foundations of data science for everyone

III: probability and statistics part 2

dr.federica bianco | *fbb.space* |  *fedhere* |  *fedhere*

this slide deck

https://slides.com/federicabianco/fds_03

1 statistics

statistics

takes us from observing a limited number of samples to infer on the population

TAXONOMY

Population: all of the elements of a "family" or class

Sample: a finite subset of the population that you observe

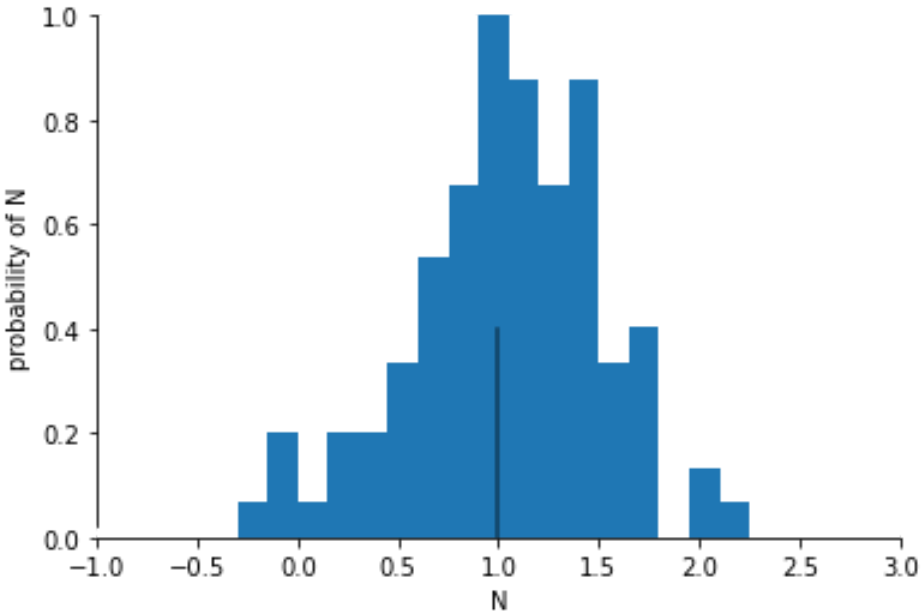
distributions: a collection of numbers with a specific shape

observational approach: a distribution represent the frequency with which we obtain a value ~x when measuring a phenomenon

HISTOGRAMS OF SAMPLES

analyst approach: a distribution represent the *probability* with which a phenomenon generates a value that we measure to be ~x

number of values measured between x and x+dx



frequency ⟶ **probability**

distributions: a collection of numbers with a specific shape

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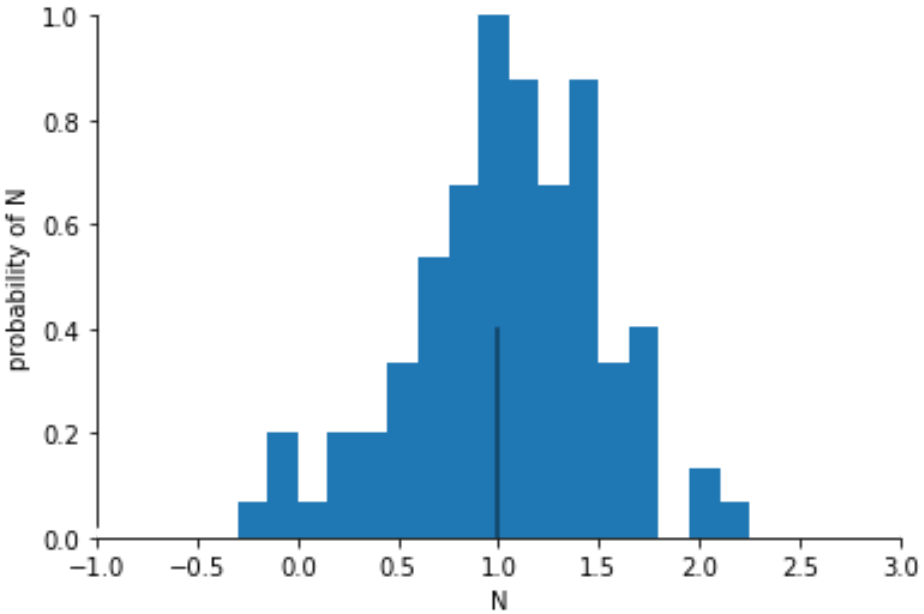

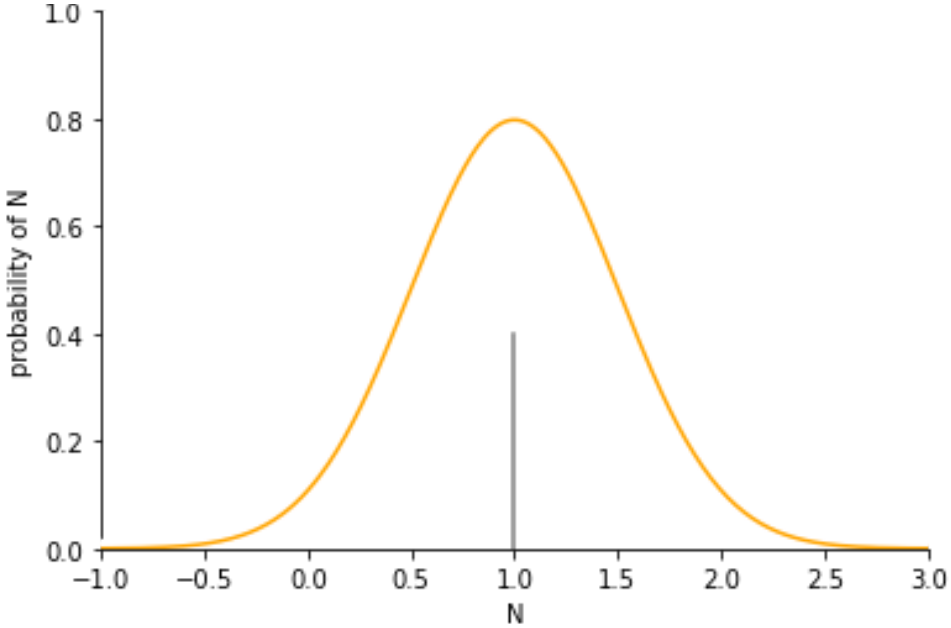

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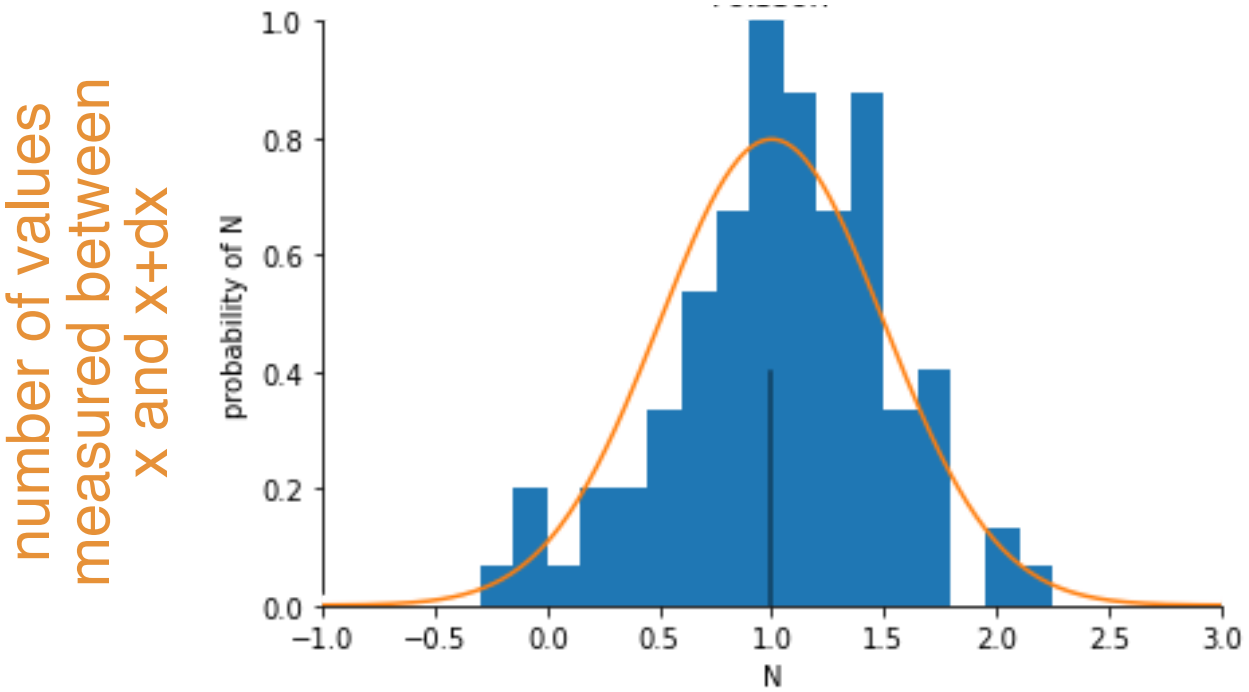


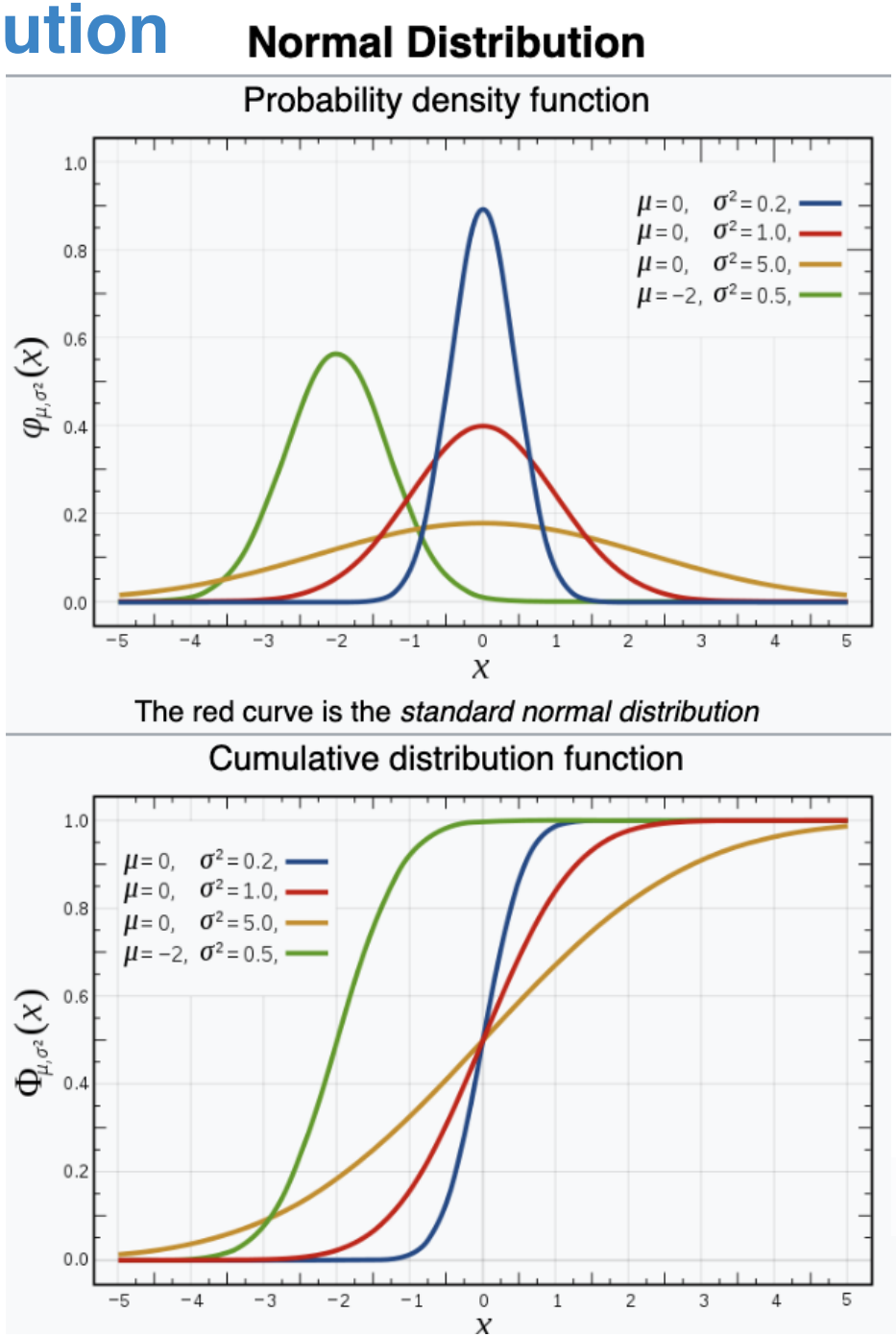
frequency → **probability**

Gaussian (or normal) distribution

most common distribution:

well behaved mathematically, symmetric, when we can we will assume our uncertainties or samples are Gaussian distributed

support: *the x values for which the distribution is defined*



Notation	$\mathcal{N}(\mu, \sigma^2)$
Parameters	$\mu \in \mathbb{R}$ = mean (location) $\sigma^2 > 0$ = variance (squared sc
Support	$x \in \mathbb{R}$
PDF	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
CDF	$\frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$
Quantile	$\mu + \sigma\sqrt{2}\,\operatorname{erf}^{-1}(2F-1)$
Mean	μ
Median	μ
Mode	μ
Variance	σ^2
Skewness	0
Ex. kurtosis	0
Entropy	$\frac{1}{2}\log(2\pi e\sigma^2)$
MGF	$\exp(\mu t + \sigma^2 t^2/2)$
CF	$\exp(i\mu t - \sigma^2 t^2/2)$
Fisher information	$\mathcal{I}(\mu,\sigma) = \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 2/\sigma^2 \end{pmatrix}$
Kullback-Leibler divergence	$D_{\mathrm{KL}}(\mathcal{N}_0 \| \mathcal{N}_1) = \frac{1}{2}\{(\sigma_0/\sigma_1)^2 + \frac{(\mu_1-\mu_0)^2}{\sigma_1^2} - 1 + 2\ln\frac{\sigma_1}{\sigma_0}\}$

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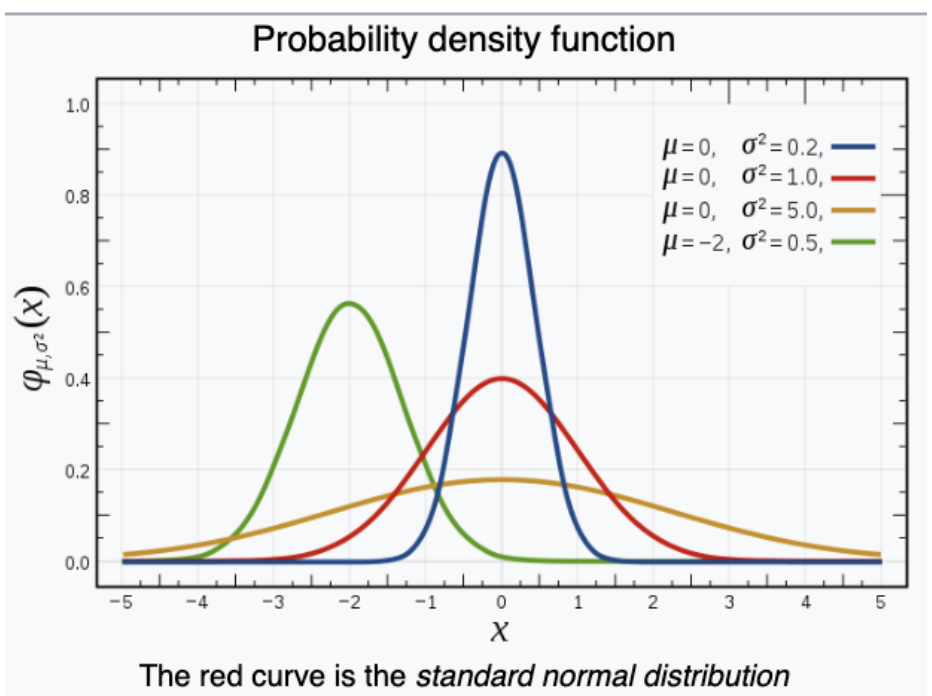

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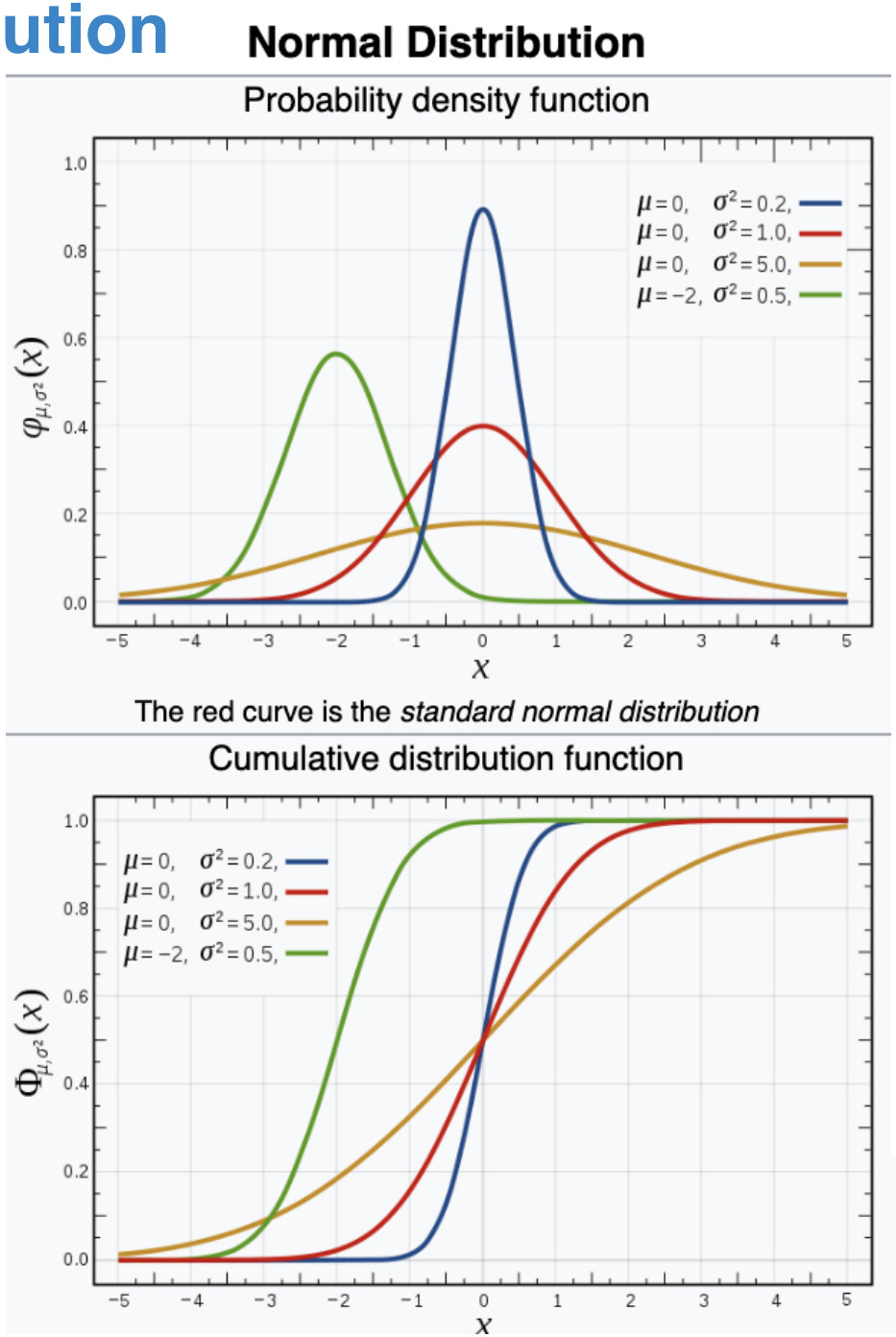
Central tendency



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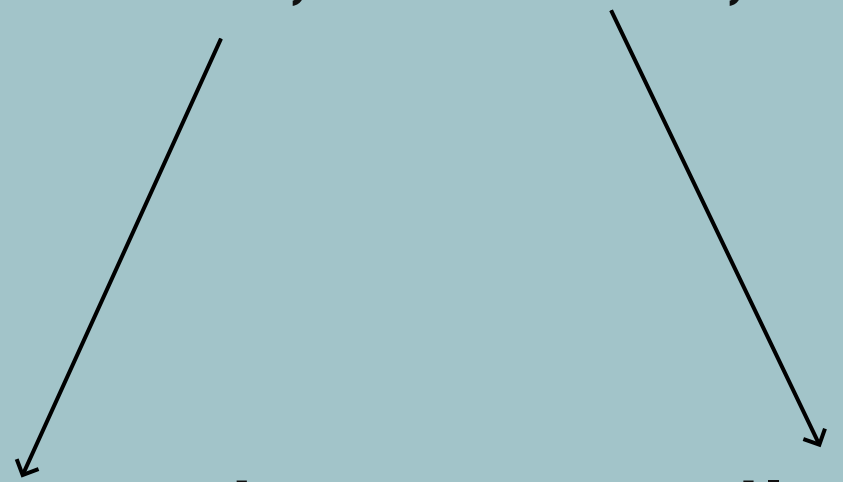
TAXONOMY

central tendency: mean, median, mode



spread : variance, interquantile range

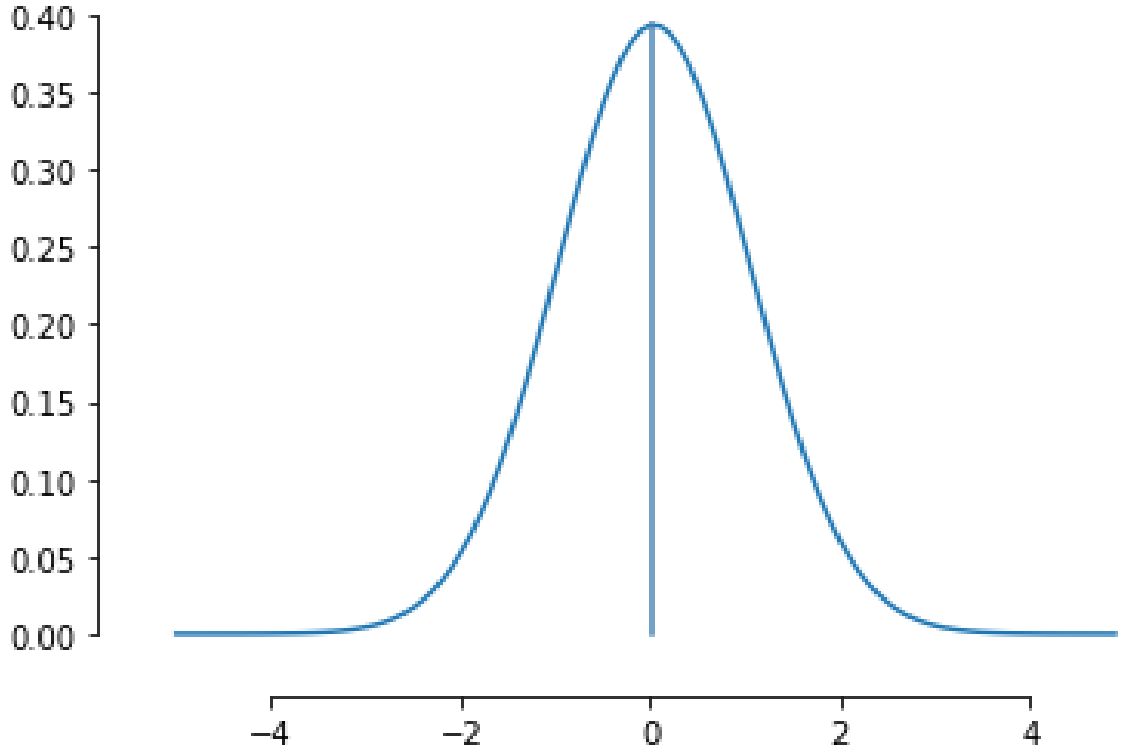
Moments and quantiles

a distribution's moments summarize its properties:

$$m_n = \int_{-\inf}^{\inf} (x - c)^n f(X) dx$$

central tendency: mean (n=1)

