## 4.1 Continuity

### Theorem 4.1: The 4 equivalent conditions

Suppose that $\exists (M,d),(N,\rho)$ as two metric spaces, $A \subset M$, $f : A \to N$ be a function. Prove that the following properties are equivalent:

i. $f$ is continuous on $A$

ii. $\forall$ convergent sequence $x_k \to x_0 \in A$, we have the image sequence $f(x_k) \to f(x_0)$. This is the same as saying

$$\lim_{k\to\infty} f(x_k) = f\left(\lim_{k\to\infty}(x_k)\right)$$

iii. $\forall U_i \in N$, the pre-image $f^{-1}(U_i) \subset A$ is open in $A$. i.e. $\exists$ open set $V \subset M$ such that $f^{-1}(U_i) = V \cap A$.

iv. $\forall$ closed set $F \subset N$, the pre-image $f^{-1}(F) \subset A$ is closed in $A$. i.e. $\exists$ closed set $G \subset M$ such that $f^{-1}(F) = G \cap A$.

Strategy: $i \to ii \to iv \to iii \to i$

$i \to ii$

**Proof.**
Suppose that $f : A \to N$ is continuous at $x_0$,
Fix a sequence $x_k \to x_0 \in A$. We want to show that $f(x_k) \to f(x_0) \in N$. That is, $\forall \varepsilon > 0, \exists N$ such that $\forall k > N, \rho(f(x_k), f(x_0)) < \varepsilon$.

Fix any $\varepsilon > 0$. By continuity, this guarantees a $\delta$ such that $\forall x \in D(x_0,\delta), f(x) \in D_\rho(f(x_0),\varepsilon)$. By convergence, $\exists N$ such that $\forall k > N, d(x_n, x_0) < \delta$. Then with the same $N$, $\forall k > N, \rho(f(x_k), f(x_0)) < \varepsilon$ which completes the proof.

**QED**

$ii \to iv$

**Proof.**
We start by fixing a closed set $F \subset N$ and its pre-image with sequential compactness. Therefore, we fix $x \in A \cap \mathrm{cl}(f^{-1}(F))$. Since closures are closed sets, $\exists x_k \in \mathrm{cl}(f^{-1}(F))$ such that

$x_k \to x$. By $ii)$, we have $f(x_k) \to f(x) \in F$. Then $f(x_k) \in \text{cl}(F) \forall x_k$. Then $f(x) \in \text{cl}(F)$. Since $F$ is closed, $\text{cl}(F) = F$.

Then $f(x) \in F$, $x \in f^{-1}(F)$, which implies $A \cap \text{cl}(f^{-1}(F)) \subset f^{-1}(F)$. By definition, $f^{-1}(F) \subset \text{cl}(f^{-1}(F)) \cap A$. Therefore, $f^{-1}(F) = \text{cl}(f^{-1}(F)) \cap A$ which makes $f^{-1}(F)$ a closed set.

**QED**

$iv \to iii$

**Proof.**
Fixing $U \subset N$ to be open, then $U^C$ is a closed set in $N$. By $iii)$, we know that $f^{-1}(U^C) \subset M$ is a closed set. Therefore, $\left(f^{-1}(U^C)\right)^C = f^{-1}(U)$ should be an open set.

**QED**

$iii \to i$

Fix $x_0 \in A$ and let $U = D_\rho(f(x_0), \varepsilon) \subset N$ is open. So $f(x_0) \in U$ which makes $x_0 \in A \cap f^{-1}U$. By $iii)$, $f^{-1}(U)$ is open in $A$. Then $\exists \delta > 0$ such that $D(x_0, \delta) \subset f^{-1}(U) = f^{-1}(D_\rho(f(x_0), \varepsilon))$. This makes the function continuous.

# 4.2 Images of Compact Set and Connected Set

## Theorem 4.2.a

**Theorem:** Suppose $f : M \to N$ is continuous. If $K \subset M$ connected, then $f(K)$ is connected.

**Proof.**
Assume to the contrary that $f(K)$ is not connected, then $\exists U, V \in N$ open sets such that they separate $f(K)$.

By Theorem 1, we know that $f^{-1}(U)$ and $f^{-1}(V)$ are open sets.

**Claim:** $f^{-1}(U)$ and $f^{-1}(V)$ separate $K$:

We know that $U \cap f(K) \neq \varnothing$, then let $a \in U \cap f(K)$, then $\exists x \in K$ such that $f(x) = a$, $a \in U$ implies that $x \in f^{-1}(U)$, therefore, $x \in f^{-1}(U) \cap K$, $f^{-1}(U) \cap K \neq \varnothing$.

Similarly, $f^{-1}(V) \cap K \neq \varnothing$.

Also, $U \cap V \cap f^{-1}(K) = \varnothing$. Suppose that $\exists a \in f^{-1}(U) \cap f^{-1}(V) \cap K$, then $\exists f(a) \in U \cap V \cap f(K)$ this is a contradiction. Therefore, $f^{-1}(U) \cap f^{-1}(V) \cap K = \varnothing$.

