Chapter 7 Analysis and Design of Linear Discrete-Time System （Sampled-data System）

7.5 Mathematical Models of Discrete-Time Systems

- **Difference Equation （差分方程）**
- **Impulse Transfer function （脉冲传递函数）**

7.5.1 Linear Time-Invariant Difference Equations

(1) Definition of difference e(kT) = e(k)

$$\lim_{T\to 0}\frac{\Delta e(k)}{T}=\frac{de(t)}{dt}$$

Forward difference

First-order $\quad \Delta e(k)=e(k+1)-e(k)$

Second-order $\quad \Delta^2 e(k)=\Delta e(k+1)-\Delta e(k)$

$$=e(k+2)-2e(k+1)+e(k)$$

$\vdots$

nth-order $\quad \Delta^n e(k)=\Delta^{n-1}e(k+1)-\Delta^{n-1}e(k)$

Backward difference

First-order $\nabla e(k) = e(k) - e(k-1)$

Second-order $\nabla^2 e(k) = \nabla e(k) - \nabla e(k-1)$

$\vdots$ $= e(k) - 2e(k-1) + e(k-2)$

nth-order $\nabla^n e(k) = \nabla^{n-1} e(k) - \nabla^{n-1} e(k-1)$

$$\lim_{T \to 0} \frac{\nabla e(k)}{T} = \frac{de(t)}{dt}$$

(2) Difference equation

The equation of the input, output and their higher order differences.

The (forward) difference equation of nth-order linear time-invariant discrete system.

$$c(k+n)+a_1c(k+n-1)+a_2c(k+n-2)+\cdots+a_{n-1}c(k+1)+a_nc(k)$$
$$=b_0r(k+m)+b_1r(k+m-1)+\cdots+b_{m-1}r(k+1)+b_mr(k)$$

The (backward) differential equation of n-order linear time-invariant discrete system.

$$c(k)+a_1c(k-1)+a_2c(k-2)+\cdots+a_{n-1}c(k-n+1)+a_nc(k-n)$$
$$=b_0r(k-n+m)+b_1r(k-n+m-1)+$$
$$\cdots+b_{m-1}r(k-n+1)+b_mr(k-n)$$

(3) To solve difference equations:

$$\begin{cases}\text{Iteration method}\\ \text{Z-transform method}\end{cases}$$

Example 1 The differential equation of a continuous system is：

$$\begin{cases} \ddot{e}(t)-4\dot{e}(t)+3e(t)=r(t)=1(t) \\ e(t)=0 \qquad (t\le 0) \end{cases}$$

Obtain the corresponding forward difference equation and its solution.

Solution.

$$\dot{e}(t)\approx\frac{\Delta e(k)}{T}=\frac{e(k+1)-e(k)}{T}\overset{T=1}{=}e(k+1)-e(k)$$

$$\ddot{e}(t)\approx\frac{\Delta^2 e(k)}{T^2}=\frac{\Delta e(k+1)/T-\Delta e(k)/T}{T}\overset{T=1}{=}e(k+2)-2e(k+1)+e(k)$$

$$\begin{array}{rl} & e(k+2)-2e(k+1)+e(k) \\ -4[& \qquad\qquad e(k+1)-e(k)] \\ +3[& \qquad\qquad\qquad\qquad e(k)] \\ \hline & e(k+2)-6e(k+1)+8e(k)=1(k) \end{array}$$

$$\boxed{\begin{cases} e(k+2)-6e(k+1)+8e(k)=1(k) \\ e(k)=0 \qquad (k\le 0) \end{cases}}$$

Solution I of the difference equation —— Iteration method

$$\begin{cases} e(k+2)-6e(k+1)+8e(k)=1(k) \\ e(k)=0 \qquad (k\le 0) \end{cases}$$

Solution: $$e(k+2)=6e(k+1)-8e(k)+1(k)$$

$$k=-1: \quad e(1)=6e(0)-8e(-1)+1(-1)=0$$

$$k=0: \quad e(2)=6e(1)-8e(0)+1(0)=0-0+1=1$$

$$k=1: \quad e(3)=6e(2)-8e(1)+1(1)=6-0+1=7$$

$$k=2: \quad e(4)=6e(3)-8e(2)+1(2)=6\times 7-8\times 1+1=35$$

$$\vdots \qquad \vdots$$

$$e^*(t)=\delta(t-2)+7\delta(t-3)+35\delta(t-4)+\cdots$$

Solution II of difference equation — Z-transform method

$$e(k+2)-6e(k+1)+8e(k)=1(k)$$

$$\begin{cases} e(k+2)-6e(k+1)+8e(k)=1(k) \\ e(k)=0 \qquad (k\le 0) \end{cases}$$

$Z:$

$$z^2[E(z)-e(0)z^0-e(1)z^{-1}]$$
$$-6\cdot z[E(z)-e(0)z^0]$$
$$+\ 8\,[E(z)]$$

$$(z^2-6z+8)E(z)=Z[1(k)]=\frac{z}{z-1}$$

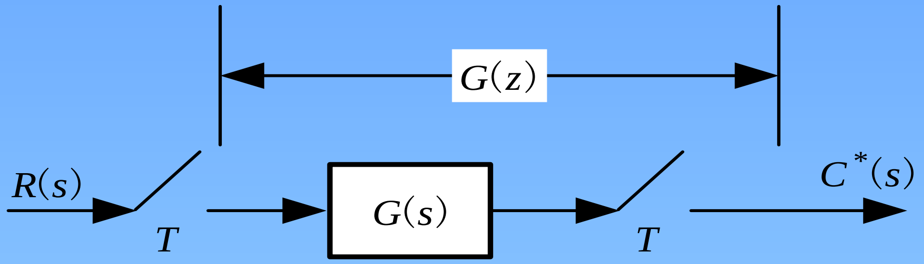
$$E(z)=\frac{z}{(z-1)(z-2)(z-4)}$$

$Z^{-1}:$ $e(n)=\sum \mathrm{Res}\left[E(z)\cdot z^{n-1}\right]$

$$=\lim_{z\to 1}\frac{z\cdot z^{n-1}}{(z-2)(z-4)}+\lim_{z\to 2}\frac{z\cdot z^{n-1}}{(z-1)(z-4)}+\lim_{z\to 4}\frac{z\cdot z^{n-1}}{(z-1)(z-2)} = \frac{1}{3}-\frac{2^n}{2}+\frac{4^n}{6}$$

$$e^*(t)=\sum_{n=0}^{\infty}e(nT)\cdot\delta(t-nT)=\sum_{n=0}^{\infty}\left(\frac{1}{3}-\frac{2^n}{2}+\frac{4^n}{6}\right)\cdot\delta(t-nT)$$

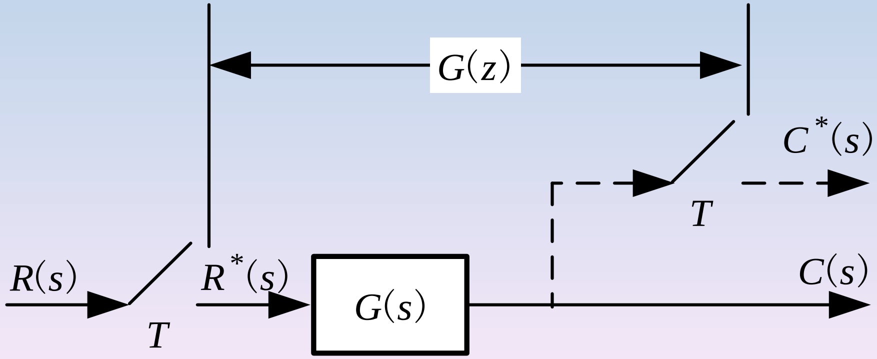
7.5.2 Mathematical Models in Complex Domain — Impulse Transfer Function（脉冲传递函数）