Gaussian (or Normal) distribution



Quantiles

measure what fraction of a distribution is within some x values

central tendency: median (50%)

Moments

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central tendency: mean (n=1)

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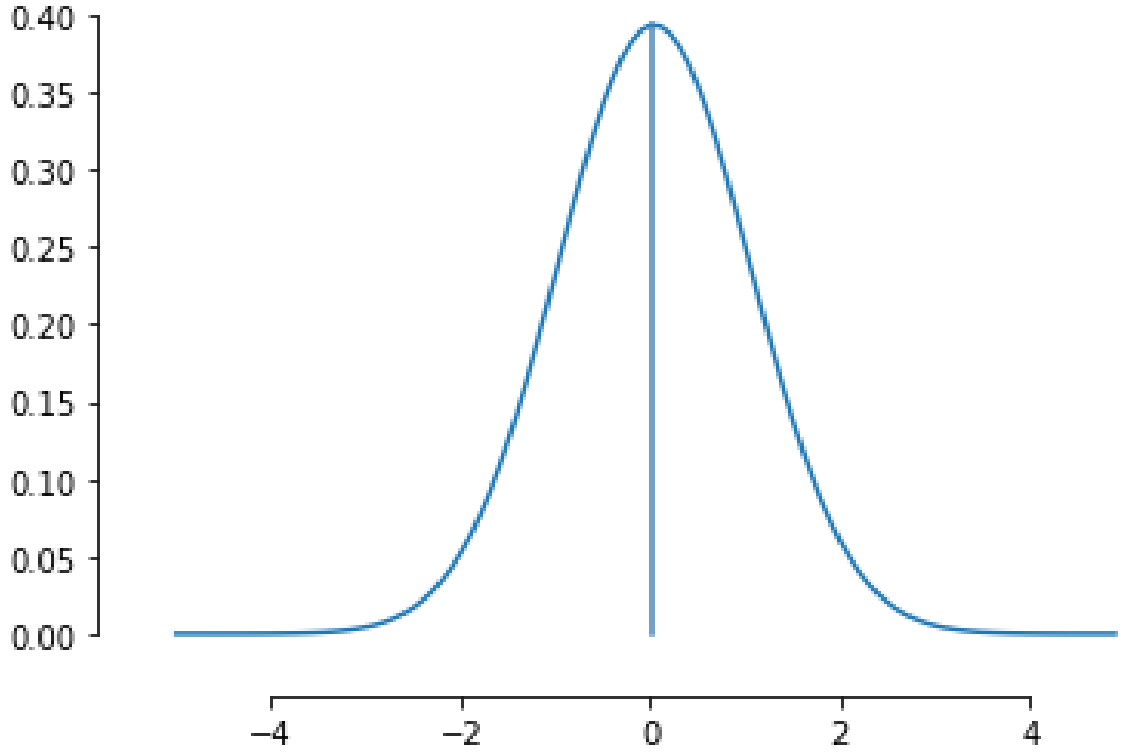

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Moments

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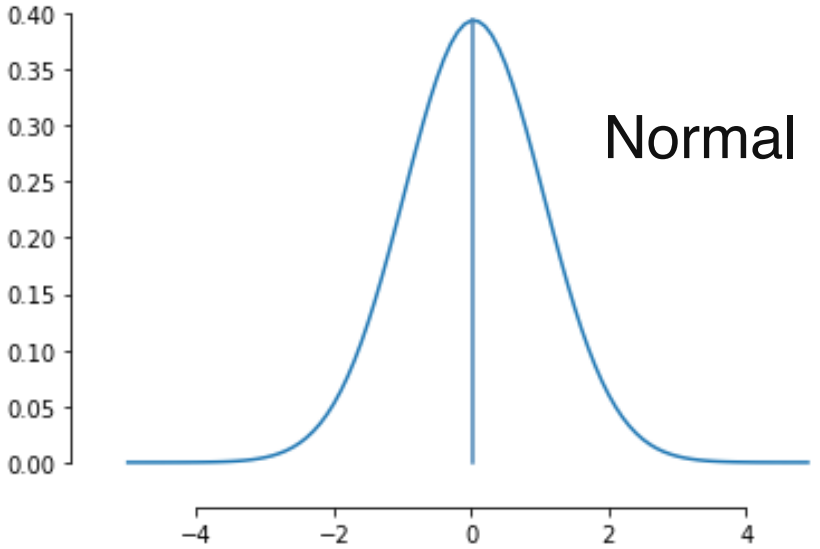
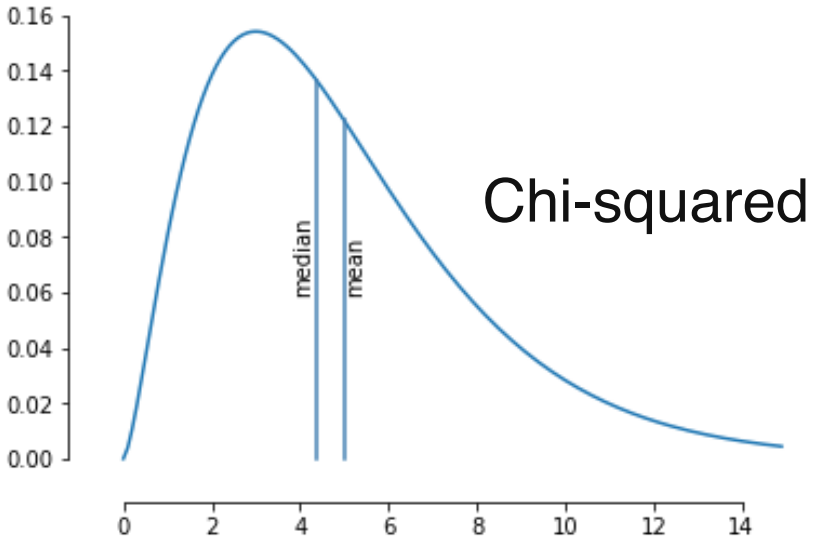
symmetric distribution: mean=median

skewed distribution: mean≠median





Quantiles

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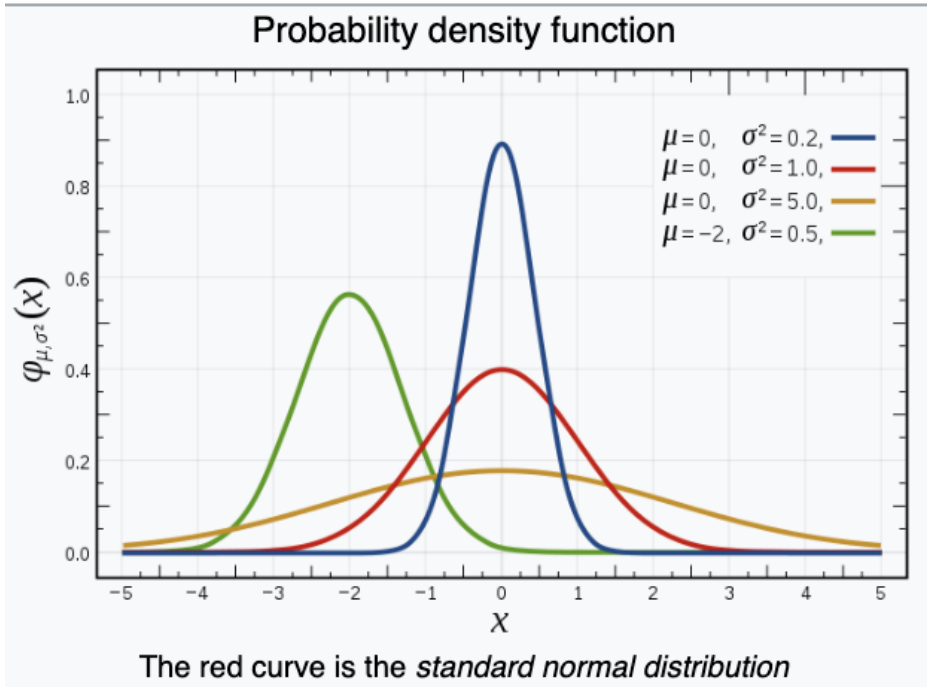
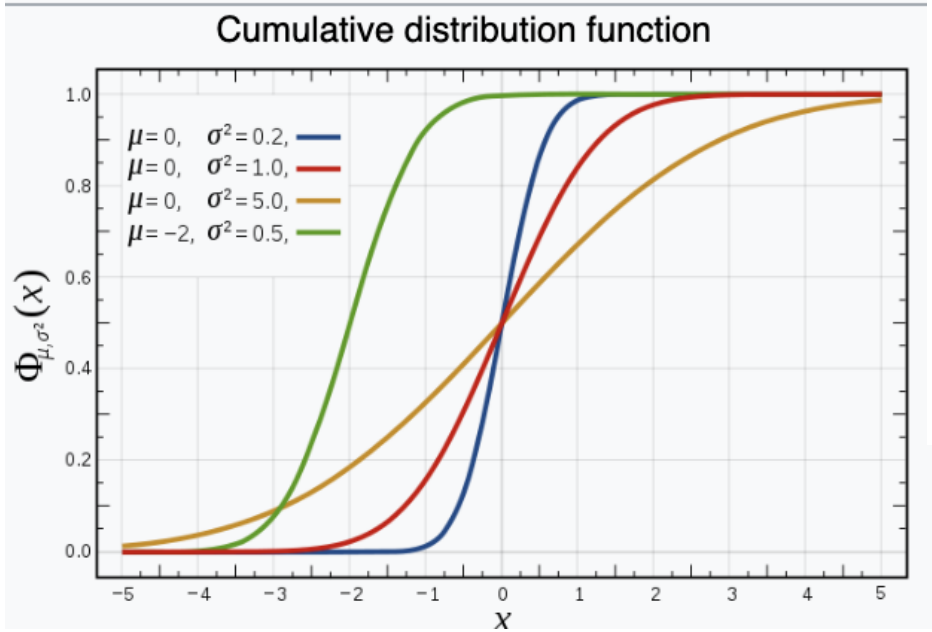
Spread

Normal Distribution



The red curve is the *standard normal distribution*



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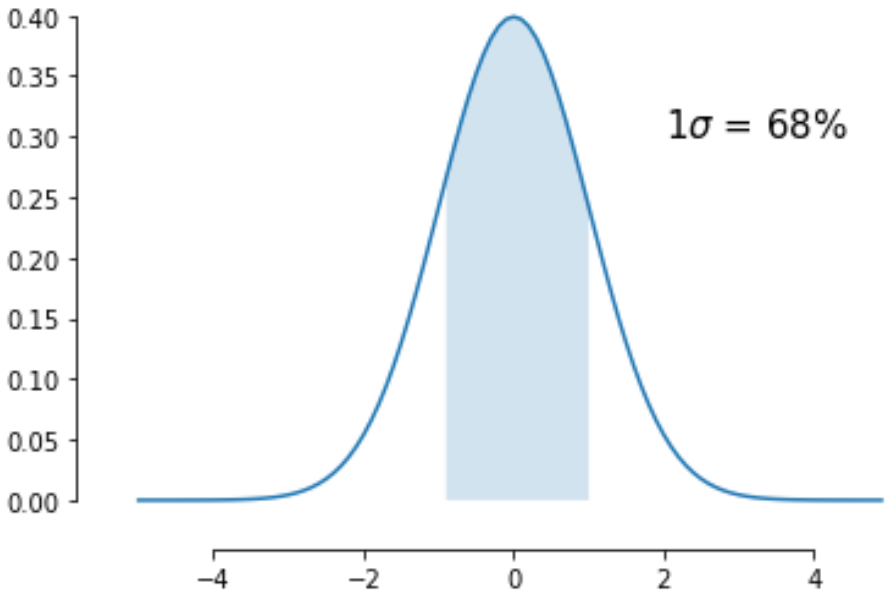
spread: **variance (n=2)**

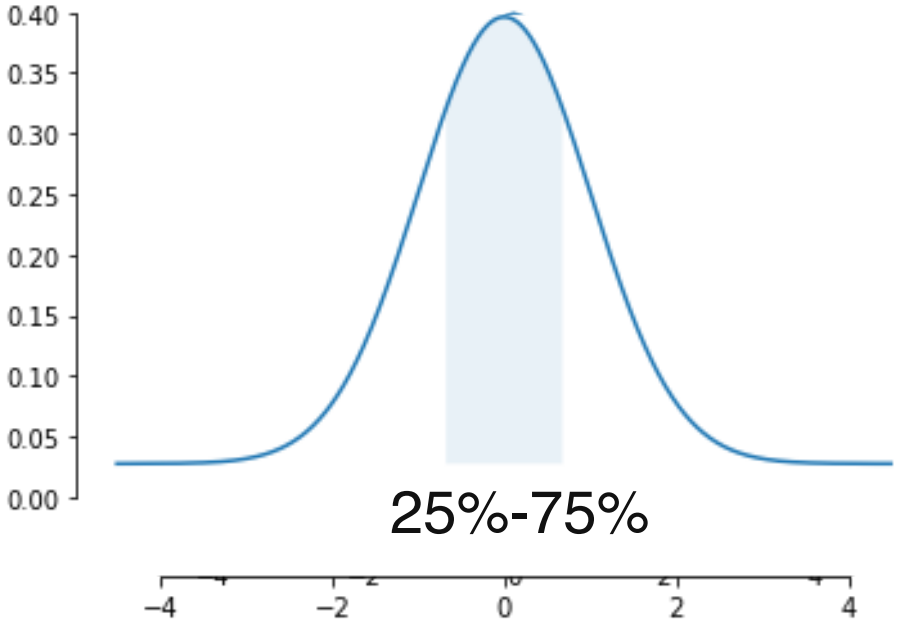
standard deviation $\sigma = \sqrt{\text{variance}}$



Quantiles

measure what fraction of a distribution is within some x values

central tendency: **quartiles (25%-75%)**



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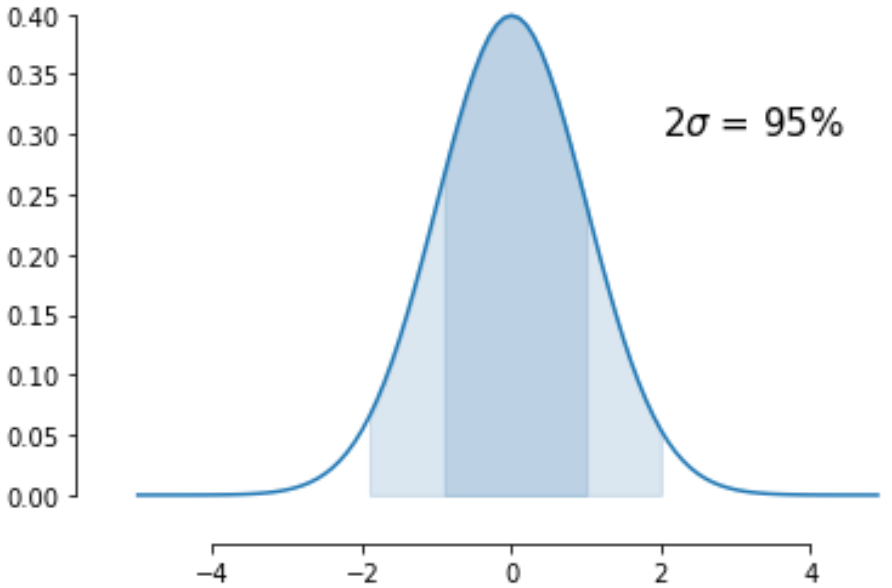
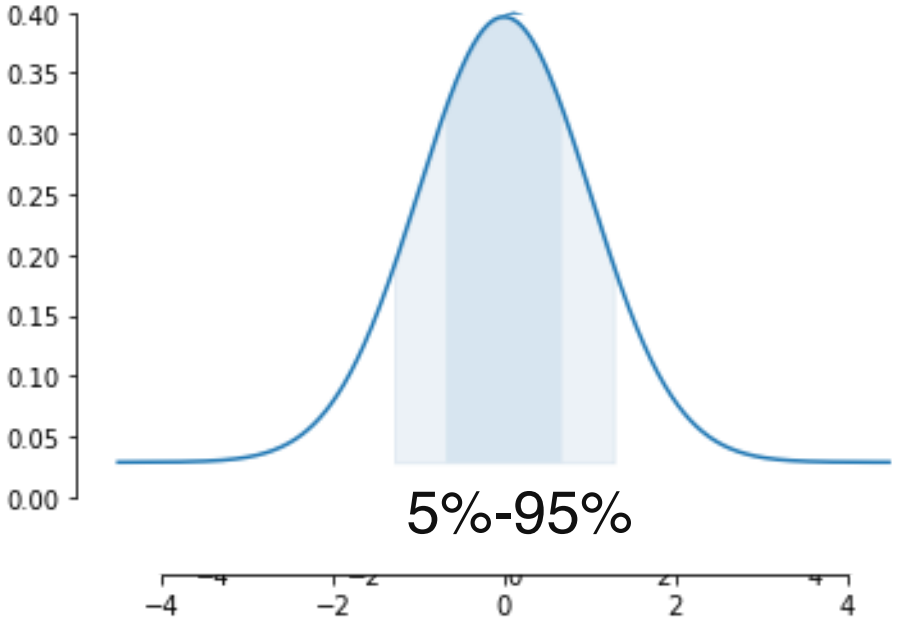


Quantiles

measure what fraction of a distribution is within some x values

central tendency: **quantiles (5%-95%, 1%-99%...)**



Binomial

I bet heads:

head = success

"given n tosses, each with a probability of 0.5 to get head"

Coin toss:

Parameters: p, n

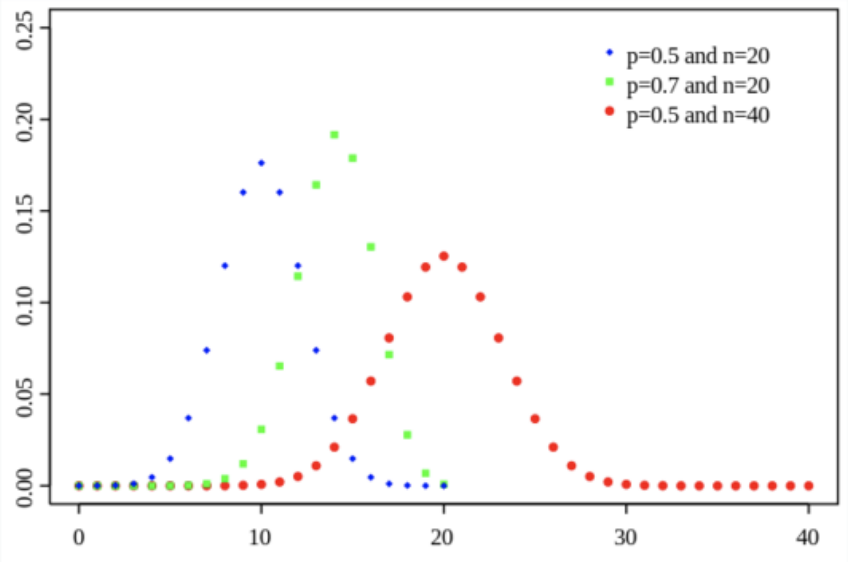
fair coin: p=0.5 n=1

Vegas coin: p=0.5 n=1

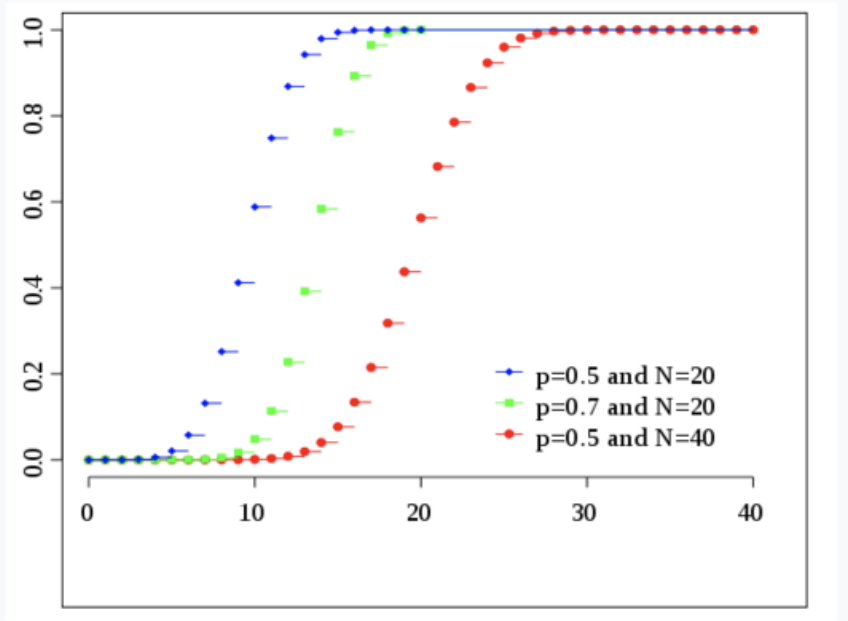
Support: integer positive
Mean: np
Variance: np*(1-p)