Thirdly, I want to show that $K \subset f^{-1}(U) \cup f^{-1}(V)$. Let $a \in K$, then $\exists x \in f(K)$ such that $f(a) = x$. Since $f(K) \subset U \cup V$, then $x \in U \cup V$, which implies that $a \in f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$. Therefore, $K \subset f^{-1}(U) \cup f^{-1}(V)$.

The above three properties make $f^{-1}(U)$ and $f^{-1}(V)$ two open sets that separate $K$. This is a contradiction to the previous assumption that $K$ is connected. Therefore, $f(K)$ should be connected.
some properties of pre-image that I am not sure if those are correct and can be used directly.

**QED**

## Theorem 4.2.b

**Theorem:** $B \subset M$ is compact implies that $f(B) \subset N$ is compact.

**Proof.**
Given a sequence $f(x_n) \in f(B)$ with all $x_n \in B$. Since $B$ is compact, then $\exists x_{n_k} \subset x_n$ such that $x_{n_k} \to x_0 \in B$.

Since $f$ is continuous, and continuous function preserves the convergence of sequence, therefore,

$$f(x_{n_k}) \to f(x_0) \subset f(B)$$

Therefore, $f(B)$ is also a compact set.

**QED**

## 4.3 Operations

### Operations

### Max/Min Value Theorem

**Theorem:** Let $K \subset M$ be compact, $f : K \to \mathbb{R}$ be continuous, then $\exists x_0, x_1 \in K$ such that

$$f(x_0) \leq f(x) \leq f(x_1) \qquad \forall x$$

**Proof.**
By theorem 4.2.b, we know that $f(K)$ is compact in $\mathbb{R}$. By Bolzano-Weierstrass theorem, $f(K)$ is close and bounded. Therefore by boundedness, $\exists \inf(f(k)), \sup(f(k))$. By $f(K)$ being closed, $\inf(f(k)), \sup(f(k))$ as accumulation points contained in $f(K)$. Therefore, $\exists x_0, x_1 \in K$ such that $f(x_0) = \inf(f(K))$ and $f(x_1) = \sup(f(K))$

**QED**

### Intermediate Value Theorem

**Theorem:** Let $K \subset M$ be connected and $f : K \to \mathbb{R}$ be continuous, suppose $x, y \in K$, $f(x) < f(y) \in \mathbb{R}$, then $\forall$ intermediate values $f(x) \leq c \leq f(y), \exists z \in K$ such that $f(z) = c$.

**Proof.**
Since $K$ is connected and $f$ is continuous, then $f(K)$ is also connected in $\mathbb{R}$

**Lemma 1:** connected sets in $\mathbb{R}$ are intervals:

$\Rightarrow$

Suppose $A \subset \mathbb{R}$ is connected and assume to the contrary that $A$ is not an interval, that is, $\exists a, b \in A$ such that $\exists c \in [a, b]$ that is not in $A$. Then I can construct two open sets $(-\infty, c)$ and $(c, \infty)$ such that the two open sets separate $A$. Therefore, $A$ is no longer connected and this is a contradiction. Therefore, $A$ is connected.

$\Leftarrow$

Suppose $A = [a, b]$ and suppose that $A$ is disconnected, then $\exists U, V \subset \mathbb{R}$ that are open sets such that $U, V$ separates $A$. Then the following three conditions, only two can be true simultaneously:

1. $A \cap U \neq \varnothing$, $A \cap V \neq \varnothing$
2. $A \cap U \cap V = \varnothing$
3. $A \subset U \cup V$

Is this the way to go?
**Lemma 2:** $A \subset \mathbb{R}$ is an interval iff $\forall a, b \in A, [a, b] \subset A$.

**QED**

## Proof of a function being Continuous with $\varepsilon - \delta$

**Statement:** Prove that $f : (0, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x^2}$ is continuous.

**Proof.**
Fix $c \in (0, \infty)$ and $\varepsilon > 0$. I fix $\delta_1 = \frac{1}{2}c$, then finding an $x \in D(c, \delta_1)$, we have

$$\begin{aligned} |x - c| &< \delta_1 \\ &< \frac{1}{2}c \\ \frac{1}{2}c < x &< \frac{3}{2}c \end{aligned}$$

Consider the expression $|f(x) - f(c)|$,

$$\begin{aligned} |f(x) - f(c)| &= \left| \frac{1}{x^2} - \frac{1}{c} \right| \\ &= \left| \frac{c^2 - x^2}{x^2 c^2} \right| \\ &= \frac{c + x}{x^2 c^2} |c - x| \end{aligned}$$

Since $x$ is bounded by$\frac{1}{2}c < x < \frac{3}{2}c$, we have the expression strictly smaller than

$$\begin{aligned}\frac{c+x}{x^2c^2}|c-x| &< \frac{\frac{5}{2}c}{\left(\frac{1}{4}c\right)^2 c^2}|c-x| \\ &< \frac{12}{c^3}\end{aligned}$$

If we would like to have the entire expression bounded by the fixed $\varepsilon$, we have:

$$\begin{aligned}\frac{12}{c^3}|c-x| &< \varepsilon \\ |c-x| &< \frac{c^3}{12}\varepsilon\end{aligned}$$

Therefore, we let $\delta_2 = \frac{c^3}{12}\varepsilon$ so $|c-x| < \delta_2$.