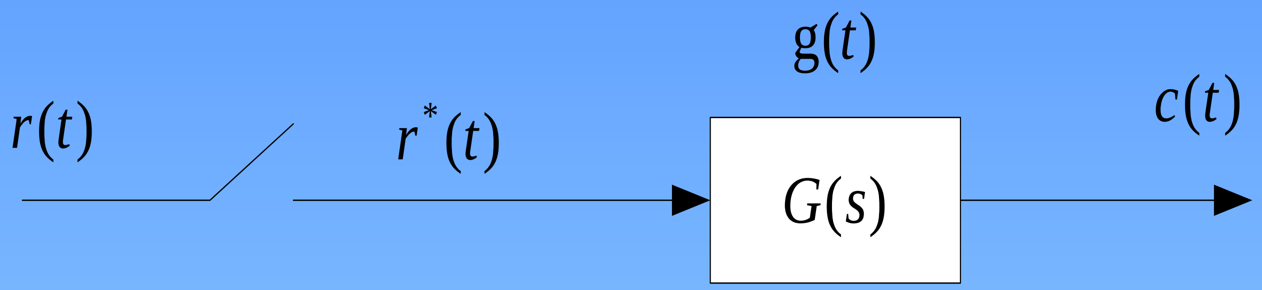


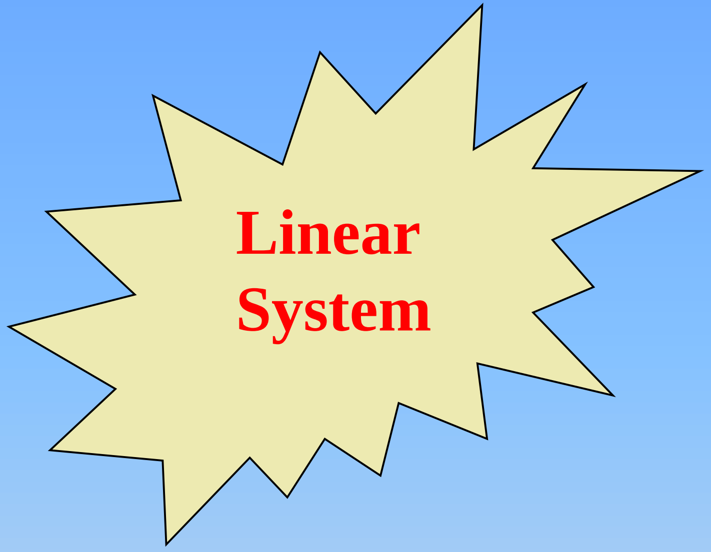
1. Definition

The ratio of the z-T. of the output to the z-T. of the input under zero initial condition.

$$G(z)=\frac{C(z)}{R(z)}$$







$$r^*(t) = \sum_{n=0}^{\infty} r(nT)\delta(t - nT)$$

$$\because r^*(t) = r(0)\delta(t) + r(T)\delta(t - T) + \cdots r(nT)\delta(t - nT) + \cdots$$

$$\therefore c(t) = r(0)g(t) + r(T)g[t - T] + \cdots + r(nT)g[t - nT] + \cdots$$

$$c(kT) = r(0)g(kT) + r(T)g[(k-1)T] + \cdots + r(nT)g[(k-n)T] + \cdots$$

$$= \sum_{n=0}^{\infty} r(nT)g[(k-n)T]$$

$$c(kT)=\sum_{n=0}^{\infty} r(nT)g[(k-n)T]$$

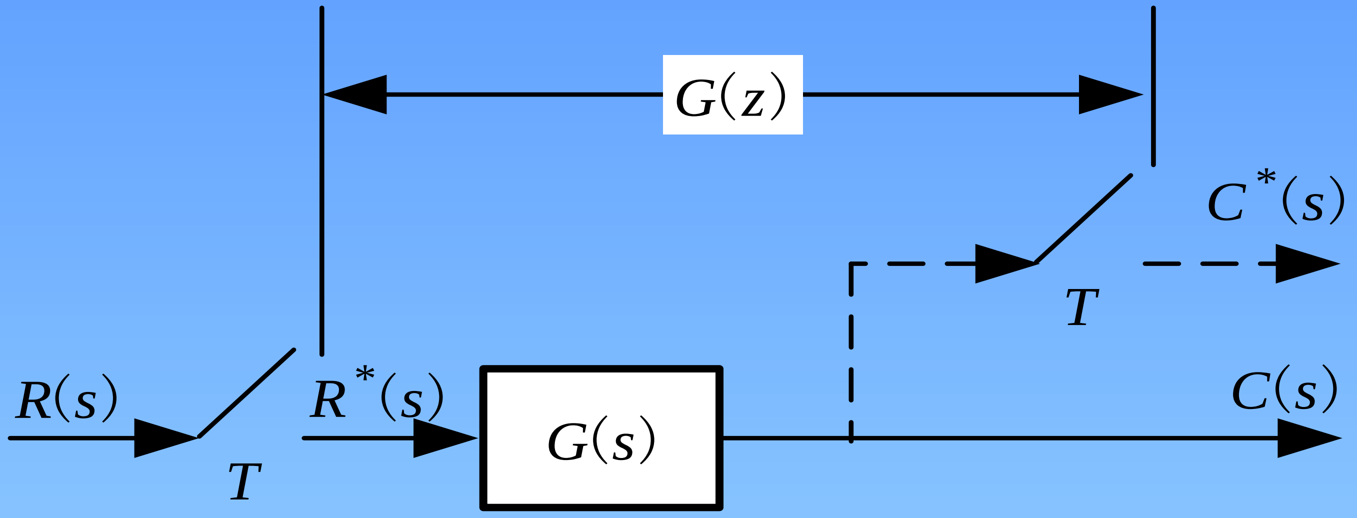
$$C(z)=\sum_{k=0}^{\infty} c(kT)z^{-k} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} r(nT)g[(k-n)T]z^{-k}$$

$$= \sum_{n=0}^{\infty} r(nT)z^{-n}\sum_{k=0}^{\infty} g[(k-n)T]z^{-(k-n)}$$

$$\therefore G(z)=\frac{C(z)}{R(z)}=\sum_{k=n}^{\infty} g[(k-n)T]z^{-(k-n)}=\sum_{j=0}^{\infty} g(jT)z^{-j}$$

The z-transform of unity impulse response sequence

G(Z)=Z[g(t)]=Z[G(s)]



Example 1 Consider **the discrete system shown in the figure with**

$$G(s) = \frac{1}{s(0.1s+1)}$$

Obtain the impulse-transfer function $G(z)$。

Solution:

Method I. The impulse response is:

$$g(t) = (1 - e^{-10t}) \qquad (t > 0)$$

$$g(kT) = 1 - e^{-10kT}$$

Then the impulse tranfer function is:

$$G(z) = \sum_{k=0}^{+\infty} g(kT) z^{-k} = \sum_{k=0}^{+\infty} \left(1 - e^{-10kT}\right) z^{-k}$$

$$= \frac{z}{z-1} - \frac{z}{z - e^{-10T}} = \frac{z(1 - e^{-10T})}{(z-1)(z - e^{-10T})}$$

Method II. Because $G(s) = \dfrac{1}{s} - \dfrac{1}{s+10}$

Then by G(Z)=Z[g(t)]=Z[G(s)], it derives

$$G(z) = \frac{z}{z-1} - \frac{z}{z - e^{-10T}} = \frac{z(1 - e^{-10T})}{(z-1)(z - e^{-10T})}$$

The properties of impulse transfer function:

(1) $G(z)$ is a complex function of complex variable z;

(2) $G(z)$ depends only on the structure and parameters of the system;

(3) $G(z)$ has a relation with the difference equation of the system;

(4) $G(z)$ is equal to $Z[\, g^*(t) \,]$;

(5) $G(z)$ ~ zero-pole location in z plane.

