# 2024秋 PDE 课堂笔记

## PDE

PDE: $F(x, u, Du, \cdots, D^m u) = 0$　　$x \in \mathbb{R}^n$　　F 给定，u 未知.

PDEs $\begin{cases} F_1(x, u_1, u_2, \cdots, u_m, D^{l_1} u_1, \cdots, D^{l_m} u_m) = 0 \\ \vdots \\ F_k(x, u_1, u_2, \cdots, u_m, D^{l_{k1}} u_1, \cdots, D^{l_{km}} u_m) = 0 \end{cases}$

$k \geq 2$.

若 $k > m$　超定　overdetermined
若 $k < m$　欠定　underdetermined

PDE $\begin{cases} \text{linear} \\ \text{nonlinear} \end{cases}$

- linear　关于 $u$, $D^\alpha u$ 线性: $\sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u(x) = 0$.
- nonlinear
  - semi-linear　关于最高阶导数线性的: $\sum_{|\alpha| = m} a_\alpha(x) D^\alpha u + b(x, u, \cdots, D^{m-1} u) = 0$
  - quasi-linear　最高阶和其他项可分离: $\sum_{|\alpha| = m} b_\alpha(x, u, \cdots, D^{m-1} u) D^\alpha u + f(x, u, \cdots, D^{m-1} u) = 0$
  - completely non-linear.

Ex 1. linear PDE

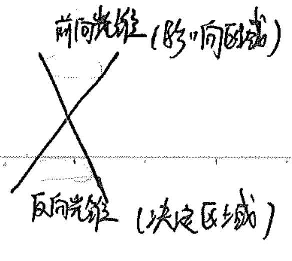
- Wave eq　$\Box_c u := (\partial_t^2 - c^2 \Delta) u = 0$　on $\mathbb{R}^{1+n}$　$\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$
- heat eq　$\partial_t u - \Delta u = 0$
- Laplace eq　$-\Delta u = 0$
- Poission eq　$-\Delta u = f(x)$
- Schrodinger eq　$i \partial_t u + \Delta u = 0$
- Tricomi eq　$(\partial_t^2 + t \Delta) u = 0$　$t \in \mathbb{R}$　$\begin{cases} t > 0 & 椭 \\ t < 0 & 双 \\ t = 0 & 退化 \end{cases}$

subsonic (椭圆)　supersonic (双曲)

- Airy eq　$\partial_t u + \partial_x^3 u = 0$

2. non-linear PDE

- 几何光学方程 (eikonal eq)　$(\partial_t u)^2 = \sum_{j=1}^n (\partial_j u)^2$
  (波动方程的特征曲面方程)
  其解: $t^2 = |x|^2$


波动方程解的奇异性 沿光锥传播.
singularity

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- 极小曲面方程 $\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=0.$

- Monge-Ampere eq $\det(D^2u)=f$
- Burger's eq $\partial_t u+u\,\partial_x u=0.$ 研究双曲PDE的试金石
- kdv eq $\partial_t u+u\partial_x u+\partial_x^3 u=0.$
- 反应扩散方程 $\partial_t u-\Delta u=f(u)$

PDEs

- Maxwell's eqs $\begin{cases}\vec{E}_t=\mathrm{curl}\,\vec{B}\\ \vec{B}_t=-\mathrm{curl}\,\vec{E}\\ \mathrm{div}\,\vec{B}=\mathrm{div}\,\vec{E}=0\end{cases}$

non-linear PDEs

- Euler eqs (无粘) 理想流体 $\begin{cases}\\ \\ \end{cases}$ 质量守恒 动量守恒 能量守恒

- Navier-Stokes eqs (考虑粘性) (没有能量守恒,动量守恒形式也有不同)

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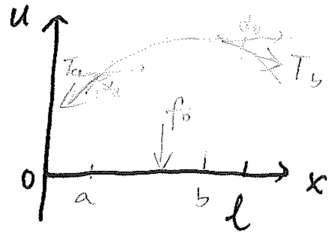
Q1: 什么叫求解PDE? PDE的适定性问题 $\begin{cases}\text{存在}\\ \text{唯一}\\ \text{连续依赖.}\end{cases}$

Q2 什么是解?

- $u\in C_c^\infty$? 解析?
- 经典解 $u\in C^k$ 若为k阶方程.
- 弱解 $\begin{cases}\text{正则性}\\ \text{存在性}\end{cases}$

Def 弦..




设长度 $l$, 外力 $f_0$, 密度 $\rho$ $x=a$ 点处张力 $T_a$

Q: 设弦作微小的横振动, 求 $t$ 时刻位移 $u=u(t,x)$

动量守恒: 动量的增量 = 冲量

动量: $mV$ $V=\partial_t u$ 冲量: $Ft$ $F<\begin{matrix} f_0 \\ T \end{matrix}$

对 $\forall [a,b]\subset[0,l]$ $\forall [t_1,t_2]$

$t_0$ 时刻动量 $\int_a^b \rho\, \partial_t u|_{t=t_0}\, dx$

$[t_1,t_2]$ 内动量的增量 $\int_a^b \rho\, \partial_t u\Big|_{t=t_1}^{t_2} dx = \int_{t_1}^{t_2}\int_a^b \partial_t(\rho\partial_t u)\, dx\, dt$

$[t_1,t_2]$ 内 $f_0$ 产生的冲量 $\int_{t_1}^{t_2}\int_a^b f_0 dx\, dt$

$[t_1,t_2]$ 内张力垂直于 $x$ 轴产生的冲量, $\int_{t_1}^{t_2}\left(T_a\cdot i_u + T_b\cdot i_u\right) dt$

$T_a\cdot i_u = |T_a|\cos(\alpha_a-\frac{\pi}{2}) = -\sin\alpha_a|T_a|$ $\sin\alpha_a \approx \tan\alpha_a = \frac{\Delta u}{\Delta x} = \partial_x u|_{x=a}$

$T_b\cdot i_u = |T_b|\cos(\frac{\pi}{2}-\alpha_b) = \sin\alpha_b|T_b|$ $\sin\alpha_b \approx \partial_x u|_{x=b}$

$$=\int_{t_1}^{t_2} -|T_a|\,\partial_x u|_{x=a} + |T_b|\,\partial_x u|_{x=b}\, dt$$

$$= \int_{t_1}^{t_2} T_0(\partial_x u)\Big|_{x=a}^{b} dt = \int_{t_1}^{t_2}\int_a^b \partial_x(T_0\partial_x u)\, dx\, dt$$

由动量守恒定律,形式上有 $\int_{t_1}^{t_2}\int_a^b \partial_t(\rho\partial_t u)\, dx dt = \int_{t_1}^{t_2}\int_a^b \left(f_0 + \partial_x(T_0\partial_x u)\right) dx\, dt$

$\Rightarrow$ $\partial_t^2 u - c^2\partial_x^2 u = f$ $t>0$ $0<x<l$

弦振动方程.

定解问题: ① PDE

② initial data

$u|_{t=0}=u_0$ $\partial_t u|_{t=0}=u_1$

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③ boundary condition

case 1 $u|_{x=0}=g_1(t)$ $u|_{x=l}=g_2(t)$ 两端点位移

case 2 $-T\partial_x u|_{x=0}=g_1(t)$, $-T\partial_x u|_{x=l}=g_2(t)$ 两端外力作用

case 3 $-T\partial_x u|_{x=0}+a_1 u|_{t=0}=g_1(t)$

$-T\partial_x u|_{x=l}+a_2 u|_{t=0}=g_2(t)$

弦无限长时,无端点. Cauchy 初值问题:

$$\begin{cases}\partial_t^2 u-a^2\partial_x^2 u=f(t,x) & t>0 \quad x\in\mathbb{R}\\ u|_{t=0}=u_0,\ \partial_t u|_{t=0}=u_1\end{cases}$$

波动方程 $\Box_c u:=(\partial_t^2-c^2\Delta)u=0$ $t\in\mathbb{R}_+$, $x\in\mathbb{R}^n$

$n=1$ 时. 弦振动方程

$n=2$ 时 膜振动方程.

$n=3$ 时 声波方程

设 $\Omega\subset\mathbb{R}^n$. 有界.开.边界为 $\partial\Omega$. $\Sigma:\mathbb{R}_+\times\partial\Omega$

① PDE $\partial_t^2 u-c^2\Delta u=f(t,x)$ $\mathbb{R}_+\times\Omega$

② initial data $u|_{t=0}=u_0$ $\partial_t u|_{t=0}=u_1$

③ boundary condition

case 1 $u|_\Sigma=g$ $g=g(t,x)$ → Dirichlet

case 2 $\frac{\partial u}{\partial \vec{n}}|_\Sigma=g$ → Neumann

case 3 $\frac{\partial u}{\partial \vec{n}}|_\Sigma+bu|_\Sigma=g$ $b>0$

解算子

$T:(u_0,u_1,g)\longmapsto u$

2. 能量守恒：热量增加 = 流入热量 + 热源产生热量.（一段时间内）

$$\int_D c\rho\left(u|_{t=t_2}-u|_{t=t_1}\right)dx\,dy\,dz = -\int_{t_1}^{t_2}dt\int_{\partial D}\vec{q}\cdot\vec{n}\,dS+\int_{t_1}^{t_2}dt\int_D \rho f_0\,dx\,dy\,dz$$

$$\vec{q}=-k\nabla u \qquad \vec{q}\cdot\vec{n}=-k\nabla u\cdot\vec{n}=-k\frac{\partial u}{\partial \vec{n}}$$

故 $$\int_D c\rho\int_{t_1}^{t_2}\partial_t u\,dt\,dx = -\int_{t_1}^{t_2}\int_{\partial D}-k\frac{\partial u}{\partial \vec{n}}\,dS\,dt+\int_{t_1}^{t_2}\int_D \rho f_0\,dx\,dt$$

$$=\int_{t_1}^{t_2}\int_D \nabla\cdot(k\nabla u)+\rho f_0\,dx\,dt$$

$$\Rightarrow\quad c\rho\,\partial_t u = k\Delta u+\rho f_0.$$

$$\Rightarrow\quad \partial_t u-\frac{k}{c\rho}\Delta u=\frac{f_0}{c}$$

初始条件 $t=0$ $\quad(\Sigma=\partial\Omega\times[0,\infty))$

边界条件 1. 已知边界温度 $u|_\Sigma = g(x,y,z,t)$

2. 已知通过边界的热量 $k\frac{\partial u}{\partial \vec{n}}\Big|_\Sigma = g(x,y,z,t)$

3. 已知周围介质温度 $g_0$ $\quad k\frac{\partial u}{\partial \vec{n}}\Big|_\Sigma=\alpha_0(g_0-u)\big|_\Sigma$

3. 质量守恒：质量增加 = 流入质量 + 源(汇)产生质量 　连续性方程

设 $t$ 时刻 $x$ 处流体密度为 $\rho(x,t)$. $\quad\vec{v}$ 为流速.

则 $$\int_D \rho|_{t=t_2}-\rho|_{t=t_1}\,dx = -\int_{t_1}^{t_2}\int_{\partial D}\rho\vec{v}\cdot\vec{n}\,dS\,dt$$

$$=\int_D\int_{t_1}^{t_2}\partial_t\rho\,dt\,dx \qquad = -\int_{t_1}^{t_2}\int_D \nabla(\rho\vec{v})\,dx\,dt$$

$$\partial_t\rho+\nabla\cdot(\rho\vec{v})=0.$$

特例 1. $\vec{v}$ 为已知常量时. $\partial_t\rho+\vec{v}\cdot\nabla\rho=0.$ $\quad$ Transport equation

2. 不可压缩流体. $\rho$ 为常数 $\nabla\cdot\vec{v}=0.$ $\quad$ 可压缩. $\rho(x,t)$ 可变

3. 不可压缩无旋流体 $\quad \mathrm{curl}\, V=0.$ 无旋场积分与路径无关. 存在势能. $\exists\varphi$ $\vec{v}=\nabla\varphi$

$$\nabla\cdot\vec{v}=\nabla\cdot\nabla\varphi=\Delta\varphi=0.$$

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理想流体：忽略粘性与热传导（无粘、绝热），有三大守恒定律. Euler 方程组.

记 $\vec{u}$ 流体速度  $\vec{n}$ 单位外法向量  $d\sigma$ 面积微元.

$[t, t+dt]$ 内经过 $d\sigma$ 的体积微元 $\vec{n}\cdot\vec{u}\,dt\,d\sigma$

流体质量 $\rho\vec{u}\cdot\vec{n}\,dt\,d\sigma$

1: $[t_1, t_2]$ 流体质量增量 $\int_\Omega \rho(t,x)\Big|_{t=t_1}^{t_2} dx$

在 $[t_1, t_2]$ 经过 $\partial\Omega$ 流入 $\Omega$ 质量 $-\int_{t_1}^{t_2}\int_{\partial\Omega} \rho\vec{u}\cdot\vec{n}\,d\sigma\,dt$

由质量守恒 $\int_\Omega \rho(t,x)\Big|_{t=t_1}^{t_2} dx + \int_{t_1}^{t_2}\int_{\partial\Omega} \rho\vec{u}\cdot\vec{n}\,d\sigma\,dt = 0$,

若 $\rho, \vec{u} \in C^1$. $\int_\Omega\int_{t_1}^{t_2} \partial_t\rho\,dt\,dx + \int_{t_1}^{t_2}\underbrace{\int_{\partial\Omega} \rho\vec{u}\cdot\vec{n}\,d\sigma}_{=\int_\Omega \operatorname{div}(\rho\vec{u})dx}\,dt = 0.$

$$\int_\Omega\int_{t_1}^{t_2} \partial_t\rho + \operatorname{div}(\rho\vec{u})\quad dt\,dx = 0.$$

$\partial_t\rho + \operatorname{div}(\rho\vec{u}) = 0$.  连续性方程.

2: $[t, t+dt]$ 经 $d\sigma$ 流入 $\Omega$ 的动量 $\rho\vec{u}\cdot\vec{n}\,d\sigma\,dt\cdot\vec{u} = \rho(\vec{u}\otimes\vec{u})\vec{n}\,d\sigma\,dt$

$[t_1, t_2]$ $\Omega$ 内动量的增量 $\int_\Omega \rho\vec{u}\Big|_{t_1}^{t_2} dx =: I_1 = \int_{t_1}^{t_2}\int_\Omega \partial_t(\rho\vec{u})\,dx\,dt$

$[t_1, t_2]$ 经过 $\partial\Omega$ 的流体动量 $-\int_{t_1}^{t_2}\int_{\partial\Omega} \rho(\vec{u}\otimes\vec{u})\vec{n}\,d\sigma\,dt =: I_2 = -\int_{t_1}^{t_2}\int_\Omega \operatorname{div}(\rho(\vec{u}\otimes\vec{u}))\,dx\,dt$

$[t_1, t_2]$ 内 作用在 $\partial\Omega$ 的压力所产生的冲量. $-\int_{t_1}^{t_2}\int_{\partial\Omega} p\vec{n}\,d\sigma\,dt =: I_3 = -\int_{t_1}^{t_2}\int_\Omega \operatorname{div}(pI)\,dx\,dt$

在 $[t_1, t_2]$ 内 $\Omega$ 的冲量 $\int_{t_1}^{t_2}\int_\Omega \rho\vec{F}(t,x)\,dx\,dt =: I_4$

$I_1 = I_2 + I_3 + I_4$  $\partial_t(\rho\vec{u}) + \operatorname{div}(\rho\vec{u}\otimes\vec{u} + pI) = \rho\vec{F}$

3: $e$: 单位质量的内能.  $\vec{F}$: 单位质量的外力

能量微元: $[t, t+dt]$ 经 $d\sigma$ 流入 $\Omega$ 的能量  $\rho(e + \frac{1}{2}|\vec{u}|^2)\,d\sigma\,dt$

$[t_1, t_2]$ $\Omega$ 内能量的增量 $I_1 = \int_\Omega \rho(e + \frac{1}{2}|\vec{u}|^2)\Big|_{t=t_1}^{t_2} dx = \int_{t_1}^{t_2}\int_\Omega \partial_t(\rho(e + \frac{1}{2}|\vec{u}|^2))\,dx\,dt$

$[t_1, t_2]$ 经过 $\partial\Omega$ 的能量. $I_2 = -\int_{t_1}^{t_2}\int_{\partial\Omega} \rho(e + \frac{1}{2}|\vec{u}|^2)\vec{u}\cdot\vec{n}\,d\sigma\,dt = -\int_{t_1}^{t_2}\int_\Omega \operatorname{div}(\rho(e + \frac{1}{2}|\vec{u}|^2)\vec{u})\,dx\,dt$

$[t_1, t_2]$ 作用在 $\partial\Omega$ 压力做的功. $I_3 = -\int_{t_1}^{t_2}\int_{\partial\Omega} p\,d\sigma\,\vec{u}\cdot\vec{n}\,dt = -\int_{t_1}^{t_2}\int_\Omega \operatorname{div}(p\vec{u})\,dx\,dt$

$\Omega$ 内部的外力做的功 $I_4 = \int_{t_1}^{t_2}\int_\Omega \rho\vec{F}(t,x)\,dx\cdot\vec{u}\,dt$

Campus

$I_1 = I_2 + I_3 + I_4$

$$\partial_t\left(\rho\left(e+\tfrac{1}{2}|\vec{u}|^2\right)\right) + \mathrm{div}\left(\rho\left(e+\tfrac{1}{2}|\vec{u}|^2\right)\vec{u} + p\vec{u}\right) = \rho\vec{F}\cdot\vec{u}$$

## 变分原理

基本想法：$P(D)u=0 \rightleftharpoons$ 求泛函的个极值问题

求变分问题 $\Omega\subset\mathbb{R}^n$ 有界. $\partial\Omega\in C^\infty$

Evans Ch8

$L = L(x,z,q): \bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ (Lagrangian)

泛函 $J: v\mapsto \int_\Omega L(x,v,Dv)\,dx$

求 $u\in M_\varphi$ s.t. $J(u)=\min\limits_{v\in M_\varphi} J(v)$ $\quad M_\varphi=\{v\in C^\infty(\Omega): v|_{\partial\Omega}=\varphi\}$.

claim: 对这样的 $u$. 其满足 Euler-Lagrange PDE

Pf: $\forall f\in C_c^\infty(\Omega)$ define $j(\tau):=J(u+\tau f)$ $\tau\in\mathbb{R}$.