Finally, we have $\delta = \min\{\delta_1, \delta_2\}$

**QED**

## 4.6 Uniform Continuity

### Proof of a function being UC with $\varepsilon - \delta$

$f : (1,2) \to \mathbb{R}$, prove that $f$ is u.c.

**Proof.**
Fix $x, y \in (1,2)$, consider the expression

$$\begin{aligned}|f(x) - f(y)| &= \left|\frac{1}{x} - \frac{1}{y}\right| \\ &= \frac{|y-x|}{|xy|} \\ &= \frac{|y-x|}{xy}\end{aligned}$$

Since $x, y \in (1,2)$, then $xy > 1$ which implies $\frac{y-x}{xy} < |y-x|$. Therefore, given $\varepsilon > 0$, we can choose $\delta = \varepsilon$ such that given $|y - x| < \delta$, we have

$$|f(x) - f(y)| < |y - x| < \delta = \varepsilon$$

Therefore, we have $f$ being uniformly continuous.

**QED**

$f : (0, 1) \to \mathbb{R}$, prove that $f$ is **not u.c.**

consider the expression

$$\begin{aligned} |f(x) - f(y)| &= \frac{|y - x|}{|xy|} \\ &= \frac{|y - x|}{xy} \end{aligned}$$

Suppose $y = 2x$, then

$$\begin{aligned} |f(x) - f(y)| &= \frac{x}{2x^2} \\ &= \frac{1}{2x} \end{aligned}$$

Choose $\varepsilon$ to be anything, we want to show that $\forall\delta$, $\exists x, y \in A$ such that $d(x, y) < \delta$ but $d(f(x), f(y)) > \varepsilon$

Fix $\varepsilon = 1$, then $\forall\delta > 0$, choose $0 < x < \min\{\delta, 1/2\}, y = 2x$.
Therefore, $|y - x| = x < \delta$, but $|f(x) - f(y)| = \frac{1}{2x} > 1 = \varepsilon$

Therefore, $f$ is not uniform continuous on $(0, 1)$.

## Proof of a function not being UC

Consider the function $f(x) = x\sin(x) : \mathbb{R} \to \mathbb{R}$ is not UC.
The equivalent statement for not $UC$ is:
$\exists\varepsilon_0 > 0$ such that $\forall\delta_n = 1/n, \exists x_n, y_n \in A$ such that $d(x_n, y_n) < \delta_n$ but $\rho(f(x), f(y)) > \varepsilon_0$.

**Proof.**

Consider $x_n = 2\pi n, y_n = 2\pi(n + 1/2n)$, then

$$|x_n - y_n| = \frac{1}{2n} < \delta_n$$

However, consider that

$$\begin{aligned}|f(y) - f(x)| &= |(2\pi(n + 1/2n)\sin(2\pi n + \pi/n) - 2\pi n \sin(2\pi n)| \\ &= |(2\pi(2\pi n + \pi/n)\sin(n + 1/2n)|\end{aligned}$$

Let $z_n = \pi/n$, then $\frac{2\pi^2}{z_n} = 2\pi n$. Note that $z_n \to 0$ as $n \to \infty$. We can transform the previous expression into:

$$\begin{aligned}|(2\pi(n + 1/2n)\sin(n + 1/2n)| &= \left|\left(\frac{2\pi^2}{z_n} + z_n\right)\sin\left(2\pi n + z_n\right)\right| \\ &= \left|\left(\frac{2\pi^2}{z_n} + z_n\right)\sin(z_n)\right| \\ &= \left|2\pi^2\frac{\sin(z_n)}{z_n} + z_n\sin(z_n)\right|\end{aligned}$$

For large enough $n$, we have the difference approaches to $2\pi^2 \neq 0$. Therefore, $\exists 0 < \varepsilon_0 < 2\pi^2$ such that $\forall \delta_n = \frac{1}{n}, d(x_n, y_n) < \delta_n$ but $\rho(f(x), f(y)) > \varepsilon_0$.

**QED**

Note that this can also be used as a counterexample to argue that with both functions u.c., one being bounded, the product is still u.c.

## Both bounded and UC implies the product is UC

Suppose that $f, g : A \to \mathbb{R}$, both are bounded and are uniformly continuous. prove that $f \cdot g$ is also uniformly continuous.