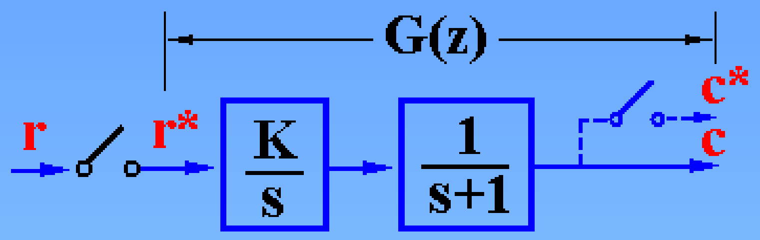
The limitation of impulse-transfer functions

(1) It can not reflect the full information of the system response under non-zero initial conditions;

(2) It is only for SISO discrete systems;

(3) It is only for linear time-invariant difference equations;

Example 2 Consider the discrete system shown in the figure (T=1). Obtain



(1) Impulse-transfer function of the system

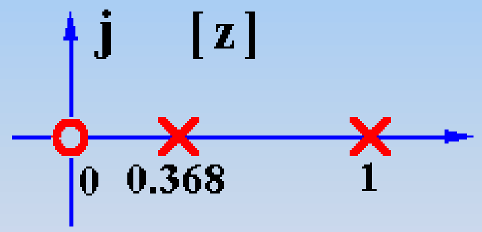
(2) Zero-poles location in z plane;

(3) Difference equation of the system.

Solution. (1) $$G(z)=\frac{C(z)}{R(z)}=Z\left[\frac{K}{s(s+1)}\right]=K\cdot Z\left[\frac{1}{s}-\frac{1}{s+1}\right]$$

$$=K\left[\frac{z}{z-1}-\frac{z}{z-e^{-T}}\right]=\frac{(1-e^{-T})Kz}{(z-1)(z-e^{-T})}=\frac{(1-e^{-T})Kz}{z^2-(1+e^{-T})z+e^{-T}}$$

$$=\frac{0.632Kz^{-1}}{1-1.368z^{-1}+0.368z^{-2}}$$

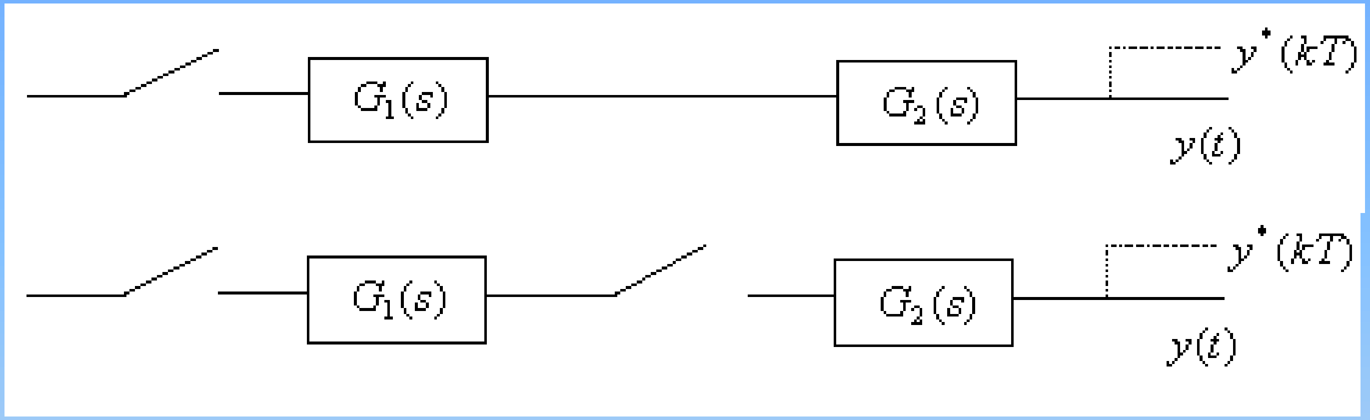


(2) Zero-poles location in z plane

(3) $$\left(1-1.368z^{-1}+0.368z^{-2}\right)C(z)=0.632Kz^{-1}R(z)$$

$$c(k)-1.368c(k-1)+0.368c(k-2)=0.632Kr(k-1)$$

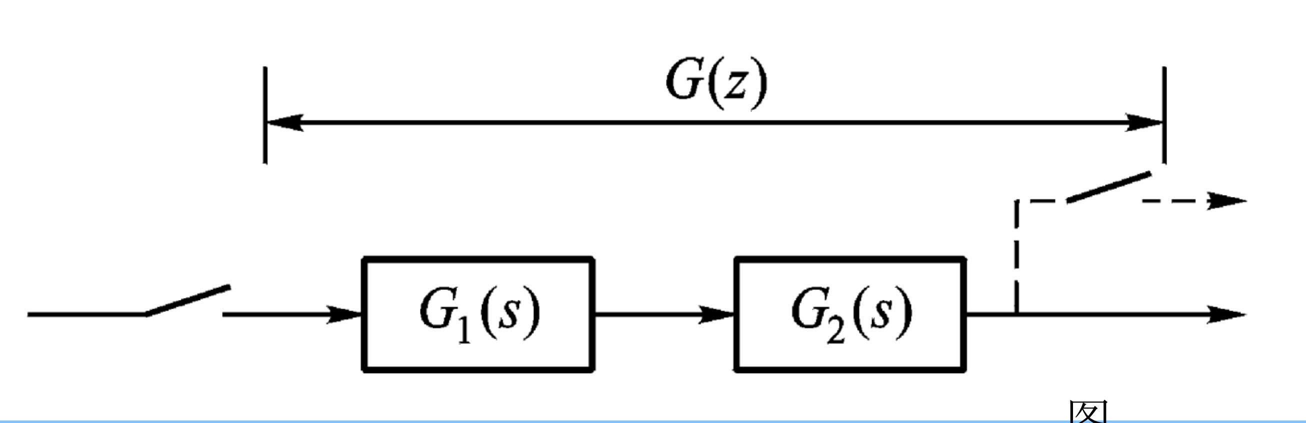
7.5.3 Impulse transfer function of Open-Loop Systems



