2.1 Open Sets

Show a particular set is open

Let $A = \{(x,y) | x^2 + y^2 < 1\} \subset \mathbb{R}^2$, show that D is open.
Proof.
Fix a point $a \in A$ and $a = (a_1, a_2)$, then $a_1^2 + a_2^2 < 1$. Fix $\varepsilon = \frac{1}{2}(1 - \sqrt{a_1^2 + a_2^2})$. Find a point $b \in D(a, \varepsilon)$.

Observe that $d(a, (0,0)) = \sqrt{a_1^2 + a_2^2}$.
Therefore, by triangular inequality, the distance between b and the origin:

$$\begin{aligned} d(b,(0,0)) &\leq d(a,b) + d(a,(0,0)) \\ &\leq d(a,b) + \sqrt{a_1^2 + a_2^2} \\ &< \varepsilon + \sqrt{a_1^2 + a_2^2} \\ &< \frac{1}{2}(1 - \sqrt{a_1^2 + a_2^2}) + \sqrt{a_1^2 + a_2^2} \\ &< \frac{1}{2} < 1 \end{aligned}$$

Therefore, $b \in A$, $D(a, \varepsilon) \subset A$. A is open. **QED**

2.3 Closed Sets

2.4-6 Accumulation Point, Closure, Boundary

Circle of Implication

The following are equivalent

1. A is closed
2. $\mathrm{AC}(A) \subset A$
3. $A = \mathrm{cl}(A)$
4. $\mathrm{bd}(A) \subset A$

$1 \Rightarrow 2$

Proof.

Suppose A closed, then A^C is open. Let $a \in \mathrm{AC}(A)$, show that $a \in A$.

Suppose not, that $a \notin A$, then $a \in A^C$.
By definition, A^C is open. Therefore, $\exists \epsilon > 0$ such that $D(a, \varepsilon) \subset A^C$.
Since $a \in \mathrm{AC}(A)$, then $\forall \varepsilon, \exists b \in A$ such that $b \in D(a, \varepsilon)$.
Therefore

$$b \in D(x, \varepsilon) \subset A^C$$

and

$$b \in A \cap A^C$$

This is a contradiction, and $a \in A$ **QED**

$2 \Rightarrow 3$

Proof.

Suppose that $\mathrm{AC}(A) \subset A$, show that $A = \mathrm{cl}(A)$. This is equivalent as showing $A = A \cup \mathrm{AC}(A)$, but $\mathrm{AC}(A) \subset A$. Therefore. this is proven.

QED

$3 \Rightarrow 4$

Proof.

Suppose that $A = \mathrm{cl}(A)$, show that $\mathrm{bd}(A) \subset A$
By definition, $\mathrm{bd}(A) = \mathrm{cl}(A) \cap \mathrm{cl}(A^C)$. Therefore, $\mathrm{bd}(A) \subset \mathrm{cl}(A) = \mathrm{AC}(A) \cup A$.

QED

$4 \Rightarrow 1$

Suppose that $\mathrm{bd}(A) \subset A$, show that A is closed.

Proof.

Suppose that A is not closed. Then A^C is not open. By definition
$\exists x \in A^C$ such that $\forall \varepsilon > 0, D(x, \varepsilon)$ is not a subset of A^C
Therefore, $D(x, \varepsilon) \cap A \neq \varnothing, \exists y \in D(x, \varepsilon) \cap A$. Therefore, $x \in \mathrm{AC}(A)$.

$x \in \mathrm{AC}(A)$ implies that $x \in \mathrm{cl}(A)$, and $x \in A^C$ implies that $x \in \mathrm{cl}(A^C)$. Therefore,

$$x \in \mathrm{cl}(A) \cap \mathrm{cl}(A^C)$$

Which implies that $x \in \mathrm{bd}(A)$. By previous assumption, $\mathrm{bd}(A) \subset A$. Therefore, $x \in A \cap A^C$. This is a contradiction, and therefore, A should be closed.

QED

2.7 Convergence in General Metric Spaces

Closed set iff limit in the set

Claim: A set A is closed iff $\forall$ sequence $x_k \in A$ that converges in M, the limit must be in A.

$\Rightarrow$

Proof.

Suppose that $A \subset M$ is closed. Suppose x_n is a sequence that converges in M. $\exists x$ as the limit.
Now suppose that the limit is not in A, then $x \in A^C$

By definition, A closed is equivalent as saying A^C is an open set. Therefore for x,

$$\forall \varepsilon > 0, D(x, \varepsilon) \subset A^C$$

However, $x_n \to x$ implies that $\forall \varepsilon, \exists N$ such that $\forall n > N, d(x_n, x) < \varepsilon$. Therefore, for a particular ε, we will have some x_n such that $x_n \in D(x, \varepsilon) \subset A^C$. But the entire sequence is in A.

Therefore $x_n \in A \cap A^C$. This is a contradiction, and $x \in A$.

QED

$\Leftarrow$

Proof.

Suppose that all sequences in A converges in limits that lie in A, we want to show that A is a closed set.

Now suppose that A is not a closed set, then A^C is not an open set.
Then, $\exists x \in A^C$ such that

$$\forall \varepsilon > 0, D(x, \varepsilon) \not\subset A^C$$

Which implies that $\forall \varepsilon > 0, D(x, \varepsilon) \cap A \neq \varnothing$, x is a limit point.

Therefore, we construct $\varepsilon_k = 1/k$. Then, $\exists x_k \in D(x, \varepsilon_k) \cap A$. We can therefore construct a sequence with $k = 1, 2, \cdots$.

