_n A_n)=+\infty=\sum_n\mu^*(A_n)$

(7) $\mu^*(A)=+\infty$ 时. i.e. $A\in\mathcal{M}-\mathcal{M}_F$. $A\cap K\in\mathcal{M}_F$ 而 $\mathbb{R}^N=\bigcup_{n=1}^\infty K_n$. σ-紧.
K_n compact

$A=\bigcup(A\cap K_n):=\bigcup_n A_n$

$=\bigcup_{n=1}^\infty B_n$ $B_1=A_1$. $B_2=A_1-A_2$. $B_n\in\mathcal{M}_F$. disadjoint.

$\sum_n\mu(B_n)(\ge\mu^*(A))=\infty$.

$\exists$ compact $C_n\subset B_n$. $\mu(C_n)>\mu(B_n)-\frac{\varepsilon}{2^n}$

$\mu(\bigcup_{j=1}^J C_j)\ge\sum_{j=1}^J\mu(B_j)-(1-2^{-(J+1)})\varepsilon>\sum_{j=1}^J\mu(B_j)-\varepsilon$.

故 $\mu_*(A)=+\infty$. $\mu_*(A)=\mu^*(A)$. $\mu(A)=\infty$

rmk 1 $E\in\mathcal{M}$. $\forall\varepsilon>0$ $\exists$ close $F_\varepsilon\subset E$. open $G_\varepsilon\supset E$. s.t. $\mu(G_\varepsilon-F_\varepsilon)<\varepsilon$.
(靠σ紧性).

2. $\exists$ G_δ-set $A\supset E$ and F_σ-set $B\subset E$ s.t. $\mu(A-B)=0$.

(8) $l(f)=\int_{\mathbb{R}^N}f\,d\mu$ $f\in C_c(\mathbb{R}^N)$.

: 只需证 $l(f)\le\int_{\mathbb{R}^N}f\,d\mu$. (若其成立 代入 $-f$. 得 $-l(f)\le\int_{\mathbb{R}^N}-f\,d\mu$
$l(f)\ge\int_{\mathbb{R}^N}f\,d\mu$)

先假设 $|f|\le a$ $\forall\varepsilon>0$. $\exists n\in\mathbb{N}$ s.t. $\frac{2a}{n}<\varepsilon$. $-a<-a+\frac{2a}{n}<-a+2\cdot\frac{2a}{n}<\cdots<a$

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Cor 1 For any Radon measure μ on $(\mathbb{R}^N, \mathcal{B}^N)$.
$C_c(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N, d\mu)$ $1 \le p < \infty$. Hence L^p is separable.

Pf: It suffices to show that $\forall E \in \mathcal{B}(\mathbb{R}^N)$ with $\mu(E) < \infty$

χ_E can be approximated by $C_c(\mathbb{R}^N)$ in L^1

$\forall \varepsilon > 0$. 由E内外正则. $\exists\ F \subset E \subset U$. F闭. U开. $\mu(U \setminus F) < \varepsilon$.

By Uryshon's lemma $\exists\ f \in C_c(\mathbb{R}^N)$ $F \prec f \prec U$. $\int |\chi_E - f|\, d\mu \le \mu(U \setminus F) < \varepsilon$.

Riesz Rep. Thm Ver 2 $C_0(\mathbb{R}^N)' \cong M(\mathbb{R}^N, \mathcal{B}^N)$.

for any bdd linear functional F on $(C_0(\mathbb{R}^N), \|\cdot\|_0)$

there exists a unique Radon measure $\mu = \mu_F \in M(\mathbb{R}^N, \mathcal{B}^N)$ s.t.

$F(f) = \int_{\mathbb{R}^N} f\, d\mu \quad \forall f \in C_0$ with $\|F\| = \|\mu\|$ $(= |\mu|(\mathbb{R}^N))$

Pf: 1 Uniqueness $\mu = \mu_1 - \mu_2$ $\int_{\mathbb{R}^N} f\, d\mu = 0 \quad \forall f \in C_0$ $d\mu = h\, d|\mu|$. $|h| = 1$ a.e. μ.

$|\mu|(\mathbb{R}^N) = \int_{\mathbb{R}^N} |h|^2\, d|\mu| = \int_{\mathbb{R}^N} h\bar{h}\, d|\mu| = \int h(\bar{h} - f)\, d|\mu| \le \int |\bar{h} - f|\, d|\mu| \quad \forall f \in C_0$

$\bar{h} \in L^1(d\mu)$ $\to 0$,

2. define $P(f) = \sup\{ |F(g)| : |g(x)| \le f(x) \}$ $\forall f \in C_c^+$

① $f \in C_c$. $f = f^+ - f^-$ $P(f) := P(f^+) - P(f^-)$

② $\forall f \ge 0$. $P(f) \ge 0$. $P(f) \le \|F\| \cdot \|f\|_0$ Assume w.l.o.g. $\|F\| = 1$.

$f_2 \ge f_1 \Rightarrow P(f_2) \ge P(f_1)$. by definition.

