Probability and Statistical Inference

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5 Sigma-Algebra

5.1 Motivation

The Banach-Tarski paradox introduces the fundamental problem in measure theory with uncountable latent space Ω. To be more specific, it is our inability to properly define a measure with the aforementioned axioms.

Proposition 5.1.1 Non-existence of extension of length to all subsets of $\mathbb{R}$:

There does not exist a function μ with all the following properties:

a). $\mu : \mathscr{P}(\mathbb{R}) \to [0, \infty]$

b). $\mu(I) = \ell(I)$ $\forall$ open interval I on $\mathbb{R}$

c). Countable additivity: $\mu(\bigcup_k A_k) = \sum_k \mu(A_k)$ for all disjoint $A_k \subset \mathbb{R}$

d). Translation invariant: $\mu(t + A) = \mu(A)$ $\forall A \subset \mathbb{R}$ and $t \in \mathbb{R}$.

Proof: 5.5
The only condition we can relax is a). Instead of the entire power set, we define the measure to be only on a subset of the power set, defined as "σ-algebra:

5.2 Setup

Theorem 5.2.1 Sigma Algebra
A subset $\mathscr{F} \subseteq \mathscr{P}(X)$ is called a **sigma algebra** if:

1. $\varnothing, X \in \mathscr{F}$,
2. If $A \in \mathscr{F}$, then $\overline{A} \in \mathscr{F}$,
3. If $(A_i)_{i\in\mathbb{N}} \subseteq \mathscr{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathscr{F}$.

Definition 5.2.2 (Measurable Set). If $A \in \mathscr{F}$, then A is called an **$\mathscr{F}$-measurable set**.

Remark 5.1 *Examples of sigma algebras:*

- ***Trivial sigma algebra:*** $\mathscr{F} = \{\varnothing, X\}$.
- ***Full power set:*** $\mathscr{F} = \mathscr{P}(X)$.

5.3 Properties

Proposition 5.3.1 Intersection of Sigma Algebras:
The countable intersection of sigma algebras is a sigma algebra. If $\mathscr{F}_i$ is a sigma algebra on X for $i \in I$, then:

$$\bigcap_{i\in I} \mathscr{F}_i \text{ is also a sigma algebra.}$$

Definition 5.3.2 (Sigma Algebra Generated by a Set).
For any $\mathcal{M} \subseteq \mathscr{P}(X)$, the smallest sigma algebra containing $\mathcal{M}$ is denoted $\sigma(\mathcal{M})$ and is called the **sigma algebra generated by** $\mathcal{M}$.

1. Collect all large $\mathscr{F}$ as sigma algebras such that $\mathcal{M} \subseteq \mathscr{F}$.
2. Take their intersection:

$$\sigma(\mathcal{M}) = \bigcap_{\mathcal{M}\subseteq\mathscr{F}} \mathscr{F}.$$

Example 5.3.3 Let $X = \{a, b, c, d\}$ and $\mathcal{M} = \{\{a\}, \{b\}\}$. Then:

$$\sigma(\mathcal{M}) = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}\}.$$

Theorem 5.3.4 closure property of σ-algebras
For a sigma-algebra $\mathscr{F}$, if $A \in \mathscr{F}$, then $\sigma(A) \subseteq \mathscr{F}$.

Proposition 5.3.5 Sigma Algebra on subsets:
If a σ-algebra on a larger space is defined, then any subset of that space has a σ-algebra by intersecting everything with that subset.

5.4 Borel σ-field on $\mathbb{R}$

Definition 5.4.1 (Borel Sigma Algebra).
Let X be a topological space.
The Borel sigma algebra $\mathscr{B}(X)$ is the sigma algebra **generated by all open sets** of X.

Lemma 5.4.2 Compactness:
All compact subset has a finite measure w.r.t. Borel sigma algebra.

Remark 5.2 *The triplet "(Set, σ-algebra, measure)": $(X, \mathscr{F}, \mu)$ is called a* ***measure space****.*

Remark 5.3 *The Borel sigma algebra is particularly useful in analysis and probability theory:*

- *For continuous random variables, the pre-image of an open set under a continuous mapping $f : \mathbb{R} \to \mathbb{R}$ is measurable since it belongs to $\mathscr{B}(\mathbb{R})$.*
- *$\mathscr{B}(\mathbb{R})$ is the natural sigma algebra for defining measures, such as the Lebesgue measure.*

Definition 5.4.3 (Borel σ-field on $\mathbb{R}^d$). The Borel σ-field on $\mathbb{R}^d$, denoted $\mathscr{B}^d$, is the smallest σ-algebra on $\mathbb{R}^d$ containing all Cartesian products of univariate Borel sets:

$$\prod_{i=1}^{d}(a_i, b_i).$$

5.5 Proof on Proposition 5.1.1:

Proof.
Construction:
Define the interval $I = (0, 1]$ with an equivalence relation $x \sim y$ if $x - y \in \mathbb{Q}$. That is:

$$[x] = \{x + r \mid r \in \mathbb{Q}, x \in I\}.$$

This partitions I into disjoint sets.
Pick $A \subseteq I$ with:

i) $\forall x, y \in A$, $x \sim y \implies x = y$,

ii) For each $x \in I$, if $x \in [x]$ for some x, then $x + r \in A$, where $A_i = x + A$.

Claim 5.5.1:
Disjointness of Shifts: If $A_n = A + r_n$, where r_n is an enumeration of $\mathbb{Q} \cap (-1, 1)$, then:

$$A_n \cap A_m = \varnothing \quad \text{for } n \neq m.$$

Suppose $x \in A_n \cap A_m$. Then:

$$x \in A + r_n \quad \text{and} \quad x \in A + r_m.$$

This implies:

$$x = a + r_n \quad \text{and} \quad x = a' + r_m \implies r_n - r_m \in \mathbb{Q}.$$

By the construction of A, this forces $r_n = r_m$, which is a contradiction. Thus $A_n \cap A_m = \varnothing$ for $n \neq m$.

Claim 5.5.2:
[Covering of $(0, 1]$]

$$(0, 1] \subseteq \bigcup_{n \in \mathbb{N}} A_n \subseteq (-1, 2).$$

(i) The first inclusion is given since all A_n serve as a partition of $(0, 1]$,

(ii) By construction, for all $x \in \bigcup_{n \in \mathbb{N}} A_n$, there exists $i \in \mathbb{N}$ such that $x \in A + r_i$. Since $A \subseteq (0, 1]$, we conclude:

$$x \in \mathbb{R} \cap (-1, 2).$$

By property (ii), $\mu(x + A) = \mu(A)$ for all $x \in \mathbb{R}$. By Claim 2:

$$\mu((0, 1]) \leq \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \mu((-1, 2)).$$

We know $\mu((0, 1]) = C < \infty$. Then:

$$\mu((-1, 2)) = \mu((-1, 0]) + \mu((0, 1]) + \mu((1, 2]) = 3C.$$

Thus:

$$C \leq \sum_{n=1}^{\infty} \mu(A_n) \leq 3C.$$

If $\mu(A) > 0$, this leads to a contradiction such that an item bounded above by a finite value $3C$ diverges to infinity. Therefore:

$$\mu(A) = 0.$$

$\square$