5.1 Uniform Convergence

Prove a function converges pointwise

Consider the function $f_k[0,1] \to \mathbb{R}$, $f_k = \begin{cases} 0, x \in \left[\frac{1}{k}, 1\right] \\ -kx+1, x \in \left[0, \frac{1}{k}\right] \end{cases}$

Show that the function converges to $f = 0$.

Proof.
Fix $x \in [0,1]$, then by Archimedean principle, $\exists K$ such that $\frac{1}{K} < x$. Therefore, $\forall k > K, f_k(x) = 0 = f$. Therefore, $f_k \to f$ pointwise.

However, f_k does not converge to f uniformly.

Fix $\varepsilon_0 = \frac{1}{3}$. We can easily show that by fixing $x = \frac{1}{2k}$ thus $x < \frac{1}{k}$. We have $f_k(x) = \frac{1}{2}$ and $f(x) = 0$. Therefore, $|f_k(x) - f(x)| > \varepsilon_0$.

QED

Uniform Convergence Preserves Continuity

Statement: Let $f_k(x) : A \to N$ be a sequence of continuous function and let $f_k(x) \to f(x)$ uniformly. Prove that f is continuous on A.

Proof.
Fix f_k and $x \in A$. We know that f_k is continuous. Therefore, fixing $\varepsilon > 0, \exists \delta$. Fixing $y \in D(x, \delta)$, we have $\rho(f(x), f(y)) < \varepsilon/3$.

Also from uniform convergence, with the fixed $\varepsilon > 0, \exists K$ such that $\forall k > K, \rho(f_k(x) - f(x)) < \varepsilon/3 \ \forall x$. We have the following triangular inequality

$$\begin{aligned}\rho(f(y), f(x)) &\leq \underbrace{\rho(f_k(y), f(x)) + \rho(f(y), f_k(x))}_{\text{Bounded by uniform convergence}} + \underbrace{\rho(f_k(y), f_k(x))}_{\text{Bounded by continuity}} \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &< \varepsilon\end{aligned}$$

Therefore, f is continuous.

QED

5.2 Cauchy Criterion

Uniformly Cauchy $\Leftrightarrow$ Uniform Convergence

Statement: Let $(N\rho)$ be a complete GMS, $f_k : A \to N$. Then $f_k \to f$ uniformly on A iff f_k is uniformly Cauchy, i.e. $\forall \varepsilon > 0, \exists L$ such that $\forall j, k > L, \rho(f_j(x), f_k(x)) < \varepsilon \ \forall x \in A$.

$\Leftarrow$

This direction is rather trivial.

Proof.

Suppose that f_k uniformly converges to f, then $\forall \varepsilon, \exists K$ such that $\forall j, k > K$,

$$\rho(f_k, f) < \varepsilon/2$$
$$\rho(f_j, f) < \varepsilon/2$$

Therefore,

$$\begin{aligned}\rho(f_k, f_j) &\leq \rho(f_k, f) + \rho(f_j, f) \\ &< \varepsilon/2 + \varepsilon/2 \\ &< \varepsilon\end{aligned}$$

Therefore, f_k uniformly Cauchy. **QED**

$\Rightarrow$

Proof.
Step 1: Show the existence of point-wise limit
Fix $\tilde{x} \in A, \{f_n(\tilde{x})\}$ is a sequence in N. Since f_n uniformly Cauchy, $\{f(\tilde{x})\}$ is a Cauchy sequence in N. Assuming N is complete, $\exists f(\tilde{x}) \in N$ such that $\{f(\tilde{x})\} \to f(\tilde{x})$.

Step 2: Show $f_n(x) \to f(x)$
From uniformly Cauchy, I know that $\forall \varepsilon > 0, \exists L$ such that $\forall i, j > L$,

$$\rho(f_i(x), f_j(x)) < \varepsilon/2$$

Since $\forall x \in A, f_j(x) \to f(x), \exists L_x$, in particular, picking $L_x \geq L$, such that $\forall j > L_x, \rho(f_j(x), f(x)) > \varepsilon/2$. Therefore,

$$\begin{aligned} \rho(f_n, f(x)) \leq & \underbrace{\rho(f_n, f_{L_x}(x))}_{\text{From Uniformly Cauchy}} + \underbrace{\rho(f_{L_x}(x), f(x))}_{\text{From pointwise limit}} \\ & < \varepsilon/2 + \varepsilon/2 \\ & < \varepsilon \end{aligned}$$

Therefore, $f_n \to f$ uniformly.