③ Linearity. $P(f_1 + f_2) = P(f_1) + P(f_2)$. $f_1, f_2 \in C_c^+$

$\forall \varepsilon > 0$. $\exists\ g_1, g_2 \in C_c$ s.t. $P(f_j) < |F(g_j)| + \varepsilon$. $j = 1, 2$

$P(f_1) + P(f_2) < |F(g_1)| + |F(g_2)| + 2\varepsilon = \alpha_1 F(g_1) + \alpha_2 F(g_2) + 2\varepsilon = F(\alpha_1 g_1 + \alpha_2 g_2) + 2\varepsilon$

$|\alpha_j| = 1$. $\alpha_j = \overline{\operatorname{sgn} F(g_j)}$ $\le P(f_1 + f_2) + 2\varepsilon$.

故 $P(f_1) + P(f_2) \le P(f_1 + f_2)$.

反过来. $\forall g \in C_c$ s.t. $|g| \le f_1 + f_2$. $g_j(x) = \begin{cases} \dfrac{f_j g}{f_1 + f_2} & f_1(x) + f_2(x) \ne 0 \\ 0 & \text{else} \end{cases}$ $|g_j| \le f_j$

$g_1 + g_2 = g$.

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$|F(g)| = |F(g_1+g_2)| = |F(g_1)+F(g_2)| \le |F(g_1)| + |F(g_2)| \le P(f_1) + P(f_2)$

故 $P(f_1+f_2) \le P(f_1)+P(f_2)$.

记得 $P(f_1+f_2) = P(f_1) + P(f_2)$.

即, Given $F \in C_0'$ $\exists P$ 为 C_c 上正线性泛函 s.t. $|F(f)| \le P(|f|) \le \|f\|_0$ $\forall f \in C_c$.

3. Applying Riesz Rep. Thm Ver 1. $\exists$ (positive) Borel measure λ s.t.

$$P(f) = \int f \, d\lambda \quad \forall f \in C_c$$

why this?

and $\lambda(\mathbb{R}^N) \underset{\uparrow}{=} \lim_{n\to\infty} \lambda(\bar{B}_n) \le \limsup_n \int f_n \, d\lambda \qquad f_n \succ \bar{B}_n$

测到 $\mathbb{R}^N$ 的一些 $\le \limsup_n P(f_n) \le \|f_n\|_0 = 1$.

$|F(f)| \le \int_{\mathbb{R}^N} |f| \, d\lambda = \|f\|_{L^1(d\lambda)} \quad f \in C_c$ $\Rightarrow F$ bdd in $L^1(d\lambda)$

$\therefore F \in (L^1(d\lambda))' \cong L^\infty(d\lambda)$: $\exists g \in L^\infty(d\lambda)$, $\|g\|_{L^\infty} \le 1$ s.t. $\|F\|_{(L^1)'} \le 1$

且 $F(f) = \int_{\mathbb{R}^N} fg \, d\lambda \quad \forall f \in C_c$

Now let $d\mu = g \, d\lambda$ i.e. $F(f) = \int_{\mathbb{R}^N} f \, d\mu$

(Note $d|\mu| = |g| \, d\lambda$)

Then $|\mu|(\mathbb{R}^N) = \int_{\mathbb{R}^N} |g| \, d\lambda \left(= \sup_{\|h\|_{L^\infty}=1} \int_{\mathbb{R}^N} gh \, d\lambda\right) \ge \sup\{|F(f)| : f \in C_c, \|f\|_0 \le 1\}$

$= \|F\| = 1$

然而 $\int_{\mathbb{R}^N} |g| \, d\lambda \le \lambda(\mathbb{R}^N) \le 1$. 故 $\int_{\mathbb{R}^N} |g| \, d\lambda = 1$.

Extension 1 LCH + σ-compact

(Locally Compact Hausdorff). $X = \bigcup_n K_n$.

1. Uryshon lemma : $K \prec f \prec G$

2. Partition of unity : subject to $K \subset \bigcup_{j=1}^n G_j$ $f_j \prec G_j$ $\sum_j f_j = 1$ on K.

C_c 的 完备化 C_0

Extension 0. ① $\Omega \subset \mathbb{R}^N$: domain

$C_c(\Omega) \qquad C_0(\Omega) \qquad C_0(\Omega)' = M(\Omega)$

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② compact $\overline{\Omega} \subset \mathbb{R}^N$ $\quad C(\overline{\Omega})' = M(\overline{\Omega})$

$(0,1)$ $\quad d\mu = \frac{1}{t}dt$

$$\int f(t)\frac{1}{t}dt < \infty \qquad f \in C_c((0,1))$$

③ {Radon measures} = dual of $C_c(\Omega)$, where the convergence on $C_c(\Omega)$ is $f_n \to f_0$ $n\to\infty$

1° $\exists$ compact $K \subset\subset \Omega$ s.t. $\operatorname{supp} f_n \subset K$ $\forall n$

2° $f_n \rightrightarrows f_0$ on K

Extension 3 $\quad X$: metric space $\qquad$ Dunford-Schwartz

$BC(X)'$ = regular and finitely additive "measure" with finite total variation.