(1) There is no sampler/switch between two components

$$G(s) = G_1(s)G_2(s)$$

$$G(z) = Z\left[G_1(s)G_2(s)\right] = G_1G_2(z)$$



图

Example 3 **Consider the discrete system shown in the above figure，**

where

$$G_1(s)=\frac{1}{s+a} \qquad G_2(s)=\frac{1}{s+b}$$

Obtain the open-loop impulse transfer function。

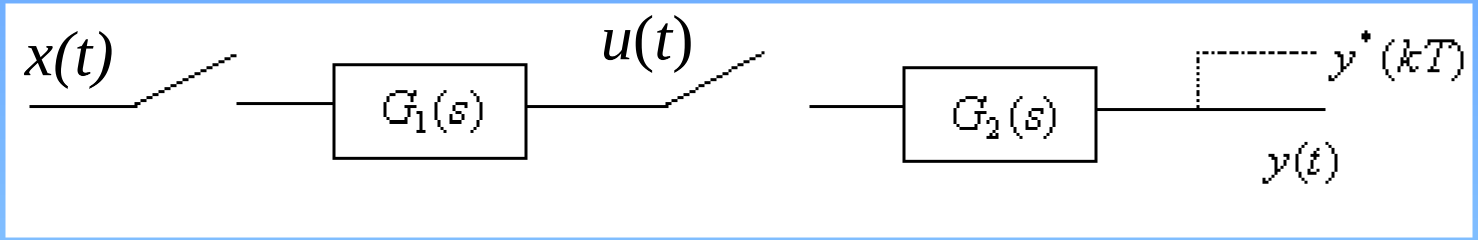
solution:

$$G_1(s)G_2(s) = \frac{1}{b-a}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

$$G(z) = G_1G_2(z)$$

$$= \frac{1}{b-a}\left[\frac{z(e^{-aT} - e^{-bT})}{(z-e^{-aT})(z-e^{-bT})}\right]$$

（2）There is a sampler/switch between two components

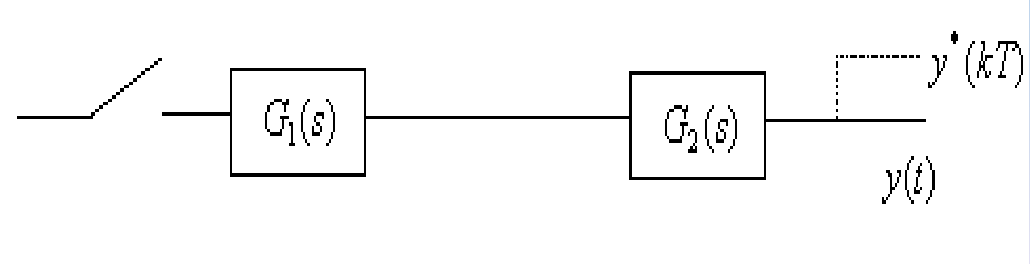


$$U(z)=G_1(z)R(z)$$

$$Y(z)=G_2(z)U(z)=G_1(z)G_2(z)R(z)$$

$$\therefore G(z)=G_1(z)G_2(z)$$

注 $G_1(z)G_2(z)\neq G_1G_2(z)$



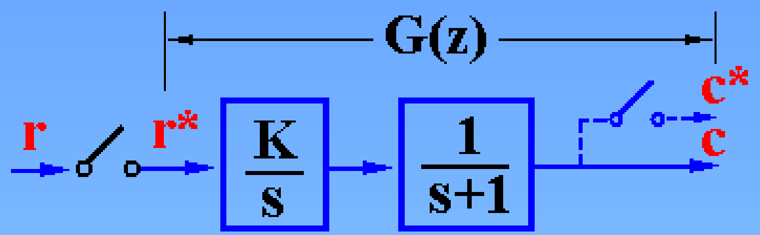
$$G(z)=Z[G_1(s)G_2(s)]=G_1G_2(z)$$

Example 2 Consider the discrete system shown in the figure (T=1). Obtain

(1) Impulse-transfer function of the system

(2) Zero-poles location in z plane;

(3) Difference equation of the system.

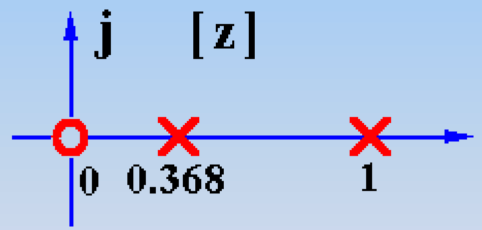


Solution. (1)

$$G(z)=\frac{C(z)}{R(z)}=Z\left[\frac{K}{s(s+1)}\right]=K\cdot Z\left[\frac{1}{s}-\frac{1}{s+1}\right]$$

$$=K\left[\frac{z}{z-1}-\frac{z}{z-e^{-T}}\right]=\frac{(1-e^{-T})Kz}{(z-1)(z-e^{-T})}=\frac{(1-e^{-T})Kz}{z^2-(1+e^{-T})z+e^{-T}}$$

$$=\frac{0.632Kz^{-1}}{1-1.368z^{-1}+0.368z^{-2}}$$



(2) Zero-poles location in z plane

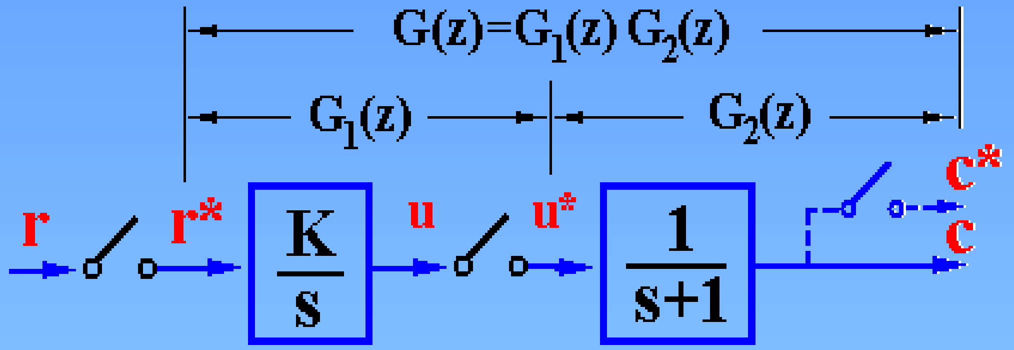
(3) $$\left(1-1.368z^{-1}+0.368z^{-2}\right)C(z)=0.632Kz^{-1}R(z)$$

$$c(k)-1.368c(k-1)+0.368c(k-2)=0.632Kr(k-1)$$

(1) Switch between factors

$$G(z) = G_1(z)G_2(z) = Z\left[\frac{K}{s}\right]\cdot Z\left[\frac{1}{s+1}\right]$$

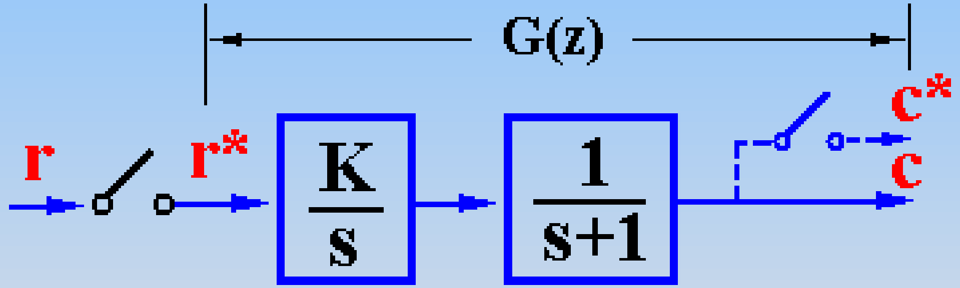
$$= \frac{Kz}{z-1}\cdot\frac{z}{z-e^{-T}} = \frac{Kz^2}{(z-1)(z-e^{-T})}$$



(2) No switch between factors

$$G(z) = Z[G_1(s)\cdot G_2(s)] = G_1G_2(z)$$

$$= K\left[\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right] = \frac{(1-e^{-T})Kz}{(z-1)(z-e^{-T})}$$



Note: the zeros of G(z), the poles of G(z).