Binomial distribution

Probability mass function



Cumulative distribution function



Notation	$B(n, p)$
Parameters	$n \in \{0, 1, 2, \ldots\}$ – number of trials $p \in [0, 1]$ – success probability for each trial
Support	$k \in \{0, 1, \ldots, n\}$ – number of successes
pmf	$\binom{n}{k} p^k (1-p)^{n-k}$
CDF	$I_{1-p}(n-k, 1+k)$
Mean	np
Median	$\lfloor np \rfloor$ or $\lceil np \rceil$
Mode	$\lfloor (n+1)p \rfloor$ or $\lceil (n+1)p \rceil - 1$
Variance	$np(1-p)$
Skewness	$\frac{1-2p}{\sqrt{np(1-p)}}$
Ex. kurtosis	$\frac{1-6p(1-p)}{np(1-p)}$
Entropy	$\frac{1}{2}\log_2(2\pi enp(1-p)) + O\left(\frac{1}{n}\right)$ in shannons. For nats, use the natural log in the log.
MGF	$(1-p+pe^t)^n$
CF	$(1-p+pe^{it})^n$
PGF	$G(z) = [(1-p)+pz]^n$
Fisher information	$g_n(p) = \frac{n}{p(1-p)}$ (for fixed n)

Gaussian (or normal) distribution

most common distribution:

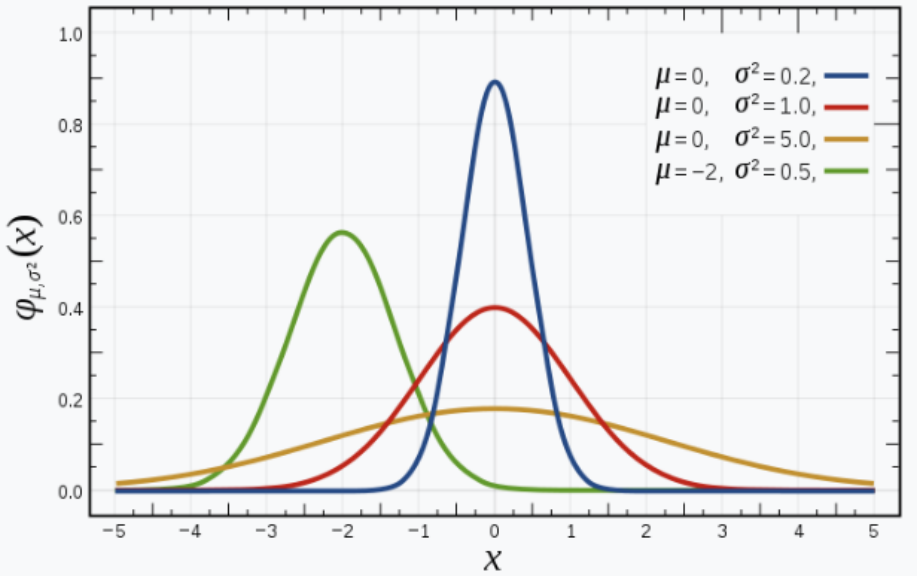
well behaved mathematically, symmetric, when we can we will assume our uncertainties or samples are Gaussian distributed

Parameters: *μ, σ*

Support: natural numbers
Mean: μ
Variance: σ

Normal Distribution

Probability density function



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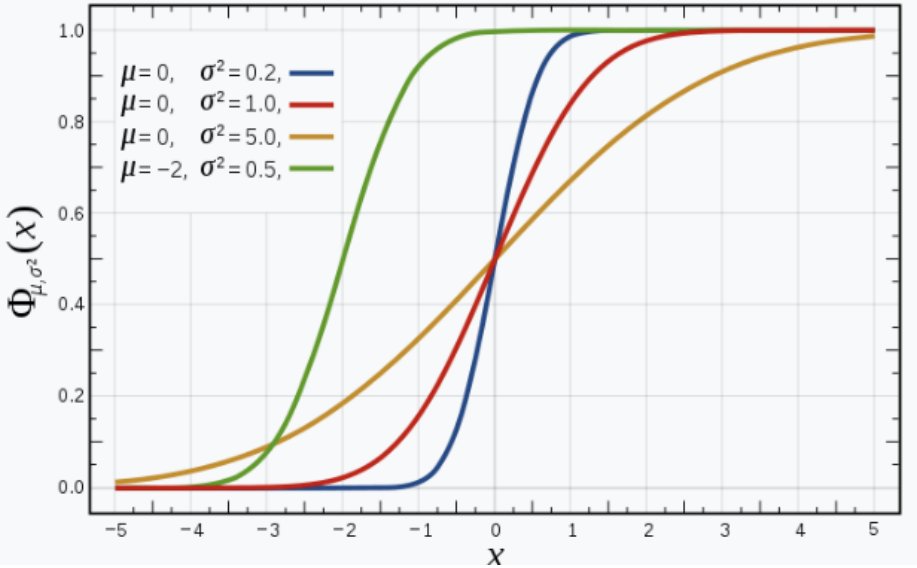

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Poisson

Shut noise/count noise

The innate noise in natural steady state processes (star flux, rain drops...)

Parameters: λ

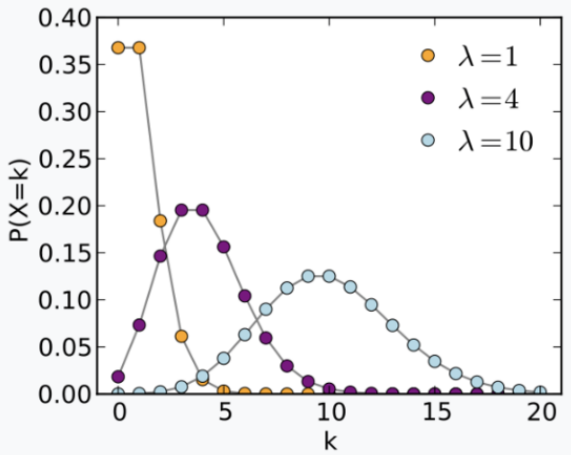
Support: natural numbers

Mean: λ

Variance: λ

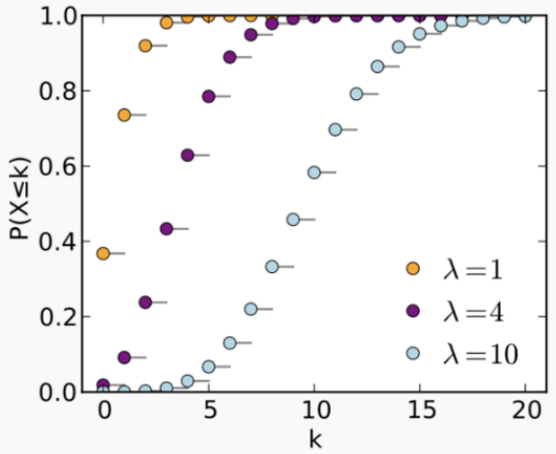
Poisson

Probability mass function



The horizontal axis is the index *k*, the number of occurrences. λ is the expected number of occurrences, which need not be an integer. The vertical axis is the probability of *k* occurrences given λ. The function is defined only at integer values of *k*. The connecting lines are only guides for the eye.

Cumulative distribution function



The horizontal axis is the index *k*, the number of occurrences. The CDF is discontinuous at the integers of *k* and flat everywhere else because a variable that is Poisson distributed takes on only integer values.

Notation	$\mathrm{Pois}(\lambda)$
Parameters	$\lambda > 0$, (real) – rate
Support	$k \in \{0, 1, 2, \ldots\}$
pmf	$\frac{\lambda^k e^{-\lambda}}{k!}$
CDF	$\frac{\Gamma(\lfloor k+1 \rfloor, \lambda)}{\lfloor k \rfloor !}$, or $e^{-\lambda} \sum_{i=0}^{\lfloor k \rfloor} \frac{\lambda^i}{i!}$, or $Q(\lfloor k+1 \rfloor, \lambda)$ (for $k \geq 0$, where $\Gamma(x, y)$ is the upper incomplete gamma function, $\lfloor k \rfloor$ is the floor function, and Q is the regularized gamma function)
Mean	λ
Median	$\approx \lfloor \lambda + 1/3 - 0.02/\lambda \rfloor$
Mode	$\lceil \lambda \rceil - 1, \lfloor \lambda \rfloor$
Variance	λ
Skewness	$\lambda^{-1/2}$
Ex. kurtosis	λ^{-1}
Entropy	$\lambda[1 - \log(\lambda)] + e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k \log(k!)}{k!}$ (for large λ) $\frac{1}{2}\log(2\pi e\lambda) - \frac{1}{12\lambda} - \frac{1}{24\lambda^2} - \frac{19}{360\lambda^3} + O\left(\frac{1}{\lambda^4}\right)$
MGF	$\exp(\lambda(e^t - 1))$
CF	$\exp(\lambda(e^{it} - 1))$
PGF	$\exp(\lambda(z - 1))$
Fisher information	$\frac{1}{\lambda}$

Moments and frequentist probability

a distribution's moments summarize its properties:

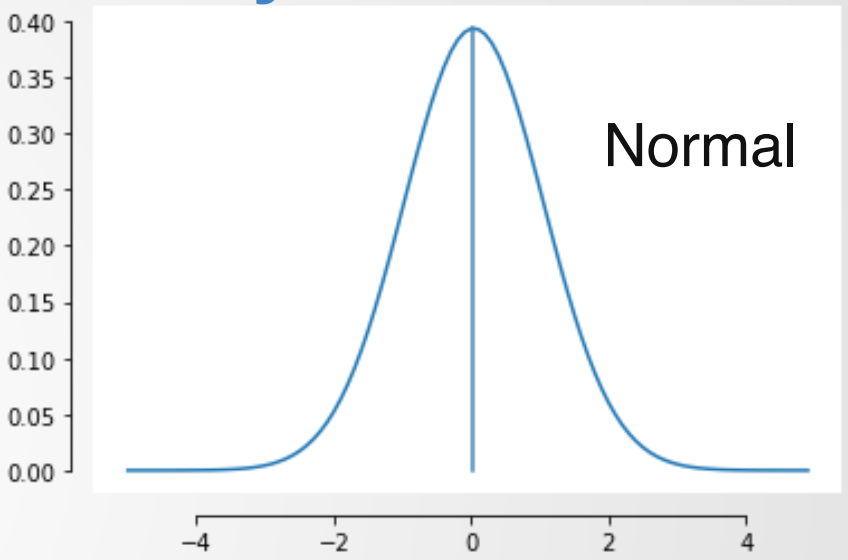
$$m_n = \int_{-\inf}^{\inf} (x - c)^n f(X) dx$$

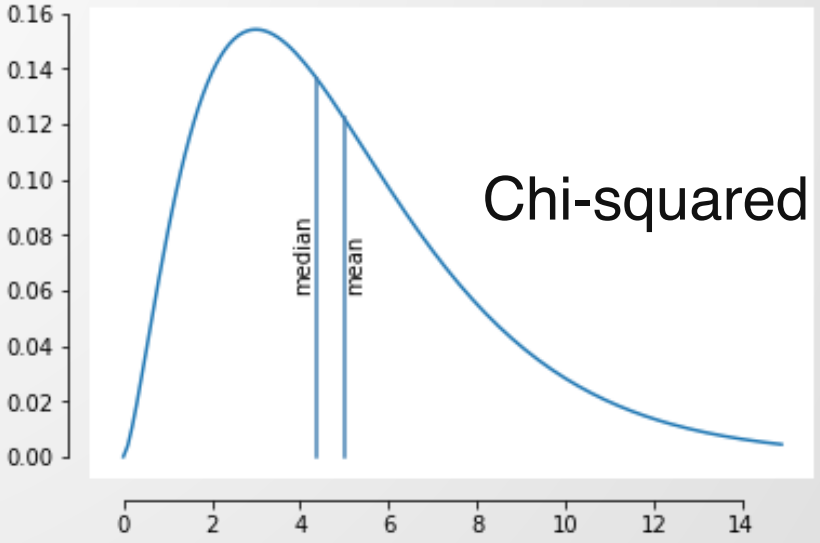
central tendency: mean (n=1), median, mode (peak)

spread: standard deviation (variance n=2), quantiles

symmetry: skewness (n=3)

cuspiness: kurtosis (n=4)





coding time!

https://colab.research.google.com/

https://github.com/fedhere/FDSfE_FBianco/blob/master/statistics/distributionParametersDemo.ipynb

2 are they the same?

questions we need statistics to answer

Preamble: kinds of descriptive questions

- What is the highest/lowest value?
  - *what is the most viewed video?*
  - what is the average number of views?

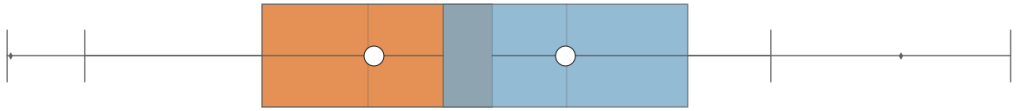
```
videos.describe()
```

	category_id	views	likes	dislikes	comment_count
count	40949.000000	4.094900e+04	4.094900e+04	4.094900e+04	4.094900e+04
mean	19.972429	2.360785e+06	7.426670e+04	3.711401e+03	8.446804e+03
std	7.568327	7.394114e+06	2.288853e+05	2.902971e+04	3.743049e+04
min	1.000000	5.490000e+02	0.000000e+00	0.000000e+00	0.000000e+00
25%	17.000000	2.423290e+05	5.424000e+03	2.020000e+02	6.140000e+02
50%	24.000000	6.818610e+05	1.809100e+04	6.310000e+02	1.856000e+03
75%	25.000000	1.823157e+06	5.541700e+04	1.938000e+03	5.755000e+03
max	43.000000	2.252119e+08	5.613827e+06	1.674420e+06	1.361580e+06

```
1 videos["views"].mean()
2 videos["views"].median()
3
4 --> 2360784.6382573447
5 --> 681861.0
6
```

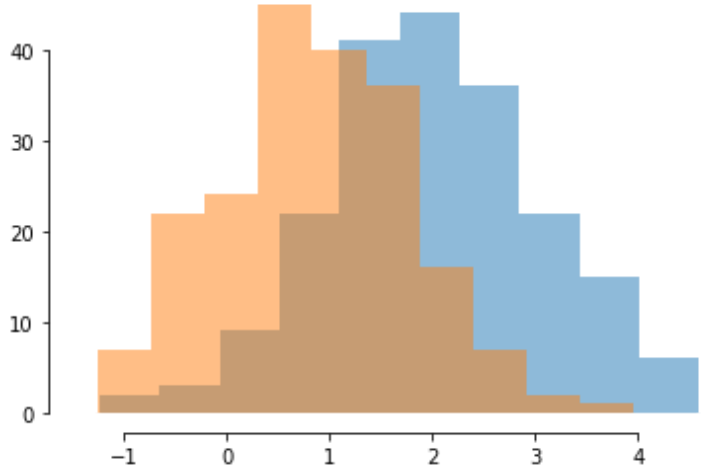
Preamble: kinds of analytical questions

- Are two measurements the same?



  - *is the amount of rain in Wilmington this year the same as last year?*

- Are two distributions the same?



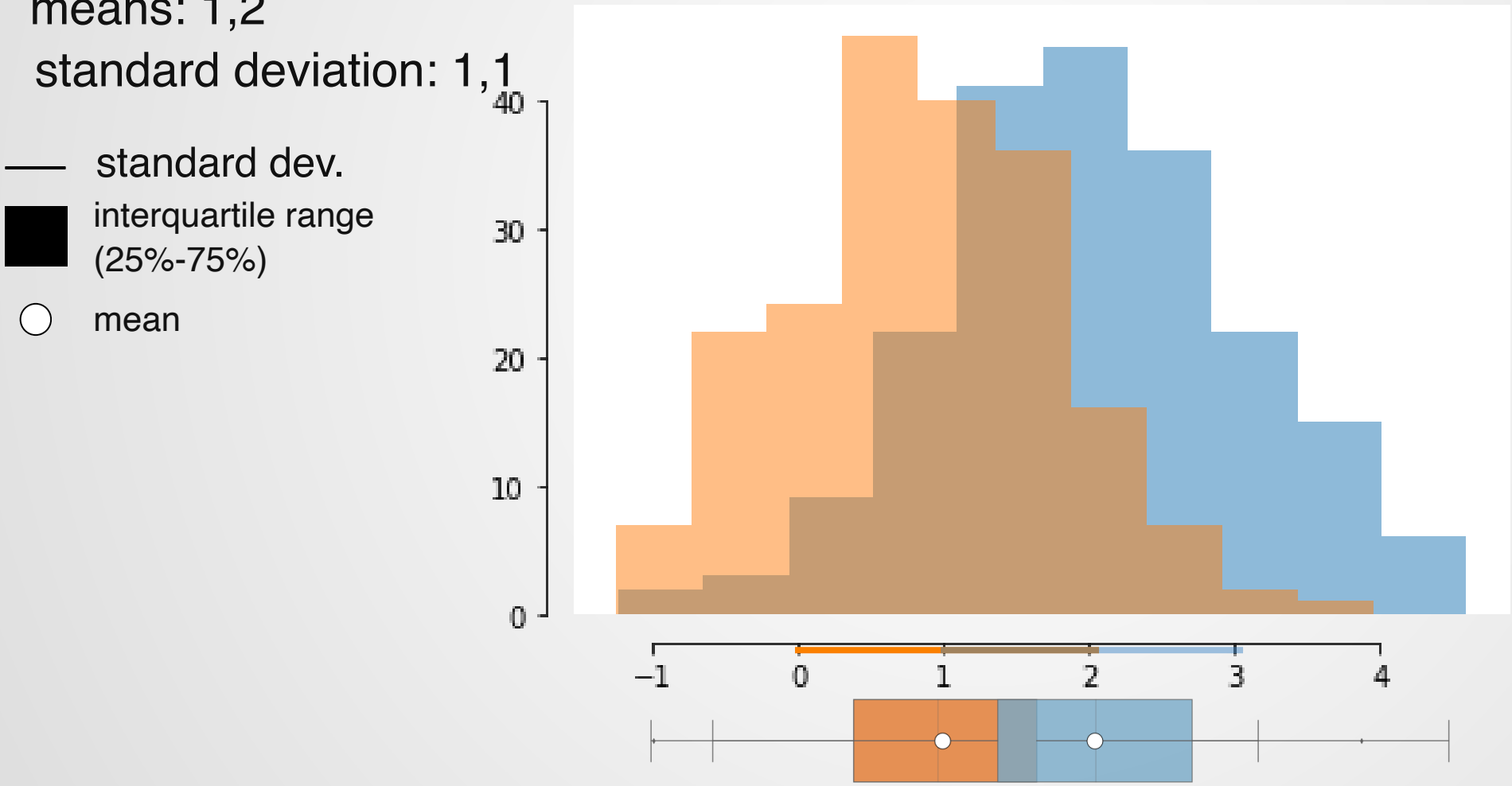
  - *is the age distribution of CityBike users the same among registered Male customers and registered Female customers?*

measuring differences between distributions

means: 1,2

standard deviation: 1,1

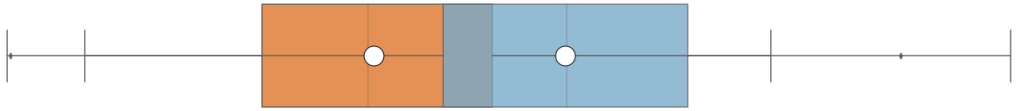
are these distributions the same?



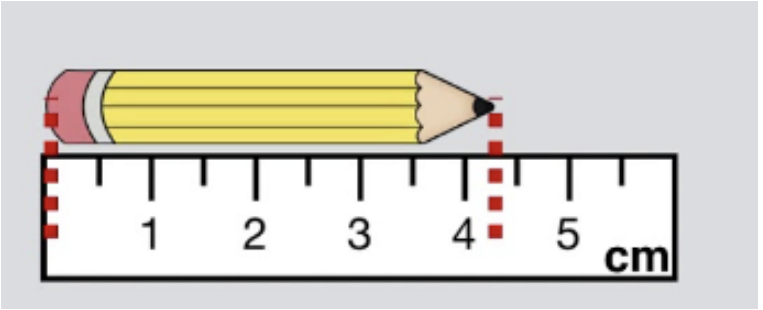
if distributions have the same measured means within 1 (or *n*) standard deviation they should be considered "the same"

21 measurement uncertainty and measurement samples

- Are two measurements the same?

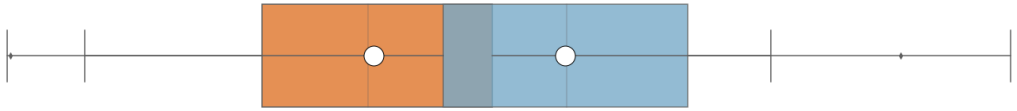


  - *is the amount of rain in Wilmington this year the same as last year?*
  - *are two pencils the same length*



uncertainty because of the limitations of the measuring tool

- Are two measurements the same?



