3.1 Compactness

Bolzano-Weierstrass

Statement: A is compact iff A is sequentially compact

Lemmas

(a) If $A \subset M$ is compact, then A is closed

Proof.

Suppose $A \subset M$ is compact. Fix $x \in A^C$, for $n = 1, 2, \ldots$.
Let $U_n = \{y \in M | d(y, x) > \frac{1}{n}\}$. Then $U_n, n = 1, 2, \ldots$ is an open cover.

$$A \subset \cup_{i=1}^{\infty} U_n$$

Proof:
Let $a \in A$, then $a \neq x$, $d(a, x) > 0$. By Archimedean principle, $\exists N$ such that $1/N < d(a, x)$, then $a \in U_N$, which implies that $a \in \cup_{i=1}^{\infty} U_n$, and $A \subset \cup_{i=1}^{\infty} U_n$

Then A is compact which implies that $\exists$ finite subcover, $\exists N$ such that

$$A \subset \cup_{i=1}^{N} U_i = U_N$$

Then, $D(x, N) = U_N^C \subset A^C$. This implies that A^C is open, which further implies that A is closed.

QED

(b) If $B \subset M$ is closed and M is compact, then B is compact

Proof.

We first establish $\{U_i | i \in I\}$ be an open cover for B. Assuming B is closed, then B^C is open.
Then $\{U_i, B^C | i \in I\}$ is an open cover for M. Then M being a compact set indicating that $\exists N$ such that $\{U_i, B^C | i \in \{1, 2, \ldots, N\}\}$ is a finite subcover for

M. Then, taking away B^C on both sides. That makes $\{U_i | i \in \{1, 2, \ldots, N\}\}$ a finite subcover for B. Therefore, B is compact.

QED

$\Rightarrow$

Proof.
Step 1: Set up and reduction

Suppose that A is compact, show that A is sequentially compact, that is, fix $\{x_k\} \in A$, show that $\exists$ a subsequence that converges to A.

Suppose to the contrary:
Suppose that A is not sequentially compact, which implies that x_k has no converging subsequence.
WLOG: Assume all points are distinct in $\{x_k\}$.

Step 2: Claim 1: For all $k = 1, 2, \ldots, \exists r > 0$ such that $x_j \notin D(x_k, r) \forall j \neq k$

Proof:
Suppose otherwise that $\exists k$ such that $\forall \varepsilon > 0, \exists x_j \in D(x_k, \varepsilon)$. We fix $\varepsilon = \frac{1}{m}$ and obtain $\{x_{j_m}\}$ such that $x_{j_m} \in D(x_k, \frac{1}{m})$. Then $x_{j_m} \to x_k$ which is a contradiction.

Step 3: Finishing

Consider $B = \{x_k | k = 1, 2, \ldots\} \subset A$. By Claim 1, there should not be any accumulation point, therefore, $\mathrm{AC}(B) = \varnothing$. Therefore, B contains all its accumulation points which implies that B is closed.

By Lemma b), if a set is closed and its underlying set is compact, then that set is compact. Therefore, B is compact.

On the other hand, $\{U_k\}_{k=1}^{\infty}$ with $U_k = \{x_k\}$ is an open cover. But it does not have a finite subcover for B. Therefore, this is a contradiction, and $\exists$ a converging subsequence for x_k, which implies that A is sequentially compact.

QED

$\Leftarrow$

Suppose that $A \subset M$ is sequentially compact. Show that A is compact.

Proof.

Let $\{U_i\}_{i \in I}$ be an open cover for A.

Claim: (2.1) $\exists r > 0$ such that $\forall y \in A, D(y, r) \subset U_i$ for some i.

Proof:

Suppose otherwise that $\forall r > 0, \exists y \in A$ such that $D(y, r) \not\subset U_i$ $\forall i \in I$. A is sequentially compact implies that $\exists y_{n_k} \subset y_n$ such that $y_{n_k} \to z \in A$. Since $y_{n_k} \in A$ and U_i are open covers, then $\exists z \in U_{i_0}$ for some $i_0 \in I$. Since all U_i are open sets, then $\exists \varepsilon > 0$ such that $D(z, \varepsilon) \subset U_{i_0}$. Due to convergence, for $\varepsilon/2, \exists$ large enough K such that $d(z, y_{n_K}) < \varepsilon/2$. Therefore, $D(y_{n_K}, \frac{1}{n_K}) \subset D(z, \varepsilon) \subset U_{i_0}$. This is a contradiction to the previous assumption. Therefore, this claim is true.