Claim: $\{x_k\} \to x$
This is because $d(x, x_k) < \varepsilon_k = 1/k$. By sandwich theorem, this limit goes to 0 when k goes to infinity. Therefore, x_k converges to x.

By previous assumption, if a sequence converge, the limit should be in the same set. Therefore, $x \in A$. However, this is a contradiction.

Therefore, A should be a closed set.

QED

2.8 Completeness in General Metric Spaces

Cauchy $\Rightarrow$ Bounded

Fix $\varepsilon = 1$. Then x_n is Cauchy implies that $\exists N$ such that

$$\forall n, m > N, d(x_n, x_m) < 1$$

Therefore, fix $x_0 \in M$, let

$$R = \max\{d(x_0, x_1), \cdots, d(x_N, x_{N-1})\} + 1$$

The later terms are all bounded by $\varepsilon = 1$. Then, $d(x_n, x_0) < R\ \forall n = 1, 2, \cdots$.

Cauchy Sequence and Convergence Subsequence

Claim: $\exists x_{n_k}$ as a subsequence of a Cauchy sequence x_n. If $x_{n_k} \to x$, then $x_n \to x$.

Proof.

Given ε, since x_n is Cauchy, $\exists N$ such that

$$\forall n, m > N, d(x_n, x_m) < \varepsilon/2$$

Since $x_{n_k} \to x$ as $k \to \infty, \exists K$ such that

$$\forall k \geq K, d(x_{n_k}, x) < \varepsilon/2$$

Since n_k and n grow together, $\exists \bar{k}$ large enough such that $n_{\bar{k}} > N$.
Therefore,

$$d(x_n, x_{n_{\bar{k}}}) \leq \varepsilon/2$$

Therefore, by triangular inequality:

$$\begin{aligned} d(x_n, x) &\leq d(x_n, x_{n_k}) + d(x, x_{n_k}) \\ &< \varepsilon/2 + \varepsilon/2 \\ &< \varepsilon \end{aligned}$$

Therefore, $x_n \to x$

QED

Prove that Cauchy sequences have at most 1 cluster point

Proof.

Suppose (M, d) as a GMS, and $\{x_n\} \in M$ as a Cauchy sequence. Suppose otherwise that $\exists x_1, x_2, x_1 \neq x_2$ such that both are cluster points.

By proposition, $\exists \{x_{\phi(n)}\} \subset \{x_n\}$ such that $x_{\phi(n)} \to x_1$, and $\exists \{x_{\psi(n)}\} \subset \{x_n\}$ such that $x_{\psi(n)} \to x_2$.

We obtain three propositions

(a) $\forall \varepsilon > 0, \exists N_1$ such that $\forall n > N_1, d(x_{\phi(n)}, x_1) < \varepsilon/3$

(b) $\forall \varepsilon > 0, \exists N_2$ such that $\forall n > N_2, d(x_{\psi(n)}, x_1) < \varepsilon/3$

(c) From $\{x_n\}$ being Cauchy, we know that $\forall \varepsilon > 0, \exists N_3$ such that $n, m > N, d(x_n, x_m) < \varepsilon/3$

Therefore, we define $N = \max\{N_1, N_2, N_3\}$. We obtain the following inequality:

$$\begin{aligned} d(x_1, x_2) &\leq d(x_1, x_{\phi(n)}) + d(x_{\phi(n)}, x_2) \\ &\leq d(x_1, x_{\phi(n)}) + d(x_{\phi(n)}, x_{\psi(n)}) + d(x_{\psi(n)}, x_2) \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &< \varepsilon \end{aligned}$$

Therefore, $x_1 = x_2$

QED

Complete $\Rightarrow$ Closed

Claim: Suppose that A in an general metric space is complete, then A is a closed set.

Proof.

Suppose that A is a complete set. And assume to the contrary that A is not a closed set, then A^C is not an open set. Therefore, $\exists x \in A^C$ such that $\forall \varepsilon > 0, D(x, \varepsilon) \not\subset A^C$.

Then the same ε-neighbourhood should overlap with A:

$$D(x, \varepsilon) \cap A \neq \varnothing$$

Therefore, we can find elements in this intersection.

Construction: Define $\varepsilon_k = 1/k$. Then, $\exists x_k \in D(x, \varepsilon) \cap A$.

Claim: $\{x_k\}$ is Cauchy $\forall \varepsilon_k > 0$, $\exists x_n, x_m$ (note that they need not be distinct), such that

$$d(x_n, x) < \varepsilon/2$$

$$d(x_m, x) < \varepsilon/2$$

Then by triangular inequality,

$$\begin{aligned} d(x_n, x_m) &\leq d(x, x_m) + d(x, x_n) \\ &< \varepsilon/2 + \varepsilon/2 \\ &< \varepsilon \end{aligned}$$

We can let $N = \min\{n, m\}$. Eventually, we have this N such that $\forall n, m \geq N, d(x_n, x_m) < \varepsilon$. Therefore x_k is Cauchy.

Claim: $x_k \to x$

Observe the distance between the sequence and x:

$$
\begin{aligned}
0 \leq d(x_k, x) < \varepsilon_k = 1/k \\
\lim_{k\to\infty} 0 \leq \lim_{k\to\infty} d(x_k, x) < \lim_{k\to\infty} 1/k \\
0 \leq \lim_{k\to\infty} d(x_k, x) < 0
\end{aligned}
$$

By previous assumption, A is complete meaning that if a sequences in the set converge, the limit is in the set. Therefore, $x \in A$. However, it is previously assumed that $x \in A^C$.

This is a contradiction, and A should be closed.

QED