**Proof.**

We know that both functions are bounded, therefore, $\exists M, N > 0$ such that $|f(x)| < M$ and $|g(x)| < N$ $\forall x$. From uniform continuity, we know that fix $\varepsilon, \exists \delta_1$ and $\delta_2$ such that $\forall x, y \in A$, $|x - y| < \delta_1, |x - y| < \delta_2$ implies that

$$|f(x) - f(y)| < \frac{\varepsilon}{2N}$$

$$|g(x)-g(y)| < \frac{\varepsilon}{2M}$$

Let $\delta = \min\{\delta_1, \delta_2\}$, observe the product

$$\begin{aligned}
|f(x)g(x)-f(y)g(y)| &= |f(x)g(x)-f(y)g(x)+f(y)g(x)-f(y)g(y)| \\
&\le |f(x)g(x)-f(y)g(x)|+|f(y)g(x)-f(y)g(y)| \\
&\le |g(x)|\cdot|f(x)-f(y)|+|f(y)|\cdot|g(x)-g(x)| \\
&< M\cdot\frac{\varepsilon}{2M}+N\cdot\frac{\varepsilon}{2N} \\
&< \varepsilon
\end{aligned}$$

Therefore, given $\varepsilon, \exists\delta$ such that $\forall x, y \in A$, the product is uniformly continuous.

**QED**

## Continuous and Compact Domain imply UC

**Proof.**
Fix $\varepsilon$, need to show that $\exists\delta > 0$ such that $\forall x, y \in A, d(x, y) < \delta$, we have $\rho(f(x), f(y)) < \varepsilon$. Since $f$ is continuous on $A$. Then fix $\varepsilon$, $\exists\delta_x$ such that $\forall y \in A, d(x, y) < \delta_x$ implies that $\rho(f(x), f(y)) < \frac{\varepsilon}{2}$. Note that $\{D(x, \delta_x)|x \in A\}$ is an open cover for $A$. Since $A$ is compact, then $\exists$ a finite subcover: $\{D(x_k, \frac{\delta_{x_k}}{2})|x_k \in A, k = 1, 2, \ldots, N\}$. We then construct $\delta = \min\{\frac{\delta_{x_1}}{2}, \frac{\delta_{x_2}}{2}, \ldots, \frac{\delta_{x_N}}{2}\}$.

Fix $x \in A$, then $x \in D(x_i, \frac{\delta_{x_i}}{2})$ for some $i$. Choose $y \in A$ such that $d(x, y) < \delta$. Therefore with the triangular inequality:

$$\begin{aligned}
d(y, x_i) &\le d(x, y) + d(x, x_i) \\
&< \delta + \frac{\delta_{x_i}}{2} \\
&< \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} \\
&< \delta_{x_i}
\end{aligned}$$

Therefore $y \in D(x_i, \delta_{x_i})$. By continuity, we know that $d(x, x_i) < \delta_{x_i} \Rightarrow \rho(f(x), f(x_i)) < \frac{\varepsilon}{2}$,

$d(y, x_i) < \delta_{x_i} \Rightarrow \rho(f(y), f(x_i)) < \frac{\varepsilon}{2}$. Therefore by another triangular inequality:

$$\begin{aligned}\rho(f(y), f(x)) &\leq \rho(f(x), f(x_i)) + \rho(f(y), f(x_i)) \\ &< \varepsilon/2 + \varepsilon/2 \\ &< \varepsilon\end{aligned}$$

We therefore have $\delta$ for the fixed $\varepsilon$ such that $\forall x, y \in A, d(x, y) < \delta, \rho(f(x), f(y)) < \varepsilon$. Hence uniformly continuous.

**QED**

**Alternative Proof: Bolzano-Weierstrass**

**Proof.**

The theorem states that $(M, d), (N, \rho)$ are general metric spaces and $A \subset M$, $f : A \to N$, if $f$ is continuous and $A$ is compact, then $f$ is uniformly continuous.

I proceed with the proof by assuming that $f$ is continuous, $A$ compact, but $f$ is not uniformly continuous.

It is proven in the previous question that $f$ is not uniformly continuous iff $\exists\varepsilon$ and two sequences $x_n, y_n$ such that $d(x_n, y_n) < 1/n$ but $\rho(f(x_n), f(y_n)) > \varepsilon$. Since $A$ is a compact set, then $\exists$ converging subsequences $x_{n_k} \subset x_n$, $y_{n_k} \subset y_n$ such that $x_{n_k} \to x_0 \in A$ and $y_{n_k} \to y_0 \in A$.

Since $f$ is continuous and by previous proposition, continuous function preserves convergence, then

$$\begin{aligned}\lim_{n\to\infty} d(x_{n_k}, y_{n_k}) &< \lim_{n\to\infty} 1/n \\ d(x_0, y_0) &\leq 0 \\ x_0 &= y_0\end{aligned}$$

But

$$
\begin{aligned}
\lim_{n\to\infty} \rho(f(x_{n_k}), f(x_{n_k})) &\geq \lim_{n\to\infty} \varepsilon \\
\rho(f(\lim_{n\to\infty} x_{n_k}), f(\lim_{n\to\infty} x_{n_k})) &\geq \varepsilon \\
\rho(f(x_0), f(y_0)) &\geq \varepsilon > 0
\end{aligned}
$$

Therefore, with $x_{n_k}$ and $x_{n_k}$ converges together as $x_0 = y_0$, there is still a gap between $f(x_0)$ and $f(y_0)$ larger than $\varepsilon$, which implies that $f$ is not continuous. This is a contradiction to the previous assumption that $f$ is continuous.