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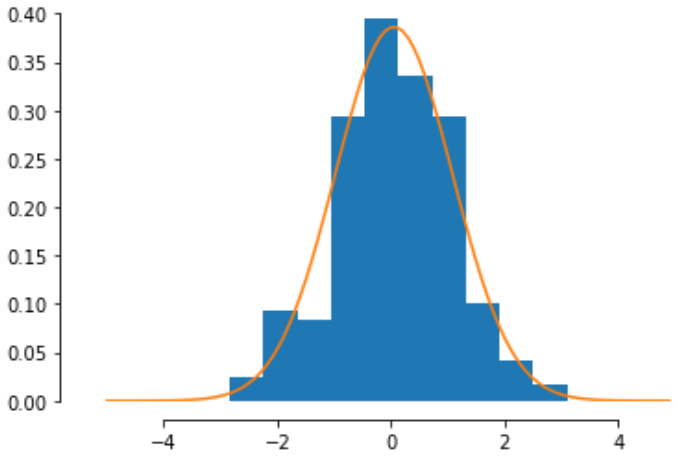
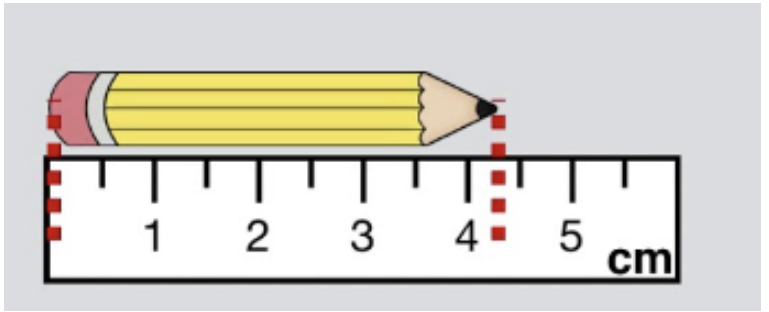
Take N measurements,
they will all be a bit different





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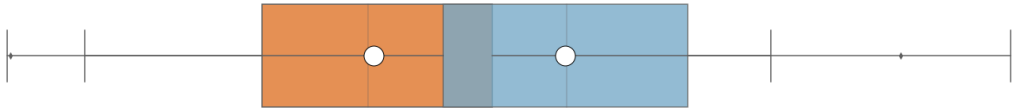

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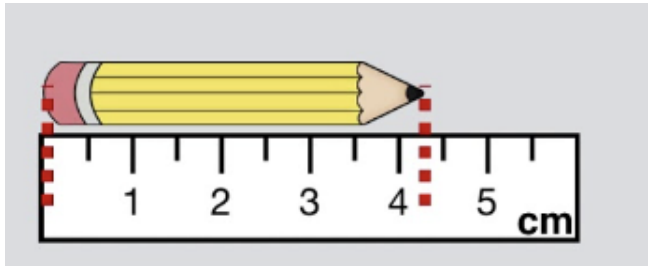

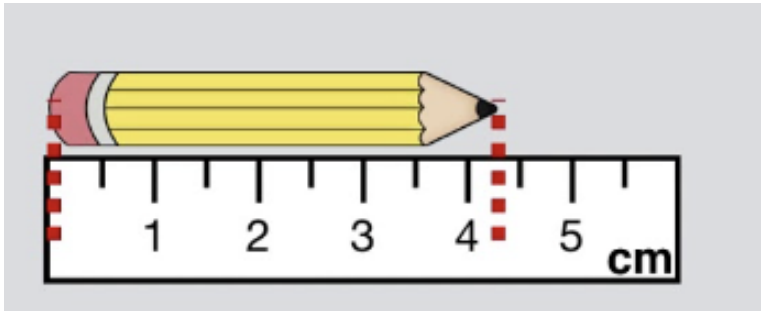

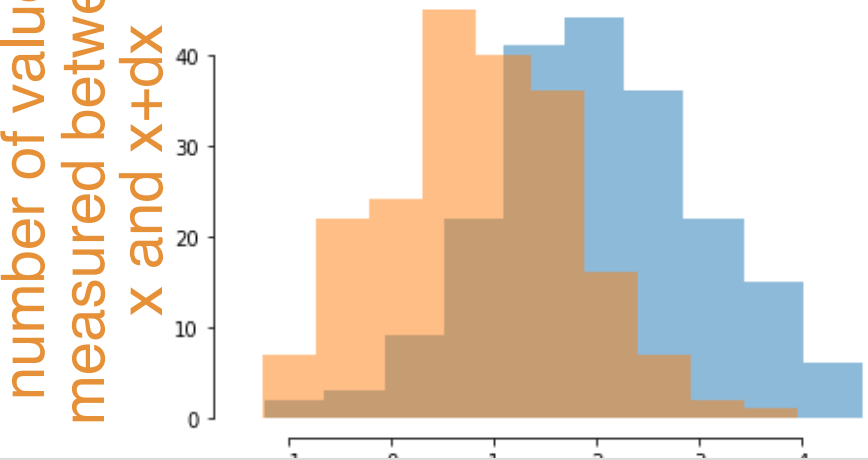

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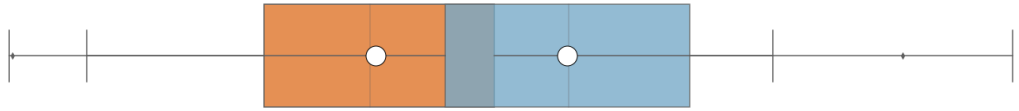

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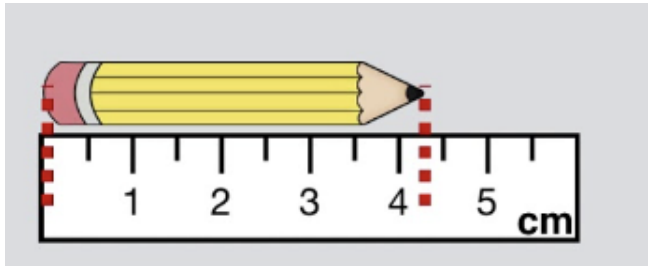
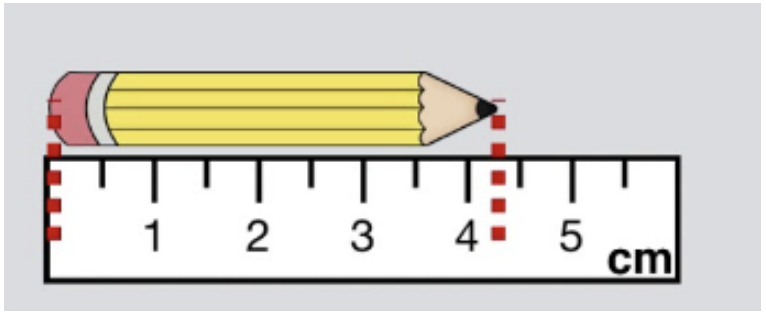
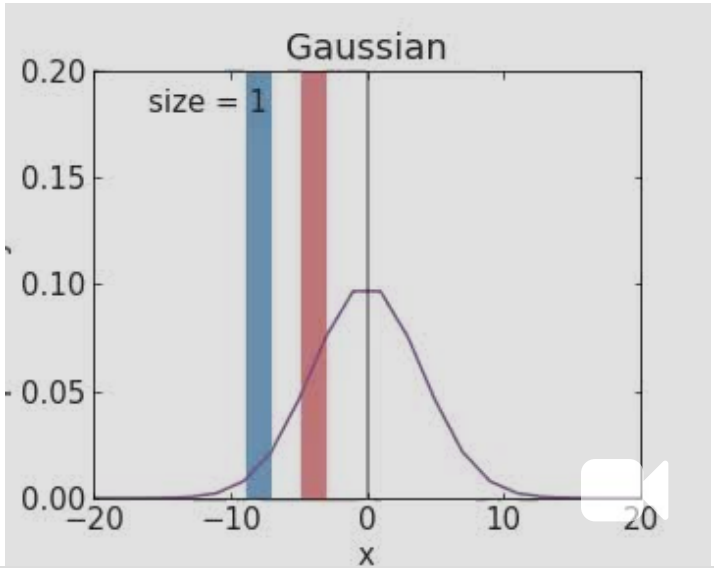






number of values measured between x and x+dx

uncertainty because of the limitations of the measuring tool

Methods and philosophy [edit]

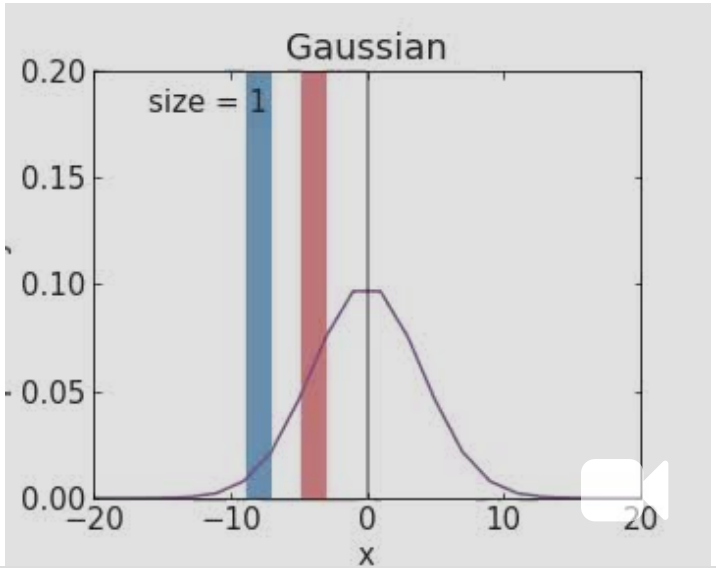
The rankings of national happiness are based on a Cantril ladder survey. Nationally representative samples of respondents are asked to think of a ladder, with the best possible life for them being a 10, and the worst possible life being a 0. They are then asked to rate their own current lives on that 0 to 10 scale.[15] The

wikipedia

- Are two measurements the same?
  - *are two countries just as happy?*
  - *are two pencils the same length?*

Take N measurements,
they will all be a bit different

Country	Region	Happiness Score	Standard Error
Denmark	Western Europe	7.527	0.03328
Norway	Western Europe	7.522	0.03880



number of values measured between x and x+dx

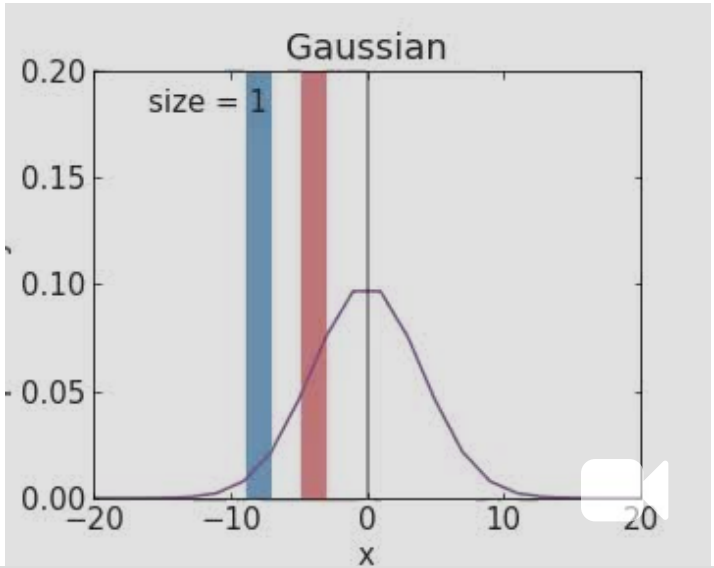
intrinsic variance in the phenomenon

KEY CONCEPT:

the larger the number of samples from the distribution the more similar the distribution of our sample is to the actual "generative process": i.e. the histogram will look more and more like the actual distribution curve

Take N measurements, they will all be a bit different

number of values measured between x and x+dx



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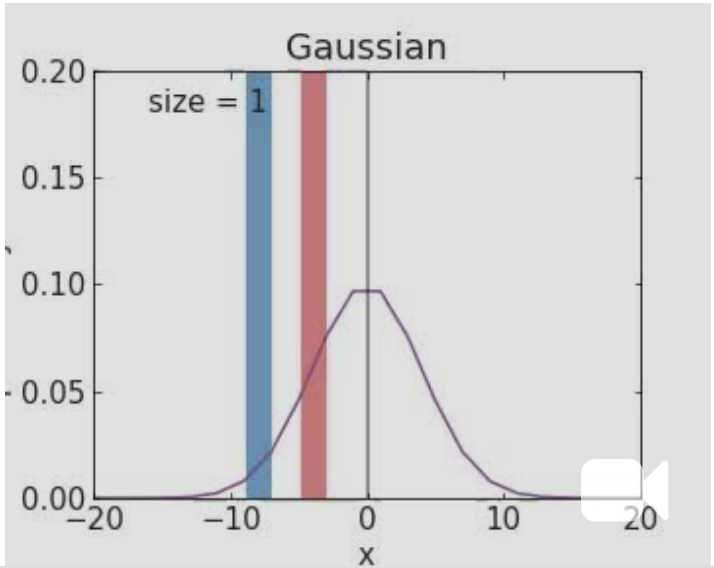

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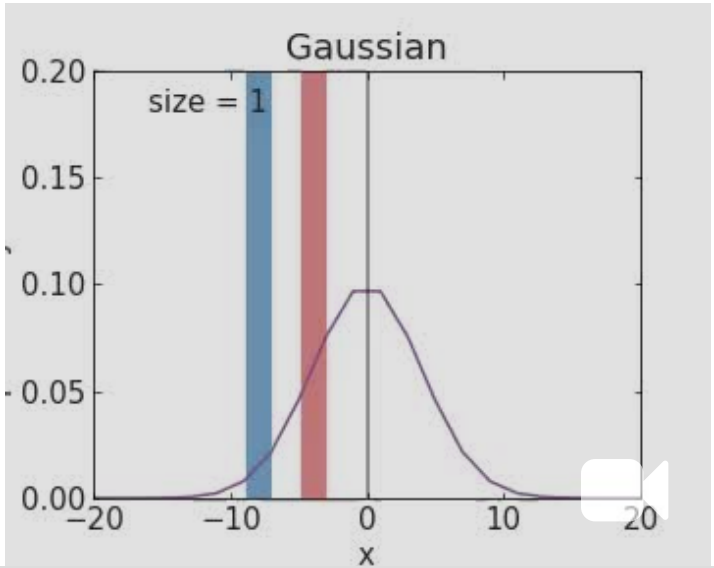
THEREFORE

It is easier to tell if two distributions are the same when the samples are large

Take N measurements,
they will all be a bit different

number of values measured between x and x+dx



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THEREFORE

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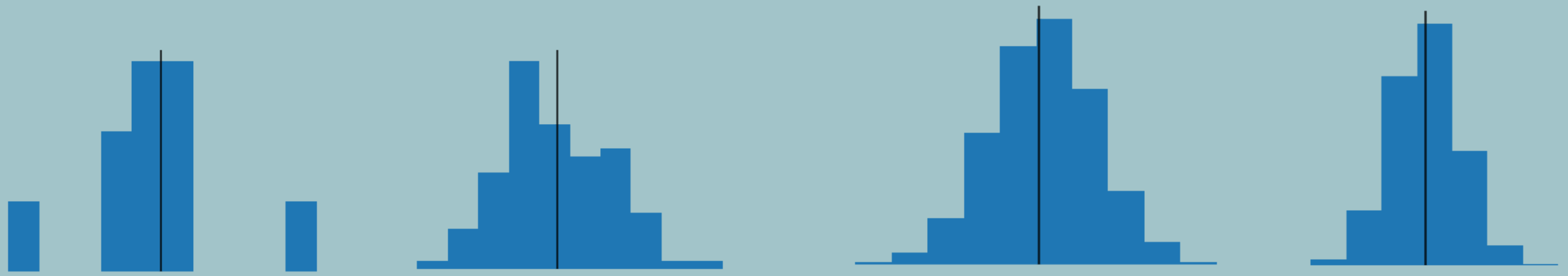
THEREFORE

I need statistical tests that acknowledge the size of the sample when I compare distributions

Law of Large Numbers

As the size of a _____ tends to infinity the mean of the sample tends to the mean of the _____



Central Limit Theorem

Laplace (1700s) but also: Poisson, Bessel, Dirichlet, Cauchy, Ellis

Let $x_1 \ldots x_N$ be an N-elements sample from a population whose distribution has

mean **μ** and standard deviation **σ**

In the limit of N -> infinity

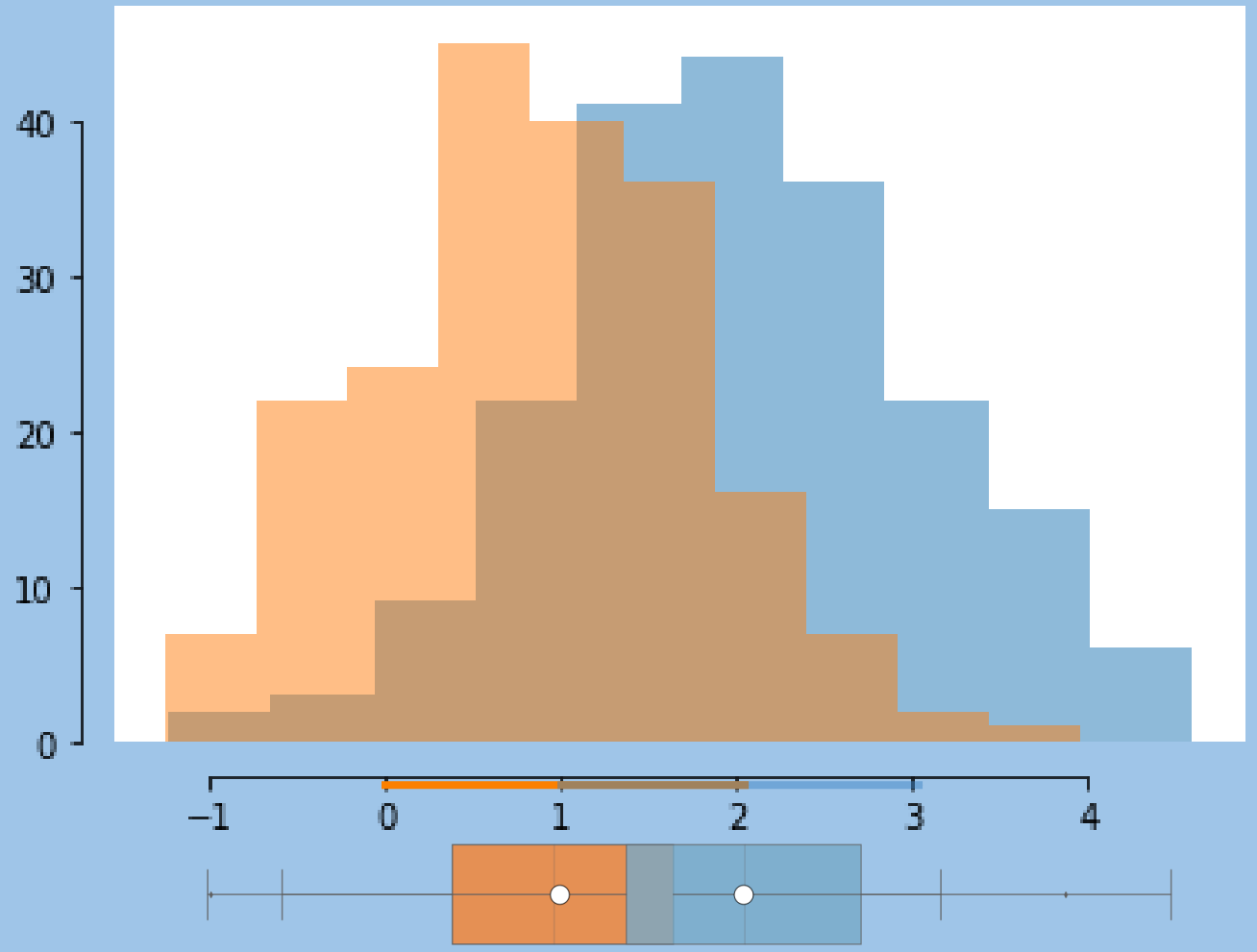
the sample mean $\bar{x}$ approaches a Normal (Gaussian) distribution with mean μ and standard deviation σ

regardless of the distribution of X

$$\bar{x} \sim N\left(\mu, \sigma/\sqrt{N}\right)$$

Easy way to assess if two number are the same:

Are the mean farther than the standard deviations?



standard deviation: 1,1

standard dev.

interquartile range (25%-75%)

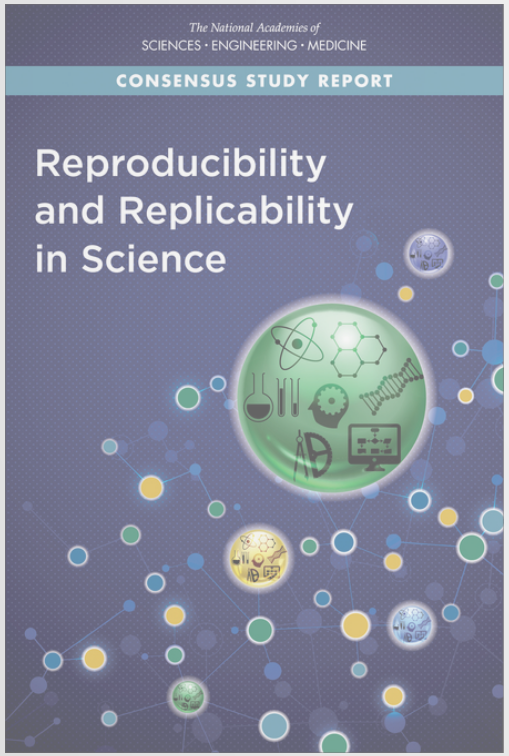
mean

3

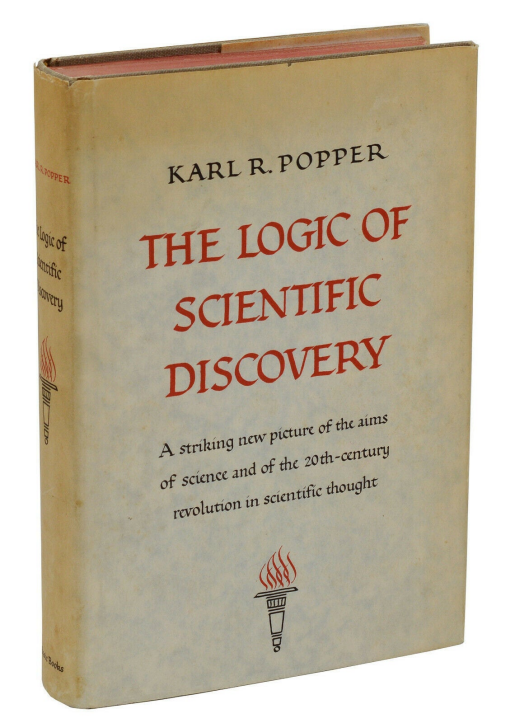
the principle of Falsifiability

3 General principles of "good" science

Reproducibility



Falisifiability



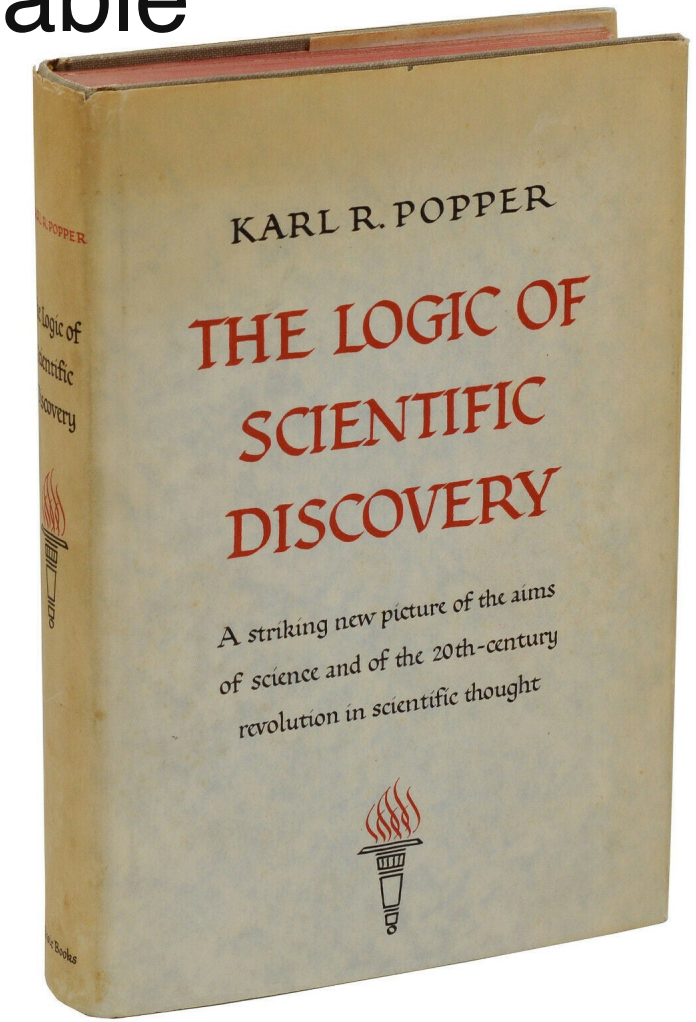
Parsimony

the *demarcation* problem:

science hypotheses needs to be falsifiable

My proposal is based upon an *asymmetry* between verifiability and **falsifiability**; an asymmetry which results from the logical form of universal statements. For these are never derivable from singular statements, but can be contradicted by singular statements.

