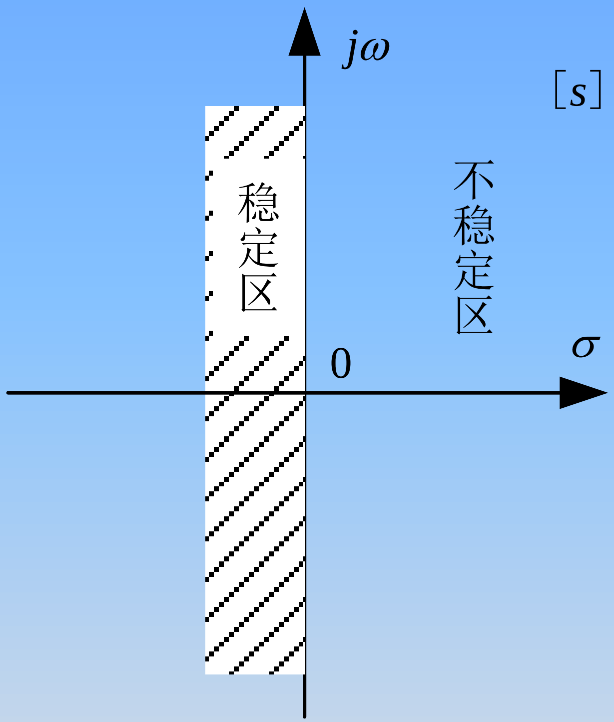
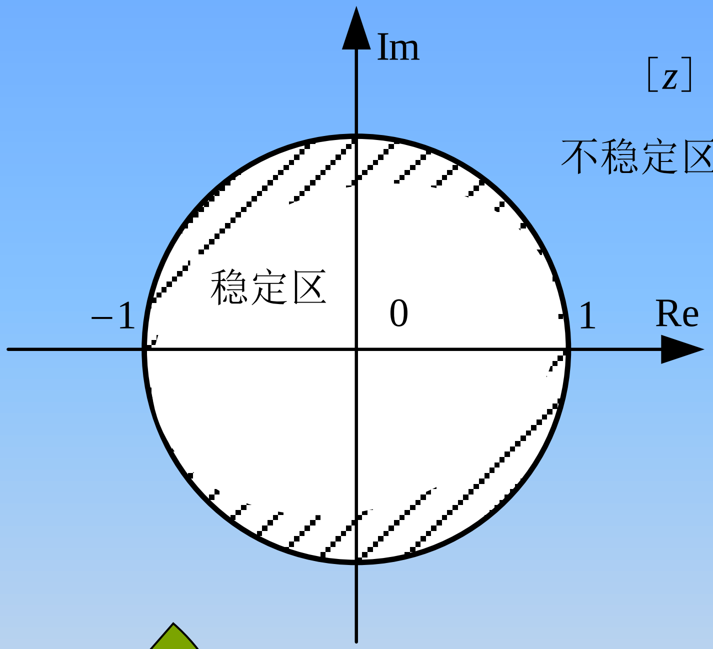
7.6 Performance Analysis of Discrete-Time Systems

- Stability
- Dynamic Performance
- Steady-state Errors

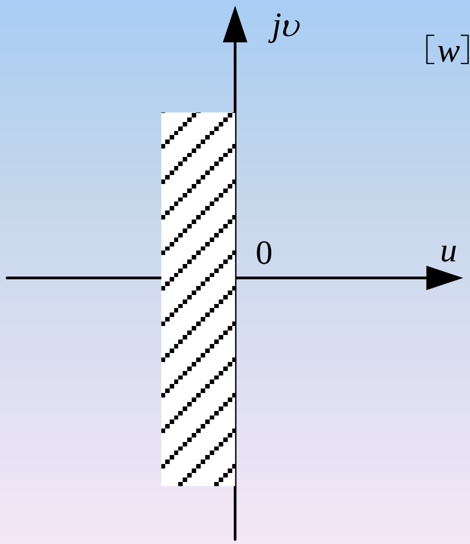
we’ve learned three methods to determine the stability of a discrete-time systems.