Note that $\partial(u+\tau f)=\partial u=\varphi$ $\Rightarrow u+\tau f\in M_\varphi$

$\rightsquigarrow j(\cdot)$ has a minimum at $\tau=0$.

$\rightsquigarrow j'(0)=0$

$$j(\tau)=\int_\Omega L(x,u+\tau f, Du+\tau Df)\,dx$$

$$j'(\tau)=\int_\Omega\left(L_z(x,u+\tau f,Du+\tau Df)\cdot f + \sum_{j=1}^n L_{q_j}(x,u+\tau f,Du+\tau Df)\,\partial_j f\right)dx$$

$$=\int_\Omega L_z(x,u+\tau f,Du+\tau Df)\cdot f\,dx - \sum_{j=1}^n\int_\Omega f\cdot\partial_{x_j}\left(L_{q_j}(x,u+\tau f,Du+\tau Df)\right)dx$$

$$=\int_\Omega f\cdot\left(L_z(x,u+\tau f,Du+\tau Df) - \sum_{j=1}^n\partial_{x_j}\left(L_{q_j}(x,u+\tau f,Du+\tau Df)\right)\right)dx$$

$j'(0)=0$ 又对 $\forall f\in C_c^\infty(\Omega)$ 成立

$$\Rightarrow L_z(x,u,Du)-\sum_{j=1}^n\partial_{x_j}\left(L_{q_j}(x,u,Du)\right)=0.$$

$L_z=\nabla\cdot L_q$ Euler-Lagrangian 方程.

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9.12 膜平衡问题 $\Omega \subset OXY$ $\partial\Omega = \gamma \cup \Gamma$

最小势能原理 $u = u(x,y)$ $u|_\gamma = \varphi$ 外力 $<$ 膜内 $f$ ，$\Gamma$ 上 $p$

总势能 = 应变能($I_1$) - 外力做功($I_2$)

忽略高阶项.

$$I_1 = T\Delta\sigma = T\int_\Omega \left(\sqrt{1+|\nabla v|^2} - 1\right) dx\,dy = \frac{T}{2}\int_\Omega |\nabla v|^2 dx\,dy$$

$$I_2 = \int_\Omega f v\, dx\,dy + \int_\Gamma p(s)v(s)\,ds$$

$$J(v) = \frac{T}{2}\int_\Omega |\nabla v|^2 dx\,dy - \int_\Omega f v\, dx\,dy - \int_\Gamma p(s)v(s)\,ds$$ 其中 $v \in M_\varphi := \{w \in C^1(\bar\Omega) \mid w|_\gamma = \varphi\}$

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Lagrangian $L(x, z, q) = \frac{T}{2}q^2 - fz$

$\rightsquigarrow$ Euler-Lagrangian $-f = T\nabla q = T\Delta z$ $\qquad -\Delta u = f.$

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$u$ is a minimizer. define $j(\varepsilon) := J(u+\varepsilon w)$ $\forall w \in M_0 = \{w \in C^1(\bar\Omega) \mid w|_\gamma = 0\}$

$$j(\varepsilon) = J(u+\varepsilon w) = \frac{T}{2}\int_\Omega |\nabla(u+\varepsilon w)|^2 dx\,dy - \int_\Omega f\cdot(u+\varepsilon w)\,dx\,dy - \int_\Gamma p\cdot(u+\varepsilon w)\,ds$$

$$= \frac{T}{2}\int_\Omega (u_x+\varepsilon w_x)^2 + (u_y+\varepsilon w_y)^2\,dx\,dy - \int_\Omega f\cdot(u+\varepsilon w)\,dx\,dy - \int_\Gamma p\cdot(u+\varepsilon w)\,ds$$

$$j'(\varepsilon) = \frac{T}{2}\cdot 2\cdot\int_\Omega (u_x+\varepsilon w_x)w_x + (u_y+\varepsilon w_y)w_y\,dx\,dy - \int_\Omega f w\,dx\,dy - \int_\Gamma p w\,ds$$

$$j'(0) = T\int_\Gamma w\frac{\partial u}{\partial n}ds - T\int_\Omega w\cdot\Delta u\,dx\,dy - \int_\Omega f w\,dx\,dy - \int_\Gamma p w\,ds$$

$$= \int_\Omega (-T\Delta u - f)\,w\,dx\,dy + \int_\Gamma \left(T\frac{\partial u}{\partial n} - p\right) w\,ds$$

choose $w \in C_c^\infty(\bar\Omega)$. $\Rightarrow T\Delta u + f = 0$ $\Rightarrow T\frac{\partial u}{\partial n} - p = 0$

$\Leftrightarrow$ $u$ s.t. $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_\gamma = 0 \\ T\frac{\partial u}{\partial n} = p & \text{on } \Gamma \end{cases}$ $\qquad u \in C^2(\Omega) \cap C^1(\bar\Omega)$

变分法 (nonlinear functional analysis)

Ex Consider a functional

$$J(v) := \frac{1}{2}\int_\Omega \sum_{j,k=1}^n a^{jk}(x)(\partial_j v)(\partial_k v)\,dx - \int_\Omega v f\,dx$$

and $J(u) = \min_{v \in M} J(v)$.

Lagrangian $L(x,z,q) = \frac{1}{2}\sum_{j,k=1}^{n} a^{jk}(x) q_j q_k - zf$

E-L equ: $-f = \sum_{j=1}^{n} \partial_j (L_{q_j}) = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \partial_j (a^{jk}(x) q_k) = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \partial_j (a^{jk}(x)\partial_k u)$

关注PDE最高阶项系数 $(a^{jk}(x))_{1\le j,k\le n}$

rmk: 若 $(a^{jk}(x))$ 为一致椭圆 (ie. $\exists\, 0<\lambda<\Lambda<\infty$ $\lambda|\xi|^2 \le \sum_{j,k} a^{jk}\xi_j\xi_k \le \Lambda|\xi|^2$)

则 minimizer 存在.

linear PDE 的分类.

1° 2阶两个自变量 equ

$a_{11}\partial_x^2 u + 2a_{12}\partial_x\partial_y u + a_{22}\partial_y^2 u + b_1\partial_x u + b_2\partial_y u + cu = f$ $\quad a_{ij} = a_{ij}(x,y)$

记 $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ $(A = A(x,y))$ $(A\ne 0)$ 有实特征值 $\lambda_1\ \lambda_2$ $\quad \lambda = \lambda(x,y)$

在某点 $(x_0,y_0)$ 处 elliptic $|A(x_0,y_0)|>0 \Leftrightarrow \lambda_1,\lambda_2\ne 0$, 同号

hyperbolic $|A(x_0,y_0)|<0 \Leftrightarrow \lambda_1,\lambda_2\ne 0$, 异号

parabolic $|A(x_0,y_0)|=0 \Leftrightarrow \lambda_1 = 0 \ne \lambda_2$

在 $\Omega$ 处 elliptic $|A(x,y)|>0$ $\quad \forall (x,y)\in\Omega$

hyperbolic $|A(x,y)|<0$

parabolic $|A(x,y)|=0$

e.g. $\partial_x^2 u + \partial_y^2 u = f(x,y)$ → elliptic

$\partial_t^2 u - \partial_x^2 u = f(t,x)$ → hyperbolic

$\partial_x u - \partial_y^2 u = f(x,y)$ → parabolic

2° 2阶多变量 equ

$Lu := \sum_{j,k=1}^{n} a_{jk}(x)\partial_j\partial_k u + \sum_{l=1}^{n} b_l(x)\partial_l(x) + c(x)u = f(x)$ $\quad x\in\mathbb{R}^n$. $n\ge 2$.

$A = A(x) = (a_{jk})_{n\times n}$ 对称实矩阵必存 $n$ 个实特征根 $\lambda_1\ \lambda_2 \cdots \lambda_n$

elliptic $\lambda_1,\cdots,\lambda_n \ne 0$ 且同号 $\quad$ e.g. $-\Delta u = f$

hyperbolic $\lambda_1\cdots\lambda_n \ne 0$. 且1个异号. $(n-1)$ 个同号 $\quad$ 所有一阶双曲方程都是双曲方程. 有限传播

parabolic $\lambda_1\cdots\lambda_n$ 1个为0 $(n-1)$个同号 $\quad$ e.g. $\partial_t u - \Delta u = f$

3° 高阶PDE

$P(x,D_x)u = Lu := \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha u = f(x)$

$= \sum_{|\alpha|=m} a_\alpha(x) D_x^\alpha u + \sum_{|\beta|\le m-1} a_\beta(x) D_x^\beta u$

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$p(x, D_x) = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha$ $\quad p(x,\xi) \quad p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$

elliptic $\forall \xi \in \mathbb{R}^n \setminus 0.\ \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \ne 0.$

hyperbolic (in the direction of $x_n$) $\quad \forall \xi' = (\xi_1 \cdots \xi_{n-1}) \in \mathbb{R}^{n-1} \setminus 0, \quad \xi = (\xi', \xi_n)$

$\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha = 0$ 关于 $\xi_n$ 有 $m$ 个实根.

(严格双曲：$m$个不同实根)

parabolic $\partial_t u - \underbrace{\sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha u}_{\text{椭圆}} = f$

9.19 波动方程.

1. Transport equation $\quad \partial_t u + \vec{b}\cdot \nabla_x u = 0. \quad t>0,\ x\in \mathbb{R}^n$ $\qquad$ PDE

Recall vector field $X = X(t,x) = X_0\partial_t + \sum_{j=1}^n X_j \partial_j$

$X = (X_0, X_1, \cdots, X_n)$

则TE可表为 $Xu = 0 \qquad X = (1, \vec{b})$

integral curve $\gamma = \gamma(s)$ of the vector field $X \qquad \dot\gamma(s) = X(\gamma(s))$

$Xu = 0 \Rightarrow \dot\gamma(s) = 0, \quad \gamma(s) \equiv const \qquad \gamma(s) = (t_0+s, x_0+s\vec{b})$

Along the integral curve $\gamma(s)$, the function

define $v(s) := u(t_0+s, x_0+s\vec{b}) \qquad \forall (t_0, x_0) \in \mathbb{R}_{>0} \times \mathbb{R}^n$

s.t. $\dot v(s) = \partial_t u|_{\gamma(s)} + \vec{b}\cdot\nabla_x u|_{\gamma(s)} = 0$ $\qquad$ ODE

$\Rightarrow v(s) = const \quad$ along $\gamma = \gamma(s)$

(2. 特征线方法 (PDE → ODE) $\quad$ Q. ODE 的解的存在唯一性 )

考虑 $\begin{cases} \partial_t u + \vec{b}\cdot\nabla_x u = 0 \\ u|_{t=0} = g(x) \end{cases}$

$u(t_0, x_0) = v(0) = v(-t_0) = u(0, x_0 - t_0\vec{b}) = g(x_0 - t_0\vec{b})$

2、 Recall ODE 局部解的存在唯一性.

(*) $\frac{dx}{dt} = f(t,x) \quad (t,x) \in \Omega \subset \mathbb{R}^{1+n}$ 若 $f, \partial_x f \in C^0(\Omega)$. 则 (*)的初值问题存在唯一解.

特别地 $f \in C^1$. 也有. 以下 $f \in C^1$

向量场 $X = (1, f(t,x))$ $\quad \partial_t u + f(t,x)\partial_x u = 0$ 初值 $u(0,x) = \varphi(x)$

$\Rightarrow \exists\, x = x(t)$ 满足 $\begin{cases} \dot{x} = f \\ x(0) = x_0 \end{cases}$

(也称 characteristic)

$\Rightarrow$ Along the integral curve $\gamma = \gamma(t) = (t, x(t))$

$$\frac{du(t,x(t))}{dt} = \partial_t u + \frac{dx(t)}{dt} \cdot \partial_x u = 0.$$

$$\Rightarrow u(t,x(t)) = u(0,x(0)) = u(0,x_0)$$

Ex: $n=1$ $f=a$. $\begin{cases} \partial_t \rho + a\partial_x \rho = 0 \\ \rho(0,x) = \rho_0(x) \end{cases}$

$1^{st}$ 求特征线 $x = x(t)$ s.t. $\begin{cases} \frac{dx(t)}{dt} = a \\ x(0) = c \end{cases}$ $\Rightarrow x(t) = c + at$

$2^{nd}$ Along the characteristic $x = x(t)$. $\rho = \rho(t, x(t))$

$$\frac{d\rho}{dt} = \partial_t \rho + \frac{dx(t)}{dt}\partial_x \rho = 0.$$

$\Rightarrow \rho(t,x(t)) = \text{const} = \rho(0,x(0)) = \rho_0(x_0) = \rho_0(x(t) - at)$
$= \rho(t, x_0 + at)$

$\rho(t,x) = \rho_0(x - at)$.

若为 $\quad \partial_t \rho + a\partial_x \rho = \varphi(t,x)$.

则 $\frac{d\rho}{dt} = \varphi(t,x) \quad \Rightarrow \rho(t,x) = \rho_0(x-at) + \int_0^t \varphi(s,x)\,ds$

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1阶 PDE $\partial_t u + a(t,x)\partial_x u = g(t,x)u + h(t,x)$

where $a \in C^1$

Along the characteristic curve $x(t;x_0)$ $\begin{cases} \frac{dx(t)}{dt} = a(t,x(t)) \\ x(0) = x_0 \end{cases}$ $Q_2$ 反解 $x_0$ $= x_0(t,x)$

$$\frac{d}{dt} u(t,x(t)) = g(t,x(t))\,u(t,x(t)) + h(t,x(t))$$

$$u(t,x(t)) = \underset{\varphi(x_0)}{\underset{\shortparallel}{u(0,x(0))}} + \int_0^t \cdots$$

3. Burgers's equ ("nice" toy)

双曲:守恒律方程 $\partial_t(\cdots) + \nabla_x\cdot(\ ) = \cdots$

$\partial_t u + \partial_x\left(\frac{u^2}{2}\right) = 0$, 即 $\partial_t u + u\cdot\partial_x u = 0$ $t>0$. $x\in\mathbb{R}$ 初值 $u(0,x) = \varphi(x)$

suppose $u \in C^1(\mathbb{R}_{\geq 0}\times\mathbb{R})$ we shall apply the method of characteristics

$u \in C^1 \rightsquigarrow \begin{cases} \frac{dx(t)}{dt} = u(t,x(t)) \\ x(0) = x_0 \end{cases}$

along the characteristic $(t,x(t))$, $u$ s.t.

$\left(\frac{d}{dt}x(t)\right) = \frac{d}{dt}(u(t,x(t))) = 0$ $u(t,x(t)) = \text{const} = u(0,x(0)) = \varphi(x_0)$

$\Rightarrow \frac{d}{dt}x(t) = u = \varphi(x_0)$ $x(t)$ 是关于 $t$ 的直线

$\Rightarrow x(t;x_0) = x_0 + \varphi(x_0)t$ (#)

$Q_2$: 反解 $x_0 = x_0(t,x)$

用 $y = y(t,x)$ 表示 $x_0$ $\frac{\partial x}{\partial y}(t,y) = 1 + \varphi'(y)t \neq 0$. 若能保证 导数不为0, $\Leftrightarrow t \neq \frac{1}{\varphi'(y)}$

(#) $x(t,y) = y + \varphi(y)t$

若 $\varphi$, $\varphi'$ 均为 $C^1$, 有界. 则由 (#) 可反解 $y$ (反函数定理, 且 $y \in C^1$). $(0<t<T)$.

$T = \inf\left(-\frac{1}{\varphi_0'}\right)$

Campus

Q: $u$ 是否为 $C^1$, as assumed? for $0 \le t \le T$

$\Leftarrow$ $\varphi \in C^1$, $\varphi$, $\varphi'$ bdd. $\qquad T$: life-span

$u(t,x) = u(t, x(t,y)) = \varphi(y) \quad y = y(t,x) \qquad \varphi'(y) = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial y}$

$\rightsquigarrow \partial_x u \big|_{x=y(t)} = \varphi'(y) \frac{\partial y}{\partial x} = \frac{\varphi'(y)}{1 + t\varphi'(y)} \in C^0$

If $\varphi' \geq 0$ ($\varphi$ non-decreasing) $\quad 1 + t\varphi'(y) > 0 \quad$ for $t \in \mathbb{R}_{\geq 0}$

then $u$ is $C^1$ globally

若∃$\varphi'(y) < 0$ $\qquad$ 如 $\varphi \in C^\infty$ 且 $\varphi \not\equiv 0$

If not, blow-up at $t = T$ $\qquad$ 故Burgers' eq.大振幅会爆破

$\hookrightarrow \exists\, x_1 < x_2 \in \mathbb{R} \quad \varphi(x_1) > \varphi(x_2)$ 从 $x_1$, $x_2$ 出发的特征线会相交

推荐 Smollet 反应扩散方程 Ch 15-19

9.24 在 $\mathbb{R}_{\geq 0} \times \mathbb{R}$ 上波动方程的初值问题

$$(\#)\begin{cases} \Box_c u := (\partial_t^2 - c^2 \partial_x^2) u = F(t,x) & t > 0,\ x \in \mathbb{R} \\ u|_{t=0} = \varphi & \partial_t u|_{t=0} = \psi \end{cases}$$

1. 解的表达式 ($n=1$)

wave op. $\Box_c$ linear

线性叠加原理. 若 $u_1, u_2, u_3$ 分别是如下Cauchy问题的解

$$(\#1)\begin{cases} \Box_c u_1 = 0 \\ u_1|_{t=0} = \varphi \\ \partial_t u_1|_{t=0} = 0 \end{cases} \qquad (\#2)\begin{cases} \Box_c u_2 = 0 \\ u_2|_{t=0} = 0 \\ \partial_t u_2|_{t=0} = \psi \end{cases} \qquad (\#3)\begin{cases} \Box_c u_3 = F(t,x) \\ u_3|_{t=0} = 0 \\ \partial_t u_3|_{t=0} = 0 \end{cases}$$

则 $u = u_1 + u_2 + u_3$ 为原Cauchy问题 (#) 的解

定义解算子 $M: C^1([0,T] \times \mathbb{R}^n) \to C^2([0,T] \times \mathbb{R}^n)$

$$\psi \longmapsto M(\psi)$$

s.t. $u_2 = M(\psi)$ 为(#2)的解

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$u_1 = \partial_t(M(\varphi))$