—Karl Popper, *The Logic of Scientific Discovery*



the *demarcation* problem:

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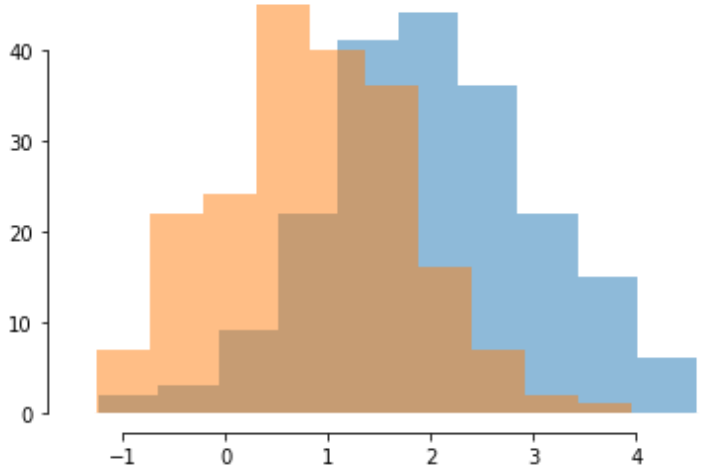
—Karl Popper, *The Logic of Scientific Discovery*

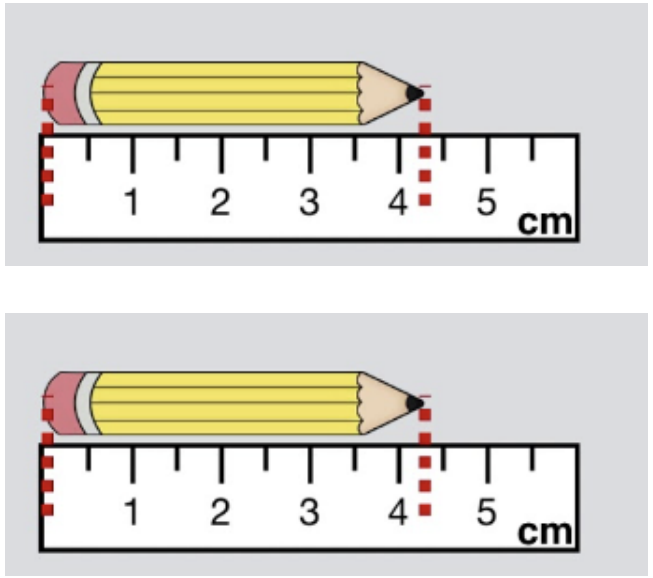
I need to see only 1 black swan to tell that the statement that all swans are white is not true. But even if I dont see a black one it does not mean all swans are white

the *demarcation* problem:

science hypotheses needs to be falsifiable

But what happens when I have distributions of measurements?





Beyond any reasonable doubt

same concept guides prosecutorial justice

guilty beyond reasonable doubt

in a probabilistic sense, all hypotheses we make are possible

We will reject a hypothesis if its probability is lower than a predefined threshold

4 hyposthesis testing

4 hyposthesis testing

We do not "prove our hypothesis"

we falsify the opposite of our hypothesis

Null
Hypothesis
Rejection
Testing

The NULL hypothesis is typically what I want to reject: its the way I think the world does NOT work

the pencils are the NOT same length (tho i think they are)

the earth is NOT round (spoiler alert... it is!!)

1 formulate your prediction

Null Hypothesis

Null
Hypothesis
Rejection
Testing

The ALTERNATIVE hypothesis is the complete opposite of the NULL

$$P(A) + P(\bar{A}) = 1$$

if *all alternatives* to our model are ruled out, then our model must hold

2 identify all alternative outcomes

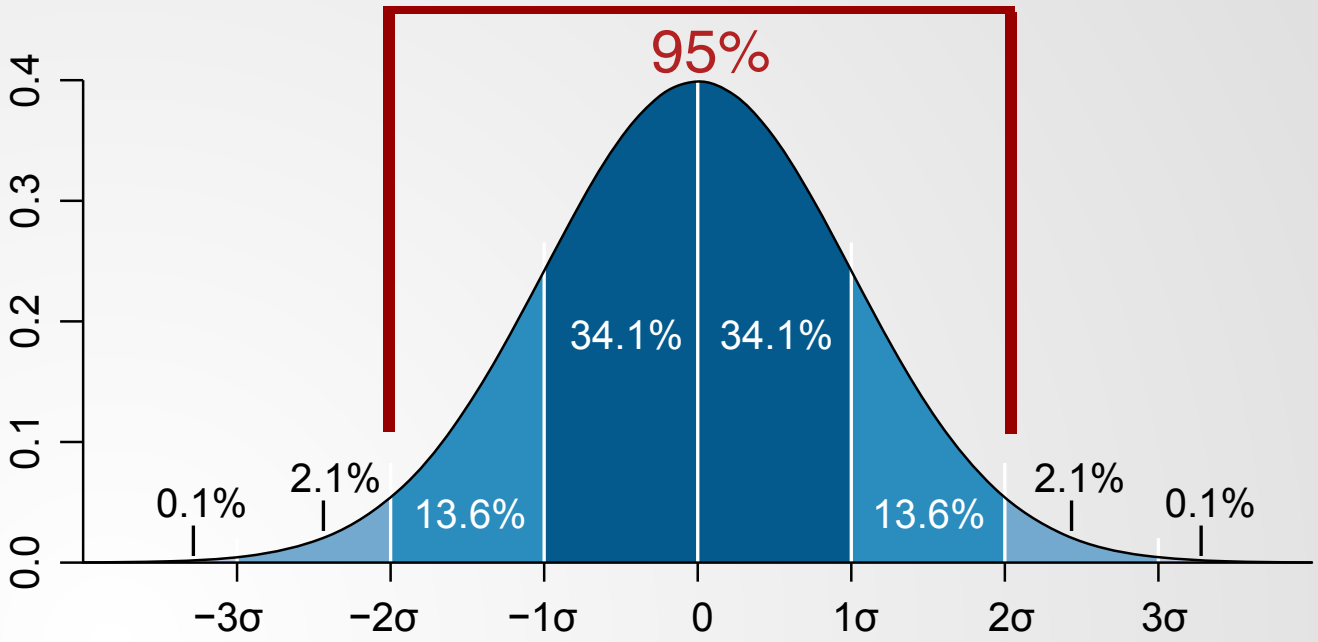
Alternative Hypothesis

Null

Hypothesis

Rejection

Testing



3

set confidence threshold

2σ confidence level

0.05 p-value

95% α threshold

Null

Hypothesis

Rejection

Testing

pivotal quantities

4

find a measurable quantity which under the Null has a known distribution

Z-test

The distribution of sample means for (independent) samples extracted from a population

with mean $\boldsymbol{\mu}$ and standard deviation $\boldsymbol{\sigma}$ is

Normally distributed

$$\bar{X} \sim N(\mu = 0, \sigma = 1)$$

Z-test

If I measure the mean of two samples

(the samples of pencil measurements)

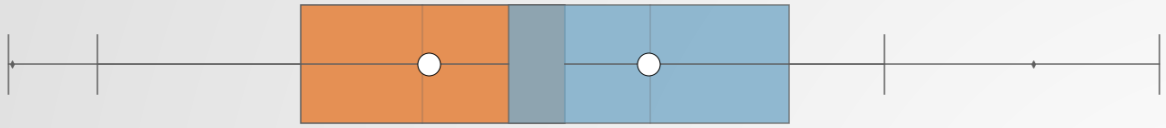
I expect the difference to be a number drawn from a standard normal:

Highest prob. is 0

Prob that the number is within 1σ of the mean is 68%

$$\bar{X} \sim N(\mu = 0, \sigma = 1)$$

Z-test



If the means are within 1-sigma of each other that means that I cannot rule out that the distributions are the same to a 1-sigma level (p-value 0.32)

If the means are not 1-sigma of each other that means that I can rule out that the distributions are the same to a 1-sigma level (p-value 0.32)

$$\bar{X} \sim N(\mu = 0, \sigma = 1)$$

Z-test

Is the mean of a sample *with known variance* the same as that of a known population?

pivotal quantity

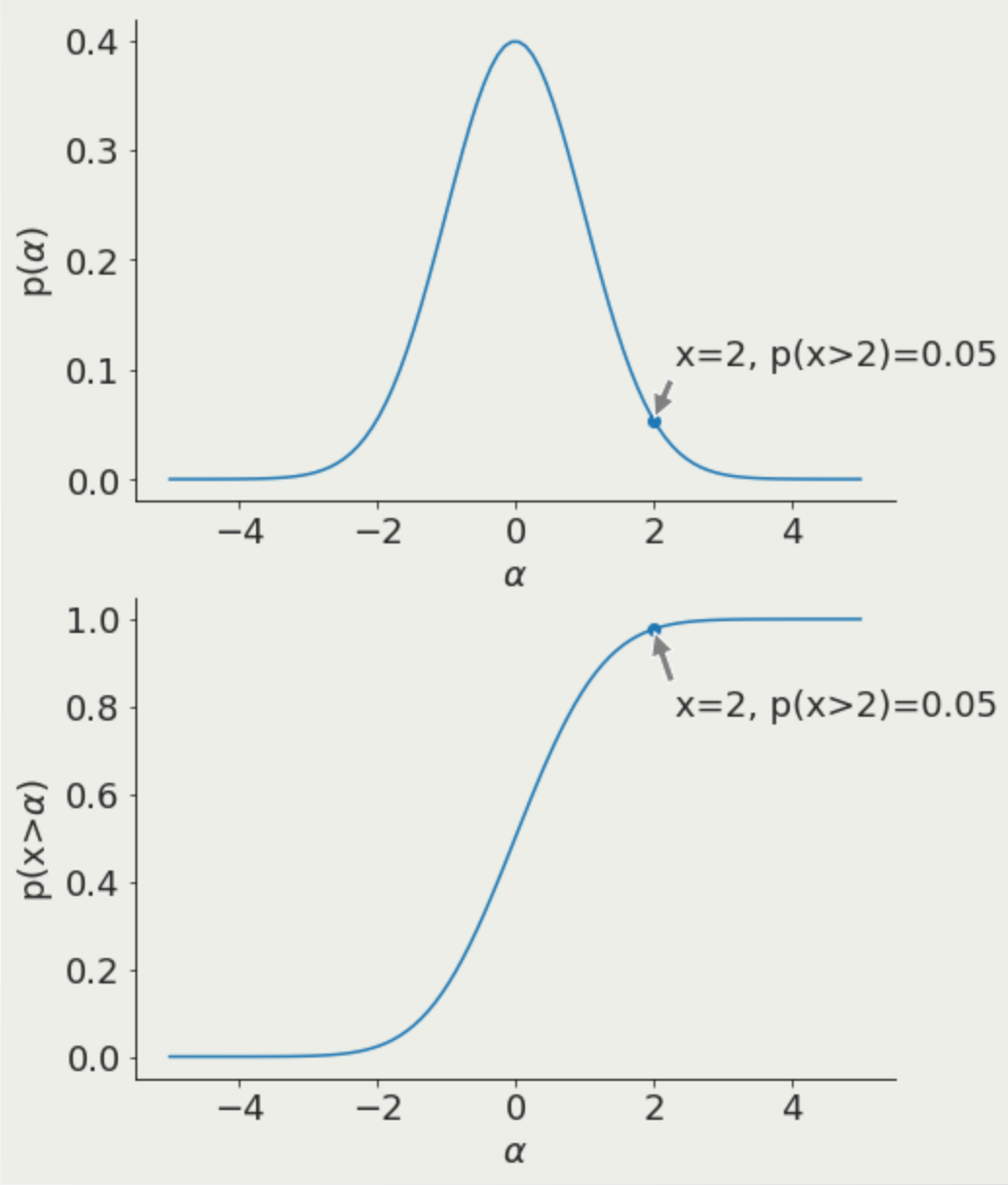
$$Z = (\bar{X} - \mu_0)/s$$

sample mean

population mean

sample variance = $\sigma_0/\sqrt{n}$

$$Z \sim N(\mu = 0,\ \sigma = 1)$$



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pivotal quantity

$$Z = (\bar{X} - \mu_0)/s$$

sample mean

population mean

sample variance = $\sigma_0/\sqrt{n}$

$$Z \sim N(\mu = 0,\ \sigma = 1)$$

The Z test provides a trivial interpretation of the measured quantity: the Z value is exactly the distance for the mean of the standard distribution of possible outcomes *in units of standard deviation*

so a result of 0.13 means we are 0.13 standard deviations to the mean (p>0.05)

Z-test

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population mean

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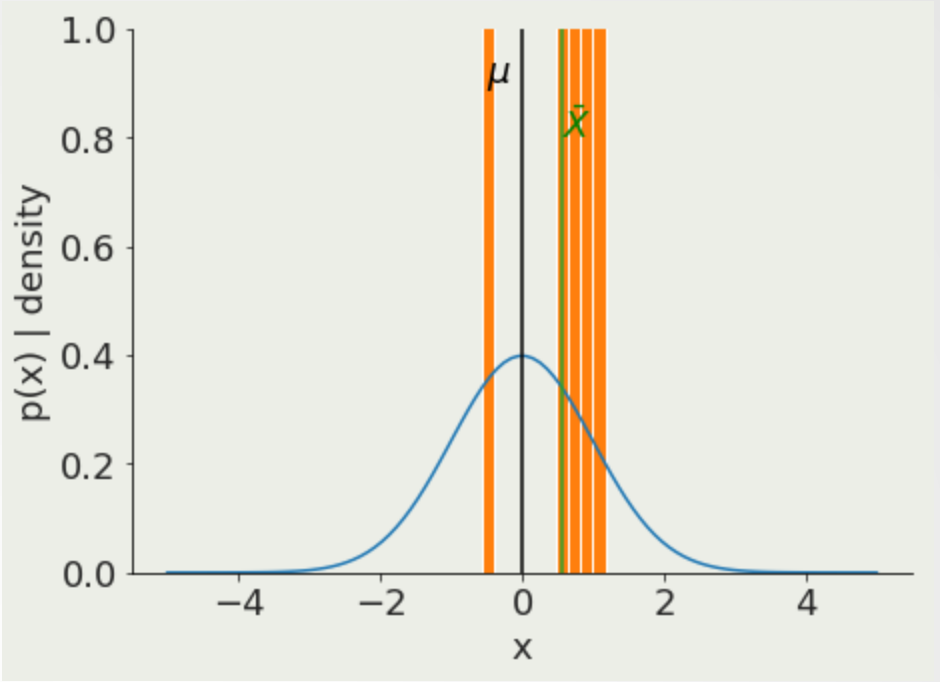


why do we need a test? why not just measuring the means and seeing it they are the same?

formulate the Null as the comprehensive opposite of your theory

model → **prediction**

"Under the Null Hypothesis" = if the proposed model is false

this has a low probability of happening

data

does not falsify alternative → **everything but model is rejected**

falsifies alternative → **model holds**

Key Slide

low probability event happened

1 formulate your prediction (NH)

2 identify all alternative outcomes (AH)

3 set confidence threshold (*p*-value)

4 find a measurable quantity which under the Null has a known distribution (pivotal quantity)

5 calculate the pivotal quantity

6 calculate probability of value obtained for the pivotal quantity under the Null

Key Slide

if probability < *p*-value : reject Null

Null

Hypothesis

Rejection

Testing

pivotal quantities

quantities that under the Null Hypotheis follow a known distribution

4

if a quantity follows a known distribution, once I measure its value I can what the probability of getting that value actually is! was it a likely or an unlikely draw?

Null

Hypothesis

Rejection

Testing

pivotal quantities

quantities that under the Null Hypotheis follow a known distribution

also called "statistics"

e.g.: *χ2 statistics:* difference between prediction and reality squared

Z statistics: difference between means

K-S statistics: maximum distance of cumulative distributions.

Null

Hypothesis

Rejection

Testing

pivotal quantities

quantities that under the Null Hypotheis follow a known distribution

4

$$p(\text{pivotal quantity}|NH) \sim p(NH|D)$$

Null

Hypothesis

Rejection

Testing

pivotal quantities

5

calculate it!

Null

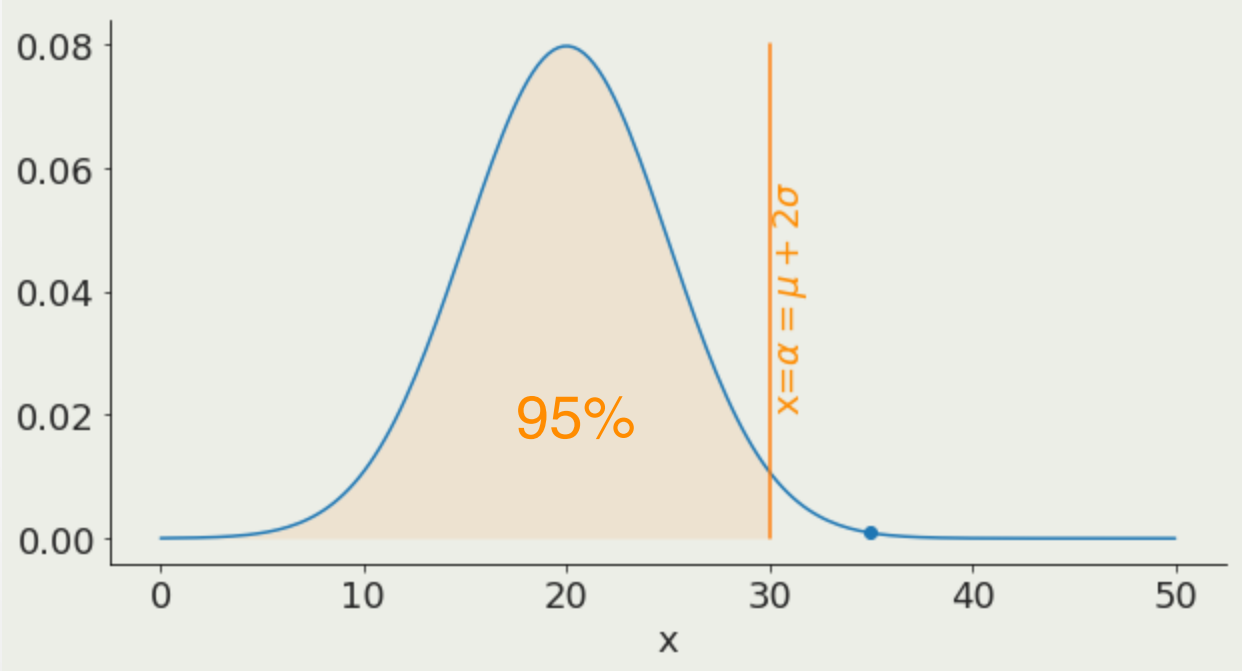
Hypothesis

Rejection

Testing

what is α?