$f$ has to be uniformly continuous.

**QED**

## Bounded Continuous Does not Imply UC

Counterexample: $f(x) = \sin(x^2)$.

**Proof.**
Fix $\varepsilon_0 = 1$, in order to have $|\sin(x^2) - \sin(y^2)| \geq 1$, we can have $x^2 = n\pi$ and $y^2 = n\pi + \frac{1}{2}\pi$, then $x = \sqrt{n\pi}, y = \sqrt{n\pi + \frac{1}{2}\pi}$

$$
\begin{aligned}
|y - x| &= \left|\sqrt{n\pi + \frac{1}{2}\pi} - \sqrt{n\pi}\right| \\
&= \frac{n\pi + \frac{1}{2}\pi - n\pi}{\left|\sqrt{n\pi + \frac{1}{2}\pi} + \sqrt{n\pi}\right|} \\
&< \frac{\pi}{2\left|\sqrt{n\pi + \frac{1}{2}\pi} + \sqrt{n\pi}\right|} \\
&< \frac{1}{n\pi}
\end{aligned}
$$

Therefore, for large enough $n$, we can find close enough $y, x$ such that their distance is no less than $\varepsilon_0 = 1$. Therefore, $f(x) = \sin(x^2)$ is not uniformly continuous.

**QED**

### Bounded Slope implies Lipschitz

Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable and $\exists M > 0$ such that $|f'(x)| \leq M \ \forall x \in (a, b)$. Then $f$ is Lipschitz and hence UC.

**Proof.**
By mean value theorem, for any $x, y \in (a, b)$, WLOG, $y > x, \exists c \in [x, y]$ such that $f(y) - f(x) = f'(c)(y - x)$. This implies that

$$|f(y) - f(x)| = |f'(c)| \cdot |y - x| \leq M \cdot |y - x|$$

The inequality shows that $f$ is Lipschitz.

**QED**

## 4.7 Differentiation

### Differentiable implies Continuous

Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable on $(a, b)$, show that $f$ is continuous on $(a, b)$.
**Proof.**

$$\begin{aligned}
\lim_{x \to x_0} (f(x) - f(x_0)) &= \lim_{x \to x_0} \frac{(f(x) - f(x_0))}{x - x_0}(x - x_0) \\
&= \lim_{x \to x_0} \frac{(f(x) - f(x_0))}{x - x_0} \lim_{x \to x_0} (x - x_0) \\
&= f'(x) \cdot 0 = 0
\end{aligned}$$

Therefore $\lim_{x \to x_0} f(x) = f(x_0)$, the limit exists and equals to the functional value. Therefore $f$ is continuous.

**QED**

**Alternative**

**Proof.**

We know that $f$ being differentiable has the $\varepsilon_\delta$ definition: $\forall |\Delta x| < \delta \Rightarrow |\Delta f - f'(x)\Delta x| < \varepsilon|\Delta x|$ Therefore by triangular inequality:

$$\begin{aligned} |\Delta f - f'(x)\Delta x| &< \varepsilon|\Delta x| \\ |\Delta f| &< \varepsilon|\Delta x| + |f'(x)\Delta x| \\ &< |\Delta x|(\varepsilon + |f'(x)|) \end{aligned}$$

$(\varepsilon + |f'(x)|)$ is a constant implies that $f$ is locally Lipschitz, which implies that $f$ is continuous at $x$. **QED**

## Chain Rule

Show that $\frac{d}{dx} g(f(x)) = g'(f(x)) \cdot f'(x)$

For notation, let $y = g(x), z = g(y), h(x) = g(f(x))$.

Need to show that $h'(x) = g'(f(x)) \cdot f'(x)$.
Or: Given $\varepsilon > 0$, $\exists \delta > 0$ such that $|\Delta x| < \delta$ such that

$$|h(x + \Delta x) - h(x) - \Delta x \cdot g'(f(x))f'(x)| < \varepsilon|\Delta x|$$

**Proof.**
Let $h(x + \Delta x) - h(x) = \Delta h, f(x + \Delta x) - f(x) = \Delta f$
From left hand side: we know that

$$\begin{aligned} &|h(x + \Delta x) - h(x) - \Delta x \cdot g'(f(x))f'(x)| \\ &= |\Delta h - g'(f(x))\Delta f + g'(f(x))\Delta f - \Delta x \cdot g'(f(x))f'(x)| \\ &\leq \underbrace{|\Delta h - g'(f(x))\Delta f|}_{I} + \underbrace{|g'(f(x))| \cdot |\Delta f - f'(x)\Delta x|}_{II} \qquad \text{Tri-ineq} \end{aligned}$$

Idea:
I would make $I, II < \frac{1}{2}\varepsilon$.