$u_3 = \int_0^t M(F(\tau,\cdot))(t-\tau,x)\,d\tau$

Pf: ① $u_1$ : $\Box_c \partial_t = (\partial_t^2 - c^2\partial_x^2)\partial_t$

$\partial_t \Box_c = \partial_t(\partial_t^2 - c^2\partial_x^2)$

$\Box_c u_1 = \Box_c \partial_t(M(\varphi)) = \partial_t \Box_c (M(\varphi)) = 0$

$u_1|_{t=0} = \partial_t(M(\varphi))|_{t=0} = \varphi$

$\partial_t u_1|_{t=0} = \partial_t^2 (M(\varphi))|_{t=0} = c^2\partial_x^2 M(\varphi)|_{t=0} = 0.$

② $u_3$ :

$u_3|_{t=0} = \int_0^0 \cdots d\tau = 0$

$\partial_t u_3|_{t=0} = M(F(t,\cdot))(0,x)|_{t=0} + \int_0^t \partial_t M(F(\tau,\cdot))(t-\tau,x)\,d\tau|_{t=0} = 0$

$\partial_t^2 u_3 - c^2\partial_x^2 u_3 =$

$\partial_t u_3 = M(F(t,\cdot))(0,x) + \int_0^t \partial_t M(F(\tau,\cdot))(t-\tau,x)\,d\tau$

$= \int_0^t \partial_t M(F(\tau,\cdot))(t-\tau,x)\,d\tau$

$\partial_t^2 u_3 = \partial_t M(F(t,\cdot))(0,x) + \int_0^t \partial_t^2 M(F(\tau,\cdot))(t-\tau,x)\,d\tau$

$= F(t,x) + \int_0^t \partial_t^2 M(F(\tau,\cdot))(t-\tau,x)\,d\tau$

$\partial_x^2 u_3 = \int_0^t \partial_x^2 M(F(\tau,\cdot))(t-\tau,x)\,d\tau$

$\Rightarrow \partial_t^2 u_3 - \partial_x^2 u_3 = F$ □

解(#2)

$\Box_c u = (\partial_t^2 - c^2\partial_x^2)u = (\partial_t + c\partial_x)(\partial_t - c\partial_x)u$

令 $V(t,x) = (\partial_t - c\partial_x)u$ 则 $V$ 满足 $\begin{cases}(\partial_t + c\partial_x)V = 0 \\ V|_{t=0} = \partial_t u|_{t=0} - c\partial_x u|_{t=0} = \psi\end{cases}$

一阶PDE. 由特征线方法 $\dot{x}(t) = c \Rightarrow x(t) = x_0 + ct \Rightarrow x_0 = x - ct$

$V(t,x) = V(0,x(0)) = \psi(x_0) = \psi(x-ct)$

$u$ 满足 $\begin{cases} (\partial_t - c\partial_x)u = \psi(x-ct) \\ u|_{t=0} = 0 \end{cases}$ 特征线 $\dot{x}(t) = -c$ $\quad x(t) = x_0 - ct$, $x_0 = x + ct$

$$\frac{d}{dt}\left(u(t, x(t))\right) = \psi(x(t) - ct) = \psi(x_0 - 2ct)$$

$$u_2 = M(\psi) = u(t, x(t)) = \int_0^t \psi(x_0 - 2c\tau)\, d\tau = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds$$

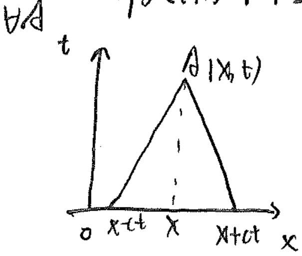
$$\Rightarrow u_1 = \partial_t M(\varphi) = \frac{1}{2}\left(\varphi(x+ct) + \varphi(x-ct)\right)$$

$$u_3 = \int_0^t M(F(\tau,\cdot))(t-\tau, x)\, d\tau = \int_0^t \frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\tau, s)\, ds\, d\tau$$

9.26 有限传播速度

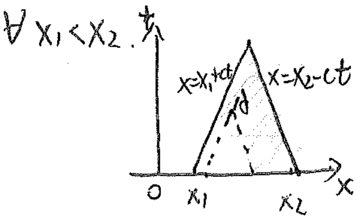
齐次(#)中 $F \equiv 0$. 的解 $u(t,x) = \frac{1}{2}\left(\varphi(x+ct) + \varphi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds$

$\forall A$



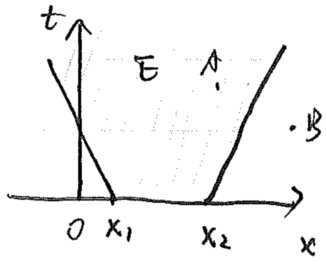

称 $[x-ct, x+ct]$ 为点 $A(x,t)$ 的依赖区间

$\forall x_1 < x_2$.




记 $\triangle$ 为这个闭三角形区域. $\forall A \in \triangle$ $A$ 的依赖区间 $\subset [x_1, x_2]$

称 $\triangle$ 为 $[x_1, x_2]$ 的决定区域




$\forall A \in E$. $A$ 的依赖区间 $\cap [x_1, x_2] \neq \phi$

$\forall B \notin E$. $B$ 的依赖区间 $\cap [x_1, x_2] = \phi$

称 $E$ 为 $[x_1, x_2]$ 的影响区域

波沿着特征线传播
(特征锥)

一般双曲方程解的特点,

波的奇性沿着特征线(特征锥)传播

microlocal analysis

解的正则性. 若 $\varphi \in C^k(\mathbb{R}), \psi \in C^{k-1}(\mathbb{R})$ $k \geq 2$. 则 $u \in C^k(\overline{\mathbb{R}}_+ \times \mathbb{R})$
(正则性没有提升)

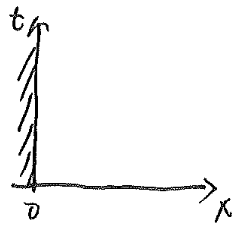
解的结构 $u(t,x) = f(x+ct) + g(x-ct)$

在 $\mathbb{R}_+ \times \mathbb{R}^n$ 上波动方程 Cauchy 问题 $\begin{cases} \Box_c u = F(t,x) & t>0. \ x \in \mathbb{R}^n \\ u|_{t=0} = \varphi, \ \partial_t u|_{t=0} = \psi & x \in \mathbb{R}^n \end{cases}$

$\rightsquigarrow$ reduce to the case in $\mathbb{R}_+ \times \mathbb{R}_+$ (通过球面平均)

In $\mathbb{R}_+ \times \mathbb{R}^n$ $\begin{cases} n \text{ odd} \\ n \text{ even (降维, 至奇数维)} \end{cases}$

半无界问题 $\begin{cases} \Box_c u = (\partial_t^2 - c^2 \partial_x^2) u = F(t,x) & t>0. \ x>0 \\ u|_{t=0} = \varphi \quad \partial_t u|_{t=0} = \psi & x>0. \\ u|_{x=0} = g(t) \end{cases}$



令 $v(t,x) = u(t,x) - g(t)$

$\begin{cases} \Box_c v = \Box_c u - g''(t) = F(t,x) - g''(t) & t>0. \ x>0. \\ v|_{t=0} = \varphi - g(0) \quad \partial_t v|_{t=0} = \psi - g'(0) \\ v|_{x=0} = 0 \end{cases}$

故不妨只考虑 $g=0$ 的情形
故可做奇延拓 (正则性不变)

将 $x$ 奇延拓 $\tilde{F}(t,x) = \begin{cases} F(t,x) & x \geq 0 \\ -F(t,-x) & x \leq 0 \end{cases}$ $\tilde{\varphi}(x) = \begin{cases} \varphi(x) & x \geq 0 \\ -\varphi(-x) & x \leq 0. \end{cases}$ $\tilde{\psi}(x) = \begin{cases} \psi(x), & x \geq 0 \\ -\psi(-x), & x \leq 0 \end{cases}$

现考虑 $\begin{cases} \Box_c \tilde{u} = \tilde{F}(t,x) & t>0 \ x \in \mathbb{R} \\ \tilde{u}|_{t=0} = \tilde{\varphi}, \ \partial_t \tilde{u}|_{t=0} = \tilde{\psi} \end{cases}$ 易知 $\tilde{u}(t,-x) = -\tilde{u}(t,x)$

由 $\tilde{u}(t,0) = 0 \Rightarrow u = \tilde{u}|_{\overline{\mathbb{R}}_+ \times \mathbb{R}_+}$ 为原问题的解

$x \geq ct$ 时, $u(t,x) = \frac{1}{2}(\varphi(x+ct) + \varphi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\tau,s)\,ds\,d\tau$

$0 \leq x \leq ct$ 时 $u(t,x) = \frac{1}{2}(\varphi(x+ct) - \varphi(ct-x)) + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(s)\,ds + \frac{1}{2c}\int_0^{t-\frac{x}{c}} \int_{c(t-\tau)-x}^{x+c(t-\tau)} F(\tau,s)\,ds\,d\tau$
$+ \frac{1}{2c}\int_{t-\frac{x}{c}}^{t} \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\tau,s)\,ds\,d\tau$

高维: 仍可如前. 有 $u = u_1 + u_2 + u_3$. 可只通过 $u_2$, 得到 $u$.

In $\mathbb{R}_+ \times \mathbb{R}^3$.

Set $(A_r(h))(x) := ⨍_{S^2} h(x+rw)\, d\sigma(w) \quad \forall r>0$

$\tilde{u}(r;t,x) = A_r(u(t,\cdot))(x)$

show $V(t,r) := r\tilde{u}(r;t,x)$ s.t.

$$\begin{cases} \Box_c V = (\partial_t^2 - c^2\partial_r^2)V = 0 \\ V|_{t=0} = 0 \quad \partial_t V|_{t=0} = r\,(A_r(\psi))(x) \quad r>0 \\ V|_{r=0} = 0 \end{cases}$$

见 evans

$\Rightarrow V(t,r) = \cdots \quad \Rightarrow u(t,x) = \lim_{r\to 0} \tilde{u}(r;t,x) = \cdots$

$V(t,r) = \frac{r}{4\pi}\int_{S^2} u(x+rw,t)\, d\sigma(w)$ , $\partial_t^2 V(t,r) = \frac{r}{4\pi}\int_{S^2} \partial_t^2 u(x+rw,t)\, d\sigma(w)$

$\partial_r V(t,r) = \frac{1}{4\pi}\int_{S^2} u(x+rw,t)\, d\sigma(w) + \underline{\frac{r}{4\pi}\int_{S^2} D_y u(x+rw,t)\cdot w\, d\sigma(w)}$

$= \frac{r}{4\pi}\int_{\partial B(x,r)} D_y u(y,t)\cdot w\, d\frac{\sigma(y)}{r^2}$

$= \frac{1}{4\pi}\int_{S^2} u(x+rw,t)\, d\sigma(w) + \frac{1}{4\pi r}\int_{B(x,r)} \Delta_y u(y,t)\, dy$

$\partial_r^2 V(t,r) = \frac{1}{4\pi r^2}\int_{B(x,r)} \Delta u(y,t)\, dy - \frac{1}{4\pi r^2}\int_{B(x,r)} \Delta_y u(y,t)\, dy + \frac{1}{4\pi r}\int_{\partial B(x,r)} \Delta_y u(y,t) dy$

$= \cancel{\frac{1}{4\pi}}\, r ⨍_{\partial B(x,r)} \Delta_y u(y,t)\, d\sigma(y)$

$$\begin{cases} \Box_c V = \partial_t^2 V(t,r) - c^2\partial_r^2 V(t,r) = r⨍_{\partial B(x,y)} \partial_t^2 u(y,t)\, d\sigma(y) - r⨍_{\partial B(x,r)} c^2\Delta_y u(y,t)\, d\sigma(y) \\ \qquad = r⨍_{\partial B(x,y)} (\partial_t^2 - c^2\Delta_y)u(y,t)\, d\sigma(y) = 0. \\ V|_{t=0} = r⨍_{\partial B(x,r)} u(y,0)\, d\sigma(y) = r⨍_{\partial B(x,r)} \varphi(y)\, d\sigma(y) = 0. \\ \partial_t V|_{t=0} = r⨍_{\partial B(x,r)} \partial_t u(y,0)\, d\sigma(y) = r⨍_{\partial B(x,r)} \psi(y)\, d\sigma(y) \\ V|_{r=0} = 0 \end{cases}$$

故 $0\le r\le ct$ 时. $V(t,r) = \frac{1}{2c}\int_{ct-r}^{ct+r} \xi ⨍_{\partial B(x,\xi)} \psi(y)\, d\sigma(y)\, d\xi$

$\lim_{r\to 0}\frac{V(t,r)}{r} = u_2(t,x) = \quad t⨍_{\partial B(x,ct)} \psi(y)\, d\sigma(y)$

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由与前完全一样的讨论，只不过将 $\mathbb{R}^1$ 换成 $\mathbb{R}^3$.  [Kirchhoff]

$$\begin{cases} \Box_c u = f(t,x) & \mathbb{R}_{>0}\times\mathbb{R}^3 \\ u|_{t=0} = \varphi(x) & \mathbb{R}^3 \\ u_t|_{t=0} = \psi(x) & \mathbb{R}^3 \end{cases}$$

的解为 $u(t,x)$

$$= \partial_t\left(t ⨍_{\partial B(x,ct)} \varphi(y)\, d\sigma(y)\right) + t ⨍_{\partial B(x,ct)} \psi(y)\, d\sigma(y) + \int_0^t (t-\tau) ⨍_{\partial B(x,c(t-\tau))} f(\tau,y)\, dy\, d\tau$$

$n=2$ 时.
$$\begin{cases} \Box_c u = f(t,x) & \mathbb{R}_{>0}\times\mathbb{R}^2 \\ u|_{t=0} = \varphi(x) & \mathbb{R}^2 \\ u_t|_{t=0} = \psi(x) & \mathbb{R}^2 \end{cases}$$

$\bar{u}(t,\bar{x}) = \bar{u}(t,x_1,x_2,x_3) := u(t,x) = u(t,x_1,x_2)$

$\bar{\varphi}(\bar{x}) := \varphi(x)$  $\bar{f}(t,\bar{x}) := f(t,x)$

$\bar{\psi}(\bar{x}) := \psi(x)$

故
$$\begin{cases} \Box_c \bar{u} = \bar{f}(t,\bar{x}) & \mathbb{R}_{>0}\times\mathbb{R}^3 \\ \bar{u}|_{t=0} = \bar{\varphi}(\bar{x}) & \mathbb{R}^3 \\ \bar{u}_t|_{t=0} = \bar{\psi}(\bar{x}) & \mathbb{R}^3 \end{cases}$$

$$t ⨍_{\partial B(\bar{x},ct)} \bar{\psi}(\bar{y})\, d\sigma(\bar{y}) = \frac{t}{4\pi(ct)^2} \int_{\partial B(\bar{x},ct)} \psi(y_1,y_2)\, d\sigma(\bar{y})$$

$$= \frac{t}{4\pi(ct)^2} \int_{B(x,ct)} 2\cdot\psi(y)\, ct \frac{dy}{\sqrt{(ct)^2-(y-x)^2}} = \frac{1}{2\pi c} \int_{B(x,ct)} \frac{\psi(y)}{\sqrt{(ct)^2-(y-x)^2}}\, dy$$

故
$$u(t,x) = \frac{1}{2\pi c}\, \partial_t\left(\int_{B(x,ct)} \frac{\varphi(y)}{\sqrt{(ct)^2-(y-x)^2}}\, dy\right) + \frac{1}{2\pi c} \int_{B(x,ct)} \frac{\psi(y)}{\sqrt{(ct)^2-(y-x)^2}}\, dy$$
$$+ \frac{1}{2\pi c} \int_0^t \int_{B(x,c(t-\tau))} \frac{f(\tau,y)}{\sqrt{(c(t-\tau))^2-(y-x)^2}}\, dy\, d\tau$$

Campus

(对于齐次波动方程)

Existence 存在性 若 $\varphi \in C^{[\frac{n}{2}]+2}(\mathbb{R}^n)$, $\psi \in C^{[\frac{n}{2}]+1}(\mathbb{R}^n)$ $n \geq 2$.

则 Cauchy problem
$$\begin{cases} \square_c u = (\partial_t^2 - c^2\Delta)u = 0 & \mathbb{R}_+ \times \mathbb{R}^n \\ u|_{t=0} = \varphi \quad \partial_t u|_{t=0} = \psi & \mathbb{R}^n \end{cases}$$

存在唯一解 $u \in C^2([0,\infty)\times\mathbb{R}^n)$ 存在性由表达式可以看出来.

有限速度传播 若 $c=1$. $\varphi, \psi \in C_c^\infty$. 不妨设支在 $B(0,M)$ 上.

$n=3$ 时. 若 $||x|-|t||>M$. 则 $u(t,x)=0$. 声波 惠更斯原理. — | 决定区域. 影响区域

$n=2$ 时. 若 $\inf\limits_{\substack{0<\tau<t \\ w\in S^1}} |x-\tau\cdot w|>M$. 则 $u(t,x)=0$. 波的弥漫性.

△Decay (as $t\to\infty$)

$2\nmid n$ : $|u(t,x)| \lesssim (1+t)^{-\frac{n-1}{2}}$

$2|n$ : $|u(t,x)| \lesssim (1+t)^{-\frac{n-1}{2}}(1+|t-|x||)^{-\frac{n-1}{2}}$

能量方法 $E(t)=\frac{1}{2}\int_{B(x_0,t_0-t)}((\partial_t u)^2+|\nabla_x u|^2)(t,x)\,dx$

$$\begin{cases} \square u = F \\ u|_{t=0}=u_0 \quad u_t|_{t=0}=u_1 \end{cases}$$

10.10 t时刻的能量:

1°. $\underset{\text{multiplier}}{(\partial_t u)}\square u = \partial_t(e_0(u)) + \sum_{j=1}^n \partial_j(e_j(u)) + R$.

$E(t) := \int_{\mathbb{R}^n} e_0(u)(t,x)\,dx$.