Exercise: **Consider** $G_1(s)=\dfrac{1}{s}$, $G_2(s)=\dfrac{10}{s+10}$, **obtain G(z).**

Solution:

If there is no switch between the components,

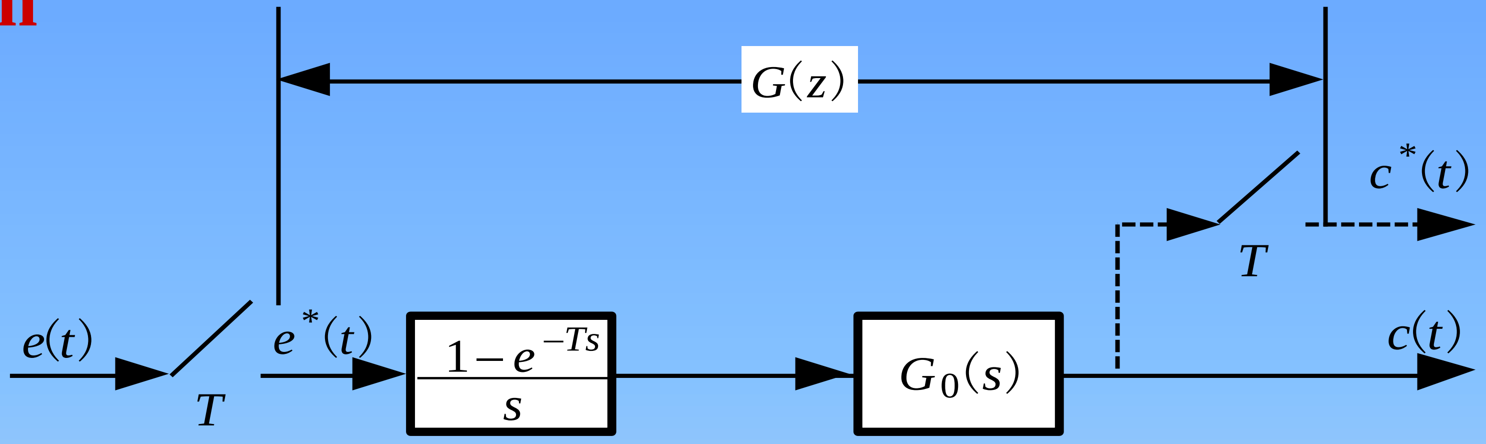
$$G(z)=G_1G_2(z)=Z[\frac{10}{s(s+10)}]=\frac{z(1-e^{-10T})}{(z-1)(z-e^{-10T})}$$

If there is a sampler between the components,,

$$G(z)=G_1(z)G_2(z)=Z[\frac{1}{s}]Z[\frac{10}{s+10}]$$

$$=\frac{z}{z-1}\frac{10z}{z-e^{10T}}=\frac{10z^2}{(z-1)(z-e^{-10T})}$$

（3）ZOH in the system



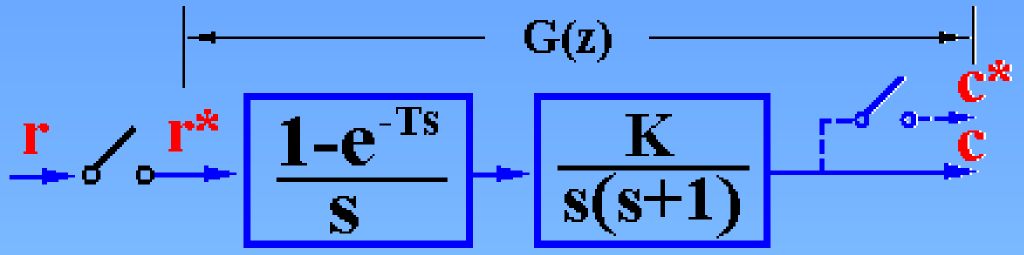
$$C(z)=Z[\frac{1-e^{-Ts}}{s}G_0(s)]R(z)=Z[\frac{1}{s}G_0(s)-\frac{e^{-Ts}}{s}G_0(s)]R(z)$$

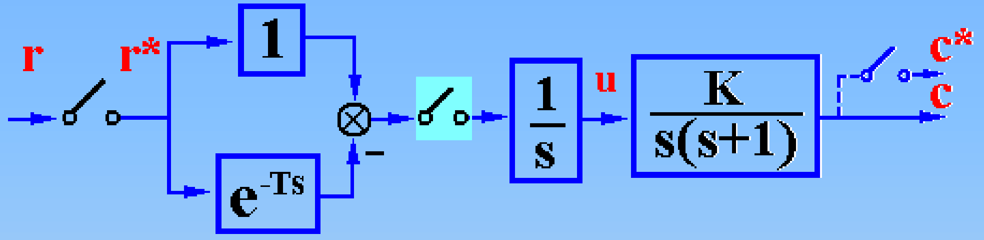
$$Z\left[\frac{e^{-Ts}}{s}G_0(s)\right]=z^{-1}Z\left[\frac{G_0(s)}{s}\right]$$

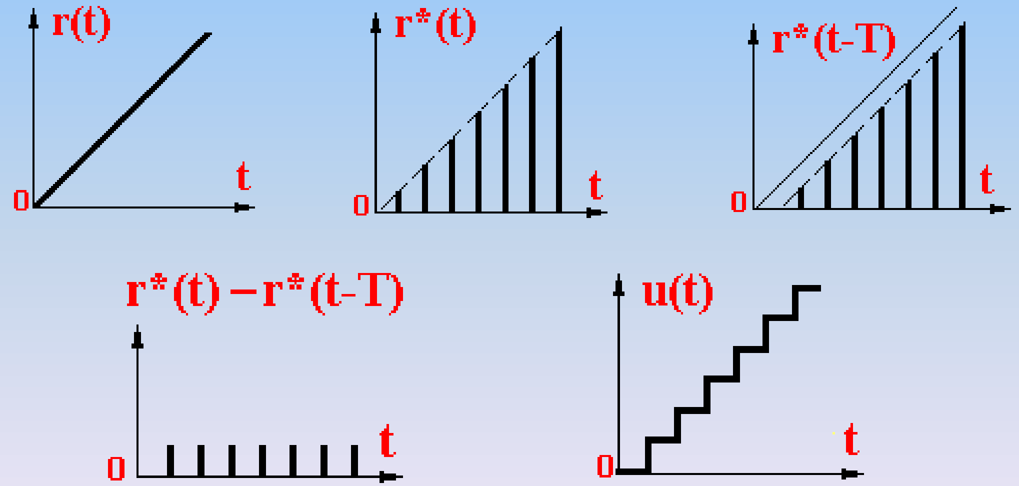
$$C(z)=(1-z^{-1})Z[\frac{G_0(s)}{s}]R(z)$$

$$G(z)=\frac{C(z)}{R(z)}=(1-z^{-1})Z[\frac{G_0(s)}{s}]$$

Example 4 Consider the discrete system shown in the following figure, obtain its impulse transfer function.