α is the x value corresponding to a chosen threshold



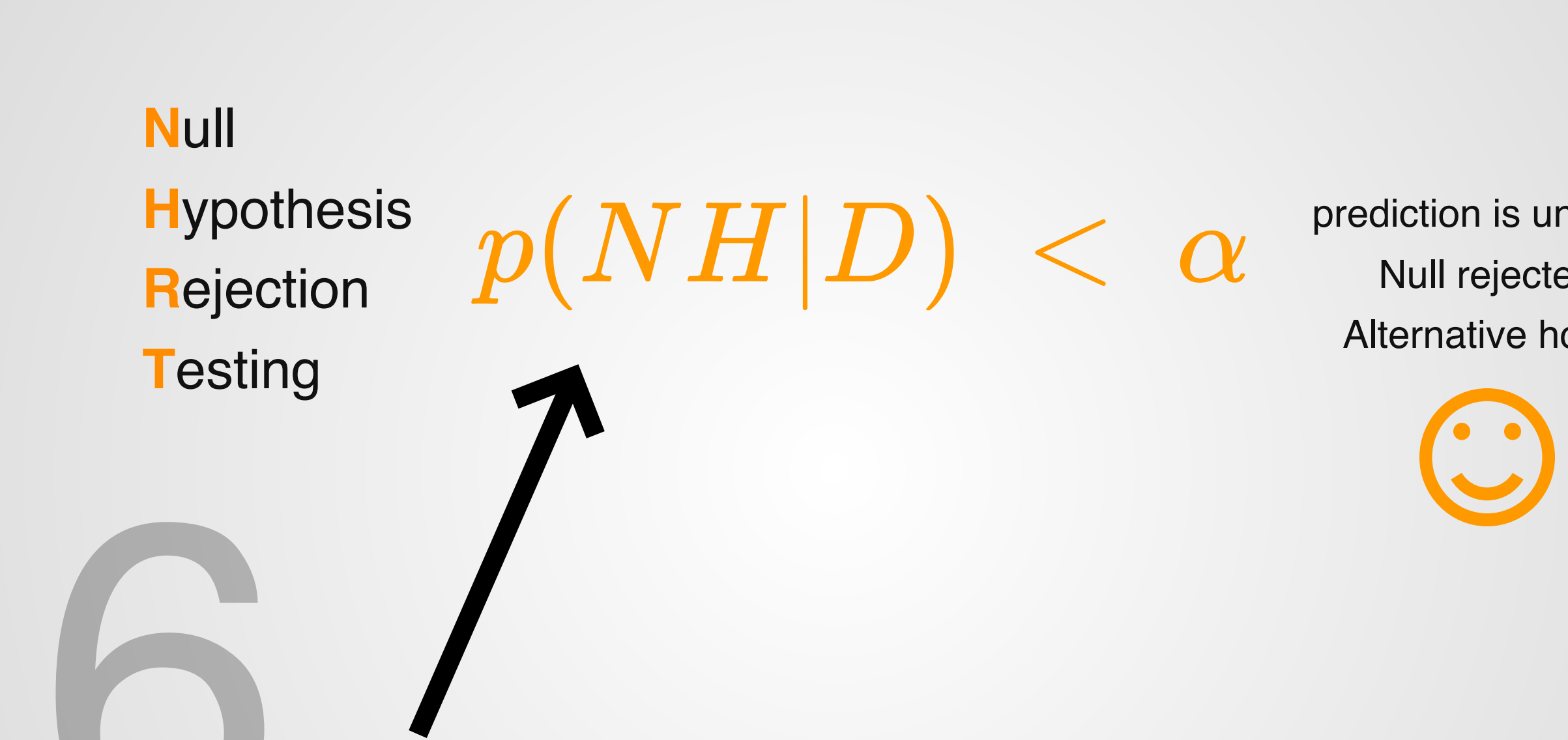
6

test data against
alternative outcomes

Null
Hypothesis
Rejection
Testing

$$p(NH|D) < \alpha$$

prediction is unlikely
Null rejected
Alternative holds



6

test data against
alternative outcomes

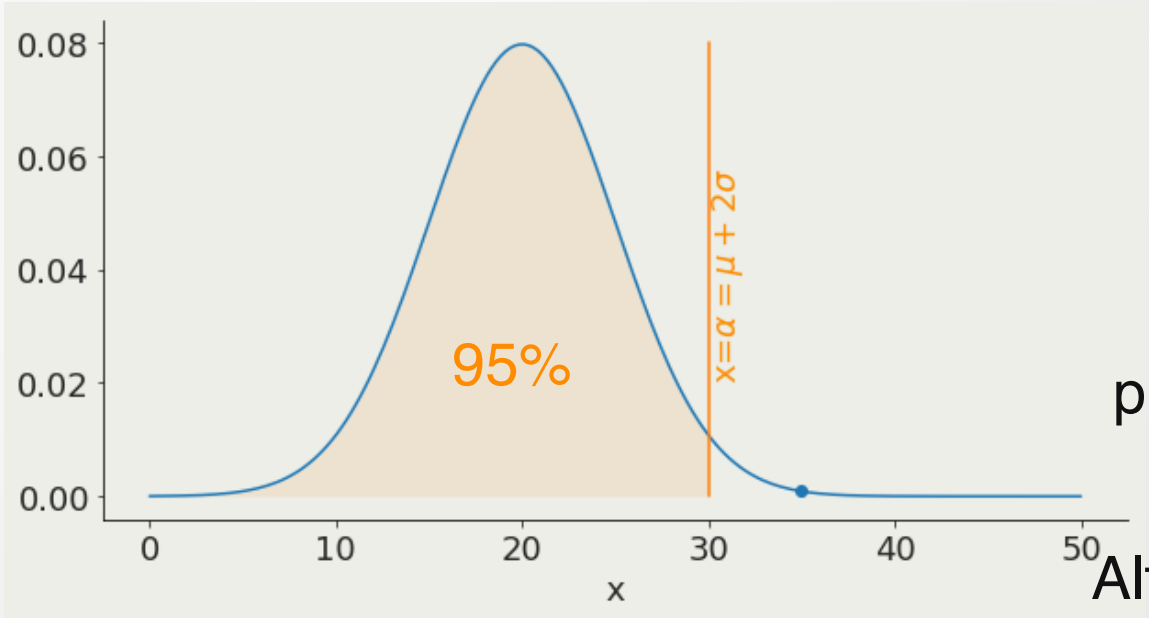
Null
Hypothesis
Rejection
Testing

6

test data against
alternative outcomes

$$p(NH|D) < \alpha$$

prediction is unlikely
Null rejected
Alternative holds



$$p(NH|D) \geq \alpha$$

prediction is likely
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Alternative rejected

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6

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$$p(NH|D) < \alpha$$

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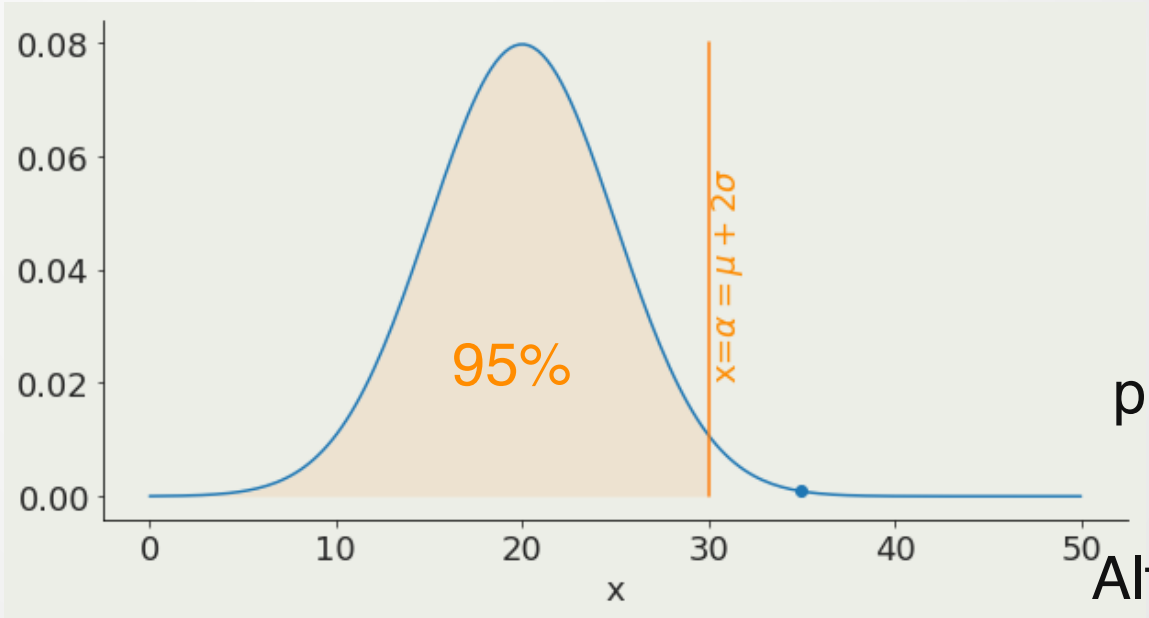


$$p(NH|D) \geq \alpha$$

prediction is likely
Null holds
Alternative rejected

K-S test

Kolmogorof-Smirnoff :

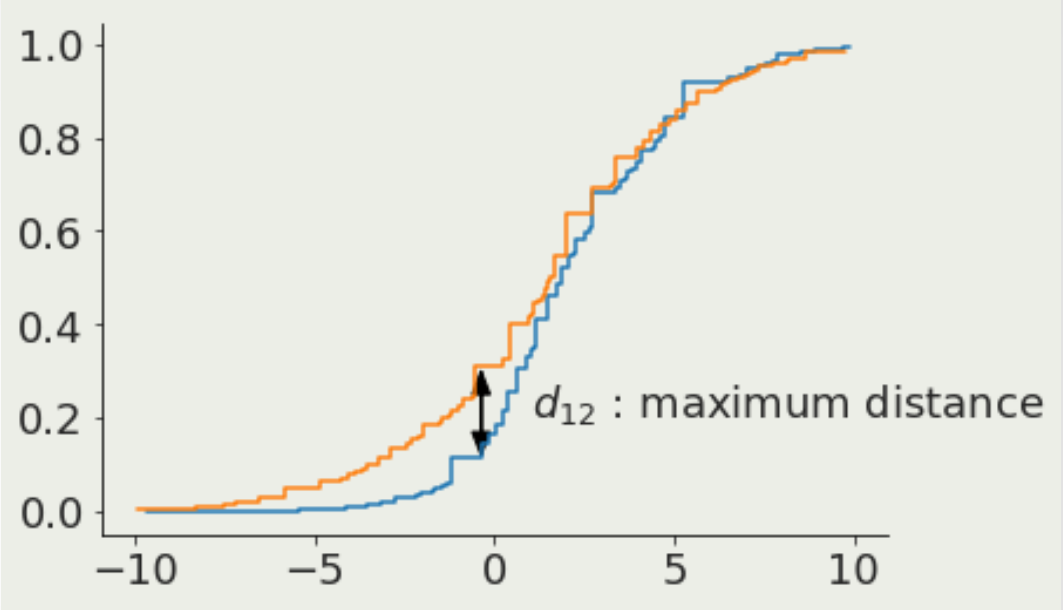
do two samples come from the same parent distribution?

pivotal quantity

$$d_{12} \equiv max_x \left| C_1(x) - C_2(x) \right|$$

Cumulative distribution 1

Cumulative distribution 2



$$P(d > observed) = 2\sum_{j=1}^{\infty}(-1)^{j-1}e^{-2j^2x^2}\sqrt{\frac{N_1 N_2}{N_1+N_2}D}$$

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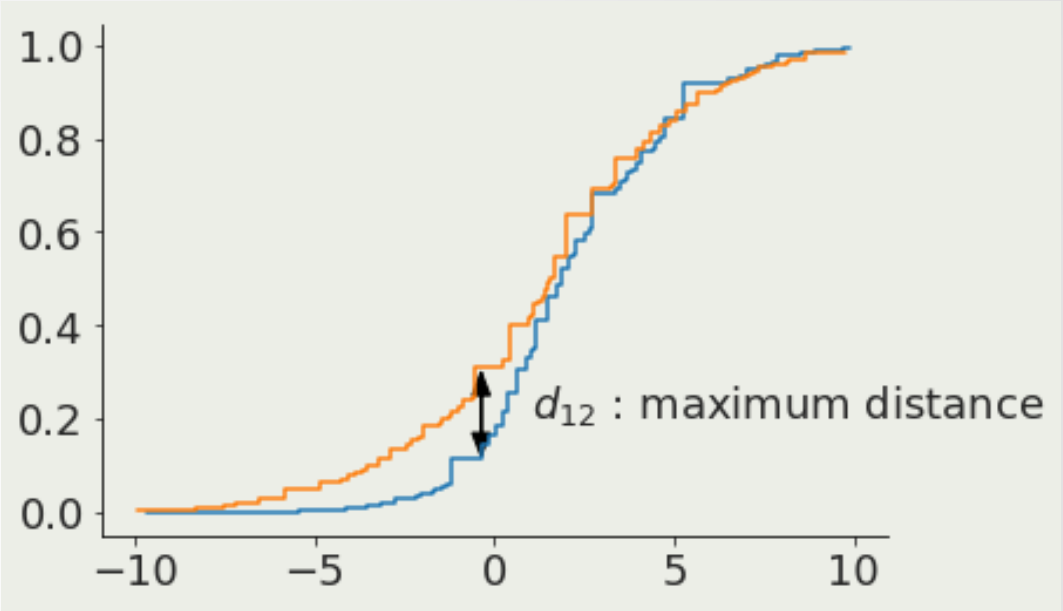


$$P(d > observed) =$$

```
sp.stats.ks_2samp(x, y)
```

executed in 7ms, finished 14:45:10 2019-09-09

```
Ks_2sampResult(statistic=0.4, pvalue=0.312852676016955 8)
```

5

p-value hypothesis testing

Moments and frequentist probability

Imagine that I take a measurements of a quantity that is expected to be normally distributed with mean 0 and stdev 1

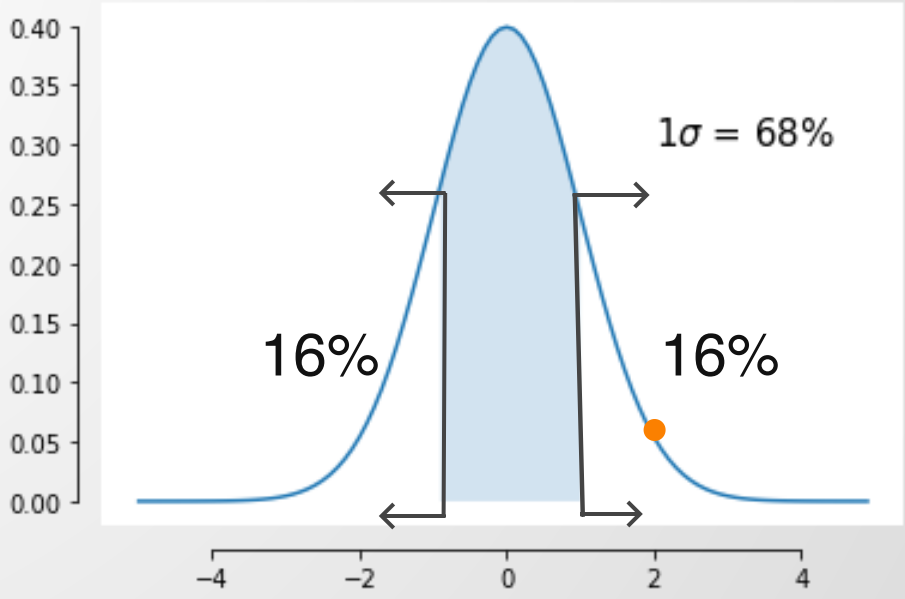
what is the probability that I would measure 1.5?

The probability of measuring any one value is mathematically *0*... however I can say that

the probability of measuring something between *-1σ* and *1σ* (within 1-sigma) is 68%.

So the probability of measuring something outside is 100-68 = 32%.

So if I measure something outside of [-*1σ*:*1σ*] that had a probability <32% of being measured.



Moments and frequentist probability

Imagine that I take a measurements of a quantity that is expected to be normally distributed with mean 0 and stdev 1

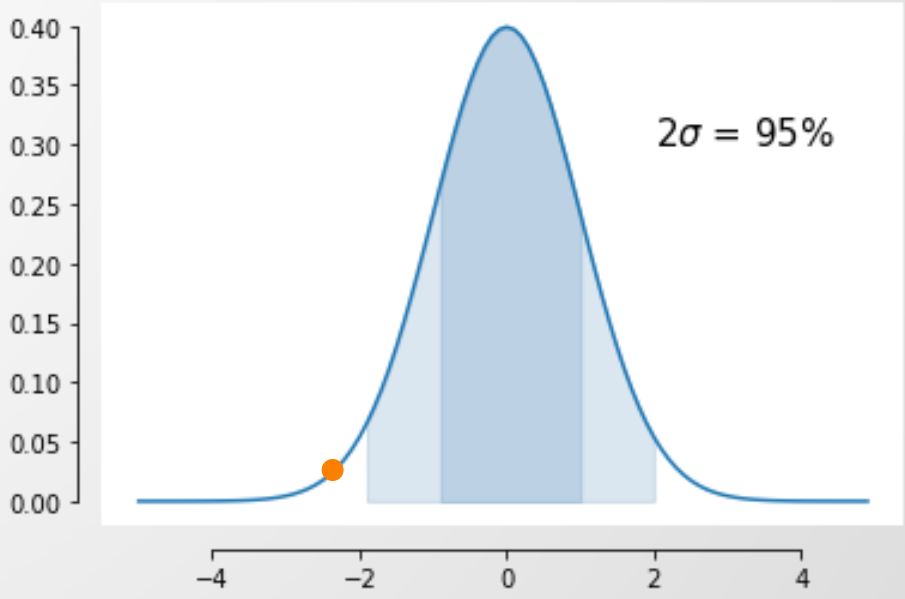
what is the probability that I would measure 1.5?

The probability of measuring any one value is mathematically *0*... however I can say that

the probability of measuring something between *-2σ* and *2σ* (within 2-sigma) is 95%.

So the probability of measuring something outside is 100-95 = 5%.

So if I measure something outside of [-2σ:2σ] that had a probability <5% of being measured.



Moments and frequentist probability

Imagine that I take a measurements of a quantity that is expected to be normally distributed with mean 0 and stdev 1

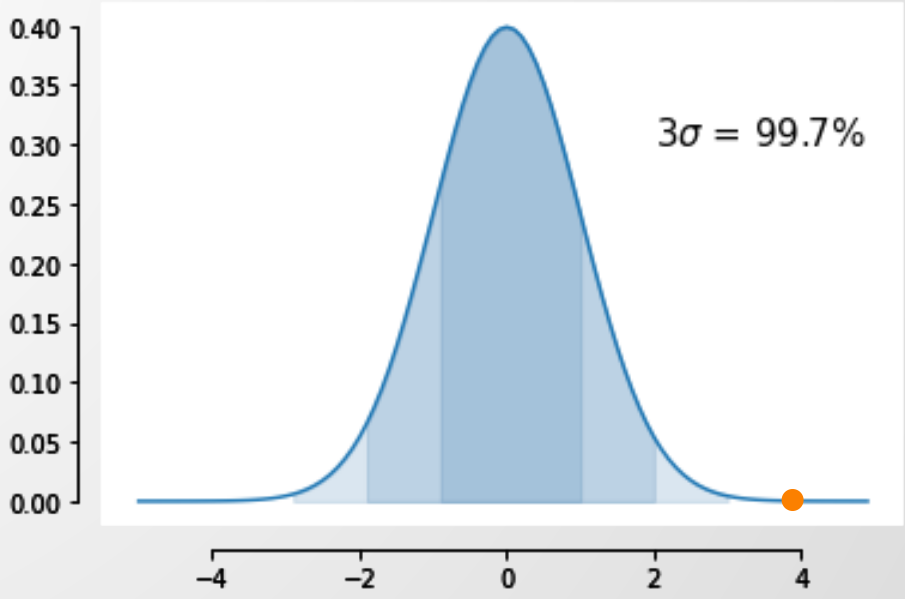
what is the probability that I would measure 1.5?

The probability of measuring any one value is mathematically *0*... however I can say that

the probability of measuring something between *-3σ* and *3σ* (within 3-sigma) is 99.7%.

So the probability of measuring something outside is 100-99.7 = 0.3%.

So if I measure something outside of [-3σ:3σ] that had a probability <0.3% of being measured.



Moments and frequentist probability

Imagine that I take a measurements of a quantity that is expected to be normally distributed with mean 0 and stdev 1

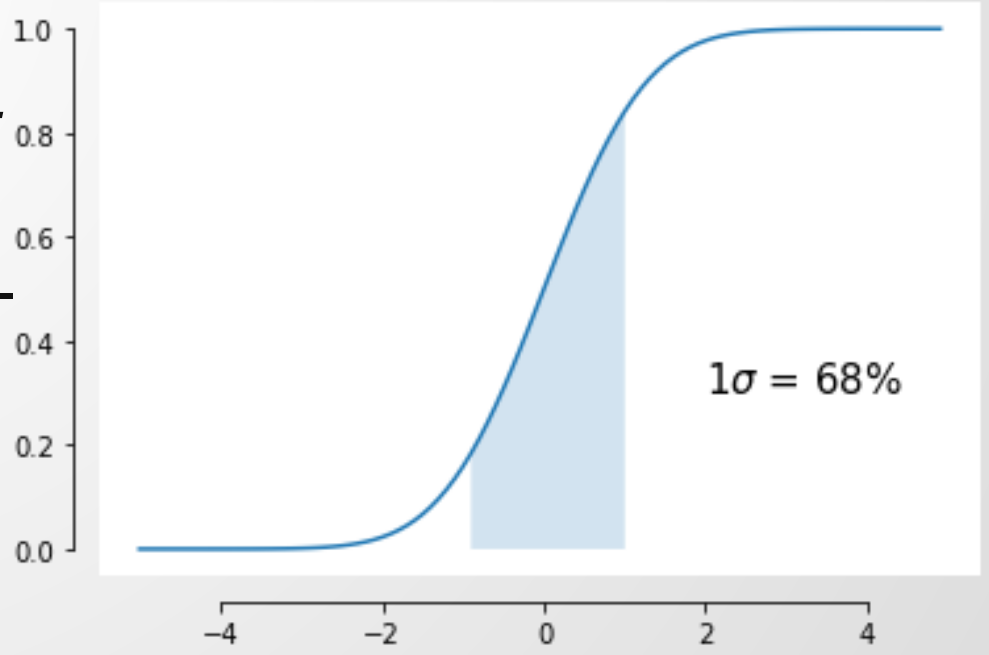
what is the probability that I would measure 1.5?

it might be easier to think about it as cumulative distributions if you are comfortable with integrals

the probability of measuring something between *-3σ* and *3σ* (within 3-sigma) is 99.7%.

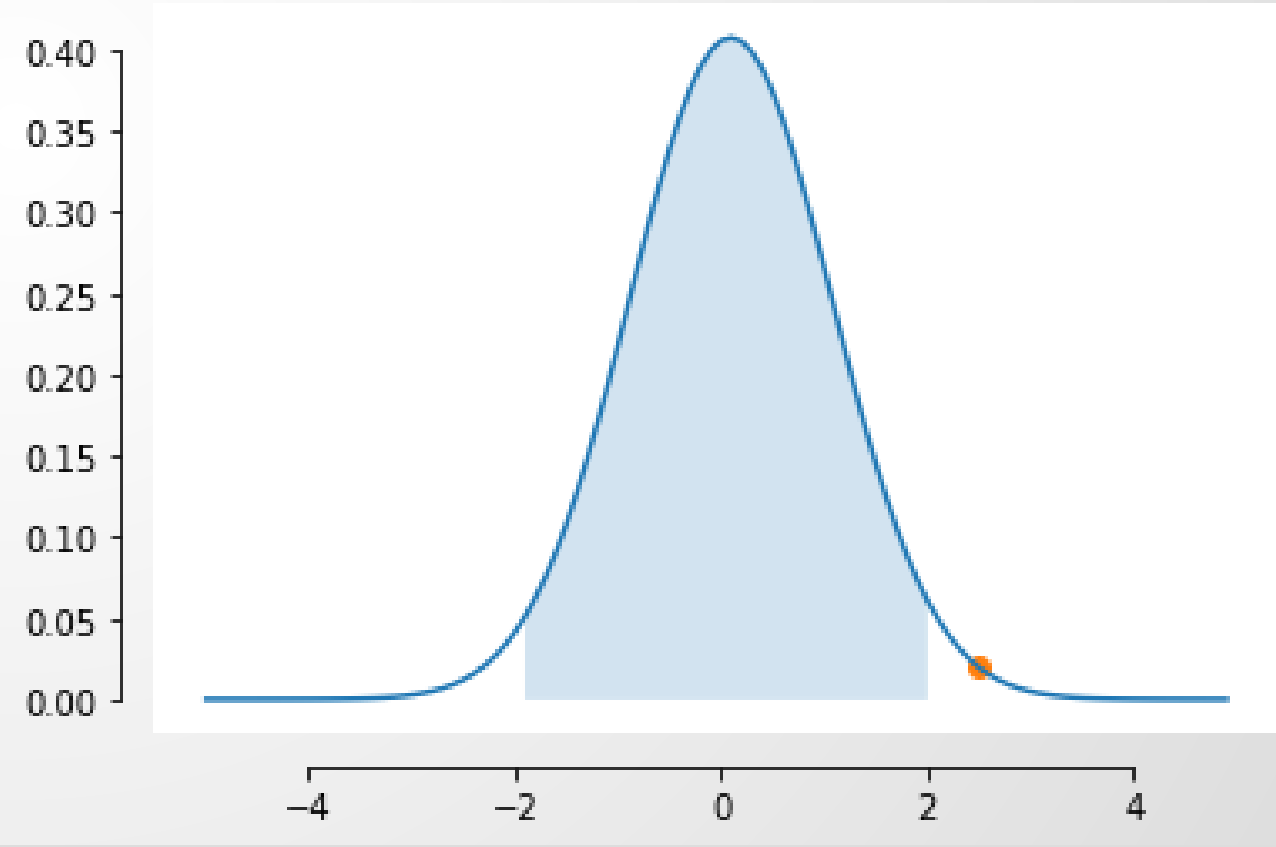
So the probability of measuring something outside is 100-99.7 = 0.3%.