Since $g$ is differentiable, we know that ratio between changes in codomain to change in domain is bounded above by some $\varepsilon_1$, i.e.

$$|\Delta g - g'(y)\Delta y| \leq \varepsilon_1|\Delta y| = \varepsilon_1|\Delta f|$$

Similarly, since $f$ is differentiable, we know that (1)

$$\begin{aligned}|\Delta f - f'(x)\Delta x| &\leq \varepsilon_2|\Delta x| \\ |\Delta f| &\leq \varepsilon_2|\Delta x| + |f'(x)\Delta x| \\ &\leq |\Delta x|(\varepsilon_2 + |f'(x)|)\end{aligned}$$

Combining these two conditions, we have

$$|\Delta g - g'(y)\Delta y| \leq \varepsilon_1(|\Delta x|(\varepsilon_2 + |f'(x)|))$$

From $f$ differentiable we also know that (2)

$$g'(f(x))|\Delta f - f'(x)\Delta x| \leq g'(f(x))\varepsilon_2|\Delta x|$$

Coming back to the proof,
Fix $\varepsilon$, choose $\varepsilon_1, \varepsilon_2$ such that

$$\begin{aligned}(1): &\qquad \varepsilon_1(\varepsilon_2 + |f'(x)|) < \frac{1}{2}\varepsilon \\ (2): &\qquad \varepsilon_2|g'(f(x))| < \frac{1}{2}\varepsilon\end{aligned}$$

Next, choose $\delta_1$ such that

$$|\Delta y| < \delta_1$$

This implies that

$$|\Delta g - g'(y)\Delta y| < \varepsilon_1|\Delta y|$$

Finally, choose $\delta$ such that $|\Delta x| < \delta$, this implies that

$$\begin{aligned}|\Delta f - f'(x)\Delta x| &< \varepsilon_2|\Delta x| \\ |\Delta f| &< \varepsilon_2|\Delta x| + |f'(x)\Delta x| = |\varepsilon_2 + f'(x)| \cdot |\Delta x| \\ &< |\varepsilon_2 + f'(x)| \cdot \delta\end{aligned}$$

Combining the two results, the LHS $\leq I + II$, then

$$\begin{aligned} LHS &\leq I + II \\ &\leq \varepsilon_1|\Delta y| + |g'(f(x))| \cdot \varepsilon_2|\Delta x| \\ &\leq \varepsilon_1(\varepsilon_2 + |f'(x)|)|\Delta x| + |g'(f(x))| \cdot \varepsilon_2|\Delta x| \\ &< \frac{1}{2}\varepsilon|\Delta x| + \frac{1}{2}\varepsilon|\Delta x| = \varepsilon|\Delta x| \end{aligned}$$

**QED**

## Rolle's Theorem

Consider the function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. If $f(a) = f(b) = 0$, then $\exists c \in (a, b)$ such that $f'(c) = 0$.

**Lemma:** If $f$ is differentiable on $(a, b)$ and has max/min at some point $c \in (a, b)$, then $f'(c) = 0$.

Suppose $f$ has a max $c \in (a, b)$, need to show that $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = 0$. Since $c$ is a max on $f$, then $\forall x \in (a, b), f(x) \leq f(c)$, therefore when $x < c, f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} < 0$, when $x > c, f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} > 0$. Collectively, $f'(c) = 0$.
Same for min.

**Proof.**
Rule out the trivial cases that $f = 0$ everywhere.
Suppose that $f \neq 0$, then by the min/max theorem, $[a, b]$ compact and $f$ continuous implies the existence of a min $c_1$ and a max $c_2 \in [a, b]$. Since $f(a) = f(b) = 0$, then either $c_1$ or $c_2$ is in $(a, b)$ because $f \neq 0$. Then whichever one in $(a, b)$ will have a derivative 0.

**QED**

## Mean Value Theorem

**Statement:** For $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on $(a, b)$. $\exists c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

**Proof.**
Construct a new function $\phi(x) : [a,b] \to \mathbb{R}$ such that

$$\phi(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right]$$

Observe that $\phi(a) = 0$ and $\phi(b) = 0$. Recall Rolle's theorem, we know that $\exists c \in [a,b]$ such that $\phi'(c) = 0$

$$\phi'(c) = f'(c) - \left[\frac{f(b) - f(a)}{b - a}\right] = 0$$
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

**QED**

## Monotonicity

**1.** $f'(x) \geq 0 \forall x \in (a,b)$ **iff** $f$ **is increasing on** $[a,b]$

$\Rightarrow$
Given $x_1, x_2 \in [a,b]$, by mean value theorem, $\exists c \in (x_1, x_2)$ such that $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$
Then since $f'(c) \geq 0$, then either $f(x_2) \geq f(x_1)$ and $x_2 > x_1$, or $f(x_2) \leq f(x_1)$ and $x_2 < x_1$. In either cases, $f$ is increasing.