QED

5.3 Integration and Differentiation with Uniform Convergence

Uniform Convergence preserves Integrability

Statement: Suppose that $f_n : [a, b] \to \mathbb{R}$ be integrable, $f_n \to f$ uniformly on $[a, b]$. Then f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

or

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx$$

Proof.

Show that f is integrable

We know that $f_n \to f$ uniformly. By definition, $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$, $|f_n - f| < \varepsilon\ \forall x$.

Therefore, fixing any $k > K$, for any particular sub-interval, if α is an upper bound for the set $\{f_k(x)|x \in [x_j, x_{j+1}]\}$. Then $\alpha + \varepsilon$ should be an upper bound for $\{f(x)|x \in [x_j, x_{j+1}]\}$. This applies to the least upper bound for any partition P and for any $k > K$:

$$\begin{aligned}
\sup\{f(x)|x \in [x_j, x_{j+1}]\} &\leq \{f_k(x)|x \in [x_j, x_{j+1}]\} + \varepsilon \\
U(f, P) &\leq U(f_k, P) + \varepsilon(b-a) \\
\inf_P\{U(f, P)\} &\leq \inf_P\{U(f_k, P) + \varepsilon(b-a)\} \\
\overline{\int_a^b} f(x)dx &\leq \overline{\int_a^b} f_k(x)dx + \varepsilon(b-a) \\
\overline{\int_a^b} f(x)dx &\leq \int_a^b f_k(x)dx + \varepsilon(b-a)
\end{aligned}$$

By similar idea,

$$\begin{aligned}
L(f, P) &\geq U(f_k, P) - \varepsilon(b-a) \\
\sup_P\{L(f, P)\} &\geq \sup_P\{(f_k, P) - \varepsilon(b-a)\} \\
\underline{\int_a^b} f(x)dx &\geq \underline{\int_a^b} f_k(x)dx - \varepsilon(b-a) \\
\underline{\int_a^b} f(x)dx &\geq \int_a^b f_k(x)dx - \varepsilon(b-a)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx\right| &\leq 2\varepsilon(b-a) \\
\lim_{\varepsilon\to 0}\left|\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx\right| &\leq \lim_{\varepsilon\to 0} 2\varepsilon(b-a)
\end{aligned}$$

$\underline{\int}_a^b f(x)dx$ and $\overline{\int}_a^b f(x)dx$ are squeezed by $\varepsilon(b-a)$,

$$\lim_{\varepsilon\to 0}\underline{\int}_a^b f(x)dx = \lim_{\varepsilon\to 0}\overline{\int}_a^b f(x)dx$$

Therefore, f is integrable.

Step 2: Show that: $\int_a^b f(x)dx = \lim_{k\to\infty}\int_a^b f_k(x)dx$
All f_k are integrable on compact domain implies that $\exists M$ such that

$$\left|\int_a^b f(x)dx\right| \le M(b-a)$$

Since f_k converges to f uniformly, given $\varepsilon > 0$, choose N such that $\forall k \ge N$ implies $|f_k(x) - f(x)| < \varepsilon/(b-a)$. Then

$$\begin{aligned}\left|\int_a^b f_k(x)dx - \int_a^b f(x)dx\right| &= \left|\int_a^b f_k(x) - f(x)dx\right| \\ &< \frac{\varepsilon}{b-a}\cdot(b-a) = \varepsilon\end{aligned}$$

Therefore, $\int_a^b f(x)dx = \lim_{k\to\infty}\int_a^b f_k(x)dx$

Eventually, I conclude that uniform convergence preserves integrability.

QED

Corollary 5.3.2

Statement: Suppose that $g_k : [a,b] \to \mathbb{R}$ are Riemann integrable an d$\sum_{k=1}^{\infty} g_k$ converges uniformly on $[a,b]$. Then,

$$\int_a^b \left(\sum_{k=1}^{\infty} g_k(x)\right)dx = \sum_{k=1}^{\infty}\left(\int_a^b g_k(x)dx\right)$$

Let $f_n = \sum_{k=1}^{n} g_k$, then $f_n \to f = \sum_{k=1}^{\infty} g_k$ uniformly. By 5.3.1, $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$.