Coro $\quad X$: compact metric space $\quad C(X)'$ = regular, contable additive (X)

$B(X)'$ = 有界,有限可加(X)上测度 $\qquad$ Alexandroff

有限可加 $\xRightarrow{正则}$ 可列可加

11.29 $\qquad$ (r: regular (T_3) ca: countable additive (measure); b: bdd)

Weak (-*) convergence of measure

1. $C(X)' = M(X) = rca(X)$, X compact metric space
2. $BC(X)' = rba(X, B(X))$ $\quad X$: metric space

$$\left|\int_X f\,d\mu\right| \le \|f\|_0 \|\mu\|$$

3. $C_0(X)' = M(X, B(X))$, $\quad X$: LCH

Weak convergence in $ca(X, \mathcal{F})$

$\mu_n \to \mu$ $\qquad$ let $\lambda(E) = \sum_{n=1}^{\infty} \frac{\mu(E)}{2^n|\mu|(X)}$ $\quad$ then $\mu_n \ll \lambda$.

$\mu_n(E) = \int_E d\mu_n = \int_E f_n\,d\lambda$. $\quad f_n \in L^1(d\lambda)$

$f_n \rightharpoonup f \Leftrightarrow \int_E f_n\,d\lambda \to \int_E f\,d\lambda \Leftrightarrow \mu_n(E) \to \mu(E)$ $\forall E \in \mathcal{F}$

$\forall E \in \mathcal{F}$ 决定了 $ca(X,\mathcal{F})$ 上一个 bdd linear functional

$$F_E: \mu \to \mu(E)$$

weak seq. compactness $\quad$ A bdd subset $K \subset ca(X, \mathcal{F})$ is seq compact iff countable additivity is uniform in $\mu \in K$

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weak-* convergence

E.g. 1 (X,ρ) : metric space

$\forall x\in X$ $\delta_{\{x\}}\in M(X)$ Then $x_n \xrightarrow{\rho} x_0$ iff $\delta_{\{x_n\}} \xrightarrow{*} \delta_{\{x_0\}}$

pf: "$\Rightarrow$" $\forall f\in C(X)$. $\delta_{\{x_n\}}(f)=f(x_n)\to f(x_0)=\delta_{\{x_0\}}(f)$ 即 $\delta_{\{x_n\}}\xrightarrow{*}\delta_{\{x_0\}}$

"$\Leftarrow$" 若 x_n 不收敛到 x_0. $\exists\,\varepsilon>0$ s.t. $\exists\,\{x_{n_j}\}_j\subset\{x_n\}_n$ $\rho(x_{n_j},x_0)>\varepsilon$.

$\overline{B(x_0,\frac{\varepsilon}{2})}\prec f\prec B(x_0,\varepsilon)$ $f(x_{n_j})=0$. $f(x_0)=1$ 与 $\delta_{\{x_{n_j}\}}\xrightarrow{*}\delta_{\{x_0\}}$ 矛盾

E.g. 2. $(\mathbb{R},\mathcal{B})$ $C_0(\mathbb{R})'=M(\mathbb{R})$. $\delta_n:=\delta_{\{n\}}\xrightarrow{*}0$

E.g. 3 $X=[0,1]$ $\mu_n=\frac{1}{n}\sum_{j=1}^{n}\delta_{\{\frac{j}{n}\}}$ $\|\mu_n\|=1$.

$\forall f\in C[0,1]$ $\int_{[0,1]}f\,d\mu_n=\frac{1}{n}\sum_{j=1}^{n}f(\frac{j}{n})\to\int_0^1 f(t)\,dt$ $\mu_n\xrightarrow{*}m$

(In fact. $\forall\mu\in M([0,1])$ there exists a seq μ_n (μ_n is the linear combination of δ-measures) s.t. $\mu_n\xrightarrow{*}\mu$)

注意到 $\mu_n(\mathbb{Q}\cap[0,1])=1$ $\forall n$. 而 $m(\mathbb{Q}\cap[0,1])=0$ 它不是自反空间

E.g. 4 $X=[0,1]$ $\delta_{\{\frac{1}{n}\}}\xrightarrow{*}\delta_{\{0\}}$ 而 $\delta_{\{\frac{1}{n}\}}((0,1))=1$ $\forall n$. $\Rightarrow$ $\delta_{\{\frac{1}{n}\}}\not\xrightarrow{*}\delta_{\{0\}}$

$\delta_{\{0\}}((0,1))=0$

Banach - Alaoglu Thm.

The unit ball of B' is weak-* compact B: Banach space

Especially if B is separable then any (strong) bdd set in B' is seq weak* compact

("vague convergence" $P_n \xrightarrow{*} P_0$: $\int f(x)\,dP_n\to\int f(x)\,dP_0$ $\forall f\in BC(X)$)

Thm (Alexandorff Portmanteau)

(X,ρ) metric space P_n, P : $(X,\mathcal{B}(X))$ 上概率测度.

Then the following statements are equivalent:

1° $P_n\overset{V}{\Rightarrow}P$: $\int_X f\,dP_n\to\int_X f\,dP$, $\forall f\in BC(X)$. vaguely convergence

2° $\limsup_n P_n(F)\le P(F)$ $\forall F$ close set in X

Campus

$\liminf_n P_n(G)\ge P(G)$ $\forall$ open G

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3° $\lim_n p_n(A) = p(A)$ $\forall A \in \mathcal{B}(X)$. s.t. $p(\partial A) = 0$. $\partial A = \bar{A} - \mathrm{int}(A)$

课上不证

Thm For $p_n, p \in P(\mathbb{R})$. $F(x) = p((-\infty, x])$.

$p_n \xrightarrow{w} p$ iff $F_n(x) \to F(x)$ $\forall$ continuity point x of F.