$$z = e^{sT}$$



$$w = \frac{z+1}{z-1}$$



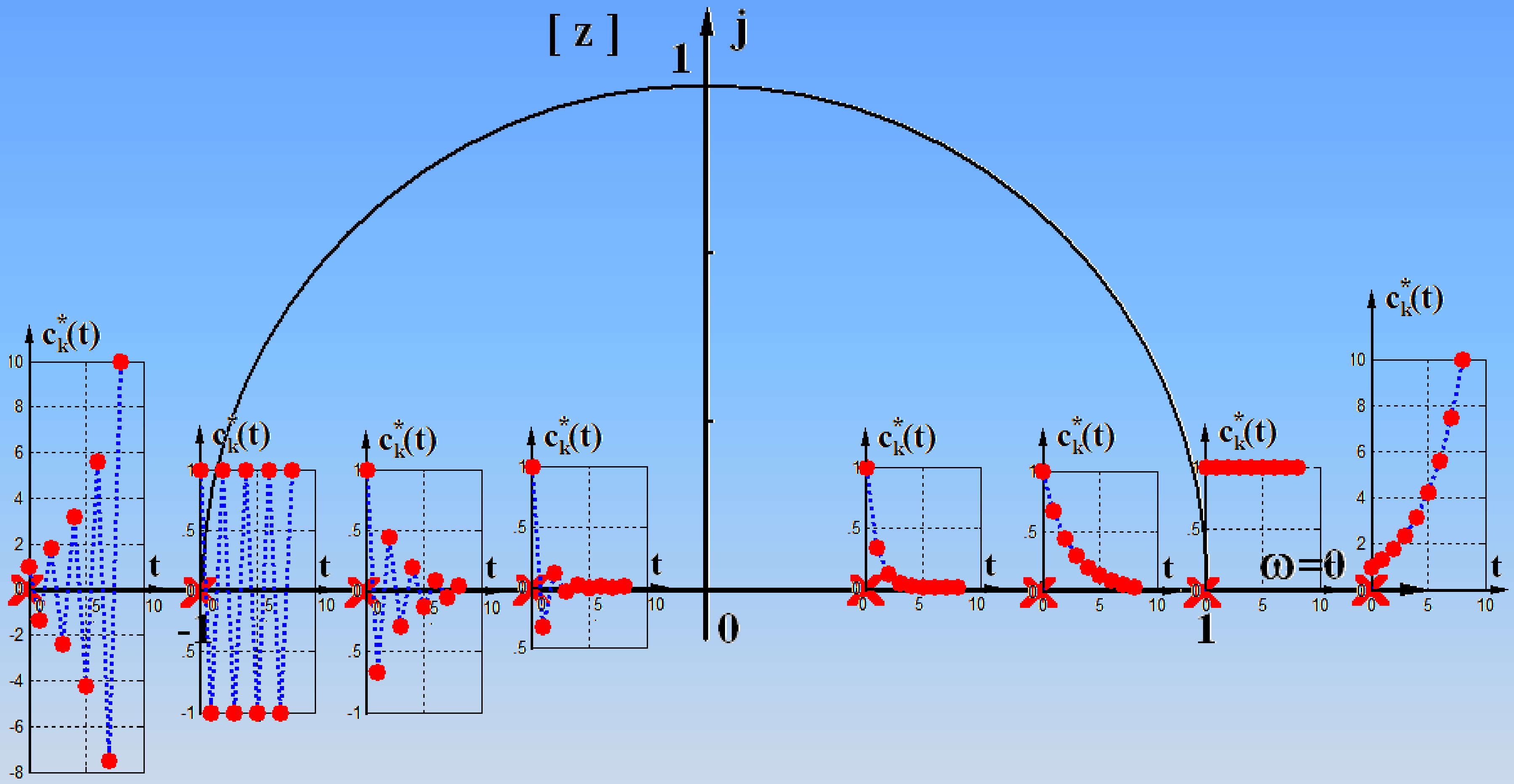
Summary

s-Domain to z-Domain Mapping

Necessary and Sufficient Condition for Stability of Linear Discrete-Time Systems

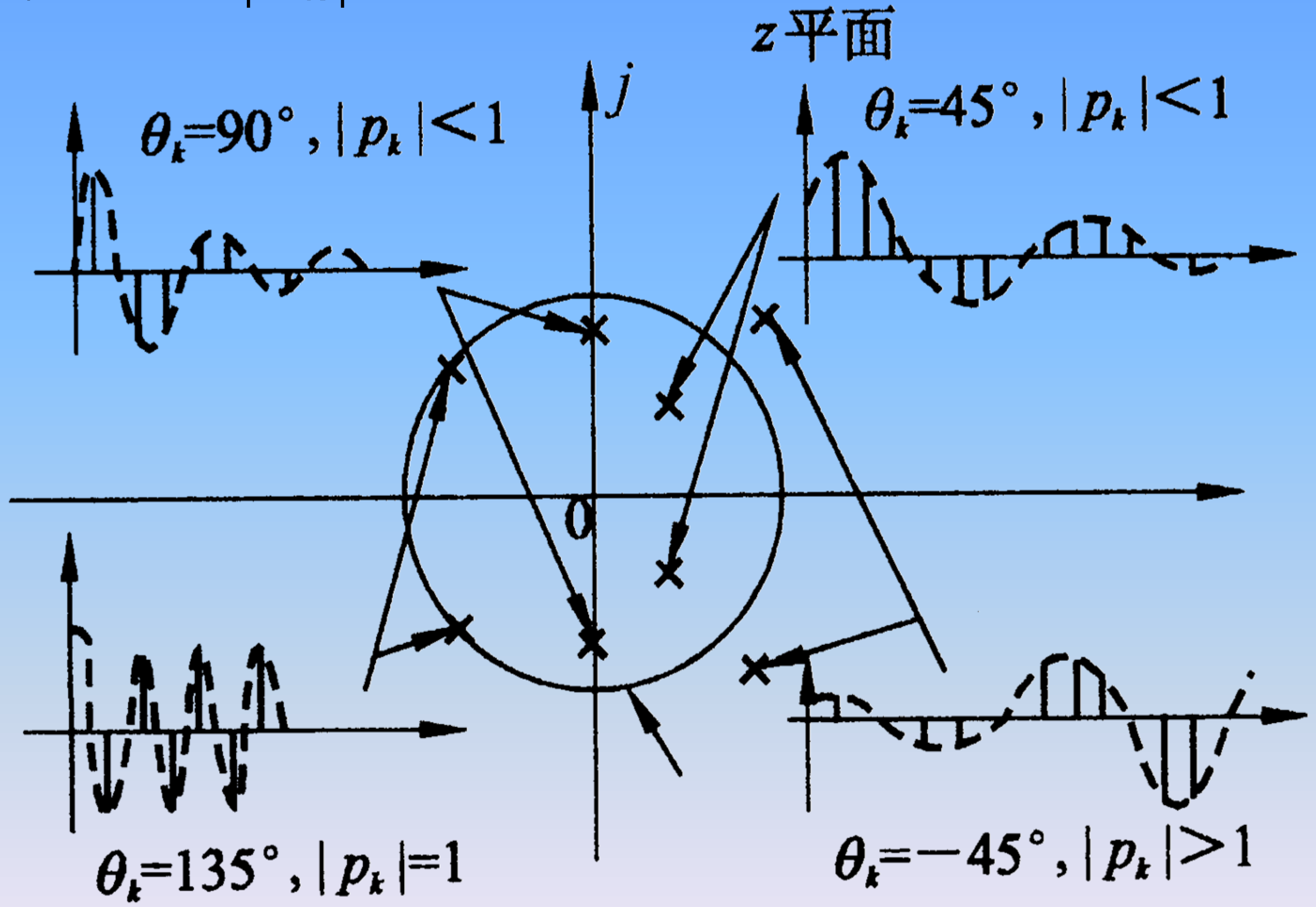
— All poles of $\Phi(z)$ lie in the unit circle of z plane