$(\partial_t u)\square u = (\partial_t u)\cdot(\partial_t^2 u - \sum_{j=1}^n \partial_j^2 u)$

$$\begin{cases} \partial_t u\cdot\partial_t^2 u = \partial_t(\frac{1}{2}(\partial_t u)^2) \\ \partial_t u\cdot\partial_j^2 u = \partial_j(\partial_t u\cdot\partial_j u) - \partial_t(\frac{1}{2}(\partial_j u)^2) \end{cases}$$

$$\Rightarrow \begin{cases} e_0(u) = \frac{1}{2}(\partial_t u)^2 + \frac{1}{2}\sum_{j=1}^n(\partial_j u)^2 \geq 0. \\ e_j(u) = -\partial_t u\cdot\partial_j u \qquad R=0. \end{cases}$$

Energy estimate : Integrate over the domain $[0,\tau]\times\mathbb{R}^n$

$$\int_0^\tau\int_{\mathbb{R}^n}(\partial_t u)\square u\,dx\,dt = \int_0^\tau\int_{\mathbb{R}^n} F(t,x)\,dt\,dx$$

$$= \underbrace{\int_0^\tau\int_{\mathbb{R}^n}\partial_t(e_0(u))\,dx\,dt}_{E(\tau)-E(0)} + \sum_{j=1}^n\underbrace{\int_0^\tau\int_{\mathbb{R}^n}\partial_j(e_j(u))\,dx\,dt}_{\text{若 } u\to 0 \text{ as } |x|\to\infty. \text{ 则此项为0. why?}}$$

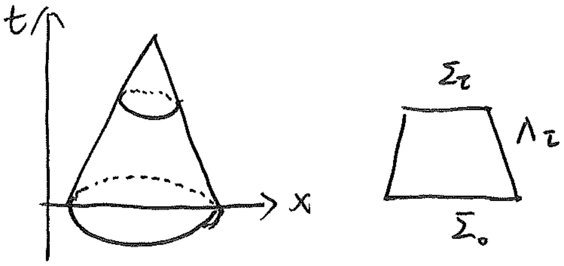
$E(\tau) = E(0) + \int_0^\tau\int_{\mathbb{R}^n} F(t,x)\,dt\,dx$

目标：证明若 $u\equiv 0$ on $B(x_0,t_0)$，则 $u\equiv 0$ on $C_{t_0,x_0}=\{(t,x)\mid 0\le t\le t_0,\ 0\le|x-x_0|\le t_0-t\}$

记 $D_\tau = C_{t_0,x_0}\cap\{0\le t\le\tau\}$

$\Sigma_\tau = \partial D_\tau \cap \{t=\tau\}$

$\Lambda_\tau = \partial D_\tau \cap \{t\neq 0,\ t\neq \tau\}$




$$\int_{D_\tau}(\partial_t u)\,\Box u\,dx\,dt = \int_{D_\tau}(\partial_t u)\cdot F(t,x)\,dt\,dx$$

$$= \int_{D_\tau}\partial_t(e_0(u))\,dx\,dt + \sum_{j=1}^{n}\int_{D_\tau}\partial_j(e_j(u))\,dx\,dt$$

$$= \int_{\Sigma_\tau} e_0(u)\,dx - \int_{\Sigma_0} e_0(u)\,dx + \int_{\Lambda_\tau}\Big(\nu_0 e_0(u) + \sum_{j=1}^{n}\nu_j e_j(u)\Big)\,d\sigma \quad \cancel{+\sum_{j=1}^{n}\int_{D_\tau}\partial_j(e_j(u))\,dx\,dt}$$

由 $|x-x_0|=|t_0-t|$ on $\Lambda_\tau$（用 $\nu=(\nu_0,\nu_1,\cdots,\nu_n)$ 记 $\Lambda_\tau$ 的外法向量。

且 $\nu_0>0$. $\nu_0^2=\sum_{j=1}^{n}\nu_j^2$ $\cancel{(\text{又}\sum_{j=0}^{n}\nu_j^2=1 \Rightarrow \nu_0=\tfrac{1}{\sqrt{2}})}$

$$\nu_0 e_0(u)+\sum_{j=1}^{n}\nu_j e_j(u)$$

$$=\frac{\nu_0}{2}\Big((\partial_t u)^2+\sum_{j=1}^{n}(\partial_j u)^2\Big)-\sum_{j=1}^{n}\nu_j\,\partial_t u\cdot\partial_j u$$

$$=\frac{\nu_0}{2}\sum_{j=1}^{n}\Big(\frac{\nu_j}{\nu_0}\partial_t u-\partial_j u\Big)^2 \geq 0.$$

$$E(\tau)=E(0)-\int_{\Lambda_\tau}\underbrace{(\cdots)}_{\geq 0}\,d\sigma+\int_{D_\tau}(\partial_t u)\cdot F(t,x)\,dt\,dx$$

$$E(\tau)\le E(0)+\int_{D_\tau}|\partial_t u|\cdot|F(t,x)|\,dt\,dx \le E(0)+\frac{1}{2}\int_0^\tau\int_{\Sigma_t}\big(|\partial_t u|^2+|F(t,x)|^2\big)\,dx\,dt$$

$$\le E(0)+\frac{1}{2}\int_0^\tau E(t)\,dt+\frac{1}{2}\int_{D_\tau}|F(t,x)|^2\,dx\,dt$$

$(Xu)\Box u = F(t,x)\,(Xu)$ where $X = X_0(t,x)\partial_t + \sum_{j=1}^{n} X_j(t,x)\partial_j$ 向量场

$\overset{\shortparallel}{\partial_t}(e_0(u)) + \sum_{j=1}^{n}\partial_j(e_j(u)) + R$

energy density 若考虑区域 D 上情况. 记 $D_\tau = D\cap\{t=\tau\}$

$e(u) := \nu_0 e_0(u) + \sum_{j=1}^{n}\nu_j e_j(u)$ (若 $\nu_0>0$. $\nu_0^2 \geq \sum_{j=1}^{n}\nu_j^2$ $\nu$ 是谁的外法向量?)

(要求 $\geq 0$.)

$E(t) := \int_{\Sigma_t} e(u)(t,x)\,dx$

分类向量场, 称 (1) time-like (类时的) $X_0^2 > \sum_{j=1}^{n} X_j^2$

(2) null (零的) $X_0^2 = \sum_{j=1}^{n} X_j^2$

(3) space-like (类空的) $X_0^2 < \sum_{j=1}^{n} X_j^2$

If $X := X_0\partial_t + \sum_{j=1}^{n} X_j\partial_j$ is non-space-like and $\nu = (\nu_0,\cdots,\nu_n)$ s.t. $\nu_0^2 \geq \sum_{j=1}^{n}\nu_j^2$

then the energy density is non-negative (证明?)

Fourier transform 将 PDE 变为 ODE
$$\begin{cases} \hat{u}_{tt}(t,\xi) + |\xi|^2\hat{u}(t,\xi) = \hat{F}(t,\xi) \\ \hat{u}(0,\xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0,\xi) = \hat{u}_1(\xi) \end{cases}$$

解这个关于 t 的 ODE, 得. 
$$\hat{u}(t,\xi) = \cos(t|\xi|)\,\hat{u}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{u}_1(\xi) + \int_0^t \frac{\sin((t-\tau)|\xi|)}{|\xi|}\hat{F}(\tau,\xi)\,d\tau$$

(都是形式上的计算)

$\Rightarrow u(t,x) = \int_{\mathbb{R}^n} \hat{u}(t,\xi)\, e^{ix\cdot\xi}\, d\xi$

No.

Date 10.22

1维波动方程初边值问题 IBVP

分离变量法 special solution $u=u(t,x)=T(t)X(x)$

$$
(\#)\begin{cases} \partial_t^2 u - a^2 \partial_x^2 u = 0 & t>0 \quad 0<x<l \\ u|_{t=0}=\varphi \quad \partial_t u|_{t=0}=\psi & 0<x<l \\ u|_{x=0}=0 \quad u|_{x=l}=0 & t>0. \end{cases}
$$

(#1) $T''(t)+\lambda a^2 T(t)=0 \quad t>0$

$$
(\#2)\begin{cases} X''(x)+\lambda X''(x)=0. \\ X(0)=0, \ X(l)=0. \end{cases}
$$

若其有非零解，则称λ为特征值，对应解X(x)为特征函数

Thm4.1 $[0,l]$上

$$
(\#3)\begin{cases} X''+\lambda X=0 & 0<x<l \\ -\alpha_1 X'(0)+\beta_1 X(0)=0 \\ \alpha_2 X'(l)+\beta_2 X(l)=0 \end{cases}
$$

其中$\alpha_i\geq 0$, $\beta_i\geq 0$, $\alpha_i+\beta_i\neq 0$.

分离变量法解方程太恶心了！上学期数学物理方法那张A4纸我现在自己都看不懂了。对其重要性深表怀疑。那些物理习题都是人为的吗？（但那门物理考试确实是简单）

10.29

1维守恒律方程 $\begin{cases}\partial_t u+\partial_x(f(u))=0 & t>0 \quad x\in\mathbb{R}. \\ u|_{t=0}=u_0\end{cases}$ 非线性波动现象，无扩散效应

例 1° Complete Euler System

2° p-system $\begin{cases}\partial_t v-\partial_x u=0 \\ \partial_t u+\partial_x(p(v))=0.\end{cases}$ $p(v)=k_0 v^{-\gamma}$ $(k_0>0. \ \gamma>1)$

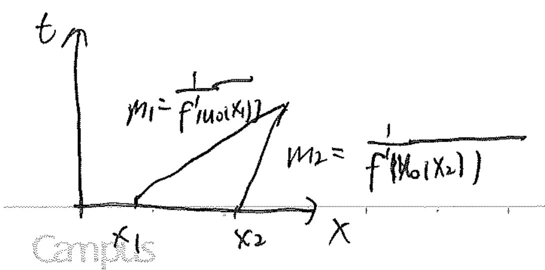
由特征线方法.

设$f\in C^1$ $\quad \partial_t u+f'(u)\partial_x u=0 \quad t>0 \quad x\in\mathbb{R}$

特征线 $\begin{cases}\dot{x}(t)=f'(u) \\ x(0)=x_0\end{cases}$

$\Rightarrow \frac{d}{dt}u(t,x(t)) \Rightarrow u(t,x(t))\equiv const$ $\quad \Rightarrow \dot{x}(t)=f'(u(0,x_0))=f'(u_0(x_0))$

$x(t)=f'(u_0(x_0))t+x_0$




若$\exists x_1<x_2$ s.t. $m_1<m_2$ 则 $f'(u_0(x_1))>f'(u_0(x_2))$
则特征线会相交.

若u为$C^1$解，则$f'(u_0(\cdot))$不减

No.

Date 2024 10 22

Sobolev Space $H^s(\mathbb{R}^n)$ for $s\in\mathbb{N}$

$H^s(\mathbb{R}^n)=\{u\in L^2(\mathbb{R}^n)\mid D^\alpha u\in L^2(\mathbb{R}^n)\ \forall\alpha\in\mathbb{N}^n \text{ with } |\alpha|\le s\}=\{u\in L^2\mid (1+|\xi|^2)^{\frac{s}{2}}\hat{u}(\xi)\in L^2(\mathbb{R}^n)\}$

$\|f\|_{H^s(\mathbb{R}^n)}=\sum_{|\alpha|\le s}\|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}=\sum_{|\alpha|\le s}(2\pi)^{\frac{n}{2}}\|\xi^\alpha\hat{f}(\xi)\|_{L^2(\mathbb{R}^n)}\approx\|(1+|\xi|^2)^{\frac{s}{2}}\hat{f}(\xi)\|_{L^2(\mathbb{R}^n)}$

用$\langle\xi\rangle$记$(1+|\xi|^2)^{\frac12}$

$H^s(\mathbb{R}^n)=\{u\in S'\mid\langle\xi\rangle^s\hat{u}\in L^2\}$

$\dot{H}^s(\mathbb{R}^n)=\{u\in S'/P\mid |\xi|^s\hat{u}\in L^2\}$ $\quad\|u\|_{\dot{H}^s(\mathbb{R}^n)}:=\||\xi|^s\hat{u}(\xi)\|_{L^2(\mathbb{R}^n)}$

模掉多项式

weak solution (in the sense of distribution) $P(y,D)u=\sum_{|\alpha|\le m}a_\alpha(y)D^\alpha u=f(y)\quad y\in\mathbb{R}^d$

$u$为弱解，若 $\langle P(y,D)u,\varphi\rangle=\langle f,\varphi\rangle\quad\forall\varphi\in C_c^\infty$.

基本解 fundamental solution of op. $P(y,D)$ $\quad u\in\mathcal{D}'$ s.t. $P(y,D)u=\delta_0$

若 $E$ 为基本解，则 $u=E*f$ 为 $P(y,D)u=f$的解：

$P(y,D)u=P(y,D)(E*f)=(P(y,D)E)*f=\delta_0*f=f$

10.31 Thm: $a_\alpha(y)=\text{const}$ 时，必存在基本解

Pf: $1^\circ$ 不失一般性，可假设主项系数为1 $\quad P(\xi)=0$ 有$m$个复根 $\lambda_1,\lambda_2\cdots\lambda_m$

则 $P(\xi)=(\xi-\lambda_1)(\xi-\lambda_2)\cdots(\xi-\lambda_m)$

$\Rightarrow\exists\tau\in\mathbb{R}\quad|\tau|\le m+1\quad$ s.t. $|\xi+i\tau-\lambda_j|>1\quad(\forall\xi\in\mathbb{R},\ \forall j)$

$\Rightarrow|P(\xi+i\tau)|=\prod_{j=1}^m|\xi+i\tau-\lambda_j|>1$

define $E$: $\quad E(\varphi):=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{\hat{\varphi}(-\xi-i\tau)}{P(\xi+i\tau)}d\xi\quad\forall\varphi\in C_c^\infty$

△与$\tau$无关

△(Paley-Wiener-Schwartz Thm: $\varphi\in C_c^\infty(\mathbb{R}^n)'$ $\Rightarrow\hat{\varphi}$ 在$\mathbb{C}^n$上解析且 rapidly decreasing)

$\mathcal{E}'$

supp $\varphi\subset B(0,M)$ $\quad|\hat{\varphi}(\xi)|\le C(k,M,\varphi)\frac{e^{M|\operatorname{Im}\xi|}}{(1+|\xi|)^k}$ 好decay (未证)

$|E(\varphi)|\le\int_{\mathbb{R}}|\hat{\varphi}(-\xi-i\tau)|d\xi\lesssim\int_{\mathbb{R}}|\hat{\varphi}(\xi)|d\xi\lesssim\|\varphi\|_{L^\infty}$ ……

故$E\in(C_c^\infty(\mathbb{R}))^*=\mathcal{E}'$ 且显然 $E\in\mathcal{D}'$

$|E(\varphi)|\le\|\hat{\varphi}\|_{L^1}\lesssim\rho_{0,0}(\hat{\varphi})+\rho_{d+1,0}(\hat{\varphi})\lesssim\|\varphi\|_{L^1}+\rho_{0,d+1}(\varphi)\lesssim\rho_{0,0}(\varphi)+\rho_{d+1,0}(\varphi)+\rho_{0,d+1}(\varphi)$

KOKUYO

No.

Date 10 · 31

$\forall \varphi \in C_c^\infty(\mathbb{R})$. $P(D)E(\varphi) = E(P(-D)\varphi) = \int_{\mathbb{R}} \frac{P(\xi+i\tau)\hat{\varphi}(-\xi-i\tau)}{P(\xi+i\tau)} d\xi = \int_{\mathbb{R}} \hat{\varphi}(-\xi-i\tau) d\xi$

$\overset{\text{全纯+Decay}}{=} \int_{\mathbb{R}} \hat{\varphi}(-\xi) d\xi = \varphi(0) = \langle \delta_0, \varphi \rangle$

故 $P(D)E = \delta_0$. $E$为$P(D)$的基本解.

$2^\circ$ $n>1$ $P(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha$ $\sum_{|\alpha|=m}' a_\alpha \neq 0$. 可做旋转使 $\xi_1^m$ 前系数不为0. 不妨假设它为1

fix $\xi' = (\xi_2, \cdots, \xi_n)$ $\exists \underset{|\tau|\le m+1}{\tau \in \mathbb{R}}$. $|P(\xi_1+i\tau, \xi')| > 1$ for all $\xi_1 \in \mathbb{R}$

fix $\tau$

由连续性. $|P(\xi_1+i\tau, \eta')| > 1$ 对 $\forall \xi_1 \in \mathbb{R}$. $\forall \eta' = (\eta_2, \cdots, \eta_n)$ 在$\xi'$的某邻域$U_j$内

$\mathbb{R}^{n-1} = \bigcup_{j=1}^\infty U_j$ $U_j \cap U_k = \phi$ $\forall j \neq k$.

$\Rightarrow \forall j$, $\exists \tau_j \in \mathbb{R}$. $|\tau_j| \le m+1$ $|P(\xi_1+i\tau_j, \xi')| > 1$ $\forall \xi_1 \in \mathbb{R}$ $\forall \xi_j \in U_j$

因为紧 可
每个点均有一个邻域的开邻域，故可被可数个互相不交之，再剔除...

定义 $E$ : $E(\varphi) := \sum_j \int_{U_j} \int_{\mathbb{R}} \frac{\hat{\varphi}(-\xi_1 - i\tau_j, -\xi')}{P(\xi_1+i\tau_j, \xi')} d\xi_1 \, d\xi'$

...也不难的问吧?

则可类似地验证 $E$是基本解

---

对某些变量做 Fourier transform, 得到基本解

lemma. 令$n=1$

$P(t,D) = \sum_{j\le m} a_j D_t^j$, $a_m(t) = 1$ $a_j \in C^\infty(\mathbb{R})$

令 $u=u(t)$ 满足 $\begin{cases} P(t,D)u = 0 & t \in \mathbb{R} \\ u(0) = \cdots = u^{(m-2)}(0) = 0 & u^{(m-1)}(0) = 1. \end{cases}$ 这是一个ODE m阶

则 $E(t) = H(t)u(t)$ 为基本解 $H$为 Heaviside function $H = \chi_{(0,\infty)}$

即 $P(t,D)E(t) = \delta_0(t)$

Pf: $H'(t) = \delta_0(t)$ $\delta_0(t) u^{(k)}(t) = u^{(k)}(0)$

由$u(0)=0$ $E'(t) = H'(t)u(t) + H(t)u'(t) = H(t)u'(t)$

$\Rightarrow$ $k=1, \cdots, m-1$ 由$u^{(k)}(0)=0$ $E^{(j)}(t) = H(t)u^{(j)}(t)$

$E^{(m)}(t) = H'(t)u^{(m-1)}(t) + H(t)u^{(m)}(t) = \delta_0(t) + H(t)u^{(m)}(t)$

$P(t,D)E(t) = \sum_{\alpha=1}^m a_\alpha D_t^\alpha E(t) = \sum_{j=1}^{m-1} a_j H(t)u^{(j)}(t) + a_m(\delta_0(t) + H(t)u^{(m)}(t))$

$= H(t)\sum_{j=1}^m a_j u^{(j)}(t) + \delta_0(t) = \delta_0(t)$

Fundamental sol. $E = E(t,x)$ s.t.