Thm (Prokhorov) A subset $\Pi \subset P(X)$ is weak* seq. compact provided Π is tight:

$\forall \varepsilon > 0$. $\exists$ compact $K_\varepsilon \subset X$ s.t. $p(K_\varepsilon) > 1-\varepsilon$ $\forall p \in \Pi$

$BC(\mathbb{R})$ is not separable

No.

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弱拓扑和强拓扑：　　　　　　　　No.　　Date 12.18

Montel空间 反例：一列 Montel空间 诱导 极限拓扑空间 [illegible]

Topology on $C_c(\Omega)$: induced limit topology: (诱导极限拓扑)

Let (X_n, τ_n) be a sequence of LCT spaces s.t.

$$X_n \subset X_{n+1}, \quad Id_n : X_n \hookrightarrow X_{n+1} \quad \text{is continous}$$

let $X := \bigcup_{n=1}^{\infty} X_n$. The induced limit topology τ on X is 其上最强拓扑 s.t.

(1) $Id_n : X_n \to X$ is continous

[illegible]

If moreover: (2) $(X_{n+1}, \tau_{n+1})|_{X_n} = (X_n, \tau_n)$

(3) X_n is a closed subspace of X_{n+1}

then (i) $(X, \tau)|_{X_n} = (X_n, \tau_n)$

(ii) $x_k \to x_0$ in (X, τ) iff $\exists X_N$ s.t. $x_k \to x_0$ in X_N

E.g. 1. $X_n = \mathbb{R}^n$. $X = \bigcup_{n=1}^{\infty} \mathbb{R}^n =: C_{0,0}$

$x_k \to x_0$ in $C_{0,0}$ iff $\exists N$ s.t. $x_k \to x_0$ in $\mathbb{R}^N$

E.g. 2. $C_c(\Omega) = \bigcup_{n=1}^{\infty} C_c(K_n)$ $\quad C_c(K_n)$: $\|\varphi\|_n = \max_{x \in K_n} |\varphi(x)|$

$\varphi_k \to \varphi_0$ in $C_c(\Omega)$ iff $\exists K \subset\subset \Omega$ s.t. $\operatorname{supp} \varphi_k \subset K$ $(\forall k)$, and $\varphi_k \rightrightarrows \varphi_0$ on K

rmk $C_c(\Omega)' \cong$ Randon Measures on Ω.

E.g. 3 $(C_c^{\infty}(\Omega), \tau) = \mathcal{D}(\Omega)$

$\bigcup_{n=1}^{\infty} C_c^{\infty}(K_n)$ $\quad P_{m,n}(\varphi) = \max_{\substack{x \in K_n \\ |\alpha| \le m}} |\partial^{\alpha} \varphi(x)|$ $\quad \operatorname{supp} \varphi \subset K_n$

Convergence in $D(\Omega)$: $\varphi_k \to \varphi_0$ in $D(\Omega)$ iff

$\exists K \subset\subset \Omega$ s.t. $\operatorname{supp} \varphi_k \subset K_n$ and $\partial \varphi_k \rightrightarrows \varphi_0$ on K $\forall \alpha$

rmk $\mathcal{D}(\mathbb{R}^N) \hookrightarrow \mathcal{S}(\mathbb{R}^N) \hookrightarrow \mathcal{E}(\mathbb{R}^N)$ 在连续嵌入情况下，空间大，对偶小

P20

$\mathcal{D}'$ E.g. 1 $f \in L^1_{loc}(\Omega) \Rightarrow f \in L^1(K)$ $\forall K \subset\subset \Omega$.

$T_f(\varphi) = \langle T_f, \varphi \rangle := \int_{\Omega} f \varphi \, dx$ $\quad \forall \varphi \in C_c^{\infty}(\Omega)$

E.g. 2 Radon measures on $\mathbb{R}^N$

$\delta_{\{0\}}(\varphi) = \int_{\mathbb{R}^N} \varphi(x) \, \delta_{\{0\}}(dx) = \varphi(0)$. $\quad \forall \varphi \in C_c^{\infty}(\Omega)$.

$\mathbb{R}$: $\delta_1(\varphi) = -\varphi'(0)$. $\delta_1 \in \mathcal{D}'(\mathbb{R})$. In fact, $\delta_1 = \dfrac{d\,\delta_{\{0\}}}{dx}$

E.g. 3 $\langle \text{p.v.}\frac{1}{x}, \varphi \rangle = \lim\limits_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x} = \lim\limits_{\varepsilon\to 0}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x}\,dx\right) = \lim\limits_{\varepsilon\to 0}\Big(\varphi(-\varepsilon)\ln\varepsilon - \int_{-\infty}^{-\varepsilon}\varphi'(x)\ln|x|\,dx$

例如，$\mathbb{R}$ $\qquad - \varphi(\varepsilon)\ln\varepsilon - \int_{\varepsilon}^{\infty}\varphi'(x)\ln x\,dx\Big)$

$$= \lim_{\varepsilon\to 0}\left(\varphi'(\theta_\varepsilon)\cdot 2\varepsilon\cdot\ln\varepsilon - \int_{|x|>\varepsilon}\varphi'(x)\ln|x|\,dx\right) = -\int_{\mathbb{R}}\varphi'(x)\ln|x|\,dx$$