Routh criterion in w domain (Generalized Routh Criterion)



[z]
j
1
0
-1
1
ω=0
$c_k^*(t)$
t

$|p_k|<1, \quad |p_k|>1$



z平面
j
0
$\theta_k=90°, |p_k|<1$
$\theta_k=45°, |p_k|<1$
$\theta_k=135°, |p_k|=1$
$\theta_k=-45°, |p_k|>1$

7.6.3 Steady-state error of discrete systems

(1) General method: obtain system response

(2) Final value theorem $\begin{cases} G(z) \to \Phi_e(z) \\ D(z) \to \text{Stability} \\ e(\infty) = \lim\limits_{z\to 1}(z-1)R(z)\Phi_e(z) \end{cases}$

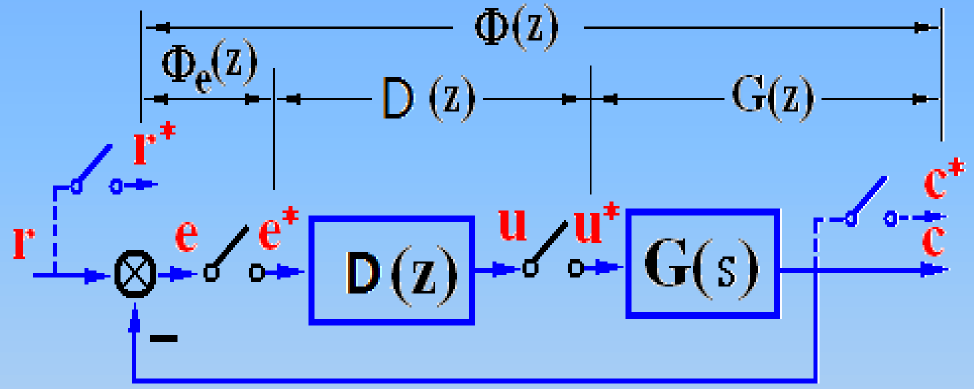
(3) Static error constant $\begin{cases} G(z) \to v,\ K_p,\ K_v,\ K_a \\ \text{Obtain } e(\infty) \end{cases}$

Chapter 7 Analysis and Design of Linear Discrete-Time System （Sampled-data System）

Design for discrete-time systems can be done in s-domain, z-domain and w-domain, respectively.

7.7.1 The Impulse Transfer Function for the Digital Controller

$$\Phi(z)=\frac{D(z)\cdot G(z)}{1+D(z)\cdot G(z)}$$



$$\Phi_e(z)=\frac{1}{1+G(z)\cdot D(z)}=1-\Phi(z)$$

$$D(z)\cdot G(z)=\frac{\Phi(z)}{1-\Phi(z)}=\frac{\Phi(z)}{\Phi_e(z)}$$

$$\boxed{D(z)=\frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}}$$

$$\Phi_e(z)=1-\Phi(z),\ \mathrm{E}(z)=\Phi_e(z)R(z)$$

7.7.2 Deadbeat Control Design 最少拍控制

Deadbeat Control Systems: Matching a particular test input within a number of steps. —— No steady-state error on the sampling point.

(典型输入作用下，能在有限拍内结束响应过程且在采样点上无稳态误差的系统。)

1. A unified description of typical test inputs

$$r(t)=\begin{cases}1(t)\\ t\\ t^2/2\end{cases}\qquad R(z)=\left\{\begin{array}{l}\dfrac{z}{z-1}=\dfrac{1}{1-z^{-1}}\\[2ex] \dfrac{Tz}{(z-1)^2}=\dfrac{Tz^{-1}}{(1-z^{-1})^2}\\[2ex] \dfrac{T^2z(z+1)}{2(z-1)^3}=\dfrac{T^2z^{-1}(1+z^{-1})}{2(1-z^{-1})^3}\end{array}\right\}\boxed{\dfrac{A(z)}{(1-z^{-1})^{\nu}}}$$

ν	$A(z)$
1	1
2	Tz^{-1}
3	$\dfrac{T^2z^{-1}(1+z^{-1})}{2}$

$$D(z)=\frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}$$

Design Idea: **Obtain D(z) by constructing $\Phi(z)$ so that the output can match the typical test signal within the minimum steps.**

