

Probability and Statistical Inference

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2 Sampling

2.1 Sample Space and Probability Models

Definition 2.1.1 (Sample Space).

The specification of a probability model requires:

1. A sample space, Ω : The set of all possible outcomes in the problem.
2. A probability assignment for these outcomes (subsets of Ω).

Consider the following examples to help understanding the construction of probability models:

Example 2.1.2 Coin Tossing

$$\Omega = \{H, T\}.$$

- If assuming the coin is balanced, we state $P(H) = P(T) = \frac{1}{2}$.
- This assumption can be extended to all subsets of Ω .
- Assumption can be justified empirically using the **Law of Large Numbers (LLN)**:

$$P(A) = \lim_{n \rightarrow \infty} f_n(A).$$

For a proper definition of "plim"

To generalize the above results:

Definition 2.1.3 (Uniform probability measure/distribution).

- $\Omega = \{\omega_1, \dots, \omega_N\}$ or countable
- **Uniform probability measure (distribution):** For any $A \subseteq \Omega$,

$$P(A) = \frac{|A|}{|\Omega|}.$$

and define

$$P(\omega_i) = \frac{1}{N}, \forall i \in \{1, 2, \dots, N\}$$

- **Warning on abuse of notation:** Note that this is not a proper probability measure, as it assigns probabilities to individual elements, not subsets.

$$P(\omega_i) \equiv P(\{\omega_i\})$$

even though $\omega_i \in \Omega$, $\{\omega_i\} \subseteq \Omega$, and $\{\omega_i\} \in \mathcal{P}(\Omega)$ as the power set.

Therefore, we have the above function P defined to be $\mathcal{P}(\Omega) \rightarrow [0, 1]$

Example 2.1.4 Fair Die

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

- Let $A = \text{"even numbers"} = \{2, 4, 6\}$. Then:

$$P(A) = \frac{|A|}{|\Omega|} = \frac{3}{6} = \frac{1}{2}.$$

Remark 2.1 *Conditioning and Specifying Ω is crucial*

Example 2.1.5 Twin paradox example

- **Assumptions:**
 - (i) Gender of newborn: $P(\text{Girl}) = P(\text{Boy}) = \frac{1}{2}$.
 - (ii) Gender of one child is independent of the gender of the other child.
- **Known:** A family has two children, one of whom is a girl.
- **Question:** What is $P(\text{both children are girls})$?

- Case (i): If we know the gender of the first child, then Ω considers only the second child

$$\Omega_i = \{G, B\}.$$

Then:

$$P(G) = \frac{1}{2}.$$

- Case (ii): If we know at least one child is a girl:

$$\Omega_{ii} = \{GG, GB, BG\}.$$

Then:

$$P(GG) = \frac{1}{3}.$$

2.2 Bernoulli Trials and Binomial Distribution

Example 2.2.1 Bernoulli Distribution

- Suppose that we have a sequence of n independent experiments

$$P(\text{Success}) = p, \quad P(\text{Failure}) = 1 - p$$

- We have the sample space:

$$\Omega = \{0, 1\}^n \quad (\text{Cartesian product of the simple } \{0, 1\}).$$

- A typical element (draw) $\omega \in \Omega$ is $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ as a sequence.
- Define $S_n : \Omega \rightarrow \mathbb{R}$, $S_n(\omega) = \sum_{i=1}^n \omega_i$ to be the count of success.

Claim 2.2.2:

Then the **independent experiment** with its probability measure:

$$P(\omega) = p^{S_n(\omega)}(1 - p)^{n - S_n(\omega)}$$

Remark 2.2 S_n is a **random variable**, a function from Ω endowed with the probability measure P and maps outcomes to $\mathcal{S} \equiv \{0, 1, \dots, n\}$.

Since the original Bernoulli trial is simple enough to be binary: $\Omega = \{0, 1\}$, the count of success can be reduced into the simple sum in the sequence. Therefore, we can then define a new probability measure (distribution) P^{S_n} on S such that:

Theorem 2.2.3 Push-Forward Measure

We can then define a new probability measure (distribution) P^{S_n} on S such that

$$\forall x \in S, P(x) \equiv \sum_{\omega \in S_n^{-1}(x)} P(\omega) = P[S_n^{-1}(x)] \equiv P^{S_n}(x)$$

Where

$$S_n^{-1}(A) \equiv \{\omega \in \Omega, S_n(\omega) \in A\} \quad \text{is the pre-image}$$

For computation purposes:

$$\forall x \in \mathcal{S}, P^{S_n}(x) = |S_n^{-1}(x)| p^x (1-p)^{n-x}$$

Intuitively speaking, the pre-image $S_n^{-1}(x)$ indicates all possible sequences ω such that they contain exactly x counts of success. We give the calculation as follows

Definition 2.2.4 (Binomial number).

$$|S_n^{-1}(x)| = \binom{n}{x} = \frac{n!}{x!(n-x)!} = \binom{n}{n-x}$$

To be the number of ways to pick x elements from n elements without order.

Definition 2.2.5 (Binomial Distribution).

Binomial Distribution: $\text{Binom}(n, p)$ is the push-forward probability measure of the Bernoulli trial with n experiments and success probability p via the random variable S_n which counts success.

$$\forall x \in \mathcal{S}, P^{S_n}(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

and

$$\sum_{n=0}^n \binom{n}{x} p^x (1-p)^{n-x} = 1$$

We say the **random variable** S_n follows the **binomial distribution**, $S_n \sim \text{Binom}(n, p)$.