So if I measure something outside of [-3σ:3σ] that had a probability <0.3% of being measured.



Moments and frequentist probability in the *falsification* framework: *p*-value

1. Set a threshold you believe corresponds to "reasonable doubt" 95% => α=0.05
2. Identify what you expect your measurement's distribution to be **if the Null hypothesis holds**
3. Measure your outcome from the data ***x***
4. If ***x*** is outside of the area of "reasonable doubt" under the Null hypothesis => the null hypothesis is rejected at ***p-value = α***, otherwise the Null cannot be rejected.



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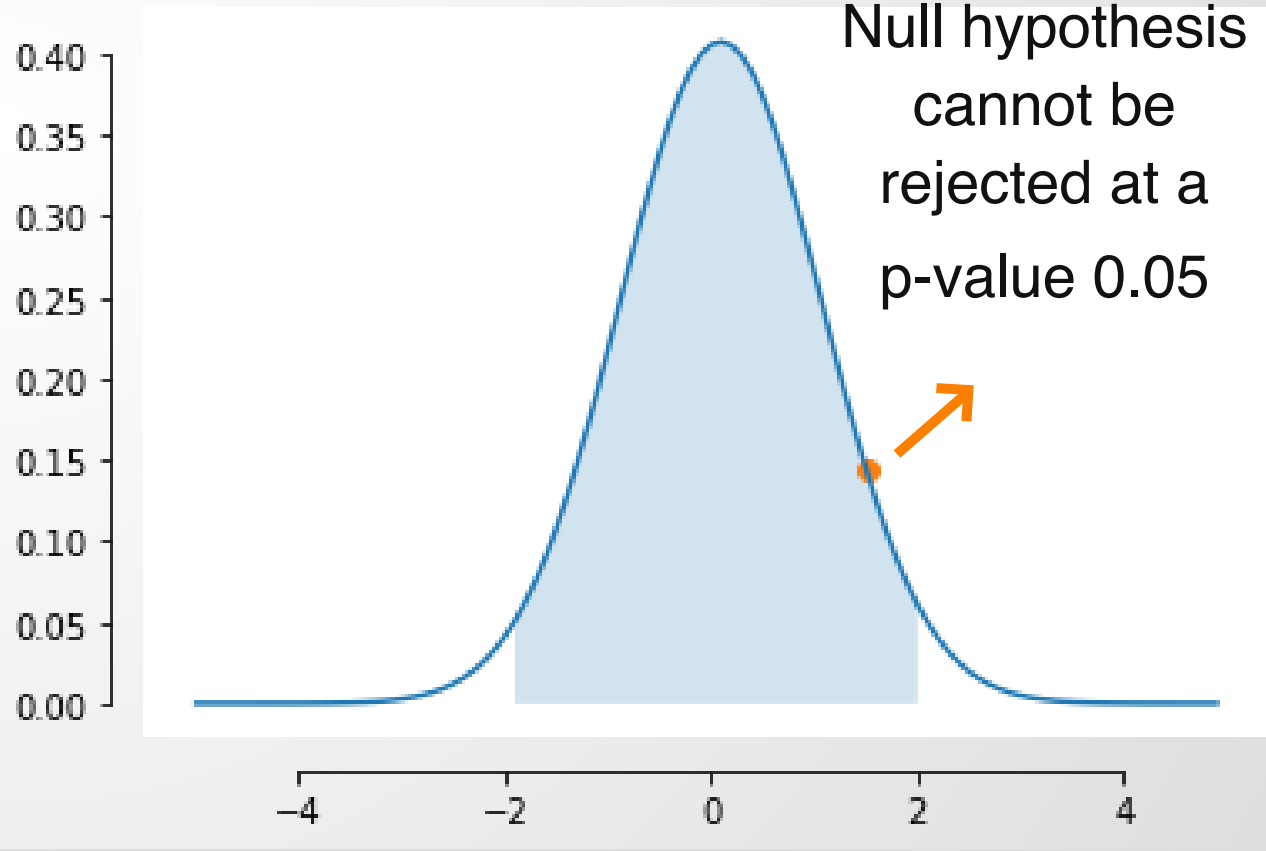
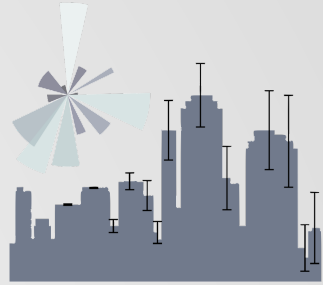


6

p-value hypothesis testing step by step

NHRT: *p*-value
Null Hypothesis Rejection Testing

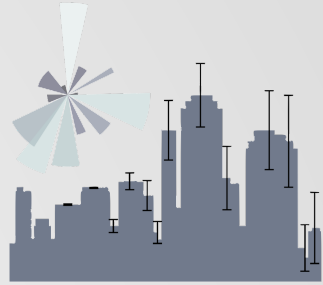


1. Set a threshold you believe corresponds to "reasonable doubt" 95% => α=0.05

its important to do this first. If we do not we may be tempted to choose a threshold that fits our result, thus always reporting rejection of null hypothesis

set up threshold α

NHRT: *p*-value
Null Hypothesis Rejection Testing



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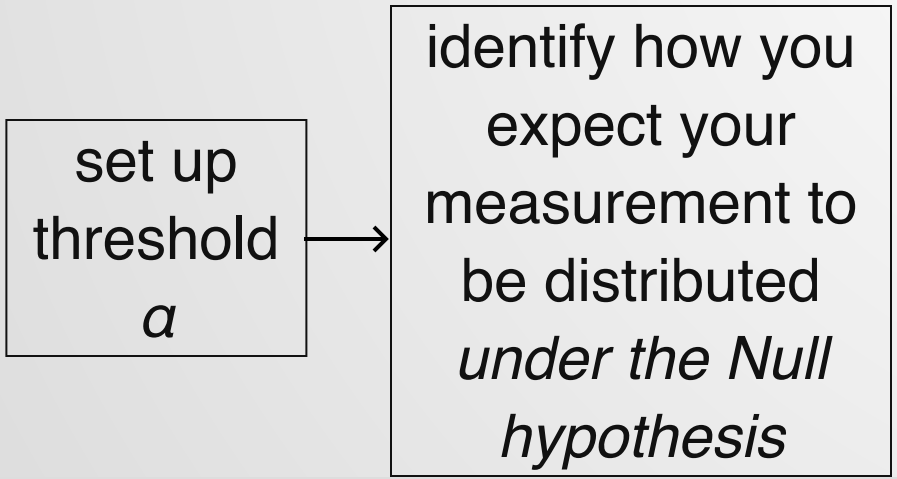
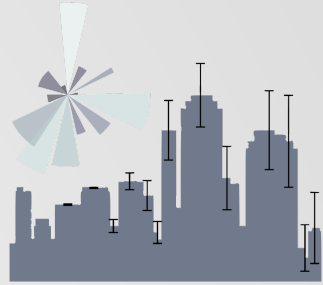


NHRT: *p*-value
Null Hypothesis Rejection Testing



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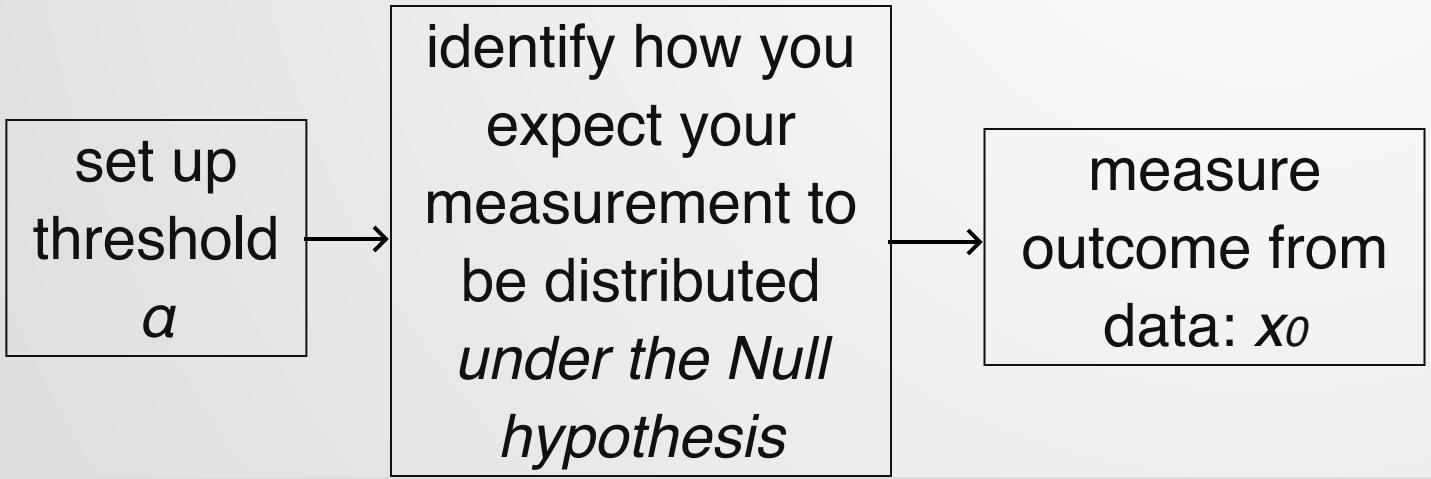
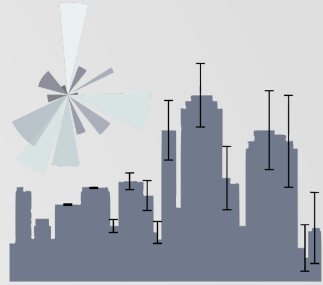


NHRT: *p*-value
Null Hypothesis Rejection Testing



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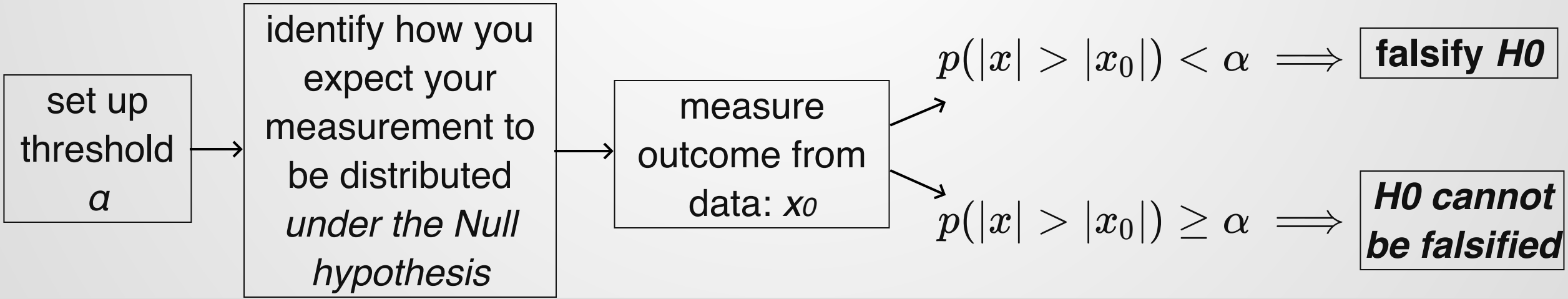
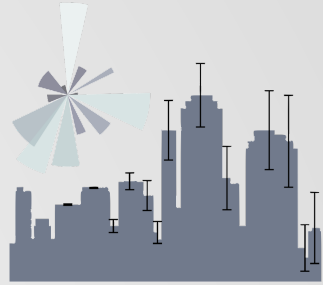


NHRT: *p*-value
Null Hypothesis Rejection Testing



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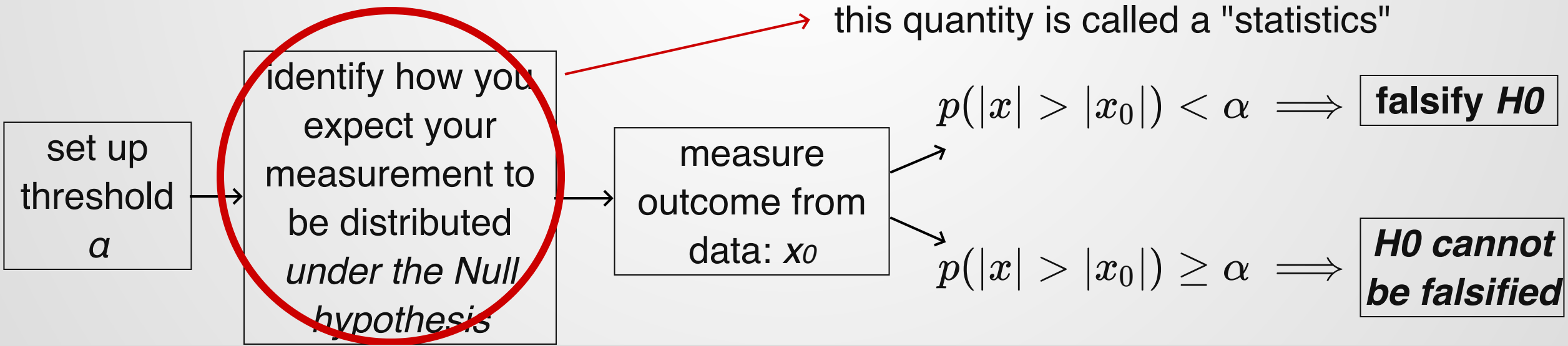
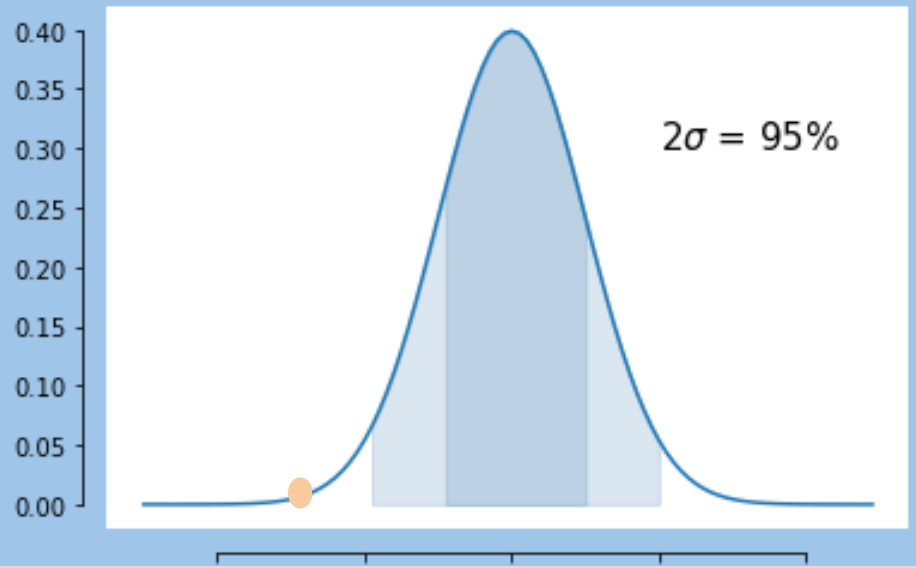


7

p-value hypothesis testing common tests

Statistical way to measure differences:

In NHRT a statistics is a quantity that relates to the data which has a known distribution under the Null Hypothesis



In NHRT a statistics is a quantity that relates to the data which has a known distribution under the Null Hypothesis

e.g.: Z statistics is Normally distributed

$Z \sim N(0,1)$

Statistics that follow a Standard Normal distribution

Z -test

Does a sample come from a known population?

$$Z = \frac{\mu - \bar{x}}{\sigma / \sqrt{N}}$$

In absence of effect (i.e. under the Null)

== the sample mean is the same as the population mean

Z is distributed according to a Gaussian N(μ=0, σ=1)

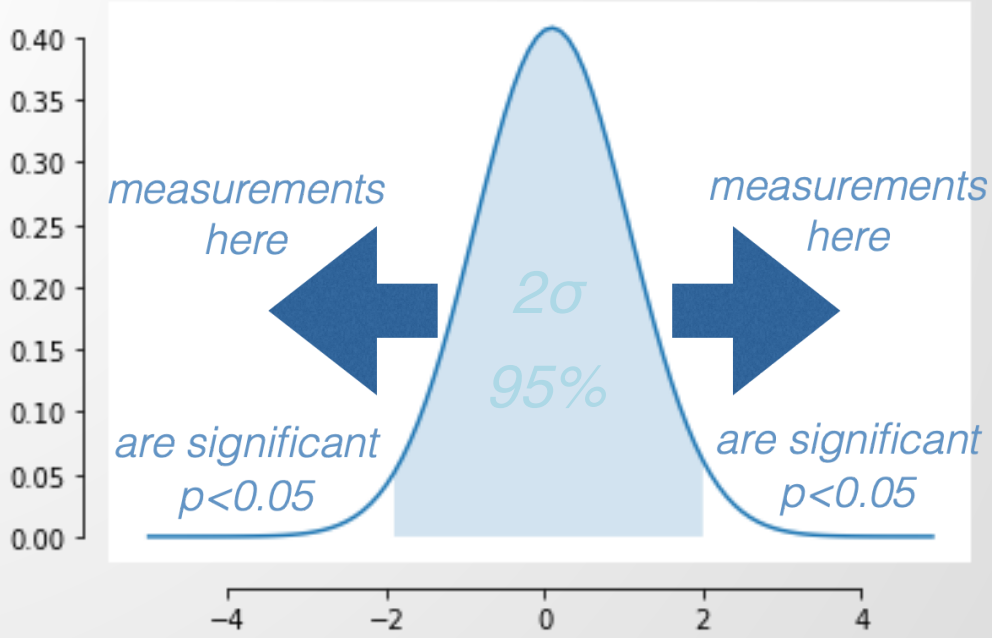
Example: new bus route implementation.

https://github.com/fedhere/PUS2022_FBianco/blob/master/classdemo/ZtestBustime.ipynb

You know the mean and standard deviation of a but travel route: that is the population

You measure the new travel time between two stops 10 times: that is your sample.

Has travel time changed?



Statistics that follow a Standard Normal distribution

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Does a sample come from a known population?

How to interpret the number you get?

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In absence of effect (i.e. under the Null)

== the sample mean is the same as the population mean

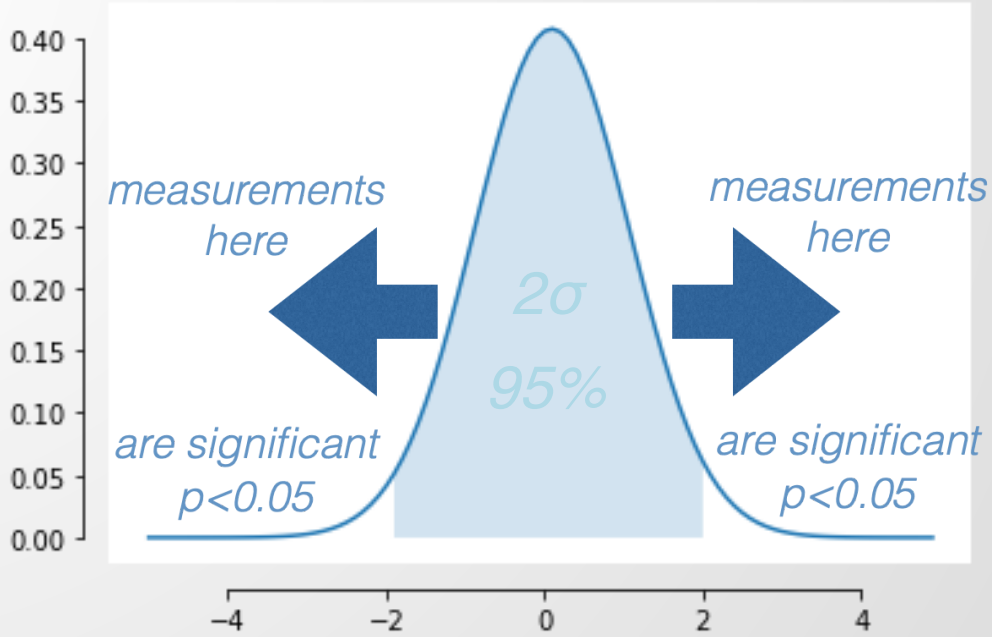
Z is distributed according to a Gaussian N(μ=0, σ=1)

The expectation is that Z will be distributed following a standard normal: a Gaussian with mean 0 std 1.

Values away from 0 are increasingly less probable.