Claim: (2.2) A is totally bounded.

Proof:

Suppose not, then $\exists \varepsilon > 0$ such that A cannot be covered by finite number of balls radius ε. Then choose $y_1 \in A, y_2 \in A \backslash D(y_1, \varepsilon), \cdots, y_{k+1} \in A \backslash \bigcup_{i=1}^{k} D(y_i, \varepsilon)$.

Observe that $d(y_i, y_j) \geq \varepsilon \forall i \neq j$. Then y_n is not Cauchy which implies that y_n does not have a converging subsequence. This is a contradiction to the previous assumption that A is sequentially compact. Therefore, Claim 2.2 is true.

Step 3: Finishing

By Claim 2.1, $\exists r$ such that $\forall y \in A, D(y, r) \subset U_i$ for some i.

By Claim 2.2, $\exists \{y_1, y_2, \cdots, y_m\} \in A$ such that $A \subset \bigcup_{i=1}^{m} D(y_i, r)$.

Also by Claim 2.1, $D(y_{i_j}, r) \subset U_{i_j}$ for some i_j.

Therefore, $\{U_{i_j}\}_{j=1}^{m}$is a finite subcover for A. A is compact.

QED

3.3 & 4 Connectedness

Path connected implies connected

Proof.

Lemma: (1) the interval $B = [a, b] \in \mathbb{R}$ is connected.

To prove this, suppose that is it not connected. Then $\exists U, V \in \mathbb{R}$ that are open sets such that

1. $B \cap U \cap V = \varnothing$
2. $B \cap U \neq \varnothing$, $B \cap V \neq \varnothing$
3. $B \subset U \cup V$.

WLOG, let $b \in V$. Let $c = \sup(U \cap [a, b])$ which exists since $U \cap [a, b] \neq \varnothing$ and is bounded above. Note that the set $U \cap [a, b]$ is closed since its complement $V \cup \mathbb{R} \backslash [a, b]$ is open. Therefore $c \in U \cap [a, b]$. Note that $c \in U$ and $b \in V$ so $c \neq b$.

Claim: neighbourhood of c intersects with $V \cap [a, b]$.
This is because c is the supremum, so any neighbourhood cannot be a strict subset of $U \cap [a, b]$.

Therefore, c is an accumulation point for $V \cap [a, b]$. Same as $U \cap [a, b]$ being closed, $V \cap [a, b]$ is also closed thus containing all its accumulation points. Therefore, $c \in V \cap [a, b]$ which implies $c \in V \cap U \cap [a, b] = \varnothing$. This is a contradiction, so $[a, b]$ is connected.

We used the fact that $[a, b]$ is a closed set. Does this mean we cannot do this when $[a, b]$ is not closed?

Now, suppose that $A \subset M$ to be path connected but not connected.
Then $\exists U, V \subset M$ with the three properties

1. $A \cap U \cap V = \varnothing$
2. $A \cap U \neq \varnothing$, $A \cap V \neq \varnothing$

3. $A \subset U \cup V$

I fix $x \in A \cap U$, $y \in A \cap V$.

With A being path connected, $\exists \phi : [0,1] \to A$ continuous such that $\phi(0) = x, \phi(1) = y$.

Let $C = \phi^{-1}(A \cap U)$ and $D = \phi^{-1}(A \cap V)$. Then $0 \in C$ and $1 \in D$.

Claim: C is closed:
Let $t_k \in C$ such that $t_k \to t \in \mathbb{R}$. Then since continuous function preserves convergence, $\phi(t_k) \to \phi(t)$. Now suppose that $\phi(t) \notin U$. Then $\phi(t) \in A \subset U \cup V$ implies that $\phi(t) \in V$. Since V is open, then $\exists \varepsilon$ such that $D(\phi(t), \varepsilon) \subset V$. But $\phi(t_k) \in U$ converges to $\phi(t)$. This is a contradiction. Therefore, C is closed. With a similar proof, D is closed.