IF there is No $\left\{\begin{array}{l}\text{Zeros}\\\text{Poles}\end{array}\right.$ of G(z) on or beyond the unit circle, except for (1, j0)

$$\mathrm{R}(z)=\frac{A(z)}{(1-z^{-1})^{\nu}}$$

$$\mathrm{E}(z)=\Phi_e(z)R(z),\quad \Phi_e(z)=1-\Phi(z)$$

$$E(z)=\Phi_e(z)\cdot R(z)=\frac{A(z)}{(1-z^{-1})^{\nu}}\Phi_e(z)$$

$$e(\infty T)=\lim_{z\to 1}(1-z^{-1})\frac{A(z)}{(1-z^{-1})^{\nu}}\Phi_e(z)=0 \quad \Rightarrow \Phi_e(z)=(1-z^{-1})^{\nu}F(z^{-1})$$

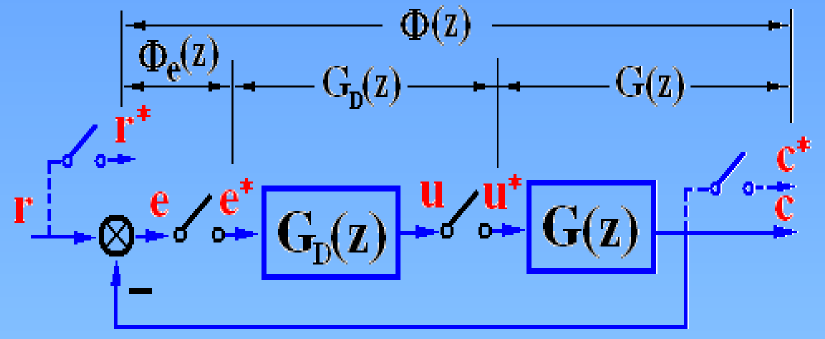
To make the D(z) simplest and of the lowest-order, we can choose $F(z^{-1})$ as 1.

$$\Phi(z)=1-\Phi_e(z)=1\text{-}(1\text{-}z^{-1})^{\nu}$$

From the design idea, we know that $e(\infty T)=0$

$$E(z)=\Phi_e(z)\cdot R(z)=\frac{A(z)}{(1-z^{-1})^{\nu}}\Phi_e(z)$$

$$e(\infty T)=\lim_{z\to 1}(1-z^{-1})\frac{A(z)}{(1-z^{-1})^{\nu}}\Phi_e(z)=0$$



$$\Phi_e(z)=(1-z^{-1})^{\nu}F(z)\overset{F(z)=1}{=}(1-z^{-1})^{\nu}$$

Hence:

$$\Phi(z)=1-\Phi_e(z)=1-(1-z^{-1})^{\nu}=b_1z^{-1}+b_2z^{-2}+\cdots+b_{\nu}z^{-\nu}$$

$$=\frac{b_1z^{\nu-1}+b_2z^{\nu-2}+\cdots+b_{\nu}}{z^{\nu}}$$

$$D(z)=\frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}$$

The rule to construct $\Phi(z)$: All poles of $\Phi(z)$ are located on the origin of z-plane.

2. Φ(z) for typical test inputs

$\Phi_e(z)=1-\Phi(z), \mathrm{E}(z)=\Phi_e(z)R(z)$

（1） for $r(t)=1(t)$

- The C.L. impulse transfer function:

$$\nu=1 \qquad \Phi(z) = z^{-1}$$

$$E(z)=1$$

The system can track the input by 1 step only。

（2） for $r(t) = t \cdot 1(t)$

$\Phi_e(z) = 1 - \Phi(z), \mathrm{E}(z) = \Phi_e(z)R(z)$

$$\mathrm{R}(z) = \frac{Tz}{(z-1)^2} = \frac{Tz^{-1}}{(1-z^{-1})^2}$$

- The C.L. impulse transfer function:

$v = 2$ $\quad$ $\Phi(z) = 2z^{-1} - z^{-2}$

$$E(z) = Tz^{-1}$$

The system can track the input by 2 step。

（3） for $r(t)=\frac{1}{2}t^2\cdot 1(t)$

- **The C.L.impulse transfer function:**

$$\nu=3 \qquad \Phi(z)=3z^{-1}-3z^{-2}+z^{-3}$$

$$E(z)=\frac{1}{2}T^2z^{-1}+\frac{1}{2}T^2z^{-2}$$

The system can track the input by 3 step。

$$D(z)=\frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}$$

Deadbeat Control Design Table

$r(t)$	$R(z)$	$\Phi_e(z)=(1-z^{-1})^v$	$\Phi(z)=1-\Phi_e(z)$	$D(z)$	t_s
$1(t)$	$\frac{1}{1-z^{-1}}$	$1-z^{-1}$	z^{-1}	$\frac{z^{-1}}{(1-z^{-1})\cdot G(z)}$	T
t	$\frac{Tz^{-1}}{(1-z^{-1})^2}$	$(1-z^{-1})^2$	$2z^{-1}-z^{-2}$	$\frac{z^{-1}(2-z^{-1})}{(1-z^{-1})^2G(z)}$	$2T$
$\frac{t^2}{2}$	$\frac{T^2z^{-1}(1+z^{-1})}{2(1-z^{-1})^3}$	$(1-z^{-1})^3$	$3z^{-1}-3z^{-2}+z^{-3}$	$\frac{z^{-1}(3-3z^{-1}+z^{-2})}{(1-z^{-1})^3G(z)}$	$3T$