$\Leftarrow$

Given $f$ increasing on $[a,b]$, then fix $x_1, x_2 \in (a,b)$ and WLOG, suppose $x_2 > x_1$. Since $f$ increasing, then $f(x_2) \geq f(x_1)$. This implies that the quotient:

$$\frac{f(x_2) \geq f(x_1)}{x_2 - x_1} \geq 0$$

Therefore $f'(x) \geq 0 \forall x \in (a,b)$.

### Inverse function Theorem

**Statement:** Suppose that either $f'(x) > 0$ or $f'(x) < 0$ $\forall x$, then $f$ is a bijection thus invertible, $f^{-1}$ exists and $(f^{-1})'(y) = \frac{1}{f'(x)}$ where $y = f(x)$.

**Proof.**
Suppose that $f : [a, b] \to \mathbb{R}$ such that $f'(x) > 0$.

Let $y = f(x)$, then $x = f^{-1}(y)$, we have the derivative

$$\begin{aligned}
(f^{-1})'(y) &= \lim_{y \to y_0} \frac{(f^{-1})'(y) - (f^{-1})'(y_0)}{y - y_0} \\
&= \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} \\
&= \left( \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \right)^{-1} \\
&= 1/f'(x)
\end{aligned}$$

**QED**

## 4.8 Integration

### Bounded, Monotonic functions are Integrable

**Proof.**

**Statement:** Suppose that $f : [a, b] \to \mathbb{R}$ is bounded and monotonically increases, then $f$ is integrable.

It is sufficient to show that $\forall \varepsilon > 0, \exists$ partition $P$ such that

$$U(f, P) - L(f, P) < \varepsilon$$

Since the domain is compact, by extreme value theorem $\exists \max f = M$ and $\min f = m$. Pick an arbitrary partition $P = \{x_0, \cdots, x_n\}$ with $x_0 = a$ and $x_n = b$. Since $f$ monotonically increases, then $f(x_i) \leq f(x)$ and $f(x_{i+1}) \geq f(x)$ $\forall x \in [x_i, x_{i+1}]$. Observe the upper sum and the lower sum.

$$\begin{aligned}
U(f,P) &= \sum_{i=0}^{n-1} \sup\{f(x)|x \in [x_i, x_{i+1}]\} \cdot (x_{i+1} - x_i) \\
&= \sum_{i=0}^{n-1} f(x_{i+1})(x_{i+1} - x_i) \\
L(f,P) &= \sum_{i=0}^{n-1} \inf\{f(x)|x \in [x_i, x_{i+1}]\} \cdot (x_{i+1} - x_i) \\
&= \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i)
\end{aligned}$$

I know that for all differences in the value of $f$, they are bounded by the difference between the max and min, i.e.

$$f(x_{i+1}) - f(x_i) \leq M - m$$

Therefore, for the difference between the upper sum and the lower sum to be bounded by $\varepsilon$, I can find the partition such that each sub-interval is shorter than $\frac{\varepsilon}{n(M-m)}$. Combining all the results Therefore,

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{i=0}^{n-1} (f(x_{i+1}) - f(x_i))(x_{i+1} - x_i) \\
&< \sum_{i=0}^{n-1} (M-m)\left(\frac{\varepsilon}{n(M-m)}\right) \\
&< \varepsilon
\end{aligned}$$

I conclude that $f$ is integrable.

**QED**

## Bounded, Finite number of discontinuities are integrable

**Statement:** $f : [a,b] \to \mathbb{R}$ is bounded and continuous at all but but finitely many points, then $f$ is integrable.

**Lemma 1:** Let $f$ be bounded, if $P$ and $P'$ are 2 partitions of $[a, b]$ and $P \subset P'$ (i.e. $P'$ is called a refinement of $P$). Then

$$L(f, P) \le L(f, P') \le U(f, P') \le U(f, P)$$

Fix any partition $P, Q$, then $P \cup Q$ should also be a partition and is a refinement of both $P$ and $Q$.
Proof needed

When partition gets finer, lower sum increases and upper sum decreases.

We know from lemma 1,

$$L(f, P) \le L(f, P \cup Q) \le U(f, P \cup Q) \le U(f, Q)$$

Then
$$\underline{\int} f = \sup_P L(f, P) \le U(f, Q)$$
The lower integral is a lower bound of $U(f, Q)$ $\forall Q$. Therefore, it is smaller than or equal to the greatest lower bound

$$\underline{\int} f = \sup_P L(f, P) \le \inf_Q U(f, Q) = \overline{\int} f$$

In order to show that $f$ is integrable, it is equivalent as showing its lower integral equals to the upper integral.

$\forall P$, we have
$$L(f, P) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(f, P)$$
Fix $\varepsilon > 0$, it is sufficient to show that $\exists P$ such that

$$U(f, P) - L(f, P) < \varepsilon$$

Suppose that $f$ is bounded, then $\exists m, M$ such that $m \le f(x) \le M$ $\forall x$. Also suppose that $f$

is continuous on $[a, b]$ except at $a_1, a_2, \ldots, a_k \in [a, b]$. Choose $P_1$ such that each sub-interval that contains some $a_i$ and has length $\leq \frac{\varepsilon}{2} \cdot \frac{1}{2k(M-m)}$.