Uniform Convergence preserves Differentiability

Statement: Let $f_n : [a,b] \to \mathbb{R}$ be differentiable, converging pointwise to $f : [a,b] \to \mathbb{R}$. Consider the sequence $\{f_n'(x)\}$, if it is

1. Continuous
2. Uniformly convergence to some function g

Then,

$$\lim_{n\to\infty} \frac{d}{dx} f_n(x) = \frac{d}{dx} \lim_{n\to\infty} f_n(x)$$
$$g(x) = f'(x)$$

Proof.
Since f_n' continuous, by Fundamental Theorem of Calculus, for some $x_0 \in [a,b]$,

$$\begin{aligned}
f_n(x) &= f_n(x_0) + \int_{x_0}^{x} f_n'(t)dt \\
\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} f_n(x_0) + \lim_{n\to\infty} \int_{x_0}^{x} f_n'(t)dt \\
f(x) &= f(x_0) + \lim_{n\to\infty} \int_{x_0}^{x} f_n'(t)dt \\
&= f(x_0) + \int_{x_0}^{x} \lim_{n\to\infty} f_n'(t)dt \quad \text{By 5.3.1} \\
&= f(x_0) + \int_{x_0}^{x} g(t)dt \\
\frac{d}{dx} f(x) &= \frac{d}{dx}\left(f(x_0) + \int_{x_0}^{x} g(t)dt \right) \\
f'(x) &= 0 + g(x)
\end{aligned}$$

QED

5.5 Space of Continuous Functions

Convergence in functional space and Uniform Convergence

Suppose that $f : A \to N$. Define $\|f\|_{\mathcal{C}_b} = \sup\{\|f(x)\|_N | x \in A\}$.

Statement: $f_k \to f$ in $\mathcal{C}_b$ iff $f_k(x) \to f(x)$ uniformly.

Proof.

$\Rightarrow$

Suppose that $f_k \to f \in \mathcal{C}_b$. Fix any $x \in A$,

$$\begin{aligned} \|f_k(x) - f(x)\|_N &= \|(f_k - f)(x)\|_N \\ &\leq \|f_k - f\|_{\mathcal{C}_b} \quad \text{As LUB} \end{aligned}$$

Since $\|f_k - f\|_{\mathcal{C}_b}$ converges to 0, then$\|f_k(x) - f(x)\|_N$ converges.

$\Leftarrow$ If fixing ε and $\|f_k(x) - f(x)\| < \varepsilon$, then taking the supremum of both sides,

$$\begin{aligned} \sup\{\|f_k(x) - f(x)\|_N | x \in A\} &< \sup\{\varepsilon | x \in A\} \\ \|f_k - f\|_{\mathcal{C}_b} &< \varepsilon \end{aligned}$$

Therefore, $f_k \to f$ in $\mathcal{C}_b$.

QED

Defining Functional Metrics

Suppose that $(M, d), (N, \rho)$ are both metric spaces, $A \subset M$. Suppose that $f, g : A \to N$. Defining the functional metrics $d_{\mathcal{C}_b}(f, g) = \sup\{\rho(f(x), g(x))_N | x \in A\}$.

Statement: $(\mathcal{C}_b(A, N), d_{\mathcal{C}_b})$ is a well-defined metric space.

Proof.

Positive Definite

This is trivial as ρ is a well-defined distance that is positive definite. All $\rho(f(x), g(x))_N \geq 0$. Therefore, the supremum has to be greater than equal to 0. Also, for $d_{\mathcal{C}_b}(f, g) = 0$. By definition, $\sup\{\rho(f(x), g(x))_N | x \in A\} = 0$, that is, $\rho(f(x), g(x))_N = 0 \forall x \in A$. This is equivalent as saying $f = g$. Therefore, $d_{\mathcal{C}_b}(f, g)$ is positive definite.

Interchangeable

This is also trivial as $\rho(f(x), g(x)) = \rho(g(x), f(x))$.

Triangular inequality

Suppose that $f, g, h \in \mathcal{C}_b$

$$d_{\mathcal{C}_b}(f, g) = \sup\{\rho(f(x), g(x))_N | x \in A\}$$

Since

$$\rho(f(x), g(x))_N \leq \rho(f(x), h(x))_N + \rho(h(x), g(x))_N$$
$$\sup\{\rho(f(x), g(x))_N | x \in A\} \leq \sup\{\rho(f(x), h(x))_N + \rho(h(x), g(x))_N | x \in A\}$$

Also,

$$\sup\{\rho(f(x), h(x))_N + \rho(h(x), g(x))_N | x \in A\} \leq \sup\{\rho(f(x), h(x))_N | x \in A\} + \sup\{\rho(g(x), h(x))_N | x \in A\}$$