3. Algorithm for Deadbeat Control Design

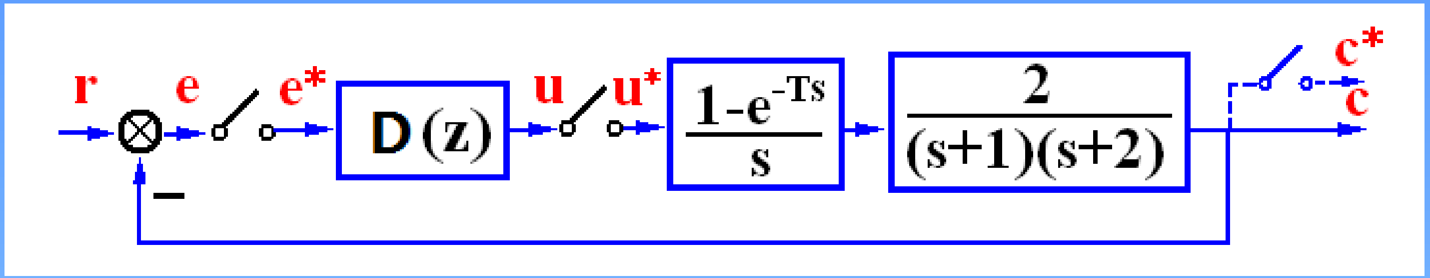
① **Obtain G(z) — Suppose there are no poles and zeros of G(z) on or beyond the unit circle.**

② **Determine $\Phi_e(z)$ for the particular test input**

$$\boldsymbol{r(t)} \quad \Rightarrow \quad R(z) = \frac{A(z)}{(1-z^{-1})^{\nu}} \quad \Rightarrow \quad \Phi_e(z) = (1-z^{-1})^{\nu}$$

③ **Obtain** $\boldsymbol{\Phi(z) = 1 - \Phi_e(z)}$

④ **Achieve** $\boldsymbol{D(z) = \dfrac{\Phi(z)}{\Phi_e(z) \cdot G(z)}}$



Example 1. Consider the system shown in the above figure (T=1). Design deadbeat controllers D(z) for r(t)=1(t)、t .

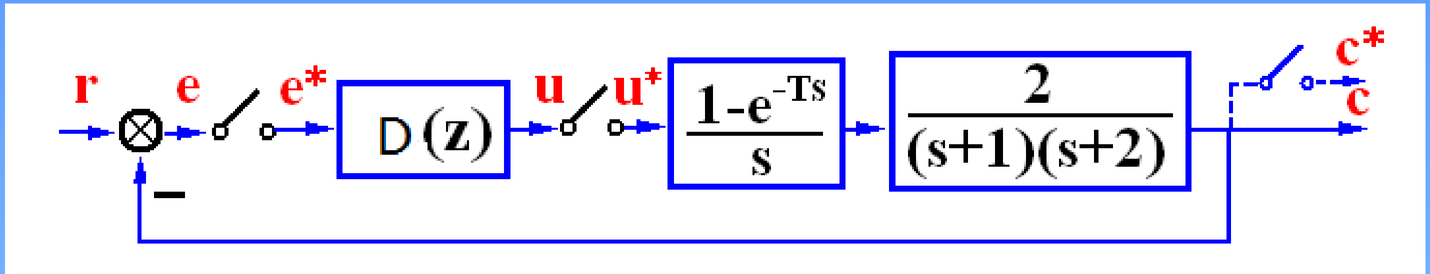
Solution.

$$G(z) = Z\left[\frac{1-e^{-Ts}}{s}\cdot\frac{2}{(s+1)(s+2)}\right] = 2(1-z^{-1})\cdot Z\left[\frac{C_0}{s}-\frac{C_1}{s+1}+\frac{C_2}{s+2}\right]$$

$$= 2\cdot\frac{z-1}{z}\cdot Z\left[\frac{1}{2}\cdot\frac{1}{s}-\frac{1}{s+1}+\frac{1}{2}\cdot\frac{1}{s+2}\right]$$

$$= \frac{z-1}{z}\left[\frac{z}{z-1}-\frac{2z}{z-e^{-T}}+\frac{z}{z-e^{-2T}}\right] = 1-\frac{2(z-1)}{z-e^{-T}}+\frac{z-1}{z-e^{-2T}}$$

$$= \frac{(1+e^{-2T}-2e^{-T})z+(e^{-3T}+e^{-T}-2e^{-2T})}{(z-e^{-T})(z-e^{-2T})} \overset{T=1}{=} \frac{0.4(z+0.365)}{(z-0.368)(z-0.136)}$$



Referring to the result for $r(t)=1(t)$ in the Design Table

$$\boxed{R(z)=\frac{z}{z-1}}$$

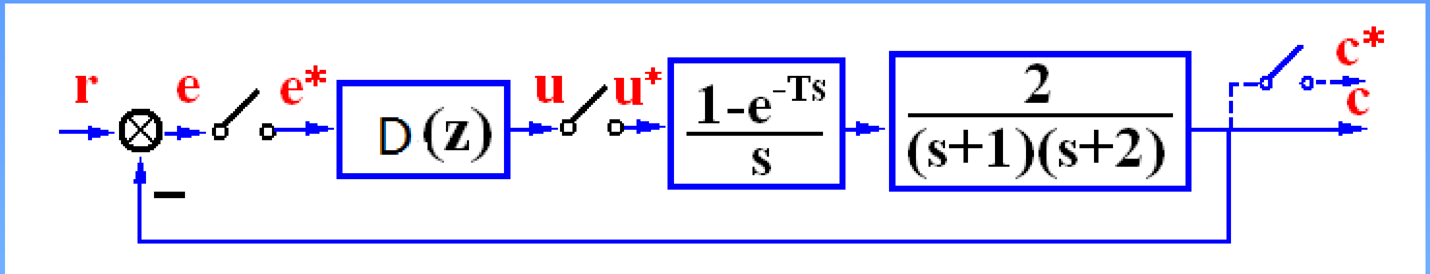
Choose $\begin{cases}\Phi_e(z)=1-z^{-1}\\ \Phi(z)=1-\Phi_e(z)=z^{-1}\end{cases}$