热算子 $P(D)E = (\partial_t - \Delta)E(t,x) = \delta_0(t)\delta_0(x)$

关于 x Fourier transform $\partial_t \hat{E}(t,\xi) + |\xi|^2 \hat{E}(t,\xi) = \delta_0(t)$

Since the sol. of the Cauchy problem (关于 t) $\begin{cases} (\partial_t + c^2|\xi|^2)u(t;\xi) = 0 \\ u(0;\xi) = 1 \end{cases}$ is $u(t;\xi) = e^{-c^2|\xi|^2 t}$

$\Rightarrow \hat{E}(t,\xi) = H(t)u(t;\xi) = H(t)e^{-c^2|\xi|^2 t}$

$E(t,x) = H(t)\prod_{j=1}^{n}\int_{\mathbb{R}} e^{ix_j\xi_j} e^{-c^2\xi_j^2 t}\, d\xi_j = H(t)\frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4c^2t}}$

11.5 Heat equation

· Cauchy problem $\begin{cases} \partial_t u - a^2\Delta u = f(t,x) & t>0 \quad x \in \mathbb{R}^n \\ u|_{t=0} = \varphi(x) & x \in \mathbb{R}^n \end{cases}$

关于 x Fourier Transform. $\begin{cases} \partial_t \hat{u}(t,\xi) + a^2|\xi|^2\hat{u}(t,\xi) = \hat{f}(t,\xi) \\ \hat{u}(0,\xi) = \hat{\varphi}(\xi) \end{cases}$

$\Rightarrow \hat{u}(t,\xi) = \hat{\varphi}(\xi)e^{-a^2|\xi|^2 t} + \int_0^t \hat{f}(\tau,\xi)e^{-a^2|\xi|^2(t-\tau)}\, d\tau$

$(e^{-a^2|\xi|^2 t})^{\vee}(x) = \int_{\mathbb{R}^n} e^{-a^2|\xi|^2 t} e^{ix\cdot\xi}\, d\xi \cancel{= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-a^2|\xi|}}$

$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-a^2t(\xi_1^2+\xi_2^2+\cdots+\xi_n^2)} e^{i(x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n)}\, d\xi_1 d\xi_2\cdots d\xi_n$

$W(x) := (4\pi)^{-\frac{n}{2}} e^{-\frac{x^2}{4}}$ $= \frac{1}{(2\pi)^n}\left(\frac{\pi}{a^2t}\right)^{\frac{n}{2}} e^{-\frac{|x|^2}{4a^2t}} = (4\pi a^2 t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4a^2t}} =: W(a^2t,x) =: W_{a^2t}(x)$

$u(t,x) = W(a^2t,x) *_x \varphi(x) + \int_0^t W(a^2(t-\tau),x) *_x f(\tau,x)\, d\tau$

$W(a^2t,\cdot) \in S(\mathbb{R}^n)$ $\quad \forall u \in S'(\mathbb{R}^n) \quad u * W(t,\cdot) \in C^\infty(\mathbb{R}^n)$

以下 a=1

记 $E(t,x)$ 为 heat eq. 基本解. $u(t,\cdot) = E(t,\cdot) *_x \varphi(\cdot) + \int_0^t E(t-\tau,\cdot) *_x f(\tau,\cdot)\, d\tau$

$(\partial_t - \Delta)E(t,x) = \delta_0(t)\delta_0(x)$ $\quad (\partial_t - \Delta)(E(t,\cdot) *_x \varphi) = ((\partial_t-\Delta)E(t,\cdot)) *_x \varphi = (\delta_0(t)\delta_0(x)) * \varphi(x)$
$= 0 \ (t>0)$

$\lim_{t\to 0^+}(W(t,\cdot) *_x \varphi) = \varphi(x)$ $\quad \forall \varphi \in L^p(\mathbb{R}^n) \quad 1 \le p < \infty$

$\forall x$ Lebesgue point of $\varphi$

No.

Date

半无界问题 $x \in \mathbb{R}^1$

第一边值问题 Dirichlet problem
$$\begin{cases} \partial_t u - \Delta u = f(t,x) & t>0 \quad 0<x<\infty \\ u|_{t=0} = \varphi(x) & 0 \le x < \infty \\ u|_{x=0} = g(t) & t>0 \end{cases}$$

令 $V(t,x) = u(t,x) - g(t)$. 则 $V|_{x=0} = 0$. 对 $V$ 满足的方程组整个奇延拓

第二边值问题
$$\begin{cases} \partial_t u - \Delta u = f(t,x) & t>0, \quad 0<x<\infty \\ u|_{t=0} = \varphi(x) & 0 \le x < \infty \\ u_x|_{x=0} = g(t) & t>0 \end{cases}$$

令 $V(t,x) = u(t,x) - g(t)x$ 则 $V_x|_{x=0} = 0$ 对 $V$ 满足的方程组整个偶延拓

第三边值问题
$$\begin{cases} \partial_t u - \Delta u = f(t,x) & t>0 \quad 0<x<\infty \\ u|_{t=0} = \varphi & 0 \le x < \infty \\ u_x|_{x=0} + \alpha u|_{x=0} = 0. \end{cases}$$

延拓 $\Phi(x) = \begin{cases} \varphi(x) & x \ge 0 \\ \text{?}(x) & x<0 \end{cases}$

考虑
$$\begin{cases} \partial_t \bar{u} - \partial_x^2 \bar{u} = 0 & t>0 \quad x \in \mathbb{R} \\ \bar{u}|_{t=0} = \Phi & x \in \mathbb{R} \end{cases} \Rightarrow \bar{u}(t,x) = \int_{\mathbb{R}^n} E(t,x-y)\,\Phi(y)\,dy$$

则
$$\bar{u}_x(t,x) = \int_{\mathbb{R}^n} \partial_x E(t,x-y)\,\Phi(y)\,dy = -\int_{\mathbb{R}^n} \partial_y E(t,x-y)\,\Phi(y)\,dy$$

$$= -\Phi(y)\big(E(t,x-y)\big)\Big|_{y=-\infty}^{\infty} - \int_{\mathbb{R}^n} -E(t,x-y)\,\Phi'(y)\,dy$$

$$= \int_{-\infty}^{0} + \int_{0}^{\infty}$$

$$\bar{u}_x|_{x=0} + \alpha u|_{x=0} = \cancel{-E(t,\ldots) + E(t,x\ldots)}$$

书上有,不记了

以下讨论均假设$u$为经典解

maximal principle $\quad Q:=\{(x,t):\ 0<x<l,\ 0<t\le T\}$ $\quad C^{2,1}=C_x^2C_t^1$

Thm1 (weak maximal principle) Suppose $u\in C^{2,1}(Q)\cap C(\bar{Q})$

s.t. $Lu:=(\partial_t-a^2\partial_x^2)u=f(t,x)\le 0$.

then, the maximal of $u$ in $\bar{Q}$ is achieved in $\Gamma:=\bar{Q}-Q$ (不包含上顶边). 即 $\max\limits_{\bar{Q}}u=\max\limits_{\Gamma}u$.

Pf: 1° $\forall P\in Q$, $f|_P\neq 0$. [反证] 若 $\exists P_0(x_0,t_0)\in Q$ s.t. $u$在$P_0$处取最大值.

则 $\partial_x u|_{P_0}=0$. $\quad \partial_x^2 u|_{P_0}\le 0$.

$\partial_t u|_{P_0}=0$, ($t_0<T$时) ($\partial_t u|_{P_0}\ge 0$ $t_0=T$时)

$f|_{P_0}=(\partial_t u-a^2\partial_x^2 u)|_{P_0}\ge 0$. 矛盾

2° $\exists P\in Q$ s.t. $f|_P=0$. $\forall\varepsilon>0$. 考虑 $V(x,t)=u(x,t)-\varepsilon t$

则 $LV=Lu-\varepsilon\le-\varepsilon<0$. 由1°. $V$的极值可在$\Gamma$取到.

$$\max_{\bar{Q}}u=\max_{\bar{Q}}(V+\varepsilon t)\le\left(\max_{\bar{Q}}V\right)+\varepsilon T=\max_{\Gamma}V+\varepsilon T$$
$$\le\max_{\Gamma}u+\varepsilon T$$

令$\varepsilon\to 0$. 得 $\max\limits_{\bar{Q}}u\le\max\limits_{\Gamma}u$. □

Cor (比较原理) 若 $Lu\ge 0$. 则 $\min\limits_{\bar{Q}}u=\min\limits_{\Gamma}u$

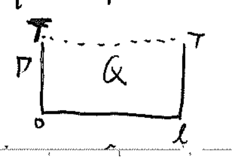
若 $Lu=0$ 则 $\max\limits_{\bar{Q}}u=\max\limits_{\Gamma}u$ 且 $\min\limits_{\bar{Q}}u=\min\limits_{\Gamma}u$.

Cor 若$u,v\in C^{2,1}(\bar{Q})$ s.t. $Lu\le Lv$ in $Q$. 且 $u|_\Gamma\le v|_\Gamma$ 则$u\le v$ in $\bar{Q}$

第一边值问题的最大模估计 $\quad$ 若 $u\in C_x^2C_t(Q)\cap C(\bar{Q})$ 满足

$$\begin{cases} Lu=u_t-a^2u_{xx}=f & (x,t)\in Q \\ u|_{t=0}=\varphi(x) & 0\le x\le l \\ u|_{x=0}=g_1(t)\quad u|_{x=l}=g_2(t) & 0\le t\le T \end{cases}$$

$Q=\{(x,t)\,|\,0<x<l,\ 0<t\le T\}$




则 $\|u\|_{L^\infty}\le\|f\|_{L^\infty}T+\max\{\|\varphi\|_{L^\infty},\|g_1\|_{L^\infty},\|g_2\|_{L^\infty}\}=:FT+B$

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考虑辅助函数 $W(x,t) = Ft + B \pm u(x,t)$

$LW = F \pm f \geq 0$. 故 $W$ 最小值在边界取得. 而 $W|_{\Gamma} \geq 0$.

故 $W(x,t) \geq 0$. $\Rightarrow$ $|u(x,t)| \leq FT + B$

Cor. 由此可得解的唯一性. Cor 解对初边值连续依赖

第二、三边值问题解的最大模估计 若 $u \in C_x^2 C_t^1(Q) \cap C_x^1 C_t^0(\overline{Q})$ 满足

$$\begin{cases} Lu = u_t - a^2 u_{xx} = f & (x,t) \in Q \\ u|_{t=0} = \varphi(x) & 0 \leq x \leq l \\ \left(-\frac{\partial u}{\partial x} + \alpha(t) u\right)\Big|_{x=0} = g_1(t) & 0 \leq t \leq T \qquad \alpha(t) \geq 0. \quad \beta(t) \geq 0. \\ \left(\frac{\partial u}{\partial x} + \beta(t) u\right)\Big|_{x=l} = g_2(t) & 0 \leq t \leq T \end{cases}$$

则 $|u(x,t)| \leq C(a,l,T)\left(\|f\|_{L^\infty} + \max\{\|g_1\|_{L^\infty}, \|g_2\|_{L^\infty}, \|\varphi\|_{L^\infty}\}\right) =: C(F+B)$

Lemma: 若 $f, \varphi, g_1, g_2 \geq 0$. 则 $u(x,t) \geq 0$ in $\overline{Q}$

Pf: 先证 $g_1 > 0$, $g_2 > 0$ 时 $u$ 在 $\overline{Q}$ 上无负最小值. 即 $u \geq 0$.

否则. 由 $f \geq 0$. 最小值在 $\Gamma$ 上取得. $P(x_0, t_0)$ ;

若 $x_0 = 0$. 则 $\frac{\partial u}{\partial x}\Big|_{P_0} \geq 0$ $\quad -\frac{\partial u}{\partial x}\Big|_{P_0} + \alpha(t) u|_{P_0} \leq 0$. 矛.

若 $x_0 = l$ 则 $\frac{\partial u}{\partial x}\Big|_{P_0} \leq 0$. $\quad \frac{\partial u}{\partial x}\Big|_{P_0} + \beta(t) u|_{P_0} \leq 0$. 矛

若 $t_0 = 0$. 与 $\varphi \geq 0$ 矛盾.

现在考虑辅助函数 $V(x,t) = u(x,t) + \varepsilon\left(2a^2 t + (x - \frac{l}{2})^2\right)$

则
$$\begin{cases} LV = Lu + \varepsilon(2a^2 - a^2 \cdot 2) = Lu \geq 0. \\ V|_{t=0} = \varphi(x) + \varepsilon(x - \frac{l}{2})^2 \geq 0 \\ \left(-\frac{\partial V}{\partial x} + \alpha(t) V\right)\Big|_{x=0} = g_1(t) + \varepsilon l + \alpha(t) \cdot \varepsilon \cdot (2a^2 t + \frac{l^2}{4}) > 0. \\ \left(\frac{\partial V}{\partial x} + \beta(t) V\right)\Big|_{x=l} = g_2(t) + \varepsilon l + \beta(t) \cdot \varepsilon \cdot (2a^2 t + \frac{l^2}{4}) > 0 \end{cases}$$

故 $V \geq 0$. $u(x,t) \geq -\varepsilon\left(2a^2 t + (x - \frac{l}{2})^2\right) \geq -\varepsilon\left(2a^2 T + \frac{l^2}{4}\right)$ 令 $\varepsilon \to 0$. 则有 $u \geq 0$.

回过头来, 引入辅助函数 $W(x,t) = Ft + B\, z(x,t) \pm u(x,t)$ , $z(x,t) = 1 + \frac{1}{l}\left(2a^2 t + (x - \frac{l}{2})^2\right)$

$z$ 满足 $Lz = 0$.

$$\begin{cases} z|_{t=0} \geq 1 \\ \left(-\frac{\partial z}{\partial x} + \alpha(t) z\right)\Big|_{x=0} \geq -\frac{\partial z}{\partial x}\Big|_{x=0} = 1 \\ \left(\frac{\partial z}{\partial x} + \beta(t) z\right)\Big|_{x=l} \geq \frac{\partial z}{\partial x}\Big|_{x=l} = 1 \end{cases}$$

故 $Lw = F + BL\delta \pm Lu = F \pm Lu \geq 0$

$w|_{t=0} = B\delta|_{t=0} \pm u|_{t=0} = B \pm \varphi(x) \geq 0$

$\left(-\frac{\partial w}{\partial x} + \alpha(t) w\right)\Big|_{x=0} \geq B \pm g_1(t) \geq 0$

$\left(\frac{\partial w}{\partial x} + \beta(t) w\right)\Big|_{x=l} \geq B \pm g_2(t) \geq 0$

由引理. $w \geq 0$. $\Rightarrow FT + B\|\delta\|_{L^\infty} \geq Ft + B\delta(x,t) \geq |u(x,t)|$

$|u(x,t)| \leq FT + B\left(1 + \frac{2a^2T}{l} + \frac{l}{4}\right)$

Thm. 记 $Q = \{(x,t) \mid -\infty < x < \infty,\ 0 \leq t \leq T\}$. 若 $u \in C^2_x C^1_t(Q) \cap C(\bar{Q})$ 满足

$$\begin{cases} u_t - a^2 u_{xx} = f(x,t) & (x,t) \in Q \\ u(x,0) = \varphi(x) & -\infty < x < \infty \end{cases}$$

且有界. 则

$\|u\|_{L^\infty(Q)} \leq T\|f\|_{L^\infty(Q)} + \|\varphi\|_{L^\infty(\mathbb{R})} =: TF + \Phi$

Pf: $\forall L > 0$. $Q_L = Q \cap \{|x| < L\}$. 设 $\|u\|_{L^\infty} = K < \infty$

在 $Q_L$ 上考虑辅助函数 $w(x,t) = Ft + \Phi + \delta_L(x,t) \pm u(x,t)$

其中 $\delta_L = \frac{K}{L^2}(2a^2t + x^2)$

$$\begin{cases} Lw = F \pm Lu = F \pm f \geq 0 \\ w|_{t=0} \geq \Phi \pm \varphi(x) \geq 0 \\ w|_{x=\pm L} \geq K \pm u(x,t) \geq 0 \end{cases}$$

由第一边值问题估计. $w(x,t) \geq 0$ in $Q_L$

$Ft + \Phi + \frac{K}{L^2}(2a^2t + x^2) \geq |u(x,t)|$ in $Q_L$

fix $(x,t)$ 再令 $L \to \infty$ 得 $Ft + \Phi \geq |u(x,t)|$

故 $|u(x,t)| \leq FT + \Phi$ $\quad \forall (x,t) \in Q$.