Therefore, C^C and D^C are both open sets. I claim that they separate $[0,1]$ with the three conditions:

1. $[0,1] \cap C^C \cap D^C = \varnothing$
   Suppose otherwise that $\exists x \in [0,1] \cap C^C \cap D^C$, then $x \in C^C$ implies that $\phi(x) \notin A \cap U$. Since U and V covers A, then $\phi(x) \in V$. Similarly, $\phi(x) \in U$. Therefore, $\phi(x) \in A \cap U \cap V = \varnothing$. This is a contradiction and therefore $[0,1] \cap C^C \cap D^C = \varnothing$.

2. $[0,1] \cap C^C \neq \varnothing, [0,1] \cap D^C \neq \varnothing$
   Let $y \in [0,1]$ such that $\phi(y) \in A \cap V$. Then since $A \cap V \cap U = \varnothing$, $\phi(y) \notin U$, which further implies that $\phi(y) \notin A \cap U$. Therefore, $y \in C^C$. By definition, $y \in [0,1]$. Therefore, $[0,1] \cap C^C \neq \varnothing$.

   Similarly, $[0,1] \cap D^C \neq \varnothing$

3. $[0,1] \subset C^C \cup D^C$
   Let $x \in [0,1]$. Therefore, $\exists \phi(x) \in A$. Since U, V covers A, then $\phi(x) \in U \cup V$. Then $\phi(x) \in A \cap U$ or in $\phi(x) \in A \cap V$. Therefore $x \in C^C$ or $x \in D^C$, $[0,1] \subset C^C \cup D^C$.

Therefore, this has shown that $[0,1]$ is a disconnected set, which is a contradiction. Therefore, A is connected.

QED

Connected in $\mathbb{R}$ and openness implies path connected

Claim: If $A \subset \mathbb{R}^n$, A is connect and is an open set, then A is path connected.

Proof.
To show this, fix x_0, and let $B = \{y \in A | x_0, y \text{ can be joined by a continuous path in } A\}$.

Claim: B is open in A.

To show this, find $y \in B \subset A$. Then $\exists\varepsilon$ such that $D(y, \varepsilon) \subset A$. From the definition of B we know that $\exists\phi_1 : [0, 1] \to A$ such that $\phi_1(0) = x_0$ and $\phi_1(1) = y$.

Fixing $x \in D(y, \varepsilon)$. In $\mathbb{R}^n$ we know that any epsilon neighbourhood is open and connected. Therefore, $\exists\phi_2 : [0, 1] \to A$ such that$\phi_2(0) = y$ and $\phi_2(1) = x$. Therefore, we define a new function $\phi : [0, 1] \to A$ such that

$$\phi(t) = \begin{cases} \phi_1(2t), t \in [0, 1/2] \\ \phi_2(2t-1), t \in [1/2, 1] \end{cases}$$

I claim that ϕ is continuous. This is because the function yields the same output around $t = 1/2$, $\phi_1(2 \cdot 1/2) = y$ and $\phi_2(2 \cdot 1/2 - 1) = y$. For else, since both pieces are continuous. Therefore, the newly defined ϕ is continuous. Also, $\phi(0) = x_0$ and $\phi(1) = x$. Therefore, x is path connected to x_0 which makes $x \in B$. Therefore, B is an open set.

Claim: B is closed in A
This is proven through sequential compactness. Suppose that$\exists y_k \in B$ such that $y_k \to y \in A$. We want to show that $y \in B$. Since $y \in A$, $\exists\varepsilon$ such that $D(y, \varepsilon) \subset A$. Then $\exists y_N$ such that $y_N \in D(y, \varepsilon)$. Then $\exists$ straight line in $\mathbb{R}^n$ such that y_N is path connected to y. Also, since $y_N \in B, y_N$ is path connected to x_0. Therefore, x_0 is connected to y which makes $y \in A$. Therefore, B is closed in A.