68% probability to get a number b/w -1 and +1

95% probability to get a number b/w -2 and +2



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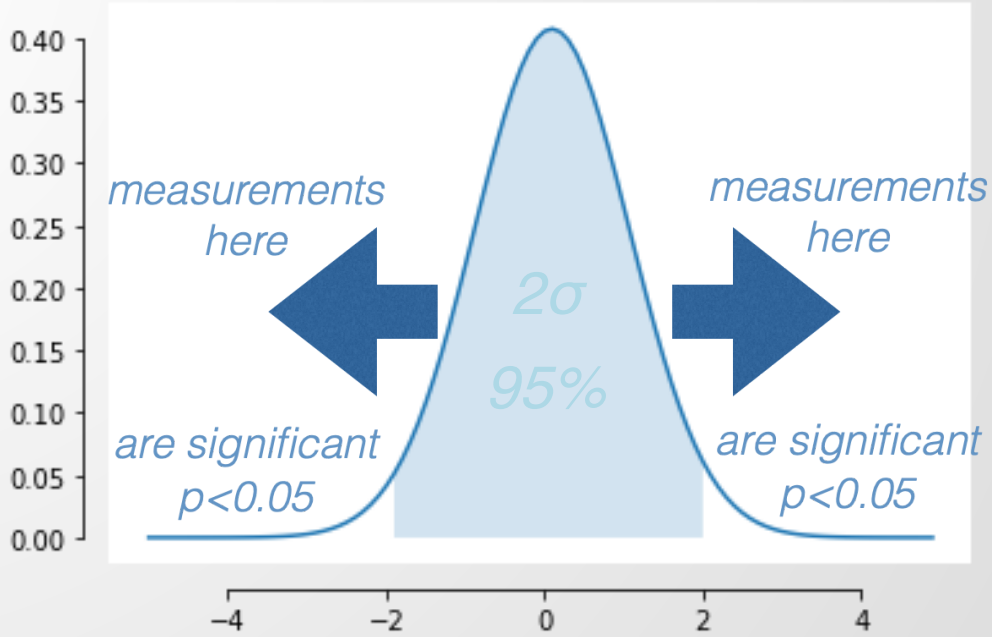
IF OUR p-value THRSHOLD IS 1-sigma that means the 68% region is between -1 and +1

=> We have less 68% probability of getting a number <-1 or > 1



Statistics that follow a Standard Normal distribution

Z -test

Are 2 proportions (fractions) the same?

$$Z = \frac{(p_0 - p_1)}{SE}$$

In absence of effect (i.e. under the Null)

== the proportions of men and women are the same

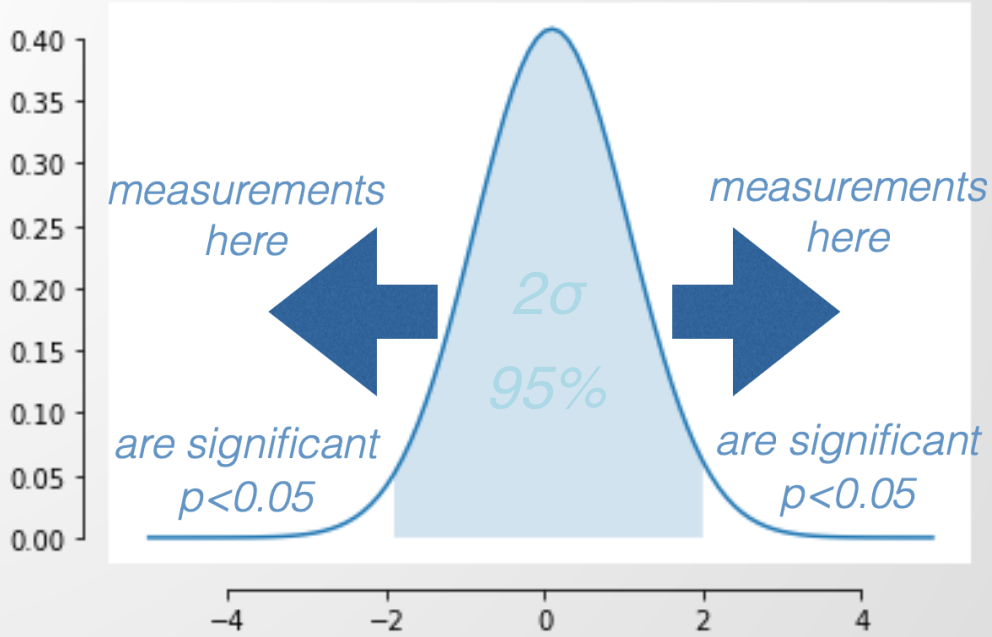
Z is distributed according to a Gaussian N(μ=0, σ=1)

Example: citibike women usage patterns

https://github.com/fedhere/PUS2020_FBianco/blob/master/classdemo/citibikes_gender.ipynb

You want to know if women are less likely than man to use citibike to commute.

You know the fraction of rides women (men) take during the week



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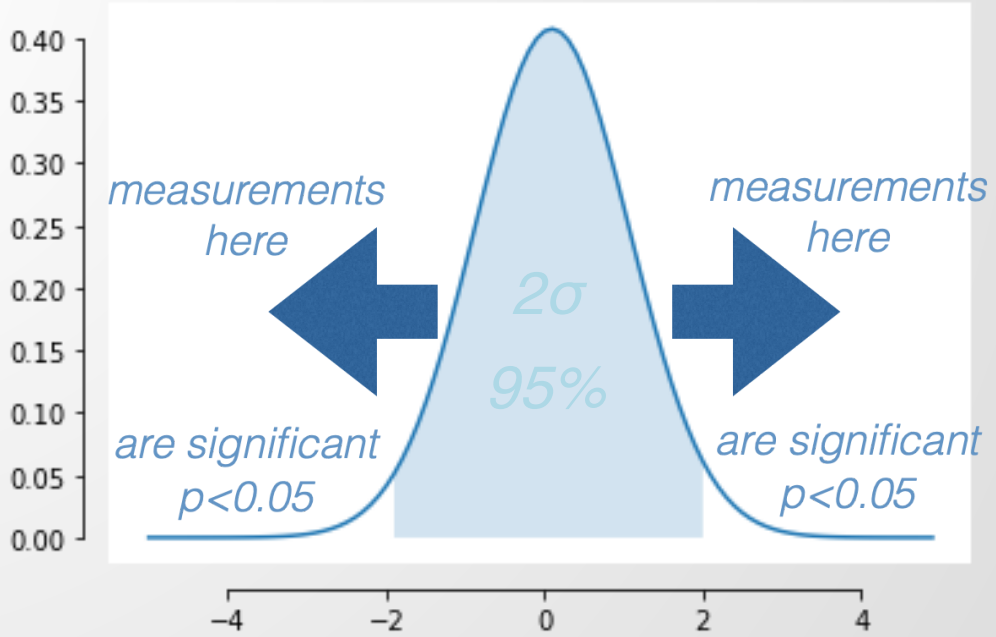
Z is distributed according to a Gaussian N(μ=0, σ=1)

$$p = \frac{p_0 n_0 + p_1 n_1}{n_0 + n_1}$$

$$SE = \sqrt{p(1-p)\left(\frac{1}{n_0} + \frac{1}{n_1}\right)}$$



Statistics that ***do not*** follow a Standard Normal distribution

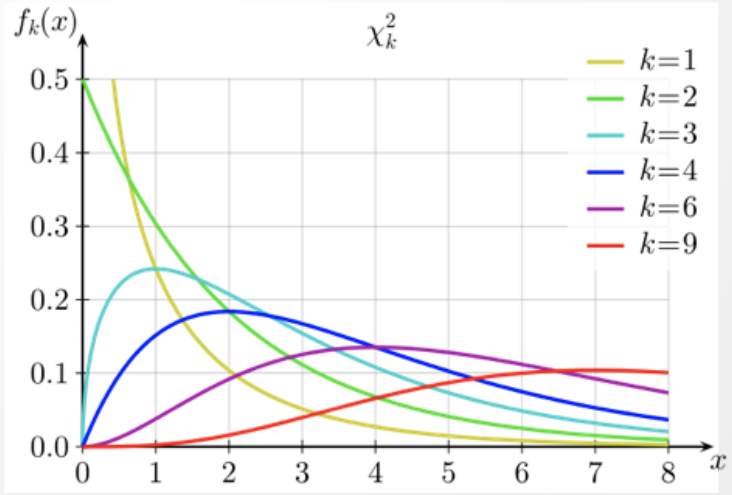
KS-test

Are 2 samples the same?

In absence of effect (i.e. under the Null)

== the samples are drawn from the same population

The KS test is chi-square distributed



Notation	$\chi^2(k)$ or χ^2_k
Parameters	$k \in \mathbb{N}_{>0}$ (known as "degrees of freedom")
Support	$x \in [0, +\infty)$
PDF	$\frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}$
CDF	$\frac{1}{\Gamma\left(\frac{k}{2}\right)} \gamma\left(\frac{k}{2}, \frac{x}{2}\right)$

Statistics that ***do not*** follow a Standard Normal distribution

KS-test

Are 2 samples the same?

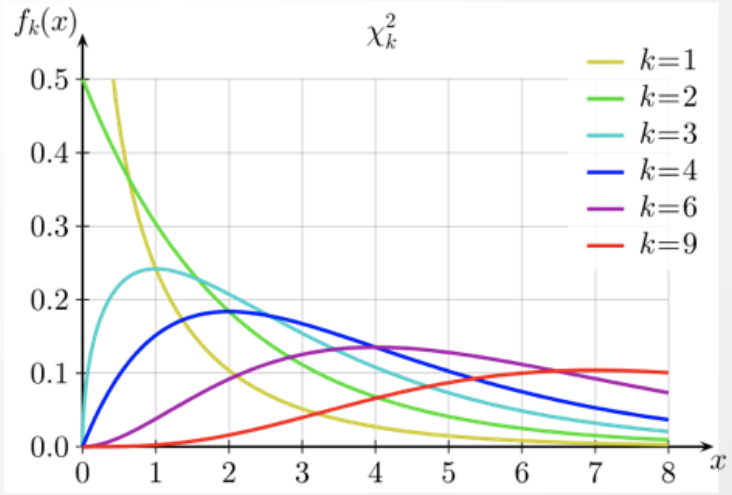
pivotal quantity

In absence of effect (i.e. under the Null)
== the samples are drawn from the same population
The KS test is chi-square distributed

$$d_{12} \equiv max_x \left| C_1(x) - C_2(x) \right|$$

Cumulative distribution 1

Cumulative distribution 2



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$$P(d > observed) = 2\sum_{j=1}^{\infty}(-1)^{j-1}e^{-2j^2x^2}\sqrt{\frac{N_1 N_2}{N_1+N_2}}d_{12}$$

Statistics that ***do not*** follow a Standard Normal distribution

KS-test

Are 2 samples the same?

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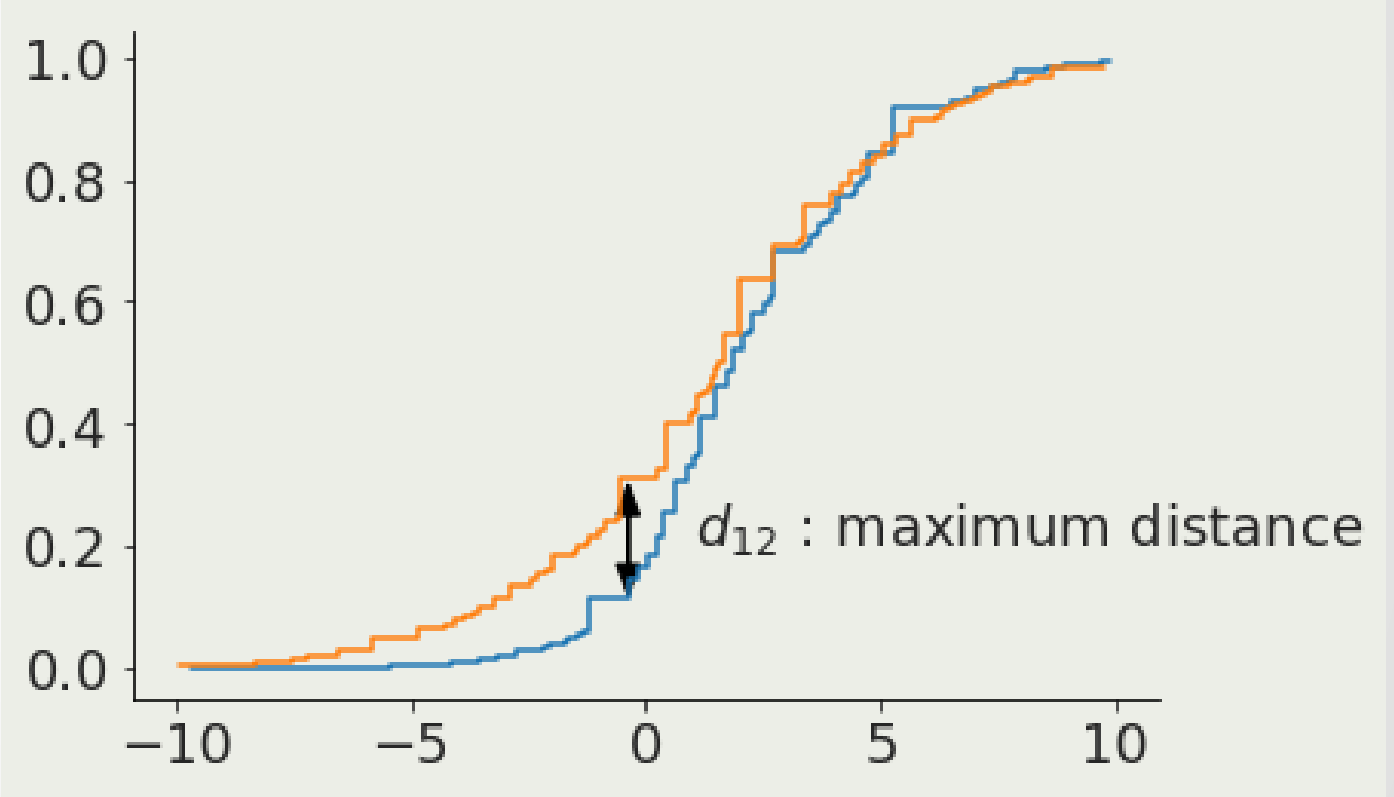
Cumulative distribution 1

Cumulative distribution 2



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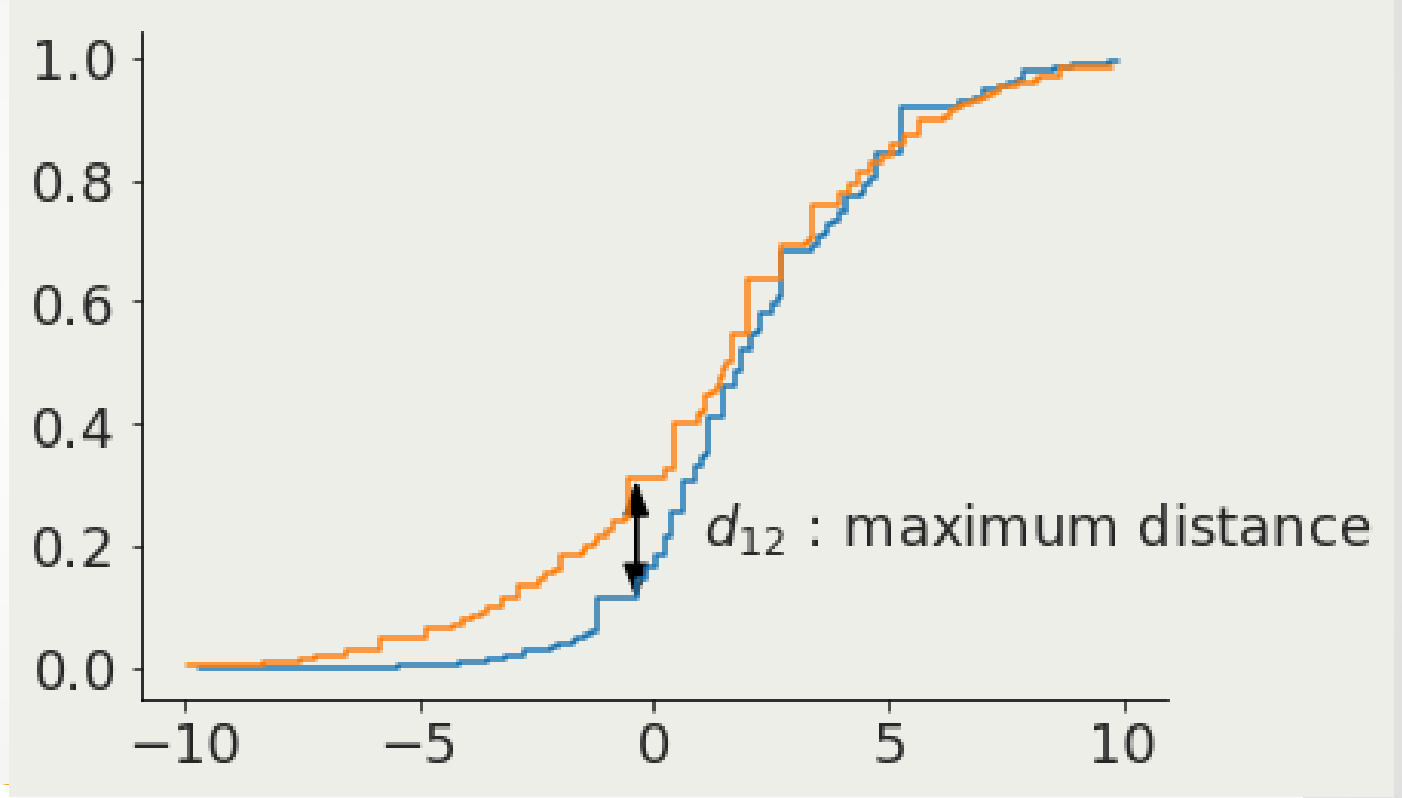

$$P(d > observed) =$$

```
sp.stats.ks_2samp(x, y)
```

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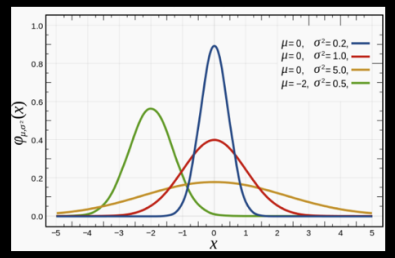
```
Ks_2sampResult(statistic=0.4, pvalue=0.3128526760169558)
```

Statistics and tests

Z statistics Gaussian

$$Z = \frac{\mu - \overline{x}}{\sigma/\sqrt{n}}$$

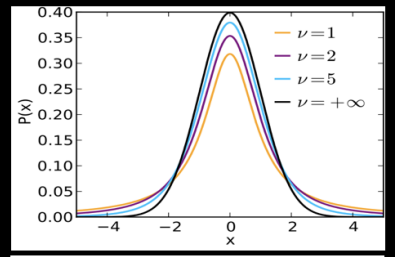


Notation	$\mathcal{N}(\mu, \sigma^2)$
Parameters	$\mu \in \mathbf{R}$ — mean (location) $\sigma^2 > 0$ — variance (squared scale)
Support	$x \in \mathbf{R}$
PDF	$\frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
CDF	$\frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$

Quantile	$\mu + \sigma\sqrt{2}\,\operatorname{erf}^{-1}(2F-1)$
Mean	μ
Median	μ
Mode	μ
Variance	σ^2

Student's t

$$t = \frac{|\bar{x}_a - \bar{x}_b|}{S_{AB}\sqrt{\frac{1}{N_A} + \frac{1}{N_B}}}$$

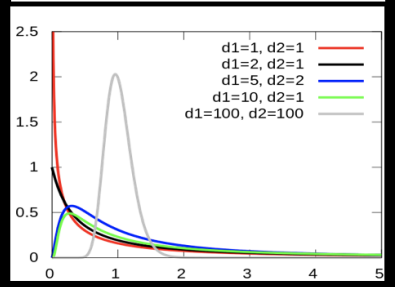


Parameters	$\nu > 0$ degrees of freedom (real)
Support	$x \in (-\infty; +\infty)$
PDF	$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$
CDF	$\frac{1}{2} + x\Gamma\left(\frac{\nu+1}{2}\right) \times \frac{{}_2F_1\left(\frac{1}{2}, \frac{\nu+1}{2}; \frac{3}{2}; -\frac{x^2}{\nu}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)}$ where ${}_2F_1$ is the hypergeometric function

Mean	0 for $\nu > 1$, otherwise undefined
Median	0
Mode	0
Variance	$\frac{\nu}{\nu-2}$ for $\nu > 2$, ∞ for $1 < \nu \le 2$, otherwise undefined

F statistics

$$F = \frac{\sum_i n_i(\overline{x}_i - \overline{x})^2/(K-1)}{\sum_{ij}(x_{ij} - \overline{x}_i)^2/(N-K)}$$

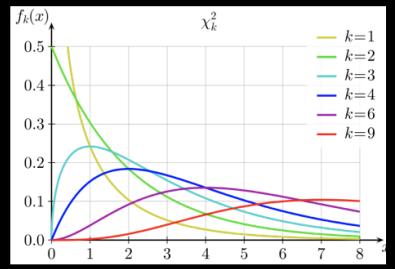


Parameters	$d_1, d_2 > 0$ deg. of freedom
Support	$x \in [0, +\infty)$
PDF	$\frac{\sqrt{\frac{(d_1 x)^{d_1}\, d_2^{d_2}}{(d_1 x + d_2)^{d_1+d_2}}}}{x\,\mathrm{B}\left(\frac{d_1}{2}, \frac{d_2}{2}\right)}$
CDF	$I_{\frac{d_1 x}{d_1 x + d_2}}\left(\frac{d_1}{2}, \frac{d_2}{2}\right)$

Mean	$\frac{d_2}{d_2 - 2}$ for $d_2 > 2$
Mode	$\frac{d_1 - 2}{d_1}\,\frac{d_2}{d_2 + 2}$ for $d_1 > 2$
Variance	$\frac{2\,d_2^2\,(d_1 + d_2 - 2)}{d_1(d_2-2)^2(d_2-4)}$ for $d_2 > 4$
Skewness	$\frac{(2d_1 + d_2 - 2)\sqrt{8(d_2-4)}}{(d_2-6)\sqrt{d_1(d_1+d_2-2)}}$ for $d_2 > 6$

Pearson's χ^2

$$\chi^2_P = \sum_i \frac{(O_i - E_i)^2}{E_i}$$



Notation	$\chi^2(k)$ or χ^2_k
Parameters	$k \in \mathbb{N}_{>0}$ (known as "degrees of freedom")
Support	$x \in [0, +\infty)$
PDF	$\frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}$
CDF	$\frac{1}{\Gamma\left(\frac{k}{2}\right)}\,\gamma\left(\frac{k}{2}, \frac{x}{2}\right)$

Mean	k
Median	$\approx k\left(1 - \frac{2}{9k}\right)^3$
Mode	$\max\{k-2, 0\}$
Variance	$2k$
Skewness	$\sqrt{8/k}$

goodness of fit χ^2 $\chi^2_F = \sum_i \frac{(m_i - x_i)^2}{e_i}$

see
Statistics in a Nutshell

5

data kinds and nomenclature

Types of Data:

Qualitative variables

No ordering

UrbanScience e.g. precinct, state, gender, Also called *Nominal, Categorical*

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Ordering is meaningful

Time, Distance, Age, Length, Intensity, Satisfaction, Number of

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discrete ● ● ● ● ● ● ● ● ● ● ● ● ● ● **continuous**

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Earthquakes (notlinear scale)

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F temperature interval size preserved

Ratio:

Car speed

0 is naturally defined

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Missing: “Prefer not to answer” (NA / NaN)

Censored: age>90

key concepts

descriptive statistics

null hypothesis rejection testing setup

pivotal quantities

Z, K-S tests

https://towardsdatascience.com/understanding-descriptive-statistics-c9c2b0641291

Understanding Descriptive Statistics

READING

Descriptive Statistics [Image 1] (Image courtesy: My Photoshopped Collection)

Statistics is a branch of mathematics that deals with collecting, interpreting, organization, and interpretation of data.