$$D(z)=\frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}=\frac{z^{-1}}{1-z^{-1}}\cdot\frac{(z-0.368)(z-0.136)}{0.4(z+0.365)}$$

$$=\frac{2.5(z-0.368)(z-0.136)}{(z-1)(z+0.365)}$$

$$C(z)=\Phi(z)R(z)=z^{-1}\cdot\frac{1}{1-z^{-1}}=z^{-1}[1+z^{-1}+z^{-2}+\cdots]=z^{-1}+z^{-2}+z^{-3}+\cdots$$

$$E(z)=\Phi_e(z)R(z)=(1-z^{-1})\cdot\frac{1}{1-z^{-1}}=1$$



For $r(t)=t$

$$\boxed{R(z)=\frac{Tz^{-1}}{(1-z^{-1})^2}}$$

Choose $\begin{cases}\Phi_e(z)=(1-z^{-1})^2 \\ \Phi(z)=1-\Phi_e(z)=2z^{-1}-z^{-2}\end{cases}$

$$D(z)=\frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}=\frac{2z^{-1}-z^{-2}}{(1-z^{-1})^2}\cdot\frac{(z-0.368)(z-0.136)}{0.4(z+0.365)}$$

$$=\frac{5(z-0.5)(z-0.368)(z-0.136)}{(z-1)^2(z+0.365)}$$

$$E(z)=\Phi_e(z)\cdot R(z)=Tz^{-1}$$

$$C(z)=\Phi(z)R(z)=(2z^{-1}-z^{-2})\cdot\frac{Tz^{-1}}{(1-z^{-1})^2}$$

$$=R(z)-E(z)=2Tz^{-2}+3Tz^{-3}+4Tz^{-4}+\cdots$$

4. G(z) has poles or zeros on or beyond the unit circle

suppose

$$G(z)=\frac{z^{-v}\prod_{i=1}^{L}(1-z_i z^{-1})}{\prod_{i=1}^{n}(1-p_i z^{-1})}$$

where z_i is the zero of $G(z)$;

p_i is the pole of G（z）.

(1)D(z) is stable realizable;

(2)$\Phi(z)$ is stable and realizable.

Then

$$D(z)=\frac{\Phi(z)}{\Phi_e(z)G(z)}=\frac{z^{v}\prod_{i=1}^{n}(1-p_i z^{-1})\Phi(z)}{\prod_{i=1}^{L}(1-z_i z^{-1})\Phi(z)}$$

$$D(z)=\frac{\Phi(z)}{\Phi_e(z)G(z)}=\frac{z^{\nu}\prod_{i=1}^{n}(1-p_i z^{-1})\Phi(z)}{\prod_{i=1}^{L}(1-z_i z^{-1})\Phi_e(z)}$$

(1) $D(z)$ is stable realizable;
(2) $\Phi(z)$ is stable and realizable.

①If there is z^{ν} in $D(z)$, $D(z)$ is un-realizable.

Thus, we have to ensure that there exists $z^{-\nu}$ in $\Phi(z)$,

which promises D(z) is realizable.

② If there is z_i on or beyond the unit circle, D(z) is unstable.

Then, those z_i will be designed as the zeros of $\Phi(z)$.

(1)$D(z)$ is stable realizable;
(2)$\Phi(z)$ is stable and realizable.

③ Note that

$$\Phi(z) = D(z)G(z)\Phi_e(z)$$

If there are p_i on or beyond the unit circle,

$\Phi(z)$ will be unstable,

***Then* those p_i will be designed as the zeros of $\Phi_e(z)$.**

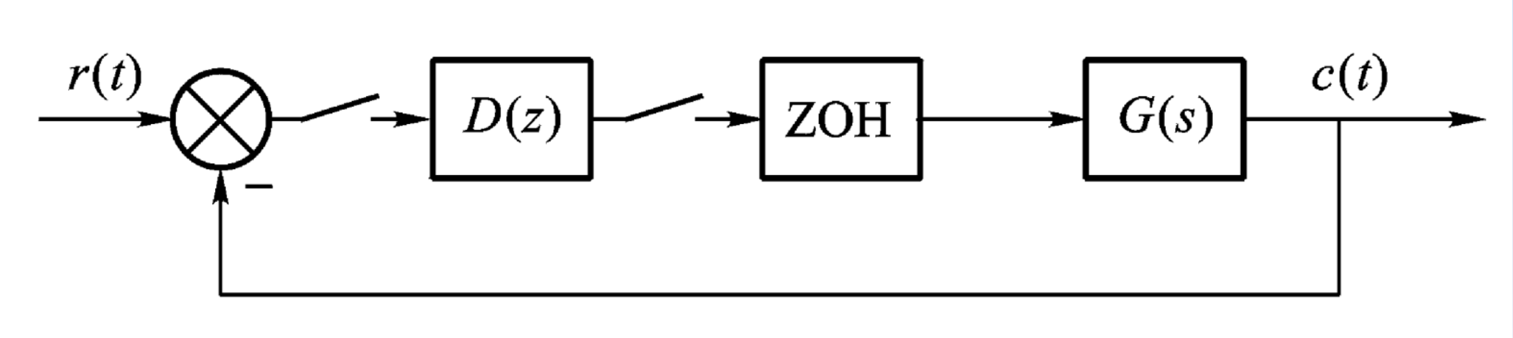
Example Given the discrete system described as in the following figure, where

$$G_0(s)=\frac{10}{s(0.1s+1)(0.05s+1)},\quad G_h(s)=\frac{1-e^{-Ts}}{s}$$

with

$$\mathbf{T = 0.2}s$$

Design a deadbeat controller for $r(t)=1(t)$



Solution: the O. L. impulse transfer function is

$$G(z)=Z[G_h(z)G_0(z)]=\frac{0.76z^{-1}(1+0.05z^{-1})(1+1.065z^{-1})}{(1-z^{-1})(1-0.135z^{-1})(1-0.0185z^{-1})}$$