Thm. $u \in C^1_x C^0_t(\bar{Q}_T) \cap C^2_x C^1_t(Q_T)$ (记 $Q_T = \{(x,t) \mid 0 < x < l,\ 0 < t \leq T\}$) 满足 $\begin{cases} Lu = f, & Q_T \\ u|_{t=0} = \varphi(x), & 0 \leq x \leq l \\ u|_{x=0} = u|_{x=l} = 0, & 0 \leq t \leq T \end{cases}$

$$\sup_{t \in [0,T]} \|u(\cdot,t)\|^2_{L^2(0,l)} + 2a^2\|u_x\|^2_{L^2([0,T]\times(0,l))}$$

$$\leq M(T)\left(\|\varphi\|^2_{L^2(0,l)} + \|f\|_{L^2([0,T]\times(0,l))}\right)$$

Pf: $\int_{Q_\tau} uf\, dx\,dt = \int_{Q_\tau} u(\partial_t u - a^2\partial_x^2 u)\, dx\,dt = \int_{Q_\tau} \partial_t(\tfrac{1}{2}u^2)\, dx\,dt - a^2\int_{Q_\tau} u\cdot\partial_x^2 u\, dx\,dt$

$= \frac{1}{2}\int_0^l u^2\Big|_{t=\tau} dx - \frac{1}{2}\int_0^l u^2\Big|_{t=0} dx + a^2\int_0^T\int_0^l (\partial_x u)^2\, dx\,dt \leq \frac{1}{2}\int_{Q_\tau}(u^2 + f^2)\, dx\,dt$

(分部积分)

$\int_0^l u^2(x,\tau)\, dx \leq \int_0^l \varphi(x)^2\, dx - 2a^2\int_{Q_\tau} (\partial_x u)^2\, dx\,d\tau + \int_{Q_\tau} u^2\, dx\,dt + \int_{Q_\tau} f^2\, dx\,dt$

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$$\int_0^l u^2(x,\tau)\,dx \le \int_0^l \varphi^2(x)\,dx + \int_{Q_\tau} f^2(x,t)\,dx\,dt + \int_{Q_\tau} u^2(x,t)\,dx\,dt$$

$$\Rightarrow \int_{Q_\tau} u^2(x,t)\,dx\,dt \le e^{\tau}\left(\int_0^l \varphi^2(x)\,dx + \int_{Q_\tau} f^2(x,t)\,dx\,dt\right)$$

再代回. $\frac{1}{2}\int_0^l u^2(x,\tau)\,dx + a^2\int_{Q_\tau}(\partial_x u)^2\,dx\,dt \le \frac{1}{2}\int_0^l \varphi^2(x)\,dx + \frac{1}{2}\int_{Q_\tau} f^2\,dx\,dt + \frac{1}{2}\int_{Q_\tau} u^2\,dx\,dt$

$$\le \frac{1}{2}(1+e^{\tau})\left(\int_0^l \varphi^2(x)\,dx + \int_{Q_\tau} f^2\,dx\,dt\right)$$

$$\le \frac{1}{2}(1+e^{T})\left(\int_0^l \varphi^2\,dx + \int_{Q_T} f^2\,dx\,dt\right)$$

故 $\int_0^l u^2(x,\tau)\,dx \le (1+e^T)\left(\int_0^l \varphi^2\,dx + \int_{Q_T} f^2\,dx\,dt\right) \quad \forall 0<\tau\le T$

$2a^2\int_{Q_T}(\partial_x u)^2\,dx\,dt \le (1+e^T)\left(\int_0^l \varphi^2\,dx + \int_{Q_T} f^2\,dx\,dt\right) \quad \forall\ \tau.$

$$\sup_{0\le t\le T}\|u(\cdot,t)\|_{L^2}^2 + 2a^2\|\partial_x u\|_{L^2(Q_T)}^2 \le 2(1+e^T)\left(\|\varphi\|_{L^2(0,l)}^2 + \|f\|_{L^2(Q_T)}^2\right)$$

# Laplace's equation

1° Harmonic function: $u$ in domain $\Omega \subset \mathbb{R}^n$ 开连通区域

$\omega_{n-1}$ 表示 $S^{n-1}$ 面积
$V_n$ 表示 $B(0,1)$ 体积

if 1) $u \in C^2(\Omega)$

2) $-\Delta u(x) = 0 \quad \forall x \in \Omega$.

Thm (mean value property for harmonic function)

$u$ harmonic in $\Omega$. $\forall B(x,r) \subset\subset \Omega$, $u(x) = \fint_{\partial B(x,r)} u(y)\,d\sigma(y) = \fint_{B(x,r)} u(y)\,dy$

Pf: by divergence thm, for any $\rho \in (0,r]$ $\int_{\partial B(x,\rho)} \frac{\partial u}{\partial \nu}(y)\,d\sigma(y) = \int_{B(x,\rho)} \Delta u(y)\,dy = 0$

On the other hand,

$$\fint_{\partial B(x,\rho)} \frac{\partial u}{\partial \nu}(y)\,d\sigma(y) = \frac{1}{\omega_{n-1}\rho^{n-1}} \int_{\partial B(x,\rho)} Du(y)\cdot\frac{y-x}{\rho}\,d\sigma(y)$$

$$= \frac{1}{\omega_{n-1}} \int_{S^{n-1}} Du(x+\rho w) w\,d\sigma(w)$$

$$= \frac{d}{d\rho}\left(\fint_{S^{n-1}} u(x+\rho w)\,d\sigma(w)\right)$$

$$= \frac{d}{d\rho}\left(\fint_{\partial B(x,\rho)} u(y)\,d\sigma(y)\right)$$

$$\Rightarrow \frac{d}{d\rho}\left(\fint_{\partial B(x,\rho)} u(y)\,d\sigma(y)\right) = 0.$$

$\forall \rho \in (0,r]$ $\fint_{\partial B(x,\rho)} u(y)\,d\sigma(y) = \fint_{\partial B(x,r)} u(y)\,d\sigma(y)$ let $\rho \to 0$, by $u \in C^2$.

$$u(x) = \fint_{\partial B(x,r)} u(y)\,d\sigma(y)$$

$$\int_{B(x,r)} u(y)\,dy = \int_0^r d\rho \int_{\partial B(x,\rho)} u(y)\,d\sigma(y) = \int_0^r \omega_{n-1}\rho^{n-1} u(x)\,d\rho = \frac{\omega_{n-1}}{n} r^n u(x) = V_n r^n u(x)$$

$$\Rightarrow u(x) = \fint_{B(x,r)} u(y)\,dy$$

—

subharmonic in $\Omega$ if $u \in C^2(\Omega)$, $-\Delta u(x) \le 0$ in $\Omega$

superharmonic in $\Omega$ if $u \in C^2(\Omega)$ $-\Delta u(x) \ge 0$ in $\Omega$

Thm subharmonic in $\Omega$. $\forall B(x,r) \subset\subset \Omega$.

$u(x) \le \fint_{\partial B(x,r)} u(y)\,d\sigma(y)$

$u(x) \le \fint_{B(x,r)} u(y)\,dy$

过程与 $-\Delta u = 0$ 时一样.

superharmonic

$u(x) \ge \fint_{\partial B(x,r)} u(y)\,d\sigma(y)$

$u(x) \ge \fint_{B(x,r)} u(y)\,dy$

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**Thm** If $u \in C(\Omega)$ s.t. mean value property, then $u \in C^\infty(\Omega)$ and is harmonic in $\Omega$

Pf: choose $\varphi \in C_c^\infty(\mathbb{R}^n)$ $\quad \operatorname{supp}\varphi = B(0,1)$ $\quad \varphi \geq 0$. $\quad \int_{\mathbb{R}^n} \varphi = 1$. $\quad \varphi$ radial.

$\varepsilon < \operatorname{dist}(x, \partial\Omega)$ $\quad u * \varphi_\varepsilon(x) = u(x) \Rightarrow u \in C^\infty(\Omega)$

$\Rightarrow \forall B(x,\rho) \subset \Omega$. $\displaystyle \fint_{B(x,\rho)} \Delta u(y)\,dy = \frac{d}{d\rho}\left(\fint_{\partial B(x,\rho)} u(y)\,dy\right) = 0$.

$\Rightarrow \Delta u = 0$, in $\Omega$.

Maximal and minimal principle

**Thm** (Strong ~) let $u \in C^2(\Omega)$

(1) let $-\Delta u \leq 0$ in $\Omega$. If $\exists y \in \Omega$ s.t. $u(y) = \sup_{\Omega} u =: M$ then $u$ is constant

(2) let $-\Delta u \geq 0$ in $\Omega$ if $\exists y \in \Omega$ s.t. $u(y) = \inf_{\Omega} u =: M$. then $u$ is constant.

[反证] 否则, $\exists y \in \Omega$ s.t. $u(y) = M$.

Pf: (1) 记 $\Omega_M = \{x \in \Omega : u(x) = M\}$. 下证 $\Omega_M$ 关于 $\Omega$ 既开又闭. 于是由 $\Omega_M \neq \varnothing$, $\Omega$ 连通, 得 $\Omega_M = \Omega$.

由 $u$ 连续性. $\Omega_M$ 闭

开: $\forall z \in \Omega$. $u(z) = M$. $\quad 0 = u(z) - M \leq \fint_{B(z,r)} (u(y) - M)\,dy \leq 0$. 故每个均取等.

$\exists r > 0$ $B(z,r) \subset\subset \Omega$. $\quad \forall y \in B(z,r)$ $u(y) = M$

(2) 在(1)中用 $-u$ 代替 $u$ 即可.

**Thm** (Weak ~) $\quad u \in C^2(\Omega) \cap C(\bar{\Omega})$

(1) $-\Delta u \leq 0$ in $\Omega$ $\quad \max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.

(2) $-\Delta u \geq 0$ in $\Omega$. $\quad \min_{\bar{\Omega}} u = \min_{\partial\Omega} u$.

(3) $\Delta u = 0$. $\quad \min_{\partial\Omega} u \leq u(x) \leq \max_{\partial\Omega} u$ $\quad$ 由此可得 Laplace's equation / Poisson's equation 唯一性

**Thm** (Liouville) $\quad \Delta u = 0$. (1) $u$ bdd in $\mathbb{R}^n$ then $u \equiv$ const. $\quad \forall r > 0$.

Pf: $\displaystyle |u(x_1) - u(x_2)| = \left|\fint_{B(x_1,r)} u(y)\,dy - \fint_{B(x_2,r)} u(y)\,dy\right| = \frac{1}{V_n r^n}\left|\int_{B(x_1,r) - B(x_2,r)} u(y)\,dy - \int_{B(x_2,r) - B(x_1,r)} u(y)\,dy\right|$

$\displaystyle \leq \frac{\|u\|_{L^\infty}}{V_n r^n}\Big(\mu\big(B(x_1,r) - B(x_2,r)\big) + \mu\big(B(x_2,r) - B(x_1,r)\big)\Big)$

$\displaystyle \mu\big(B(x_1,r) \cap B(x_2,r)\big) \geq V_n\left(\frac{r + r - |x_1 - x_2|}{2}\right)^n$

$\displaystyle \leq \frac{2\|u\|_{L^\infty}}{V_n r^n}\left(V_n r^n - V_n\left(r - \frac{|x_1 - x_2|}{2}\right)^n\right) = O\left(\frac{1}{r}\right) \to 0$ as $r \to \infty$. 故 $u \equiv$ const.

$-\Delta\Phi(x) = \delta_0(x)$

**Example** 1. $\Phi(x) = \begin{cases} \frac{-1}{2\pi}\log|x| & n=2 \\ \frac{-1}{(2-n)w_{n-1}}|x|^{2-n} & n\geq 3 \end{cases}$ harmonic on $\mathbb{R}^n\setminus 0$

$\Phi'(r) = \frac{-1}{w_{n-1}r^{n-1}}$, $\nabla r = \frac{x}{r}$

2. Poisson kernel

$$P(t,x) = C_n\frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}} \qquad C_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}$$

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**Thm (Liouville)** (2) $u \geq 0$ then $u \equiv \text{const}$

Pf: $\forall x \in \mathbb{R}^n$. 对 $\forall r > |x|$

$u(0) = \fint_{B(0,r)} u(y)\,dy$, $u(x) = \fint_{B(x,r)} u(y)\,dy$

$$|u(x)-u(0)| \leq \frac{1}{V_n r^n}\int_{B(0,r)\triangle B(x,r)}|u(y)|\,dy = \frac{1}{V_n r^n}\int_{B(0,r)\triangle B(x,r)} u(y)\,dy$$

$$\leq \frac{1}{V_n r^n}\int_{B(0,r+|x|)\setminus B(0,r-|x|)} u(y)\,dy = \frac{1}{V_n r^n}\left(V_n(r+|x|)^n - V_n(r-|x|)^n\right)u(0)$$

$$= \frac{(r+|x|)^n-(r-|x|)^n}{r^n}u(0) \leq \frac{n(r+|x|)^{n-1}}{r^n}2|x|\,u(0) \to 0 \text{ as } r\to\infty.$$

故 $u(x) = u(0)$ $\forall x \in \mathbb{R}^n$. $u \equiv \text{const}$

**Thm (Harnack Inequality)** $u \geq 0$. harmonic in the domain $\Omega \subset \mathbb{R}^n$ $\forall \Omega' \subset\subset \Omega$.

则 $\sup_{\Omega'} u \leq C(n,\Omega',\Omega)\inf_{\Omega'} u$.

Pf (by mean value property)

$\forall y \in \Omega$. $\exists R>0$ s.t. $B(y,4R) \subset \Omega$. $\forall x \in B(y,R)$. $B(x,R) \subset B(y,2R) \subset B(x,3R) \subset\subset \Omega$.

$$u(x) = \fint_{B(x,R)} u(z)\,dz = \frac{1}{V_nR^n}\int_{B(x,R)} u(z)\,dz \leq \frac{1}{V_nR^n}\int_{B(y,2R)} u(z)\,dz = 2^n u(y)$$

$$u(x) = \fint_{B(x,3R)} u(z)\,dz = \frac{1}{V_n(3R)^n}\int_{B(x,3R)} u(z)\,dz \geq \frac{1}{V_n(3R)^n}\int_{B(y,2R)} u(z)\,dz = \frac{2^n}{3^n}u(y)$$

故 $\forall x_1, x_2 \in B(y,R)$

$u(x_1) \leq 2^n u(y)$. $\frac{2^n}{3^n}u(y) \leq u(x_2)$. 故 $u(x_1) \leq 3^n u(x_2)$.

$\sup_{B(y,R)} u \leq 3^n \inf_{B(y,R)} u$.

取 $4R < d(\partial\Omega', \partial\Omega)$. $\Omega'$ 可由有限个半径为R的(球心在$\Omega'$内)开球覆盖. N个 故 $\sup_{B(y,R)} u \leq 3^{nN}\inf_{B(y,R)} u$.

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lemma let $u=u(t,x)$ s.t. $P(D_t,D_x)u=\sum_{j+|\alpha|\le m} a_{j\alpha} D_t^j D_x^\alpha u = f(t,x) \in \mathbb{R}\times\mathbb{R}^n$.
where $f \in \mathcal{E}'(\mathbb{R}\times\mathbb{R}^n)$. If the integral $V(x):=\int_{\mathbb{R}} u(t,x)\,dt$ converges for a.e. $x$, $V\in L^1_{loc}(\mathbb{R}^n)$
Then $P(0,D_x)V(x)=g(x):=\int_{\mathbb{R}} f(t,x)\,dt \in \mathcal{D}'$

$\Phi(x;x_0)=\Phi(x-x_0)$

Q: $u$ s.t.

$$\begin{cases} -\Delta u=f & \text{in } \Omega \\ u|_{\partial\Omega}=\varphi \end{cases}$$

$u\in C^2(\bar\Omega)$

$u(x_0)=?$

在 $\Omega\setminus B(x_0,\rho)$ 上

$$\int_{\Omega\setminus B(x_0,\rho)} (u\Delta\Phi - \Phi\Delta u)\,dx = \int_{\partial\Omega-\partial B(x_0,\rho)} \left(u\frac{\partial\Phi}{\partial n} - \Phi\frac{\partial u}{\partial n}\right) d\sigma$$

$$= \int_{\Omega\setminus B(x_0,\rho)} -\Phi(x-x_0)\Delta u\cdot dx \quad = \int_{\partial\Omega-\partial B(x_0,\rho)} \left(u\frac{\partial\Phi}{\partial n}(x-x_0) - \Phi(x-x_0)\frac{\partial u}{\partial n}\right) d\sigma$$

$$\int_{\partial B(x_0,\rho)} u\frac{\partial\Phi}{\partial n}(x-x_0)\,d\sigma(x) = \Phi'(\rho)\int_{\partial B(x_0,\rho)} u\,d\sigma(x) = \frac{-1}{\omega_{n-1}\rho^{n-1}}\int_{\partial B(x_0,\rho)} u\,d\sigma(x) = -⨍_{\partial B(x_0,\rho)} u\,d\sigma(x) \to -u(x_0)$$

$$\left|\int_{\partial B(x_0,\rho)} \Phi(x-x_0)\frac{\partial u}{\partial n}\,d\sigma(x)\right| = \left|\Phi(\rho)\int_{\partial B(x_0,\rho)} \frac{\partial u}{\partial n}\,d\sigma(x)\right| \le \|\nabla u\|_{L^\infty(\partial B(x_0,\rho))}\Phi(\rho)\,\omega_{n-1}\rho^{n-1}$$

$\to 0$ as $\rho\to 0$.

故 $$-\int_\Omega \Phi(x-x_0)\Delta u\,dx = \int_{\partial\Omega} u\frac{\partial\Phi}{\partial n}(x-x_0) - \Phi(x-x_0)\frac{\partial u}{\partial n}\,d\sigma + u(x_0)$$

$$-u(x_0) = \int_{\partial\Omega}\left(u\frac{\partial\Phi}{\partial n}(x-x_0) - \Phi(x-x_0)\underbrace{\frac{\partial u}{\partial n}}_{\text{未知. 希望设定.}}\right)d\sigma + \int_\Omega \Phi(x-x_0)\Delta u\,dx$$

11.26 Goal eliminate $\frac{\partial u}{\partial n}\big|_{\partial\Omega}$.
Set $G(x_0,y)=\Phi(x_0-y)+h(x_0,y)$ $\forall y\in\Omega$. (Fix $x_0$)

$$u(x_0) = -\int_\Omega (G-h)\Delta u\,dy - \int_{\partial\Omega}\left(u\frac{\partial(G-h)}{\partial n} - (G-h)\frac{\partial u}{\partial n}\right)d\sigma$$

$$= -\int_\Omega G\Delta u\,dy + \int_{\partial\Omega} G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n}\,d\sigma$$

② 希望 $G(x_0,y)|_{\partial\Omega}=0$.
即 $h(x_0,y)|_{\partial\Omega}=-\Phi(x_0-y)|_{\partial\Omega}$

$$+\underbrace{\int_\Omega h\Delta u\,dy + \int_{\partial\Omega}\left(u\frac{\partial h}{\partial n} - h\frac{\partial u}{\partial n}\right)d\sigma}_{=0 \text{ if } \Delta h=0.\ ①}$$

Green's function.