$(\ln|x|)' = \text{p.v.}\frac{1}{x}$

Thm1 A linear functional T on $\mathcal{D}(\Omega)$ belongs to $\mathcal{D}'(\Omega)$ iff $\forall K \subset\subset \Omega$ $\exists C_K \geq 0$ & $m = m_K \in \mathbb{N}$ s.t.

$$|\langle T, \varphi\rangle| \leq C_K \sup_{\substack{|\alpha|\leq m \\ x\in K}} |\partial^\alpha \varphi(x)| \quad \forall \varphi \in \mathcal{D}(\Omega)$$

Thm2 A linear functional $T \in \mathcal{D}'(\Omega)$ iff $\forall \varphi_n \to 0$ in $\mathcal{D}(\Omega)$ $\Rightarrow \langle T, \varphi_n\rangle \to 0$ as $n\to\infty$

Topology on $\mathcal{D}'(\Omega)$ $\forall T \in \mathcal{D}'(\Omega)$. $P_\varphi(T) = |\langle T, \varphi\rangle|$ $\forall \varphi \in C_c^\infty(\Omega)$.

weak $*$- topology

$\mathcal{E}'(\Omega) = \mathcal{D}'(\Omega)$ with compact support

Thm3 a distribution $T \in \mathcal{E}'(\Omega)$ iff $\exists K \subset\subset \Omega$ $C \geq 0$ $m \in \mathbb{N}$ s.t.

$$|\langle T, \varphi\rangle| \leq C \sup_{\substack{|\alpha|\leq m \\ x\in K}} |\partial^\alpha \varphi(x)| \quad \forall \varphi \in C^\infty(\Omega)$$

E.g. 1. $f \in L^1(\Omega)$ has compact support

$$T_f(\varphi) = \langle T_f, \varphi\rangle := \int_\Omega f\varphi\,dx, \quad \forall \varphi \in C^\infty(\Omega).$$
$$= \int_{\text{supp} f} f\varphi\,dx \in \mathbb{C}$$

$\varphi_n \to 0$ in $\mathcal{E}(\Omega)$ ($\partial^\alpha \varphi \rightrightarrows 0$ $\forall \alpha$ on $\forall K \subset\subset \Omega$)

E.g. 2. μ: Radon measure with compact support

define $T_\mu(\varphi) := \int_{\mathbb{R}^N} \varphi(x)\mu(dx) = \int_{\text{supp}\mu} \varphi\,d\mu$

Thm 4 A distribution $T \in \mathcal{E}'(\Omega)$ iff T is compactly supported

Support of a distribution

1. $T=0$ in open $U \subset \Omega$. $\langle T, \varphi \rangle = 0$ $\forall \varphi \in C_c^\infty(U)$ with $\operatorname{supp} \varphi \subset U$
2. $T_1 = T_2$ in U if $T_1 - T_2 = 0$ in U
3. $T_i = 0$ in U_i $i \in I \Rightarrow T = 0$ in $\bigcup_{i \in I} U_i$

Def (Support of a distribution). $T \in \mathcal{D}'(\Omega)$.

$\operatorname{supp} T = U_T^c$ U_T 为使 $T=0$ in U_T 的最大开集

Pf of Thm 3 & 4

Since $T \in \mathcal{E}'(\Omega)$ for $\varepsilon = 1$. $\exists \delta_0 > 0$ and $m_0 \in \mathbb{N}$ $K_{n_0} \subset\subset \Omega$ s.t.

$$P_{m,n}(\varphi) = \sup_{\substack{|\alpha| \le m \\ x \in K_n}} |\partial^\alpha \varphi(x)| < \delta_0 \quad \forall \varphi \in C^\infty(\Omega) \qquad (*)$$

then $|\langle T, \varphi \rangle| \le 1$

For $\forall \psi \in C^\infty(\Omega)$, if $P_{m_0,n_0}(\psi) \ne 0$ let $\varphi(x) = \frac{\delta_0}{P_{m_0,n_0}(\psi)} \psi$ Then φ s.t. $(*)$

$$|\langle T, \varphi \rangle| = \left|\langle T, \frac{\delta_0}{P_{m_0,n_0}(\psi)} \psi \rangle\right| = \frac{\delta_0}{P_{m_0,n_0}(\psi)} |\langle T, \psi \rangle| \le 1 \Rightarrow |\langle T, \psi \rangle| \le \frac{1}{\delta_0} P_{m_0,n_0}(\psi)$$

Lemma: If $P_{m_0,n_0}(\psi) = 0$ then $\langle T, \psi \rangle = 0$. Hence $\operatorname{supp} T \subset K_{n_0}$.

Pf: ψ s.t. $(*)$ $\Rightarrow |\langle T, \psi \rangle| \le 1$.