For $r(t)=1(t)$, we can design

$$\varphi_e(z)=1-z^{-1} \qquad (1)$$

$$\varphi(z)=z^{-1} \qquad (2)$$

Because there exists $z=-1.065$ (beyond the unit circle),

Thus, z should also be the zero of $\Phi(z)$

There exist z^{-1} *in G(z)* , z^{-1} should be in $\Phi(z)$, thus

$$\varphi(z)=z^{-1}(1+1.065z^{-1}) \qquad (3)$$

Because that

$$\varphi(z)=1-\varphi_e(z) \qquad (4)$$

from (3), $\varphi(z)$ is now a polynomial on z^{-1} of order 2,

To satisfy（4）， $\varphi_e(z)$ must be a polynomial on z^{-1} of order 2， thus based on (1), we redesign：

$$\varphi_e(z)=(1-z^{-1})\ (1+a_1z^{-1}) \qquad (5)$$

Where a_1 is a constant to be chosen later。

Thus multiplied by a constant b_1 to be designed later, we get

$$\varphi(z)=b_1z^{-1}(1+1.065z^{-1}) \qquad (6)$$

From（4-6），we get：

$$a_1 = 0.516 \qquad b_1 = 0.484$$

Thus,

$$\varphi_e(z) = (1 - z^{-1})\,(1 + 0.516z^{-1}) \qquad (7)$$

$$\varphi(z) = 0.484z^{-1}(1 + 1.065z^{-1}) \qquad (8)$$

Then the deadbeat controller is

$$D(z) = \frac{1 - \varphi_e(z)}{G(z)\varphi_e(z)}$$

$$= \frac{1 - (1 - z^{-1})\,(1 + 0.516z^{-1})}{\dfrac{0.76z^{-1}(1 + 0.05z^{-1})\,(1 + 0.065z^{-1})}{(1 - z^{-1})\,(1 - 0.135z^{-1})\,(1 - 0.0185z^{-1})}(1 - z^{-1})\,(1 + 0.516z^{-1})}$$

$$D(z) = \frac{0.637(1-0.0185z^{-1})\,(1-0.135z^{-1})}{(1+0.05z^{-1})\,(1+0.516z^{-1})}$$

Then the Z-transform is

$$C(z) = \varphi(z)R(z) = 0.484z^{-1}(1+1.085z^{-1})\frac{1}{1-z^{-1}}$$
$$= 0.484z^{-1} + z^{-2} + z^{-3} + \cdots + z^{-4} + \cdots$$

System can follow the input at the 2nd step.

Although the deadbeat control system tracks a particular test input accurately within a number of steps, it has the following disadvantages:

(1) It is designed only for a particular input.

(2) The output has ripples （纹波） although there are no errors on the sampling points.

(3) The control input changes drastically.

5. Ripple-free（无纹波）deadbeat control design

Ripple: though the system outputs are stable at the sampling time, they are varying between two sampling time, See p230, Fig. 7-50.

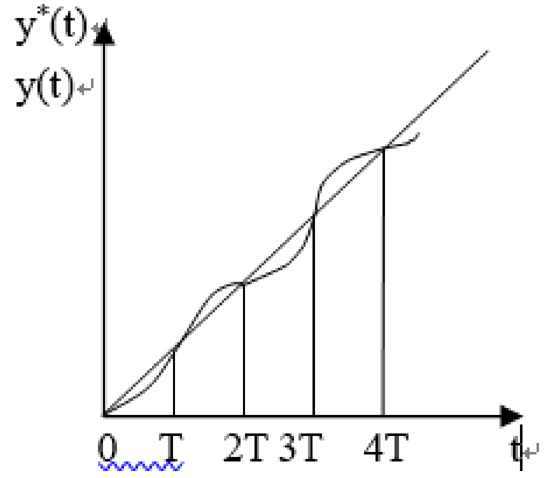
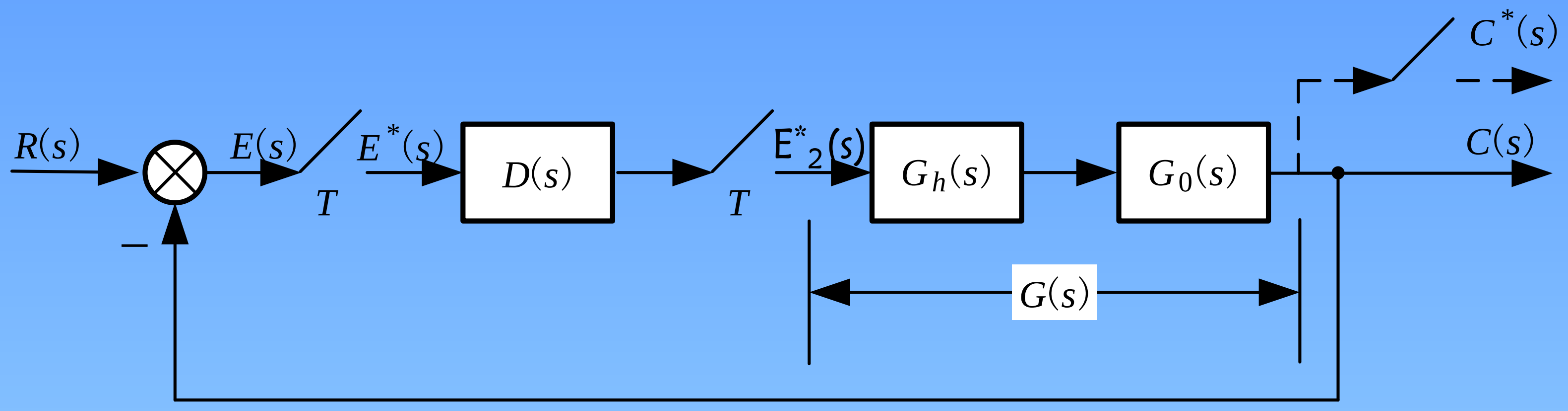