$G(x,y) = \Phi(x-y) + h(x,y)$ where $h$ s.t.

$$\begin{cases} \Delta_y h(x,y) = 0 & \forall y \in \Omega \\ h(x,y) = -\Phi(x-y) & \forall y \in \partial\Omega \end{cases}$$

(假设h也不一定存在
要求 $h \in C(\bar{\Omega})$
故 G 在边界附近 有良好的正则性

If such Green's function exists, then

$$u(x) = -\int_\Omega G \Delta u \, dy - \int_{\partial\Omega} u \frac{\partial G}{\partial n} \, d\sigma$$

① Green's function is unique (by maximal principle)

② symmetric. $G(x,y) = G(y,x)$

$\forall\, x_1 \neq x_2 \in \Omega$. set $G_k(y) = G(x_k, y)$ $\forall\, 0 < \varepsilon < \frac{|x_1 - x_2|}{2}$

$$\int_{\Omega \setminus (B(x_1,\varepsilon) \cup B(x_2,\varepsilon))} (G_1 \Delta G_2 - G_2 \Delta G_1)(y) \, dy = 0.$$

$$= \int_{\partial\Omega - (\partial B(x_1,\varepsilon) \cup \partial B(x_2,\varepsilon))} \left( G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n} \right) d\sigma = -\int_{\partial B(x_1,\varepsilon) \cup \partial B(x_2,\varepsilon)} \left( G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n} \right) d\sigma$$

$$\int_{\partial B(x_1,\varepsilon)} \left( G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n} \right) d\sigma = \int_{\partial B(x_1,\varepsilon)} \Phi(x_1, y) \frac{\partial G_2}{\partial n} - G_2 \frac{\partial \Phi}{\partial n} \, d\sigma$$

$$+ \int_{\partial B(x_1,\varepsilon)} h(x_1, y) \frac{\partial G_2}{\partial n} - G_2 \frac{\partial h(x_1,y)}{\partial n} \, d\sigma$$

$$= \int_{\partial B(x_1,\varepsilon)} \left( \Phi(x_1 - y) \frac{\partial G_2}{\partial n} - G_2 \frac{\partial \Phi}{\partial n}(x_1 - y) \right) d\sigma + \int_{B(x_1,\varepsilon)} \underbrace{(h_1 \Delta G_2 - G_2 \Delta h_1)}_{0} \, dy$$

$$= \int_{\partial B(x_1,\varepsilon)} \underbrace{\Phi(\varepsilon) \frac{\partial G_2}{\partial n}}_{\to 0 \text{ as } \varepsilon \to 0} + \underbrace{G_2 \Phi'(\varepsilon)}_{\to c\, G(x_2, x_1) \text{ as } \varepsilon \to 0} \, d\sigma$$

故 $G(x_1, x_2) = G(x_2, x_1)$

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③ $\begin{cases} 0 < G(x,y) < \Phi(x-y) + \frac{1}{2\pi}\log(\mathrm{diam}\,\Omega) & n=2. \\ 0 < G(x,y) < \Phi(x-y) & n\geq 3 \end{cases}$ (由最大模原理)

(i) $\boxed{\text{Fix } x\in\Omega.}$ $\lim_{y\to x} G(x,y) = +\infty$. $\forall\varepsilon>0$. 考虑区域 $\Omega\setminus B(x,\varepsilon)$. 关于y调和函数 $G(x,y)$ 在边界上 $\geq 0$ 故 $G(x,y)\geq 0$
$G\neq 0 \Rightarrow G(x,y)>0$ in $\Omega$.

$n\geq 3$ 时: $\Phi(x-y) - G(x,y) = h(x,y)$

由 $\begin{cases} \Delta_y h = 0 & y\in\Omega \\ h(x,y) = -\Phi(x-y) < 0 & y\in\partial\Omega \end{cases}$ $\Rightarrow h<0,\ y\in\Omega.$ 即 $\Phi(x-y) - G(x,y) > 0 \quad y\in\Omega$

$n=2$ 时: $\Phi(x-y) + \frac{1}{2\pi}\log(\mathrm{diam}\,\Omega) - G(x,y) = \frac{1}{2\pi}\log(\mathrm{diam}\,\Omega) - h(x,y)$ $y\in\partial\Omega$ 时 $= \frac{1}{2\pi}\log\frac{\mathrm{diam}\,\Omega}{|x-y|} \geq 0.$
$\Delta_y$ (·) $=0$
故 $\Phi(x-y) + \frac{1}{2\pi}\log(\mathrm{diam}\,\Omega) - G(x,y) > 0$ $y\in\Omega$.

④ $\int_{\partial\Omega} \frac{\partial G}{\partial n}(x,y)\, d\Gamma(y) = -1.$

在表达式中 取 $u\equiv 1$ 即得.

—

Green's function for the domain $\Omega$

① $\Omega = B(0,R)$ ball

猜 $h(x,y)$ 形如 $\mu\,\Phi(x^*,\lambda y)$. $x^*\notin\Omega$. 则 $y\in\partial B(0,R)$ 时

$$h(x,y) + \Phi(x,y) = \mu\,\Phi(x^*,\lambda y) + \Phi(x,y) = 0.$$

$n=2$ 时: $\mu\log|x^*-\lambda y| + \log|x-y| = \log|x^*-\lambda y|^{\mu}\cdot|x-y| = 0.$

$$|x^*-\lambda y|^{\mu}\cdot|x-y| = 1$$ 找 $x^*$ $\lambda$ $\mu$ s.t. 对 $\forall y\in\partial B(0,R)$ 左式满足.

$h(x,y) = \frac{1}{2\pi}\log\left(\frac{|x|}{R}\left(\frac{R^2}{|x|^2}x-y\right)\right) = -\Phi\left(\frac{|x|}{R}\left(\frac{R^2}{|x|^2}x-y\right)\right)$ 若取 $\mu=-1$ $\lambda=\frac{|x|}{R}$. $\frac{x^*}{\lambda}$ 为 $x$ 关于 $B(0,R)$ 的反演点. 则 可以成立.

$n\geq 3$ 时 $\mu|x^*-\lambda y|^{2-n} + |x-y|^{2-n} = 0.$ $\frac{|x-y|}{|x^*-\lambda y|} = (-\mu)^{\frac{1}{2-n}}$

若取 $\lambda=1$. $x^*$ 为 $x$ 关于 $B(0,R)$ 的反演点 $(-\mu)^{\frac{1}{2-n}} = \frac{|x|}{R}$. 则 可以成立.

$$h(x,y) = -\left(\frac{|x|}{R}\right)^{2-n}\frac{-1}{(2-n)w_{n-1}}\left|\frac{R^2}{|x|^2}x-y\right|^{2-n} = \frac{1}{(2-n)w_{n-1}}\left(\frac{|x|}{R}\right)^{2-n}\left|\left(\frac{R}{|x|}\right)^2x-y\right|^{2-n} = -\Phi\left(\frac{|x|}{R}\left(\left(\frac{R}{|x|}\right)^2x-y\right)\right)$$

$$\frac{\partial G}{\partial n_y}(x,y) = \left(-\frac{x-y}{w_{n-1}|x-y|^n}(-1) + \frac{\frac{|x|}{R}\left(\left(\frac{R}{|x|}\right)^2x-y\right)}{w_{n-1}\left|\frac{|x|}{R}\left(\left(\frac{R}{|x|}\right)^2x-y\right)\right|^n}\left(-\frac{|x|}{R}\right)\right)\frac{y}{R} = \frac{|x|^2-R^2}{w_nR|x-y|^n}$$

② $\Omega = \mathbb{R}^n\times\mathbb{R}_+$ $y_{n+1}=0$ 时.

$n=2$ 时. $|x^*-\lambda y|^{\mu}|x-y| = 1.$ $\mu=-1.$ $\lambda=1.$ $x^* = (x_1,\cdots,x_n,-x_{n+1})$ 可满足.

$n\geq 3$ 时 $\mu|x^*-\lambda y|^{2-n} + |x-y|^{2-n} = 0.$ $\mu=-1.$ $\lambda=1.$ $x^* = (x_1,x_2,\cdots,x_n,-x_{n+1})$ 可满足.

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$\Omega=B(0,R)$ 上的 Green's function $G(x,y)$. $-\frac{\partial G}{\partial n_y}(x,y)=\frac{R^2-|x|^2}{R\omega_{n-1}}|x-y|^{-n}=:P(x,y)$

$u(x)=-\int_{|y|=R}\varphi(y)\frac{\partial G(x,y)}{\partial n_y}d\sigma(y)=\frac{R^2-|x|^2}{R\omega_{n-1}}\int_{|y|=R}\varphi(y)|x-y|^{-n}d\sigma(y)$ 是否为 $\begin{cases}-\Delta u=0\\ u|_{\partial\Omega}=\varphi\end{cases}$ 的解？正则性又如何？

$P(x,y)=\frac{R^2-|x|^2}{R\omega_{n-1}}|x-y|^{-n}$ 的性质 $(|y|=R,\ x\in B(0,R))$

1. $x\neq y$ 时 $P\in C^\infty$.
2. $P$ harmonic w.r.t. $x$
3. $\int_{|y|=R}P(x,y)\,d\sigma(y)=1$
4. $\int_{\{y:|y|=R,|y-x'|>\delta\}}P(x,y)\,d\sigma(y)\to 0$ as $r\to 1$, $x=rx'\in\partial B(0,R)$. $0\le r<1$. $\delta>0$

---

确证明 $\int_{\partial\Omega}\varphi(y)P(x,y)\,d\sigma(y)\to\varphi(x_0)$ as $x\to x_0\in\partial\Omega$.

$\left|\int_{\partial\Omega}\varphi(y)P(x,y)\,d\sigma(y)-\varphi(x_0)\right|=\left|\int_{\partial\Omega}(\varphi(y)-\varphi(x_0))P(x,y)\,d\sigma(y)\right|$

$\le\int_{\{|y-x_0|\le\delta\}}|\varphi(y)-\varphi(x_0)|P(x,y)\,d\sigma(y)+\int_{\{|y-x_0|>\delta\}}|\varphi(y)-\varphi(x_0)|P(x,y)\,d\sigma(y)$

$\le\varepsilon\cdot\int_{\partial\Omega}P(x,y)\,d\sigma(y)+2\|\varphi\|_{L^\infty(\partial\Omega)}\int_{\{|y-x_0|>\delta\}}P(x,y)\,d\sigma(y)$ (由$\varphi$连续)

$=\varepsilon+2\|\varphi\|_{L^\infty}\int_{\{|y-x_0|>\delta\}}P(x,y)\,d\sigma(y)$ $\to 0$ as $x\to x_0$. (沿半径方向). 再由$\varphi$连续.可得一般的情况.

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Pf of 4: $\int_{\{y:|y|=R,|y-x'|>\delta\}}\frac{R^2-r^2R^2}{R\omega_{n-1}}|x-y|^{-n}\,d\sigma(y)$

$\left(\begin{array}{l}|x-y|^2=|x|^2+|y|^2-2|x|\cdot|y|\cos\theta=R^2(r^2+1-2r\cos\theta)\ge 2R^2r(1-\cos\theta)=r|x'-y|^2\\ |x'-y|^2=2R^2(1-\cos\theta)\end{array}\right)$

$<\int_{\partial\Omega}\frac{R^2(1-r^2)}{R\omega_{n-1}}\frac{1}{(r^{\frac12}\delta)^n}\,d\sigma(y)=\frac{R^n}{\delta^n}\frac{1-r^2}{r^{\frac n2}}\to 0$ as $r\to 1$.

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直接取 $u\equiv 1$ 代入 Poisson 积分公式即得.

~~Pf of 3. 由 $p(\cdot,y)$ harmonic. $p(0,y)= \int_{\partial B(0,R)} p(x,y)\,d\sigma(x)$~~

~~$\Rightarrow 1 = \omega_{n-1}R^{n-1}\frac{R^2}{R\omega_{n-1}}R^{-n} = \int_{\{|x|=R\}} p(x,y)\,d\sigma(x)$~~

~~由对称性. $|rx'-y| = |ry-x'|$~~

~~$p(rx',y) = \frac{R^2-r^2}{R\omega_{n-1}}|rx'-y|^{-n} = \frac{R^2-r^2}{R\omega_{n-1}}|ry-x'| = p(ry,x')$~~

---

Consider an op. $L := -\Delta + c(x)$. ($\forall x$. $c(x)\geq 0$ 且 bdd) 正算子.

maximal principle.

Thm: $u\in C^2(\Omega)\cap C^1(\bar\Omega)$. $Lu = f < 0$ in $\Omega$ 则 $u$ 在内部不能达到非负最大值. 即 $\sup\limits_{\Omega} u < \sup\limits_{\partial\Omega}(\max\{u,0\})$

Pf: 否则. $\exists x_0\in\Omega$. $u(x_0) = \sup\limits_{\Omega} u \geq 0$.

$\Rightarrow \partial_j^2 u|_{x=x_0} \leq 0 \quad \forall j$

$\Rightarrow Lu|_{x=x_0} \geq 0$ 矛盾.

Cor $Lu = f \leq 0$ 则 $u$ 的非负最大值必在 $\partial\Omega$ 上达到.
即 $\sup\limits_{\Omega} u(x) \leq \sup\limits_{\partial\Omega}(\max\{u(x),0\})$

Pf: $\forall \varepsilon > 0$.

令 $w(x) = u(x) + \varepsilon e^{ax_1}$

$Lw = Lu + \varepsilon e^{ax_1}(-a^2 + c(x))$ 由 $c(x)$ 有界. 可取 $a$ s.t. $a^2 > \|c\|_{L^\infty} \Rightarrow Lw < 0$

$\sup\limits_{\Omega} w < \sup\limits_{\partial\Omega}(\max\{w,0\})$

令 $\varepsilon\to 0$. $\sup\limits_{\Omega} u \leq \sup\limits_{\partial\Omega}(\max\{u,0\})$

Campus

Maximal principle

weak maximal principle for elliptic op.

$L = a^{jk}(x)\partial_{jk} + b^j(x)\partial_j + c(x)$

$L$ 为椭圆算子 若 $[a^{jk}(x)] > 0$. $[a^{jk}(x)]$ 最大与最小特征值分别为 $\Lambda(x)$ 与 $\lambda(x)$

一致椭圆: $\frac{\Lambda(x)}{\lambda(x)}$ bdd, $0 < \lambda(x)$

严格椭圆 $\exists \lambda_0 > 0$ s.t. $\lambda(x) \geq \lambda_0$ $\forall x \in \Omega$.

Thm (weak maximal principle)

let op. $L$ be elliptic in the bdd domain $\Omega$ and $u \in C^2(\Omega) \cap C(\bar\Omega)$

Suppose $c(x) = 0$ in $\Omega$ $\frac{b^j}{\lambda}$ is bdd in $\Omega$ Then

If $Lu \geq 0$ in $\Omega$ then $\sup\limits_{\Omega} u = \sup\limits_{\partial\Omega} u$

Pf: $1^{st}$ If $Lu > 0$. then $u$ cannot achieve 他 interior maximal in $\bar\Omega$

If not, $x_0 \in \Omega$. $u(x_0) = \sup\limits_{\Omega} u$.

$\Rightarrow Du(x_0) = 0$ $D^2u(x_0) \leq 0$, $[a^{jk}(x_0)] > 0$ $\Rightarrow Lu(x_0) = a^{jk}(x_0)\partial_{jk}u(x_0) \leq 0$ $= tr[D^2u(x_0)\cdot[a^{jk}(x_0)]]$ (非正定) But $Lu(x_0) > 0$ contradiction.

$2^{nd}$ Set $w(x) := u(x) + \varepsilon e^{\gamma x_1}$

$Lw = Lu + \varepsilon a^{11}(x)\gamma^2 e^{\gamma x_1} + \varepsilon b^1(x)\gamma e^{\gamma x_1}$

$[a^{jk}(x)]$ elliptic $\Rightarrow a^{11}(x) \geq \lambda(x)$ (取 $\xi = (1, 0, \cdots, 0)$)

$|b^1(x)| \leq b_0\lambda(x)$ $\exists b_0 > 0$

choosing $\gamma > b_0$, large s.t. $Lw > 0$ in $\Omega$.

$\Rightarrow \sup\limits_{\Omega}(u + \varepsilon e^{\gamma x_1}) \leq \sup\limits_{\partial\Omega} u + \sup\limits_{\partial\Omega}(\varepsilon e^{\gamma x_1})$ 令 $\varepsilon \to 0^+$

$\sup\limits_{\Omega} u \leq \sup\limits_{\partial\Omega} u$ □

Now suppose $c(x) \leq 0$ in $\Omega$ If $Lu \geq 0$ in $\Omega$. then $\sup\limits_{\Omega} u \leq \sup\limits_{\partial\Omega} u^+$

$L_0 := a^{jk}(x)\partial_{jk} + b^j(x)\partial_j$ If $Lu = L_0u + c(x)u \geq 0$ in $\Omega$ then $L_0u \geq -c(x)u$

$\Omega^+ = \{x \in \Omega : u(x) > 0\}$

No. "Δ"为负算子

Date 12.3

若 $\exists\, u(x)>0$

$$\sup_{\Omega} u = \sup_{\Omega^+} u = \max_{\partial\Omega^+} u = \max_{\partial\Omega} u$$

$u|_{(\partial\Omega^+ - \partial\Omega)} = 0.$

Application of weak maximal principle

(uniqueness of the sol, continuous dependence of the boundary value)

comparison principle

Thm $L$ elliptic $\Omega$ bdd domain $c(x)\le 0$ $\frac{b^i(x)}{\lambda(x)}$ bdd

① $u, v \in C^2(\Omega)\cap C(\bar\Omega)$ $\begin{cases} Lu = Lv & \Omega \\ u = v & \partial\Omega \end{cases}$ $\Rightarrow u = v$ in $\Omega$

② $u, v \in C^2(\Omega)\cap C(\bar\Omega)$ $\begin{cases} Lu \ge Lv & \Omega \\ u \le v & \partial\Omega \end{cases}$ $\Rightarrow u \le v$ in $\Omega$

Hopf's lemma

$L$ uniformly elliptic in $\Omega$. $\forall j$ $\frac{b^j}{\lambda}$ bdd in $\Omega$. $Lu \ge 0$ in $\Omega$.