$$G(z)=Z\left[\frac{1-e^{-Ts}}{s}\cdot\frac{K}{s(s+1)}\right]$$

$$=K(1-z^{-1})Z\left[\frac{1}{s^2(s+1)}\right]$$

$$=K\frac{z-1}{z}Z\left[\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1}\right]$$

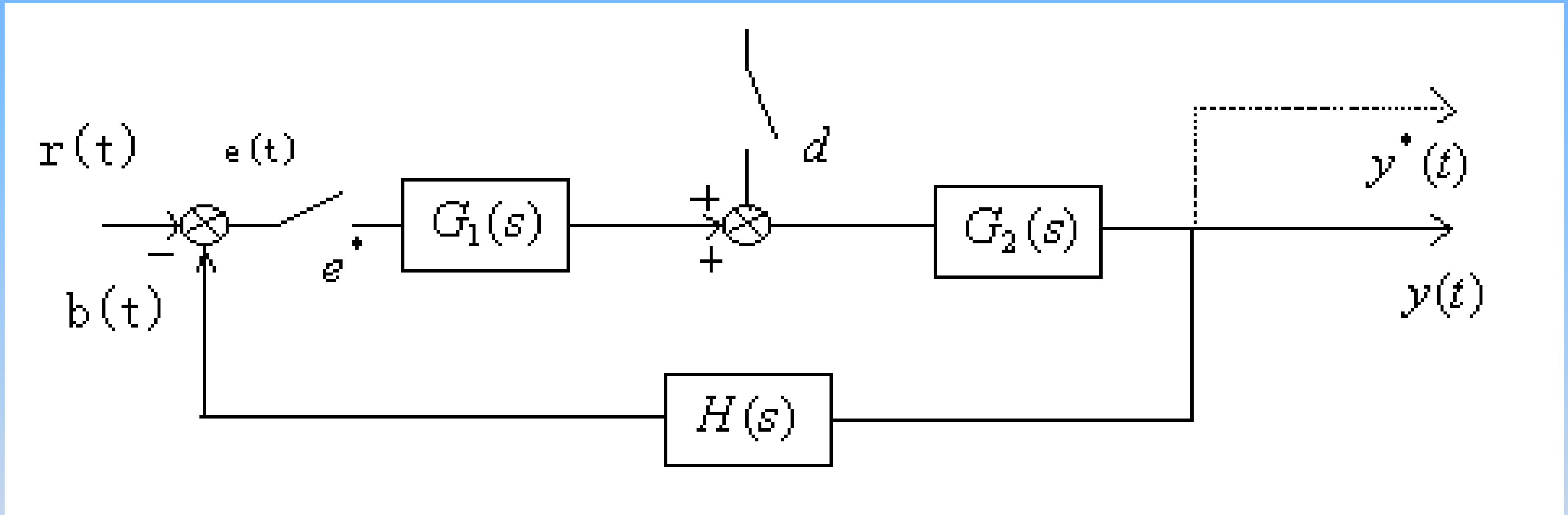
$$=K\frac{z-1}{z}\left[\frac{Tz}{(z-1)^2}-\frac{z}{z-1}+\frac{z}{z-e^{-T}}\right]$$

$$=K\left[\frac{T}{z-1}-1+\frac{z-1}{z-e^{-T}}\right]$$

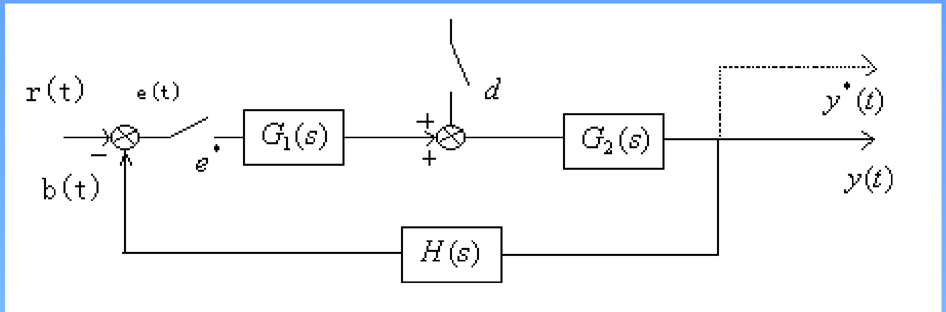
$$=K\frac{(T-1+e^{-T})z+(1-Te^{-T}-e^{-T})}{(z-1)(z-e^{-T})}$$

ZOH does not change the system order and O.-L. poles but changes the O.-L. zeros.

7.5.4、 Impulse transfer function of Closed-Loop Systems



（1）、Impulse Transfer Function for input to output.



$$d = 0$$

$$Y(z) = G_1G_2(z)E(z)$$

$$\left.\begin{aligned} & e(t) = r(t) - b(t) \\ & \Rightarrow E(z) = R(z) - B(z) \\ & B(z) = G_1G_2H(z)E(z) \end{aligned}\right\} \Rightarrow E(z) = \frac{R(z)}{1+G_1G_2H(z)}$$

Error impulse transfer function (误差脉冲传递函数):

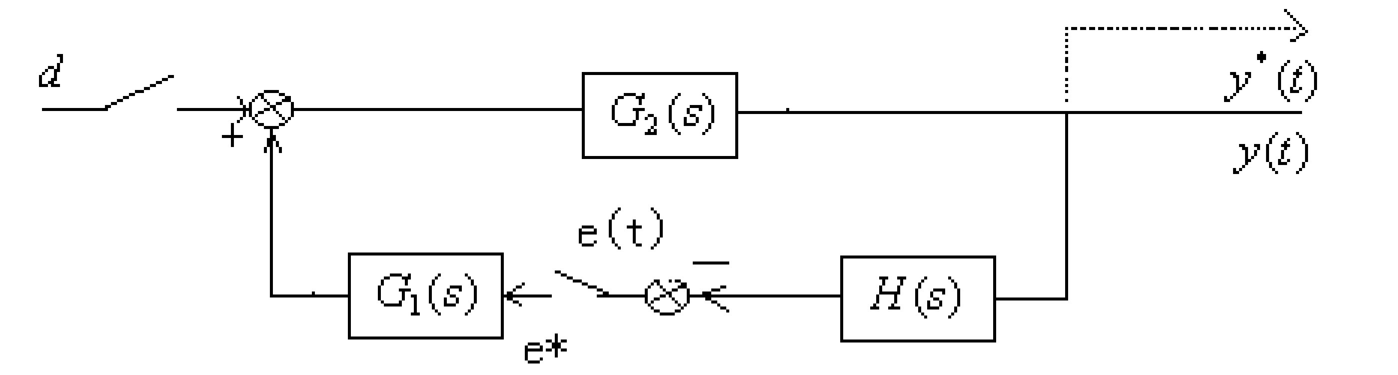
$$G_e(z) = \frac{E(z)}{R(z)} = \frac{1}{1+G_1G_2H(z)}$$

$$\Rightarrow Y(z) = G_1G_2(z)\frac{R(z)}{1+G_1G_2H(z)}$$

$$\therefore \Phi(z) = \frac{Y(z)}{R(z)} = \frac{G_1G_2(z)}{1+G_1G_2H(z)}$$

（2） Impulse Transfer Function for disturbance to output

$$r(t)=0$$



$$Y(z)=G_2(z)D(z)+G_1G_2(z)E(z)$$

$$E(z)=-[G_2H(z)D(z)+G_1G_2H(z)E(z)]$$

$$\Rightarrow E(z)=-\frac{G_2H(z)}{1+G_1G_2H(z)}D(z)$$

$$\therefore Y(z)=G_2(z)D(z)-\frac{G_1G_2(z)G_2H(z)}{1+G_1G_2H(z)}D(z)$$

$$\Rightarrow \Phi_d(z)=\frac{Y(z)}{D(z)}=G_2(z)-\frac{G_1G_2(z)G_2H(z)}{1+G_1G_2H(z)}$$

