Probability and Statistical Inference

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Theorem 0.0.1 Chebyshev's Inequality

For $f : \mathbb{R}^d \to \mathbb{R}, f \geq 0, \alpha \in \mathbb{R}$, we have $m(\{f \geq \alpha\}) < \frac{1}{\alpha} \int f$.

Proof. By monotonicity of integral,

$$\int f \geq \int_{\{f \geq \alpha\}} f$$

Also observe that

$$\int_{\{f \geq \alpha\}} f \geq \alpha \cdot m(\{f \geq \alpha\}) \quad \text{Since } f \geq \alpha \text{ a.e. on the set}$$

Associating the two inequalities:

$$\frac{1}{\alpha} \int f \geq m(\{f \geq \alpha\})$$

□

Remark 0.1 *Change the measure to any probability measure and the integration sign to an expectation, we have the typical Chebyshev's inequality in any statistics 101 class.*

Two immediate lemmas follow:

Lemma 0.0.2 :
For $f : \mathbb{R}^d \to [0, \infty]$, if $\int f < \infty$, then $f < \infty$ a.e.

Proof. Fix any $n \in \mathbb{N}$, by Chebyshev's inequality,

$$m\{f \geq n\} < \frac{1}{n} \underbrace{\int f}_{<\infty}$$

And the sequence of sets: $\{f \geq n\}_{n\in\mathbb{N}}$ are nested and $\{f \geq n\} \searrow \{f > \infty\}$. Therefore by continuity of measure:

$$\lim_{n\to\infty} m(\{f \geq n\}) \leq \lim_{n\to\infty} \frac{1}{n} \int f$$
$$m(\{f > \infty\}) \leq 0$$

Therefore, f goes to infinity on a set of measure 0, which is equivalent as f is finite almost everywhere.

□

Lemma 0.0.3 :
For $f : \mathbb{R}^d \to [0, \infty]$, if $\int f = 0$, then $f = 0$ a.e.

Proof. Similarly, fixing $n \in \mathbb{N}$, we have by Chebyshev's inequality:

$$m(\{f \geq 1/n\}) < n \int f = 0$$

Observe that the sequence of sets $\{f \geq 1/n\}_{n\in\mathbb{N}}$ is an increasing sequence of sets such that $\{f \geq 1/n\} \nearrow \{f > 0\}$. Therefore by continuity of measure:

$$\lim_{n\to\infty} m(\{f \geq 1/n\}) \leq n \int f$$
$$m(\{f > 0\}) \leq 0$$

Therefore, the set on which f is strictly larger than 0 has measure 0. This is equivalent as $f = 0$ almost everywhere.

□