$x_0 \in \partial\Omega$ s.t. $u(x_0) > u(x)$ $\forall x \in \Omega$

$\partial\Omega$ s.t. an interior sphere condition at $x_0$

$u$ is continuous at $x_0$

Then if (1) $c(x) = 0$ in $\Omega$

or (2) $c(x)\le 0$, $c/\lambda$ bdd in $\Omega$. $u(x_0)\ge 0$

or (3) $u(x_0) = 0$, $c/\lambda$ bdd in $\Omega$.

then $\frac{\partial u}{\partial \nu}\Big|_{x_0} > 0$. (if exists)

Pf: $\exists B(y,R)\subset\Omega$. s.t. $x_0 \in \partial B(y,R)$

辅助函数 $v(x) = e^{-\alpha r^2} - e^{-\alpha R^2}$ $r := |x-y| > \rho$ $0<\rho<R$

$Lv = (a^{jk}\partial_{jk} + b^j\partial_j + c)(e^{-\alpha r^2} - e^{-\alpha R^2})$

$= a^{jk}\, 4\alpha^2 (x_k - y_k)(x_j - y_j) e^{-\alpha r^2} - 2\alpha\, a^{jj} e^{-\alpha r^2} - 2\alpha\, b^j (x_j - y_j) e^{-\alpha r^2} + c e^{-\alpha r^2} - c e^{-\alpha R^2}$

12.5 $\ge 4\alpha^2 \lambda r^2 e^{-\alpha r^2} - 2\alpha n \Lambda e^{-\alpha r^2} - 2\alpha n b\, r\, e^{-\alpha r^2} + c(e^{-\alpha r^2} - e^{-\alpha R^2}) \ge e^{-\alpha r^2}\lambda\left(4\alpha^2 r^2 - 2\alpha \frac{n\Lambda}{\lambda} - 2\alpha n \frac{b}{\lambda} r + \frac{c}{\lambda}(1 - e^{-\alpha(R^2 - r^2)})\right)$

可选 $\alpha$ 足够大使 $Lv > 0$ $\forall x \in A := B(y,R) - B(y,\rho)$ $\partial A = \underbrace{\partial B(y,R)}_{v=0} \cup \partial B(y,\rho)$

$u(x) - u(x_0) < 0$ on $\partial B(y,\rho)$. $\partial B(y,\rho)$ 紧. 故有最值. 故 $\exists\,\varepsilon$ 足够小 s.t.

$u(x) - u(x_0) + \varepsilon v(\rho) < 0$. on $\partial B(y,\rho)$

Campus

$L(u(x)-u(x_0)+\varepsilon v(x)) = Lu+\varepsilon(Lv-cu(x_0)) \geq 0$ in $A$, $u(x)-u(x_0)+\varepsilon v(x)<0$ on $\partial A$

由弱极值 $w(x):=u(x)-u(x_0)+\varepsilon v(x) \leq 0$ in $A$ $\qquad w(x_0)=0 \quad w(x)\leq 0$

$\Rightarrow \frac{\partial u}{\partial \nu}(x_0) \geq -\varepsilon \frac{\partial v}{\partial \nu}(x_0) = -\varepsilon v'(R) > 0.$

Strong maximal principle

$\Omega$: domain in $\mathbb{R}^n$, $L$: uniformly elliptic in $\Omega$, $\forall j$ $\frac{b^j(x)}{\lambda(x)}$ bdd on $\Omega$

$u\in C^2(\Omega)\cap C(\bar{\Omega})$, $Lu\geq 0$ in $\Omega$, $c\leq 0$. $\frac{c}{\lambda}$ bdd.

若u在内部达到最大值,则u恒为常数.

Pf: $E=\{x\in\Omega: u(x)=M\}$ 证明E既开又闭.

闭:由连续性即得.

开: $\forall x\in E\subset\Omega$ 由$\Omega$开. $\exists r>0$ s.t. $B(x,2r)\subset\Omega$

若 $B(x,r)\subset E$ 则 x为E的内点。否则 $\exists \bar{x}\in B(x,r)\setminus E$ 已开 记 $d=d(\bar{x},E)$ 则 $d>0$, $d<r$

$B(\bar{x},d)\subset B(x,2r)\subset\Omega$. 取 $y\in\partial B(\bar{x},d)\cap E$

由强极值. $\frac{\partial u}{\partial \nu}\big|_y>0$ $\nu$为$B(\bar{x},d)$外法向量

而$y\in\Omega$ 为极值点. 故 $\frac{\partial u}{\partial \nu}\big|_y=0$ 矛盾.

故E为开集. $\Rightarrow E=\Omega$.

Application of Strong maximal principle

Example 1 (Uniqueness sol. of the Neumann bdry value problem)

$$\begin{cases} -\Delta u=f(x) & \Omega^c \\ \frac{\partial u}{\partial n}\big|_{\partial\Omega}=\varphi & \partial\Omega \\ \lim\limits_{|x|\to\infty} u(x)=0 \end{cases}$$

Pf: 若 $u_1$ $u_2$ 为其两个解 $v:=u_1-u_2$ 为 $\begin{cases} -\Delta v=0 & \Omega^c \\ \frac{\partial v}{\partial n}\big|_{\partial\Omega}=0 & \partial\Omega \\ \lim\limits_{|x|\to\infty} v(x)=0. \end{cases}$

若 $\exists x_0\in\Omega^c$ s.t. $v(x_0)>0$ 且为最大值

则 $\exists R>0$ s.t. $\forall |x|>R$ $|v(x)|<\frac{1}{2}|v(x_0)|$ 考虑 $B(x,R)\setminus\Omega$ 则v在其边界达到非负最大值

故必有 $x_0\in\partial\Omega$. 由强极值原理在$\Omega^c$上 $v(x)<M$.

故 $\frac{\partial v}{\partial n}\big|_{x_0}>0$ 矛盾

(若 $\exists x_0\in\Omega^c$ s.t. $v(x_0)<0$ 为最小值的情况类似讨论可得矛盾)

# Aprioir bounds 先验估计

Thm $u \in C^2(\Omega) \cap C(\bar{\Omega})$ solves $\begin{cases} -\Delta u = f(x) & \Omega \\ u|_{\partial\Omega} = \varphi(x) & \partial\Omega \end{cases}$ $\Omega$ bdd. $d := |\Omega|$

Then $\|u\|_{L^\infty(\bar{\Omega})} \leq \|\varphi\|_{L^\infty(\partial\Omega)} + C(n,\Omega)\|f\|_{L^\infty(\partial\Omega)}$

Pf: w.l.o.g., $0 \in \Omega$.

Set $w(x) := z(x) \pm u(x)$. $z(x) = \frac{1}{2n}\|f\|_{L^\infty(\Omega)}(d^2 - |x|^2) + \|\varphi\|_{L^\infty(\partial\Omega)}$

$$-\Delta z = +\|f\|_{L^\infty(\Omega)}$$

$-\Delta w = F \pm f \geq 0.$

$w|_{\partial\Omega} \geq \Phi \pm \varphi \geq 0.$

by weak maximal principle, $\inf\limits_{\Omega} w \geq \inf\limits_{\partial\Omega} w^- \geq 0.$

$\Rightarrow |u(x)| \leq |z(x)|$

$\|u\|_{L^\infty} \leq \|z\|_{L^\infty} \leq \frac{1}{2n}\|f\|_{L^\infty} \cdot d^2 + \|\varphi\|_{L^\infty}$

Thm (Robin bdry value problem)

$u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ solves $\begin{cases} Lu := -\Delta u + c(x)u = f(x) & \Omega \\ \left(\frac{\partial u}{\partial n} + \alpha(x)u\right)\Big|_{\partial\Omega} = \varphi(x) & \partial\Omega \end{cases}$ $\Omega$ bdd

$c(x) \geq 0$. $\alpha(x) \geq \alpha_0 > 0$. Then $\|u\|_{L^\infty(\bar{\Omega})} \leq C(n, \alpha_0, d)(\Phi + F)$

Pf: w.l.o.g. $0 \in \Omega$

Set $w(x) := z(x) + \frac{\Phi}{\alpha_0} \pm u(x)$. $z(x) = \frac{F}{2n}\left(\frac{1+d^2}{\alpha_0} + d^2 - |x|^2\right) \geq 0$

$$-\Delta z = F$$

$$Dz(x) = -\frac{F}{n}x$$

设 $\partial\Omega$ 上 $x$ 处外法向量 $\vec{n} = (\beta_1(x), \beta_2(x), \cdots, \beta_n(x))$

$\left(\frac{\partial z}{\partial n} + \alpha(x)z\right)\Big|_{\partial\Omega} = \left(-\frac{F}{n}x \cdot \vec{n} + \alpha(x)z\right)\Big|_{\partial\Omega} = \frac{F}{2n}\left(-2\sum\limits_i x_i\beta_i(x) + \alpha(x)\left(\frac{1+d^2}{\alpha_0} + d^2 - |x|^2\right)\right)\Big|_{\partial\Omega}$

$\geq \frac{F}{2n}\left(-|x|^2 - 1 + \alpha(x)\left(\frac{1+d^2}{\alpha_0} + d^2 - |x|^2\right)\right) \geq \frac{F}{2n}\left(-|x|^2 - 1 + 1 + d^2\right) \geq 0.$

$Lw = F + c(x)\left(z(x) + \frac{\Phi}{\alpha_0}\right) \pm f \geq 0.$

$\left(\frac{\partial w}{\partial n} + \alpha(x)w\right)\Big|_{\partial\Omega} = \left(\frac{\partial z}{\partial n} + \alpha(x)z + \alpha(x)\frac{\Phi}{\alpha_0} \pm \varphi(x)\right)\Big|_{\partial\Omega} \geq \left(\Phi\frac{\alpha(x)}{\alpha_0} \pm \varphi(x)\right)\Big|_{\partial\Omega} \geq 0.$

由弱极值原理. $W(x)$ 的负最小值(若存在)必在 $\partial\Omega$ 上达到. 设在 $x_0$ 处. 则 $\frac{\partial w}{\partial n}\Big|_{x=x_0} \leq 0$

故 $\left(\frac{\partial w}{\partial n} + \alpha(x)w\right)\Big|_{x_0} \leq \alpha(x)w(x_0) < 0$. 矛盾. 故 $w(x) \geq 0 \Rightarrow |u(x)| \leq \frac{\Phi}{\alpha_0} + \frac{F}{2n}\left(\frac{1+d^2}{\alpha_0} + d^2\right)$

Thm. $u \in C^2(\Omega) \cap C(\bar{\Omega})$

$$\begin{cases} -\Delta u + c(x) u = f(x) & x \in \Omega \\ u|_{\partial\Omega} = 0. \end{cases} \qquad c(x) \geq c_0 > 0.$$

then $\int_\Omega |\nabla u|^2 dx + \frac{c_0}{2} \int_\Omega u^2 dx \leq \frac{1}{2c_0} \int_\Omega f^2 dx$

Pf: 乘了u: $\int_\Omega u \cdot (-\Delta u + c(x) u) = \int_\Omega f(x) u(x) dx \leq \frac{1}{2c_0} \int_\Omega f^2 + \frac{c_0}{2} \int_\Omega u^2$

$$= -\int_\Omega u \Delta u + \int_\Omega c(x) u^2$$

$$= \int_\Omega |\nabla u|^2 - \int_{\partial\Omega} u \underbrace{\frac{\partial u}{\partial \nu}}_{=0.} d\sigma + \int_\Omega c(x) u^2$$

$$\geq \int_\Omega |\nabla u|^2 + c_0 \int_\Omega u^2$$

$$\frac{c_0}{2} \int_\Omega u^2 + \int_\Omega |\nabla u|^2 \leq \frac{1}{2c_0} \int_\Omega f^2$$

Thm $u \in C^2(\Omega) \cap C(\bar{\Omega})$

$$\begin{cases} -\Delta u + c(x) u = f(x) & x \in \Omega \\ \left(\frac{\partial u}{\partial \nu} + \alpha(x) u\right)\Big|_{\partial\Omega} = 0 \end{cases} \qquad c(x) \geq c_0 > 0. \quad \alpha(x) \geq 0,$$

then $\int_\Omega |\nabla u|^2 dx + \frac{c_0}{2} \int_\Omega u^2 dx + \int_{\partial\Omega} \alpha(x) u^2 d\sigma \leq \frac{1}{2c_0} \int_\Omega f^2 dx$

Pf: 乘了u: $\int_\Omega u(-\Delta u + c(x) u)\, dx = \int_\Omega u f\, dx \leq \frac{c_0}{2} \int_\Omega u^2 + \frac{1}{2c_0} \int_\Omega f^2$

$$= \int_\Omega |\nabla u|^2 - \int_{\partial\Omega} u \frac{\partial u}{\partial \nu} d\sigma + \int_\Omega c(x) u^2$$

$$= \int_\Omega |\nabla u|^2 + \int_{\partial\Omega} \alpha(x) u^2 d\sigma + \int_\Omega c(x) u^2$$

$$\geq \int_\Omega |\nabla u|^2 + \int_{\partial\Omega} \alpha(x) u^2 d\sigma + c_0 \int_\Omega u^2$$

$$\Rightarrow \int_\Omega |\nabla u|^2 + \frac{c_0}{2} \int_\Omega u^2 dx + \int_{\partial\Omega} \alpha(x) u^2 d\sigma \leq \frac{1}{2c_0} \int_\Omega f^2$$

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Date

12.10.

奇点可去性原理

Thm (removable singularity Thm) let $\Omega$ be a domain in $\mathbb{R}^n$ $(n\geq 2)$

Suppose $u$ is harmonic in $\Omega\setminus\{x_0\}$. and s.t. $\lim\limits_{x\to x_0}|x-x_0|^{n-2}u(x)=0$ for $n\geq 3$

$\lim\limits_{x\to x_0}\frac{u(x)}{\ln|x-x_0|}=0$ for $n=2$.

Then $u$ can be defined at $x=x_0$ s.t. $u$ is harmonic in $\Omega$.

Pf: w.l.o.g. $x_0=0$.

$\exists R>0$ s.t. $B(0,R)\subset\subset\Omega$. $u$ is contiuous in $\{x\in\Omega: 0<|x|\leq R\}$.

By Poisson's formula $\begin{cases}-\Delta V=0 & \text{in } B(0,R)\\ V=u & \text{on } \partial B(0,R)\end{cases}$ has a unique sol $V$ in $C^2(B(0,R))\cap C(\overline{B(0,R)})$

It suffices to show $u=V$ in $B(0,R)\setminus\{0\}$

set $W=V-u$

For $n\geq 3$. For $0<r<R$. $M_r:=\max\limits_{\partial B(0,r)}|W|=\max\limits_{\partial B(0,r)}|V-u|\leq\max\limits_{\partial B(0,r)}|V|+\max\limits_{\partial B(0,r)}|u|$

$\leq\underbrace{\max\limits_{\partial B(0,R)}|u|}_{\triangleq M}+\max\limits_{\partial B(0,r)}|u|$

$|W(x)|\leq M_r=M_r\cdot\frac{r^{n-2}}{|x|^{n-2}}$ on $\partial B(0,r)$

$W(x)$, $\frac{1}{|x|^{n-2}}$ harmonic in $B(0,R)\setminus B(0,r)$

$\pm W(x)-M_r\frac{r^{n-2}}{|x|^{n-2}}\leq 0$ on $\partial B(0,r)$. & $W|_{\partial B(0,R)}=0$.

by maximal principle, $|W(x)|\leq M_r\cdot\frac{r^{n-2}}{|x|^{n-2}}$ in $B(0,R)\setminus B(0,r)$

fix x. $\forall 0<r<|x|<R$ $\quad |W(x)|\leq\frac{r^{n-2}}{|x|^{n-2}}M+\frac{1}{|x|^{n-2}}r^{n-2}\max\limits_{\partial B(0,r)}|u|\to 0$ as $r\to 0$.

that is $W=0$ in $B(0,R)\setminus\{0\}$

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Date 12.10

Thm $u$ harmonic in $\Omega$, then

(1) $|Du(x)| \le \frac{n}{d(x,\partial\Omega)} \sup_{\Omega} |u|$

(2) $|D^{\alpha} u(x)| \le \left(\frac{n|\alpha|}{d}\right)^{|\alpha|} \sup_{\Omega} |u|$

(3) $u$ is analytic in $\Omega$.

Pf: (1) 由 $\forall B(x,r) \subset \Omega$. $u(x) = ⨍_{S^{n-1}} u(x+ry)\, d\sigma(y)$ $\quad Du(x) = ⨍_{S^{n-1}} Du(x+ry)\, d\sigma(y) = ⨍_{\partial B(x,r)} Du(y) d\sigma(y)$

$= ⨍_{B(x,r)} Du(y)\, d\sigma(y)$

$Du(x) = \frac{1}{V_n r^n} \int_{B(x,r)} Du(y)\, d\sigma(y) = \frac{1}{V_n r^n} \int_{\partial B(x,r)} u(y) \cdot \nu \, d\sigma(y)$

$|Du(x)| \le \frac{1}{V_n r^n} \int_{\partial B(x,r)} |u(y)|\, d\sigma(y) \le \frac{1}{V_n r^n} \sup_{\Omega} |u| \cdot w_{n-1} r^{n-1} \le \frac{n}{r} \sup_{\Omega} |u|$

故 $u(x) \le \frac{n}{d(x,\partial\Omega)} \sup_{\Omega} |u|$

(2) 归纳

(3) Taylor展开. 计算余项. 趋于 0.

Campus

Thm $\{u_k\}$ harmonic in $\Omega$. $u_k \rightrightarrows u$ in $\Omega$. then $u$ is harmonic in $\Omega$

Pf: $\forall B(x,r) \subset\subset \Omega$. $u_k(x) = ⨍_{\partial B(x,r)} u_k(y)\, d\sigma(y)$

由 $u$ 一致收敛

$$u(x) = \lim_{k\to\infty} u_k(x) = \lim_{k\to\infty} ⨍_{\partial B(x,r)} u_k(y)\, d\sigma(y)$$

$$= ⨍_{\partial B(x,r)} \lim_{k\to\infty} u_k(y)\, d\sigma(y) = ⨍_{\partial B(x,r)} u(y)\, d\sigma(y)$$

$\Rightarrow u$ harmonic

Thm (Harnack's convergence thm)

$\{u_k\}_k \nearrow$, harmonic in $\Omega$. $\exists y \in \Omega$ s.t. $\{u_k(y)\}_k$ bdd.

Then $\forall \Omega' \subset\subset \Omega$. $\{u_k\}_k$ 在其上一致收敛.

Pf: $\forall \varepsilon > 0$. $\exists K$ s.t. $\forall k_2 > k_1 > K$ $0 \le u_{k_2}(y) - u_{k_1}(y) \le \varepsilon$.

又. $u_{k_2} - u_{k_1}$ harmonic. by Harnack's ineq.

$\sup_{\Omega'} |u_{k_2} - u_{k_1}| < C(\Omega')\varepsilon$. $\Rightarrow \{u_k\}_k$ 在 $\Omega'$ 上一致收敛.

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