E(z) :

① D(z) passing through $G_2(z)$;

② Loop of E(z)itself。

There is no switch/sampler for the error signal e(t)

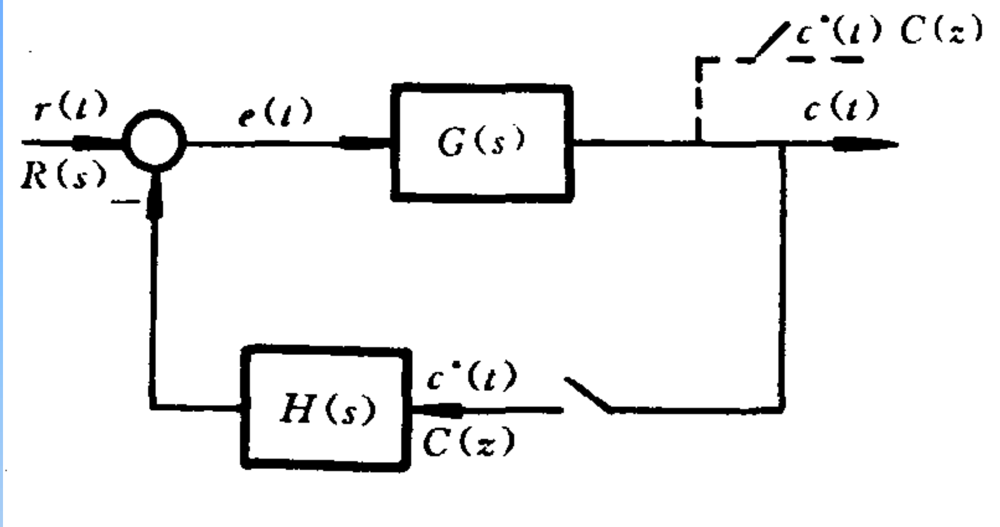


图 7-35 闭环离散系统

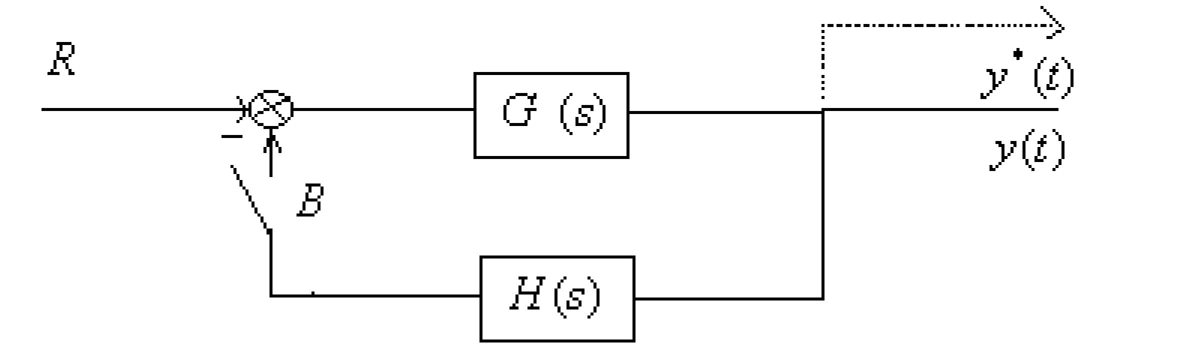
$$C(s) = G(s)R(s) - G(s)H(s)C^*(s)$$

$$C(z) = GR(z) - GH(z)C(z)$$

$$\Rightarrow C(z) = \frac{GR(z)}{1 + GH(z)}$$

Then, for this system, there exists no impulse transfer function.

Example **Consider the discrete-time system as shown in the figure，find the z-transform of the output y(t).**



Solution:

$$Y(z) = GR(z) - G(z)B(z)$$

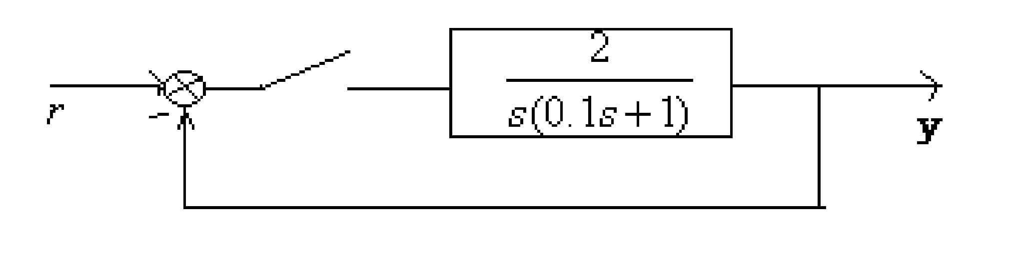
$$B(z) = GHR(z) - GH(z)B(z)$$

$$\therefore B(z) = \frac{GHR(z)}{1 + GH(z)}$$

$$\therefore Y(z) = GR(z) - \frac{G(z)GHR(z)}{1 + GH(z)}$$

There exists no impulse tranfer function.

Example **Consider the discrete-time system as shown in the figure， for T=0.1, find the unit step response of the system.**



Solution:

$$G(z)=Z[\frac{2}{s(0.1s+1)}]=\frac{2z}{z-1}-\frac{2z}{1-e^{-10T}}$$

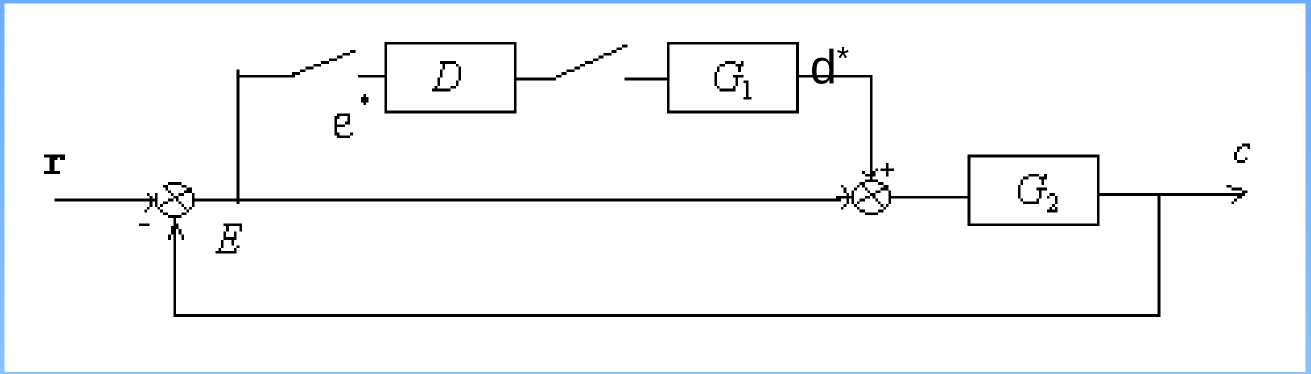
$$=\frac{2z-0.736z}{(z-1)(z-0.368)}=\frac{1.264z}{z^2-1.368z+0.368}$$

$$\therefore \Phi(z)=\frac{G(z)}{1+G(z)}=\frac{1.264z}{z^2-0.104z+0.368}$$

$$\therefore Y(z)=\Phi(z)R(z)=\Phi(z)\frac{z}{z-1} \quad =1.264z^{-1}+1.396z^{-2}+0.945z^{-3}+0.849z^{-4}+\cdots$$

$$y^*(t)=1.264\delta(t-0.1)+1.396\delta(t-0.2)+\cdots$$

Example **Consider the discrete-time system as shown in the figure，find the z-transform of the output c(t).**