But $\forall \lambda > 0$, $\lambda \psi$ s.t. $(*)$ $\Rightarrow |\langle T, \lambda \psi \rangle| \le 1 \Rightarrow |\langle T, \psi \rangle| = 0$

Derivatives of a distribution

For $T \in \mathcal{D}'(\Omega)$ define $\langle T_j, \varphi \rangle = -\langle T, \partial_j \varphi \rangle$ $\forall \varphi \in C_c^\infty(\Omega)$. Then $T_j \in \mathcal{D}'(\Omega)$

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle$$

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对 $f \in C^1(\mathbb{R}^1 \setminus \{0\})$ $\qquad \underbrace{\frac{df}{dx}}_{\text{分布意义下的导数}} = \underbrace{J_0 \delta_0}_{\text{跳跃部分}} + \underbrace{f'}_{\text{绝对连续部分}}$ $\qquad J_0 := f(0+) - f(0-)$

(recall: Lebesgue 分解)

E.g. 2 $f \in C^1(\bar{\Omega})$ 对 $x \notin \bar{\Omega}$, $f(x) = 0$ $\qquad \Omega$ $\Gamma = \partial\Omega$.

then $f \in L^\infty(\mathbb{R}^2) \subset \mathcal{D}'(\mathbb{R}^2)$

$$\langle \frac{\partial f}{\partial x_1}, \varphi \rangle = -\langle f, \partial_{x_1} \varphi \rangle = -\int_\Omega f \cdot \partial_{x_1} \varphi \, dx = -\int_\Gamma f \varphi n_1 \, ds + \int_\Omega \partial_{x_1} f \varphi \, dx$$

denote $d\mu = -f n_1 \, ds$, which is supported on Γ. Then $\frac{\partial f}{\partial x_1} = \mu + \partial_{x_1} f$ in $\mathcal{D}'(\mathbb{R}^2)$

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Especially let $f(x) \equiv 1$ $x \in \bar{\Omega}$ Then $\frac{\partial f}{\partial x_1} = n_1 ds$ $\frac{\partial f}{\partial x_2} = n_2 ds$

$\frac{\partial f}{\partial \vec{n}} (:= \nabla f \cdot \vec{n}) = ds = \delta_\Gamma$

$\mathcal{D}'(\mathbb{R})$ 1. $\forall T \in \mathcal{D}'(\mathbb{R})$ $\exists S \in \mathcal{D}'(\mathbb{R})$ s.t. $\frac{dS}{dx} = T$ in $\mathcal{D}'(\mathbb{R})$

and if $\frac{dS_1}{dx} = T$ then $\exists$ const c s.t. $S_1 = S + c$

2. $f \in C(\mathbb{R})$ s.t. $f \in AC([a,b])$ for any $[a,b] \subset \mathbb{R}$ Then $\frac{df}{dx} = f'$

Conversely, if $T \in \mathcal{D}'(\mathbb{R})$ satisfying $\frac{dT}{dx} = g \in L'_{loc}(\mathbb{R})$ then $T \in AC([a,b])$

rmk In general if $f \in L'_{loc}(\Omega)$ satisfies $\frac{\partial f}{\partial x_j} \in L'_{loc}(\Omega)$ $j = 1, 2, \cdots, N$

we call f a Sobolev function ($W^{1,1}_{loc}(\Omega)$)

§3 Local structure of distributions

1. Distribution of finite order

$C_c^m(\Omega)$ $(\mathcal{D}^m(\Omega))' = \mathcal{D}'_m(\Omega)$

$\mathcal{D}(\Omega) \hookrightarrow \mathcal{D}^m(\Omega)$ $m = 0, 1, 2, \cdots$

$\mathcal{D}'_m(\Omega) \subset \mathcal{D}'(\Omega)$

E.g. A Radon measure on Ω is a distribution of 0-order

$\delta_0 \in \mathcal{D}'(\mathbb{R})$ $\delta_0^{(k)} \in \mathcal{D}'(\mathbb{R})$ $\langle \delta_0^{(k)}, \varphi \rangle = (-1)^k \varphi^{(k)}(0)$

E.g.2 $\mathcal{E}'(\Omega) \subset \bigcup_{m=0}^{\infty} \mathcal{D}'_m(\Omega)$ with compact support

2. Local structure

Thm let $T \in \mathcal{D}'(\Omega)$ $\Omega \subset \mathbb{R}^N$ Then for any open $\omega \subset \bar{\omega} \subset\subset \Omega$.

there exists $f \in L^\infty(\omega)$ and $m \in \mathbb{N}$ s.t. $T = \frac{\partial^{mN} f}{\partial x_1^m \cdots \partial x_N^m}$ in $\mathcal{D}'(\omega)$

Pf: $T \in \mathcal{D}'(\Omega)$ for $\forall$ given $K \subset\subset \Omega$ $\exists m \in \mathbb{N}$ s.t.

$$|\langle T, \varphi \rangle| \le C \sup_{\substack{x \in K \\ |\alpha| \le m}} |\partial^\alpha \varphi(x)| \qquad \forall \varphi \in C_c^\infty(K)$$

let $K = \bar{\omega}$

Then for $\forall \varepsilon > 0$ $\exists m \in \mathbb{N}$ $\delta > 0$ s.t. $\forall \varphi \in C_c^\infty(K)$ satisfying

$\sup_{\substack{x \in K \\ |\alpha| \le k}} |\partial^\alpha \varphi(x)| < \delta$ we have $|\langle T, \varphi \rangle| < \varepsilon$ $\varphi \in C_c^\infty(K)$

2° $N=1$ let X_k $\{\psi \in C_c^\infty(K) : \psi \in \frac{d^{k+1}\varphi}{dx^{k+1}},\ \varphi \in C_c^\infty(K)\}$

equipped with $L^1(K)$-norm

Claim: If $\psi_n \to 0$ in $L^1(K)$ then $\langle T, \varphi_n \rangle \to 0$, $n \to \infty$ ([illegible]记)