Claim: $B \neq \varnothing$

This is obvious as we have fixed $x_0 \in B$, so $B \neq \varnothing$.
Then combining with the assumption that A is a connected set, we can conclude that $B = A$

QED

Two disjoint open sets in Separation

Show that $A \subset M$ is not connected iff there exists two *disjoint* open sets U, V such that $U \cap A \neq \varnothing, V \cap A \neq \varnothing$ and $A \subset U \cup V$.

Proof.

$\Leftarrow$

This direction is trivial as it is the definition of topological disconnectedness.

$\Rightarrow$

Show that if $A \subset M$ is not connected, then there exists two disjoint open sets U V such that $U \cap A = \varnothing$, $V \cap A = \varnothing$ and $A \subset U \cup V$.
Suppose that $A \subset M$ is not connected. I want to show that $\exists$ two disjoint open sets that suffice the above three conditions.
For A to be disconnected, we know that $\exists U', V$ as open sets such that

1. $A \cap U' \cap V = \varnothing$
2. $A \cap U' \neq \varnothing$ and $A \cap V \neq \varnothing$
3. $A \subset U' \cup V$

I start with the elements in $A \cap U'$. $\forall x \in A \cap U'$, define $d(x) = d(x, A \cap V) = \inf\{y \in A \cap V | d(x, y)\}$.

Claim: $d(x) > 0 \forall x$
Assume to the contrary that $\exists x_0 \in A \cap U'$ such that $d(x_0) = 0$. Then by definition of infimum, $\forall \varepsilon_k = \frac{1}{k}, \exists y_k \in A \cap V$ such that $d(x_0, y_k) < \varepsilon_k$. Since $A \cap U' \cap V = \varnothing, x_0 \in A \cap U'$ and $y_k \in A \cap V$, then $x_0 \neq y_k \forall k$. Therefore, $x_0 \in \mathrm{AC}(A \cap V)$ and there is a sequence $y_k \in A \cap V$

that converges to x_0.

Note also that $x_0 \in U$ with U being an open set meaning that $\exists \varepsilon > 0$ such that $D(x_0, \varepsilon) \subset U$. Therefore, for the above-mentioned y_k to converge to x_0, there would be $y_k \in D(x_0, \varepsilon) \subset U$, then making that $y_k \in A \cap U \cap V = \varnothing$. This is a contradiction, and all $d(x) > 0$.

Claim: $\exists U = \bigcup_{x \in A \cap U'} D(x, d(x))$ and V as the two disjoint open sets that suffices the above three conditions.

First, the two sets are disjoint. This is because all $d(x)$ are strictly positive, therefore the entire neighbourhood $D(x, d(x)) \subset A \cap U$ which is disjoint from V. Therefore, U as the entire union is disjoint from V.

Also, as the union of open sets, U is open.

Then I can review the criteria:

1. $A \cap U \cap V = \varnothing$
   Suppose otherwise that $\exists a \in A \cap U \cap V$. Since $a \in A \cap U$, this means that $D(a, d(a)) \subset A \cap U'$. Since $a \in V$ as an open set, then $\exists \varepsilon$ such that $D(a, \varepsilon) \subset V$. Collectively, these two imply that $A \cap U' \cap V \neq \varnothing$ which is a contradiction. Therefore, $A \cap U \cap V = \varnothing$

2. $A \cap U \neq \varnothing$.
   This is true as all the x are collected from $A \cap U'$ form the set U. Therefore, all x selected are in $A \cap U \cap U'$ which makes $A \cap U \neq \varnothing$.

3. $A \cap V \neq \varnothing$.
   This is true as the initial assumption when fixing V alongside U'

4. $A \subset U \cup V$.
   Fix $a \in A$. I know that $A \subset U' \cup V$. Then $a \in U'$ or $a \in V$.

   If $a \in V$ then $a \in U \cup V$ and this statement is prove.

If $a \in U'$, then $a \in A \cup U$ which makes the element a one of the x that forms $\bigcup_{x \in A \cap U'} D(x, d(x))$. Therefore, $a \in U$, which implies that $a \in U \cup V$. Therefore, $A \subset U \cup V$.

Therefore, if A is disconnected, $\exists U, V$ as two disjoint open sets such that

1. $A \cap U \cap V = \varnothing$
2. $A \cap U \neq \varnothing$, $A \cap V \neq \varnothing$,
3. $A \subset U \cup V$

QED