Solution: There exist both discrete and continuous signals, then employing L-Transform firstly，

$$C(s)=G_2(s)E(s)+G_1G_2D^*E^*$$

$$E(s)=R-C=R-G_2E-G_1G_2D^*E^*$$

$$\therefore E=\frac{R}{1+G_2}-\frac{G_1G_2}{1+G_2}D^*E^*$$

Discretize e(t)，then

$$E^*=[\frac{R}{1+G_2}]^*-[\frac{G_1G_2}{1+G_2}]^*D^*E^*$$

$$\therefore E^*=\frac{[\frac{R}{1+G_2}]^*}{1+[\frac{G_1G_2}{1+G_2}]^*D^*}$$

Take E and E* into

$$C(s)=E(s)G_2(s)+G_1G_2D^*E^*$$

$$\boldsymbol{C=\frac{G_2R}{1+G_2}-\frac{G_1G_2^2}{1+G_2}D^*E^*+G_1G_2D^*E^*}$$

$$=\frac{G_2R}{1+G_2}+\frac{G_1G_2}{1+G_2}D^*E^*$$

$$\therefore E^* = \frac{[\frac{R}{1+G_2}]^*}{1+[\frac{G_1G_2}{1+G_2}]^* D^*}$$

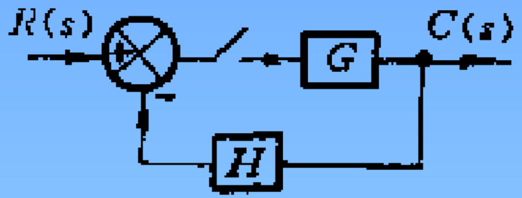
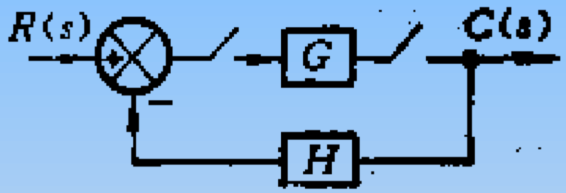
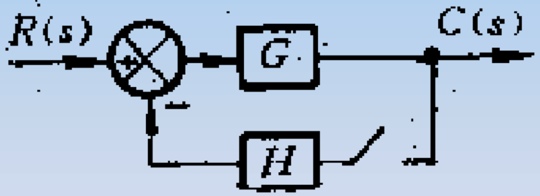
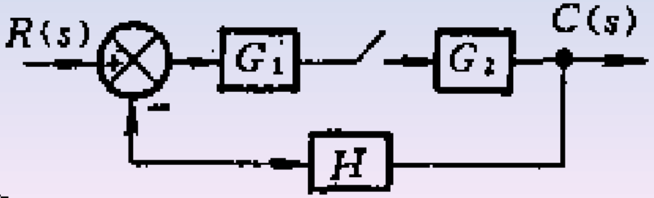
$$C^*(s) = [\frac{G_2R}{1+G_2}]^* + [\frac{G_1G_2}{1+G_2}]^* D^* E^*$$

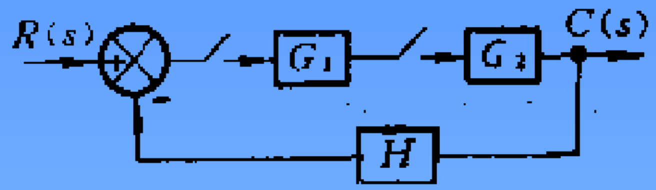
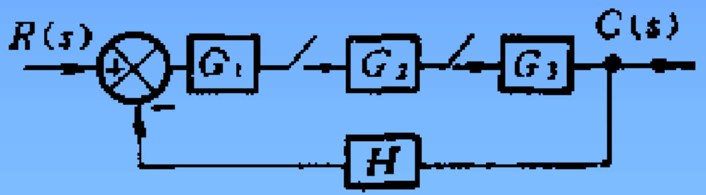
$$= \frac{[\frac{G_2R}{1+G_2}]^* + [\frac{G_1G_2}{1+G_2}]^* D^* [(\frac{G_2R}{1+G_2})^* + [\frac{R}{1+G_2})^*]}{1+[\frac{G_1G_2}{1+G_2}]^* D^*}$$

$$\because R = \frac{G_2R}{1+G_2} + \frac{R}{1+G_2} \qquad \therefore R^* = [\frac{G_2R}{1+G_2}]^* + [\frac{R}{1+G_2}]^*$$

$$\backslash \quad C^* = \frac{[\frac{G_2R}{1+G_2}]^* + [\frac{G_1G_2}{1+G_2}]^* D^* R^*}{1+[\frac{G_1G_2}{1+G_2}]^* D^*}$$

Typical diagram of C.L.discrete-time systems

	系 统 方 框 图	$C(z)$
1		$C(z)=\dfrac{G(z)}{1+HG(z)}R(z)$
2		$C(z)=\dfrac{G(z)}{1+G(z)H(z)}R(z)$
3		$C(z)=\dfrac{RG(z)}{1+HG(z)}$
4		$C(z)=\dfrac{RG_1(z)G_2(s)}{1+G_1G_2H(z)}$

5		$C(z)=\dfrac{G_1(z)G_2(z)}{1+G_1(z)HG_2(z)}R(z)$
6		$C(z)=\dfrac{G_2(z)G_3(z)RG_1(z)}{1+G_3(z)G_1G_3H(z)}$

➢ **Questions：**

① 对于两个子环节串联，采样开关的存在如何影响脉冲传函的？

② 如何建立结构图的脉冲传函？

③ 系统脉冲传递函数一定存在吗？

➢ **Tasks**

SPOC: 7.05 线性离散系统的数学模型

—脉冲传递函数 (8’)

Assignment：

p236. 7-5, 7-7, 7-8

7.5 Mathematical Models of Discrete-Time Systems

7.5.1 Linear Time-Invariant Difference Equations

(1) Definition of difference ① Forward ② Backward

(2) The difference equation and its solving method ① Iteration ② Z-transformation

7.5.2 Impulse-Transfer Function

(1) Definition (2) Properties (3) Limitation

7.5.3 Impulse Transfer Function of Open-Loop Systems

- (1) Switch between factors
- (2) No switch between factors
- (3) With ZOH

7.5.4 Impulse Transfer Function of Closed-Loop Systems

- (1) General Method
- (2) Mason's formula