$\psi_n = \frac{d^{k+1}\varphi_n}{dx^{k+1}}$

$K=[a,b]$ $\quad \frac{d\varphi_n}{dx} \to 0$ in $L^1(K) \Rightarrow \varphi_n \rightrightarrows 0$

$\varphi_n(x) = \int_a^x \varphi_n'(t)\,dt.$

$|\varphi_n(x)| \le \int_a^b |\varphi_n'(t)|\,dt \to 0$

3° Define

$$\ell(\psi) = \langle T, \varphi \rangle,\quad \psi = \frac{d^{k+1}\varphi}{dx^{k+1}} \quad \text{Then } \ell \in X_k'$$

By Hahn-Banach, ℓ can be extended as a bdd linear functional on $L^1(K)$

Hence $\exists g \in L^\infty(K)$ s.t. $\ell(\psi) = \int_K g\psi\,dx$

$\forall \varphi \in C_c^\infty(\Omega)$, $\langle T, \varphi \rangle = \ell(\psi) = \int_K g \frac{d^{k+1}\varphi}{dx^{k+1}}\,dx$ i.e. $T = (-1)^{k+1}\frac{d^{k+1}g}{dx^{k+1}}$

Let $f = (-1)^{k+1}g$

E.g. 1 $\delta_0 = \frac{dH(x)}{dx}$

Rmk $f \in L^\infty(K)$ $\tilde{f}(x) = \begin{cases} f(x) & x \in K = \bar{\omega} \\ 0 & x \in \Omega \setminus K \end{cases} \in L^\infty(\Omega)$

Define $F(x) = \int_{-\infty}^x \tilde{f}(t)\,dt \in C(\Omega)$

$T = \frac{d^{k+2}F}{dx^{k+2}}$ in $\mathcal{D}'(\omega)$ Moreover let $\eta \in C_c^\infty(\Omega)$ s.t. $\bar{\omega} \prec \eta \prec U$ open $\supset \bar{\omega}$

and $G = \eta F$ Then $G \in C_c^\infty(\Omega)$ and $G = F$ on $\bar{\omega}$

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Thm 5 Every distribution with compact support i.e. $T \in \mathcal{E}'(\Omega)$ can be represented as $T = \sum_{|\alpha| \le k} \frac{\partial^\alpha f_\alpha}{\partial x^\alpha}$ in Ω. $k \in \mathbb{N}$ $f_\alpha \in C_c(\Omega)$

Pf: $T = \frac{\partial^\beta f}{\partial x^\beta}$ in $w \subset \bar{w} \subset\subset \Omega$ $\quad K = \operatorname{supp} T \subset w$ $\quad \forall \varphi \in C_c^\infty(\Omega)$

Let $\eta \in C_c^\infty$ s.t. $K \prec \eta \prec w$ $\qquad \varphi\eta = \varphi$ in K

$\langle T, \varphi \rangle = (-1)^{|\beta|} \int_\Omega f\,\partial^\beta \varphi \qquad \forall \varphi \in C_c^\infty(\Omega)$

$\langle T, \eta\varphi \rangle = (-1)^{|\beta|} \int_\Omega f\, D^\beta(\eta\varphi) = (-1)^{|\beta|} \int_\Omega f \sum C_\beta^\alpha D^{\beta-\alpha}\eta \cdot D^\alpha \varphi$

$= (-1)^{|\beta|} \int_\Omega \sum C_\beta^\alpha D^{\beta-\alpha}\eta f D^\alpha \varphi$

$= \langle \sum_{|\alpha| \le k} D^\alpha f_\alpha, \varphi \rangle$, $f_\alpha = (-1)^{|\beta|+|\alpha|} C_\beta^\alpha D^{\beta-\alpha}\eta f$

No.

Date 2024. 12. 27 $T \in \mathcal{E}' \Rightarrow \hat{T} \in O_M$

Cor: Every $T \in \mathcal{E}'(\Omega)$ is of finite order

Thm7: If $T \in \mathcal{E}'(\mathbb{R}^n)$ s.t. $\operatorname{supp} T = \{0\}$ then $T = \sum_{|\alpha| \le k} c_\alpha \partial^\alpha \delta_0$

Multiplier space of Tempored distribution

$O_M = \{ \psi \in C^\infty \mid \exists$ polynomal $P_\alpha(x)$ s.t. $|\psi^{(\alpha)}(x)| \le |P_\alpha(x)| \}$.

Slowing increasing C^∞ space

Prop. $\psi \in C^\infty$. $\psi \in O_M \Leftrightarrow \forall \varphi \in S, \psi\varphi \in S \Leftrightarrow \forall \alpha \in \mathbb{N}^n, \varphi \in S,$ $\varphi \partial^\alpha \psi$ bdd.

Thm9 O_M is the multiplier space of S'

Thm10. $\varphi \in S$, $T \in S' \Rightarrow \varphi * T \in O_M$

Thm11. A distribution $T \in S'$ iff there exists $k \in \mathbb{N}$. $f_\alpha \in C(\mathbb{R}^n)$ $|\alpha| \le k$ which are slowly increasing s.t. $T = \sum_{|\alpha| \le k} \partial^\alpha f_\alpha$

(缓增分布是缓增函数的导数)

Paley-Wiener-Schwartz Thm

1. For $\varphi \in C_c^\infty$, $\hat{\varphi}(z) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot z} \varphi(x)\, dx$ $z = \xi + i\eta \in \mathbb{C}^n$

is an entire function in $\mathbb{C}^n$ s.t.

