Probability and Statistical Inference

Tianqi Zhang
Emory University

Apr 17th 2025

13 Lebesgue Integration

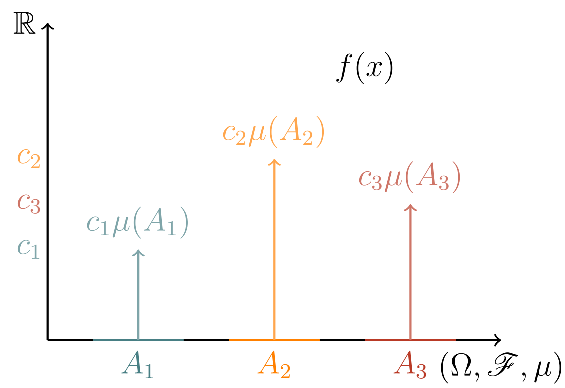
13.1 Set up:

- In $(\Omega, \mathscr{F}, \mu)$, let $f : \Omega \to \mathbb{R}$ be measurable.
- **Recall:** Indicator functions are measurable.
  - Integration of indicator functions equals the measure of the indicator function.

Definition 13.1.1 (Simple Functions).
f is said to be a simple function is f is a finite linear combinations of indicator functions:

$$f(x) = \sum_{i=1}^{n} c_i \cdot \mathbb{I}_{A_i}(x), \quad \text{for some measurable sets } \{A_i\} \text{ and constants } c_i \in \mathbb{R}.$$

Example 13.1.2 Simple functions



Definition 13.1.3 (Lebesgue Integral (for Simple Functions)).
Define the set

$$S^+ := \{f : \Omega \to \mathbb{R} \mid f \text{ is simple function, } f \geq 0\}.$$

For $f \in S^+$: define its Lebesgue integral w.r.t. its measure μ as:

$$\int_\Omega f\, d\mu := \sum_{i=1}^{n} c_i \cdot \mu(A_i), \quad \forall f \in S^+$$

Notation:

$$\int_\Omega f\, d\mu \quad \text{or} \quad \int_\Omega f(x)\, d\mu(x).$$

Proposition 13.1.4 Lebesgue Integral:

- **Linearity:** $\int_\Omega (\alpha f + \beta g)\, d\mu = \alpha \int_\Omega f\, d\mu + \beta \int_\Omega g\, d\mu, \quad \forall f, g \in S^+, \alpha, \beta \geq 0.$
- **Monotonicity:** If $f \leq g$, then $\int_\Omega f\, d\mu \leq \int_\Omega g\, d\mu$.

13.2 Lebesgue Integral

Definition 13.2.1 (Lebesgue Integral).

1. For any $f : \Omega \to \mathbb{R}$, define the set $\mathcal{S}_{\{}^{+} = \{h \in S^+ \mid h < f\}$.
2. Compute all Lebesgue integral $I(h)$ according to the previous definition.

We define

$$\int_\Omega f d\mu = \sup_{h \in \mathcal{S}_{\{}^{+}} I(h)$$

And we note that f is μ-integrable (or just integrable if the context if clear) if $\int_\Omega f d\mu < \infty$.

13.3 Important results and theorems

Lemma 13.3.1 Fatou's Lemma:
Given a measure space $(\Omega, \mathscr{F}, \mu)$ and a sequence of measurable, non-negative functions $\{f_n\}$ each maps from $(\Omega, \mathscr{F}, \mu) \to (\mathbb{R}, \mathscr{B}, \cdot)$. Define a function $f \equiv \liminf_{n\to\infty} f_n(x)$, then f is measurable and we have the following inequality:

$$\int_\Omega f\, d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n\, d\mu$$

Corollary 13.3.2 Beppo Levi's Monotone Convergence:

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and let $\{X_n\}$ be a sequence of non-negative measurable functions defined on Ω. Suppose:

$$0 \leq X_n(\omega) \leq X_{n+1}(\omega) \quad \forall \omega \in \Omega \quad \text{(monotonically increasing sequence of functions)}$$

Define the pointwise limit function:

$$X(\omega) = \lim_{n\to\infty} X_n(\omega), \quad \forall \omega \in \Omega$$

Then:

$$\lim_{n\to\infty} \int_\Omega X_n \, d\mu = \int_\Omega X \, d\mu$$

Theorem 13.3.3 Lebesgue's Dominated Convergence Theorem
Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of measurable functions mapping from $(\Omega, \mathscr{F}, \mu)$ to $(\mathbb{R}, \mathscr{B})$. Suppose:

- $f_n \to f$ pointwise almost everywhere on Ω, and
- there exists an integrable function $g \colon \Omega \to \mathbb{R}$ such that $|f_n(\omega)| \leq g(\omega)$ for all $\omega \in \Omega$ and all $n \in \mathbb{N}$.

Then all f_n and f integrable, and

$$\lim_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega \lim_{n\to\infty} f_n d\mu = \int_\Omega f d\mu$$

13.4 Expectation

Definition 13.4.1 (Expectation). Consider a measure space $(\Omega, \mathscr{F}, P)$ and a random variable $X : \Omega \to \mathbb{R}$. Its expectation is defined to be a Lebesgue integral

$$\mathbf{E}[X] = \int_\Omega X(\omega) dP(\omega)$$

Corollary 13.4.2 Jensen's Inequality:
If X is a random variable and $\varphi : \mathbb{R} \to \mathbb{R}$ is a convex function, then:

$$\varphi(\mathbb{E}[X]) \leq \mathbb{E}[\varphi(X)]$$

Special Case: For $\varphi(x) = |x|$, we obtain:

$$|\mathbb{E}[X]| \leq \mathbb{E}[|X|]$$

Intuition: The expectation of a convex function applied to X is at least as large as applying the function to the expectation of X . Convexity "pulls the curve upwards," leading to this inequality.

Corollary 13.4.3 Markov's Inequality:

For a non-negative random variable X and any $\alpha > 0$.

$$P(|X| > \alpha) \leq \frac{\mathbb{E}[|X|]}{\alpha}$$

Intuition: The probability that X exceeds some threshold α is bounded by the ratio of its expected value to α. It provides an upper bound on tail probabilities.