图 7-50 最少拍响应系统纹波

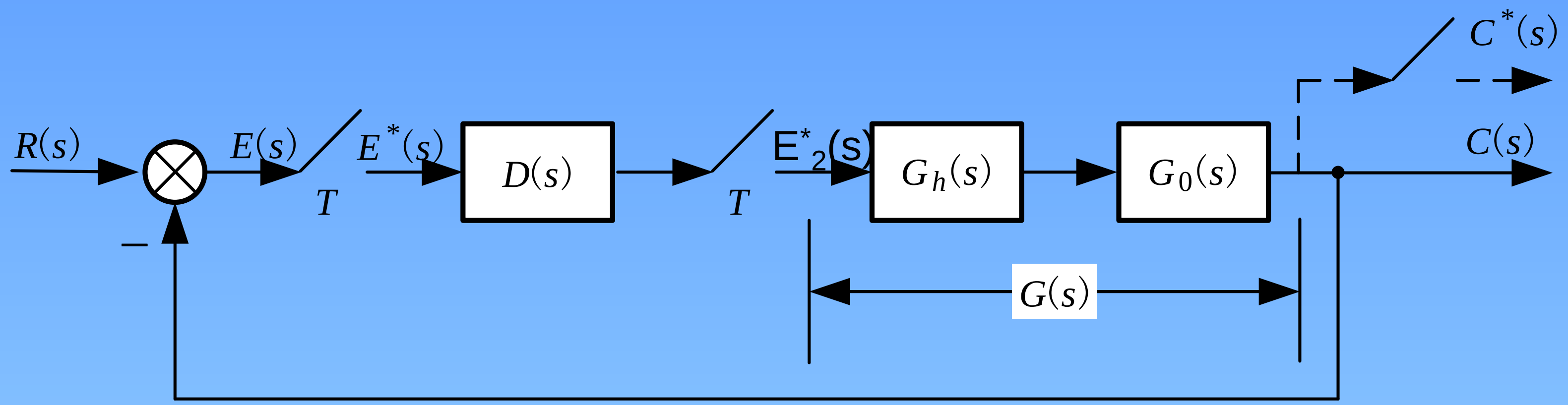
Objective: Not only tracking the input at the sampling time, the outputs are ripple-free.



$$E_2(z) = D(z)E(z)$$

虽然可设计无稳态误差最少拍系统，即E(z)可在有限拍为零，但因为D(z) 不是有限多项式 →$E_2(z)$不能在有限拍为零 → 作用在零阶保持器上，输出有纹波 → 系统输出有纹波。

解决方案：使 $E_2(z)$为z^{-1} 的有限多项式。



$$E_2(z) = D(z)E(z)$$

solution： ensure $E_2(z)$ being a polynomial on z^{-1} of a finite order.

$$D(z) = \frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}$$

Condition: $E_2(z)$ is a polynomial on $z^{-1}$of finite order。

$$E_2(z) = D(z)E(z) = D(z)\Phi_e(z)R(z), \qquad D(z)\Phi_e(z) = \frac{\Phi(z)}{G(z)}$$

➔ $D(z)\Phi_e(z)$=（*）$/z^r$, that is the zero of $G(z)$ must be a zero of $\Phi(z)$

最少拍设计中，$\Phi(z)$和$\Phi_e(z)$选取时应遵循的原则：

1. $D(z)$零点的数目不能大于极点的数目；（因果性）

2. $\Phi_e(z)$应把$G(z)$在单位圆上及单位圆外的极点作为自己的零点；（稳定性）

3. $\Phi(z)$应把$G(z)$在单位圆上及单位圆外的零点作为自己的零点；(稳定性）

4. 当$G(z)$含有z^{-1}因子时，要求$\Phi(z)$也含有z^{-1}的因子；（因果性，可实现性）

5. 因为$\Phi(z)=1-\Phi_e(z)$，他们应该是关于z^{-1}同样阶次的多项式，而且$\Phi_e(z)$还应包含常数项1。（脉冲传函间的关系）

6. 当最小拍系统还有无纹波要求时，闭环脉冲传函$\Phi(z)$的零点应抵消$G(z)$的全部零点（因为最少拍系统设计中G（z）单位圆上及单位圆外的零极点已经被补偿，因此在无纹波的设计中只需抵消$G(z)$单位圆内的零点）。

Homework:

p238 7-15, 7-16

7-15. Consider the system as shown in Fig 7-69, T=1s, design deadbeat controller D(z) for r(t)=t.

Draw $r^*(t)$, $e_1^*(t)$, $e_2^*(t)$, $x(t)$, $y(t)$ and $y^*(t)$.

7-16. Furthermore, design a ripple-free deadbeat controller for the system in 7-15.