$\exists R > 0$. $\forall n \in \mathbb{N}$. $\exists C_{n,N}$ s.t. $|\hat{\varphi}(z)| \le C(1+|z|)^{-n} e^{2\pi R|\eta|}$ (1)

conversely 也对 (?)

2. For $T \in \mathcal{E}'$ $\hat{T}(z) = \langle T_x, e^{-2\pi i z \cdot x} \rangle$ entire in $\mathbb{C}^n$

satisfying $\exists C > 0$, $R > 0$. $n \in \mathbb{N}$ s.t. $|\hat{T}(z)| \le C(1+|z|)^n e^{2\pi R|\eta|}$ (2)

Conversely, given $F(z) \in (2)$ entire in $\mathbb{C}^n$

$\exists T \in \mathcal{E}'$ s.t. $\hat{T}(z) = F(z)$

$\widehat{J_n * T} = \hat{J_n} \hat{T}$ $(\hat{T} = F(z))$

$\hat{J_n} \hat{T} \in (1)$ with R replaced by $R + \frac{1}{n}$

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By converse part of 1. C_c^∞ $\psi_n = J_n * T$ with $\operatorname{supp} \psi_n * T \subset \overline{B(0, R+\frac{1}{n})}$

But $\psi_n * T \to T$ in $\mathcal{D}'$ Hence $T \in \mathcal{E}$ with $\operatorname{supp} T \subset B(0, R+\frac{1}{n})$ $\forall n$.

random Brownian motion in $\mathbb{R}^3$ has Hausdorff dimension 2.

Lebesgue measure on $\mathbb{R}^N$ 1° $m(A+x)=m(A)$ $\quad \int_{\mathbb{R}^N} f(x+h)\,dx = \int_{\mathbb{R}^N} f(x)\,dx$

2° $\lambda>0$. $\lambda^N\int_{\mathbb{R}^N} f(\lambda x)\,dx = \int_{\mathbb{R}^N} f(x)\,dx$

convolution. 1. $f, g\in L^1$ $\quad \|f*g\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}$.

2. $f\in L^1, g\in L^p$ $\quad \|f*g\|_{L^p} \le \|f\|_{L^1}\|g\|_{L^p}$

$A_h g(\cdot)\to 0$ as $|h|\to 0$?

3. $f\in L^p$; $g\in L^{p'}$ $\quad f*g \in C_0(\mathbb{R}^N)$ (唯一个不能用LDCT的)

$$|f*g(x+h)-f*g(x)| \le \|f(x+h)-f(x)\|_{L^p}\|g\|_{L^{p'}} \to 0 \text{ as } h\to 0$$

Banach Algebra $(X,\|\cdot\|)$ multiplication $*: X\times X\to X$ s.t.

1° $(x*y)*z = x*(y*z)$

2° $x*(y+z) = x*y + x*z$

3° $\alpha(x*y) = (\alpha x)*y = x*(\alpha y)$

4° $\|x*y\| \le \|x\|\|y\|$

(若° $x*y = y*x$ 则称为交换的)

Eg 1. $X=\{T: B\to B$ bdd linear operators$\}$ 非交换 Banach Algebra，单位元 Id

2. $X=L^1(dx)$ commutable BA，无单位元

Convolution of two measures

$$\mu, \nu \in M(\mathbb{R}^N) \qquad \mu*\nu(A) = \int_{\mathbb{R}^N\times\mathbb{R}^N} \chi_A(x+y)\,\mu(dx)\,\nu(dy)$$

$$= \int_{\mathbb{R}^N\times\mathbb{R}^N}(\mu,\nu)((+)^{-1}(A)) \qquad +: \mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N,\ (x,y)\mapsto x+y$$

Then $(M(\mathbb{R}^N), \|\cdot\|, *)$ is a commutative Banach algebra with unit δ_0

$\|\mu\|<\infty$ $\quad \hat{\mu}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi}\,\mu(dx)$. $\quad |\hat{\mu}(\xi+\eta)-\hat{\mu}(\xi)| = \left|\int_{\mathbb{R}^N} e^{-ix\cdot\xi}(e^{-ix\cdot\eta}-1)\,\mu(dx)\right|$

$$\le \int_{\mathbb{R}^N} |e^{-ix\cdot\eta}-1|\,|\mu(dx)|$$

$\hat{\mu}$ 有界且一致连续 $\quad$ (LDCT)

$f\in L^1 \Rightarrow \hat{f}\in C_0$ $\quad$: $\exists f_n\in C_c$ $\quad f_n \xrightarrow{L^1} f$

$$\|\hat{f}-\hat{f}_n\|_{L^\infty} \le \|f-f_n\|_{L^1}, \quad \hat{f}_n(\xi)\to 0 \text{ as } \xi\to\infty$$

No.

Date

Thm (Amrein - ~~Be~~ Berthier)

$f \in L^2(\mathbb{R}^n)$. $E, F \subset \mathbb{R}^n$ $m(E), m(F) < \infty$. Then $\|f\|_{L^2} \leq C(E, F, n) \|f\|_{L^2(E^c)} \|\hat{f}\|_{L^2(F^c)}$

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