Therefore,

$$\sup\{\rho(f(x), g(x))_N | x \in A\} \leq \sup\{\rho(f(x), h(x))_N | x \in A\} + \sup\{\rho(g(x), h(x))_N | x \in A\}$$
$$d_{\mathcal{C}_b}(f, g) \leq d_{\mathcal{C}_b}(f, h) + d_{\mathcal{C}_b}(h, g)$$

QED

Remark: Same with Norm, i.e. if N is a normed space, then so is $\mathcal{C}_b(A, N)$

Completeness of $C_b(A, N)$

Statement: If N is complete, then $\mathcal{C}_b(A, N)$ is complete.
Use Cauchy Convergence

Proof.
Suppose that $\{f_k\} \in \mathcal{C}_b$ to be a Cauchy sequence. Then $\forall \varepsilon, \exists K$ such that $\forall i, j > K$,

$$\|f_i - f_j\|_{\mathcal{C}_b} < \varepsilon$$
$$\|f_i(x) - f_j(x)\|_N < \varepsilon \quad \forall x \in A$$

Therefore, $\{f_k\}$ is uniformly Cauchy in N. Since N is complete, this implies that f_k uniformly converges to some f. Since uniform convergence preserves continuity, then f is continuous.

Next, I need to show that f is bounded, fixing any $x \in A$, I have the following inequality:

$$\|f(x)\|_N \leq \|f_k(x) - f(x) + f_k(x)\|_N$$
$$\|f(x)\|_N \leq \|f_k(x) - f(x)\|_N + \|f_k(x)\|_N$$

Fixing any ε, $\exists K$ such that the first term is bounded by ε. Also, $f_k \in \mathcal{C}_b$ means that all f_k are bounded. Therefore, the second term $\|f_k(x)\|_N$ is finite. All implies that $\|f(x)\|_N$ is bounded above by a finite number. Therefore, $f \in \mathcal{C}_b$. Eventually, the limit is in the metric space implies that the metric space $\mathcal{C}_b(A, N)$ is complete.

QED

5.7 Contract Mapping

Proof of Contract Mapping Principle

Observe two consecutive points x_{n+1}, x_n.

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\Phi(x_n), \Phi(x_{n-1})) \\ &\leq kd(x_n, x_{n-1}) = kd(\Phi(x_{n-1}), \Phi(x_{n-2})) \\ &\leq k^2 d(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq k^n d(x_1, x_0) \end{aligned}$$

Observe any two points, that is, for some $p \in \mathbb{N}^+$, by triangular inequality:

$$
\begin{aligned}
d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\
&\leq k^n d(x_1, x_0) + \cdots + k^{n+p-1} d(x_1, x_0) \\
&\leq \sum_{j=1}^{p-1} k^{n+j} d(x_1, x_0) \\
&\leq \frac{k^n}{1-k} d(x_1, x_0) \\
\lim_{n\to\infty} d(x_n, x_{n+p}) &\leq \lim_{n\to\infty} \frac{k^n}{1-k} d(x_1, x_0) \\
&\leq 0 \quad \text{Since } k \in (0,1)
\end{aligned}
$$

Therefore, the distance between any two points when indices get sufficiently large approaches to 0. This implies that $\{x_n\}$ is Cauchy. Since M is complete, Cauchy sequence converges, i.e. $\exists x \in M$ such that $x_n \to x$.

Claim: x is a fixed point.

Observe that $d(\Phi(x), \Phi(y)) \leq kd(x, y)\ \forall x, y \in M$ implies that Φ is Lipschitz which implies continuity.
Observe the mapping:

$$
\begin{aligned}
x_{n+1} &= \Phi(x_n) \\
\lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} \Phi(x_n) \\
&= \Phi\Big(\lim_{n\to\infty} x_n\Big) \quad \text{Since } \Phi \text{ is continuous} \\
x &= \Phi(x)
\end{aligned}
$$

Therefore, x is a fixed point.

Claim: x is the unique fixed point.

Suppose not, that $\exists y \in M, y \neq x$, and $\Phi(y) = y$. Then $d(x, y) > 0$, and $d(\Phi(x), \Phi(y)) <$

$d(x, y)$ holds with strict inequality. Since both are fixed points, $\Phi(x) = x$ and $\Phi(y) = 0$. Therefore, $d(\Phi(x), \Phi(y)) = d(x, y)$. Then we have

$$d(x, y) < d(x, y)$$

which is a contradiction to the basic assumption of defining distance. Therefore, $x = y$.