Let $K$ be the union of the remaining sub-intervals. Then $K$ is compact and $f$ is continuous everywhere on $K$. By previous theorem, compact domain and continuous function implies uniform continuity. Therefore, $\exists \delta > 0$ such that $\forall x_1, x_2 \in K, |x_1 - x_2| < \delta$ implies that $|f(x_1) - f(x_2)| < \frac{\varepsilon}{2} \frac{1}{b-a}$.

**Construct a refinement of $P_1$**

Construct a $P$ such that each sub-interval containing $a_i$ has length that is smaller than $\delta$. I will call each sub-interval $I_j = [x_j, x_{j+1}]$. Define $M_j = \sup_{I_j} f(x)$ and $m_j = \inf_{I_j} f(x)$.

Observation:

1. The maximum and minimum according to this partition is still be bounded by $m$ and $M$, i.e. If $\exists a_i \in I_j$, then $m \leq m_j \leq M_j \leq M$.
2. For those $I_j$ that do not contain $a_i$, then $I_j \subset K$ on which $f$ is uniformly continuous. Therefore, $M_j - m_j = \max - \min < \frac{\varepsilon}{2} \frac{1}{b-a}$.

Finally, we have

$$\begin{aligned}
&U(f, P) - L(f, P) \\
&= \sum_j (M_j - m_j)(x_{j+1} - x_j) \\
&= \sum_{\exists a_i \in I_j} (M_j - m_j)(x_{j+1} - x_j) + \sum_{a_i \notin I_j} (M_j - m_j)(x_{j+1} - x_j) \\
&\leq (M - m) \frac{\varepsilon}{2} \frac{1}{2k(M-m)} \cdot 2k + \left( \frac{\varepsilon}{2} \cdot \frac{1}{b-a} (b - a) \right) \\
&< \varepsilon
\end{aligned}$$

## Fundamental Theorem of Calculus

### Antiderivative

Let $f : [a, b] \to \mathbb{R}$, $F$ is the antiderivative if $F'(x) = f(x) \forall x \in (a, b)$.

**Statement:** Let $f;[a,b]\to\mathbb{R}$ be continuous, then $\exists F$ as an antiderivative of $f$ and

$$\int_a^b f(x)dx = F(b)-F(a)$$

**Step 1: Show the existence of $F$**

**Proof.**
$\forall x\in[a,b]$, Let $F(x)=\int_a^x f(t)dt$.
I claim that $F$ is an antiderivative of $f$.
Fix $x\in(a,b)$, choose $h>0$ such that $[x-h,x+h]\subset(a,b)$. Then for $|\Delta x|< h$, we have:

$$\begin{aligned}
\frac{F(x+\Delta x)-F(x)}{\Delta x} &= \frac{\int_a^{x+\Delta x} f(t)dt-\int_a^x f(t)dt}{\Delta x}\\
&= \frac{1}{\Delta x}\int_x^{x+\Delta x} f(t)dt\\
\frac{F(x+\Delta x)-F(x)}{\Delta x}-f(x) &= \frac{1}{\Delta x}\int_x^{x+\Delta x} f(t)dt-f(x)\\
&= \frac{1}{\Delta x}\int_x^{x+\Delta x} f(t)dt-f(x)\left(\frac{1}{\Delta x}(x+\Delta x-x)\right)\\
&= \frac{1}{\Delta x}\int_x^{x+\Delta x} f(t)dt-\frac{1}{\Delta x}\int_x^{x+\Delta x} f(x)dt\\
&= \frac{1}{\Delta x}\int_x^{x+\Delta x} (f(t)-f(x))dt
\end{aligned}$$

With $t\in(x,x+\Delta x)$.
Given $\varepsilon>0, f$ is continuous at $x$ implies $\exists\delta>0$ such that $|t-x|<\delta$ implies $|f(t)-f(x)|<\varepsilon, \forall t\in(x-\delta,x+\delta)$.

We also know that the absolute value of the integral is less than or equal to the integral of the absolute value:

$$
\begin{aligned}
\left|\frac{1}{\Delta x}\int_x^{x+\Delta x}(f(t)-f(x))dt\right| &\leq \frac{1}{|\Delta x|}\int_x^{x+|\Delta x|}\left|(f(t)-f(x))\right|dt \\
&< \frac{1}{|\Delta x|}\int_x^{x+|\Delta x|}\varepsilon dt \\
&= \varepsilon
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left|\frac{F(x+\Delta x)-F(x)}{\Delta x}-f(x)\right| &< \varepsilon \\
\lim_{\Delta x\to 0}\frac{F(x+\Delta x)-F(x)}{\Delta x} &= f(x)
\end{aligned}
$$

i.e., $F'(x) = f(x)\ \forall x \in (a, b)$, $F$ is an antiderivative of $f$.

**Step 2: Show** $\int_a^b f(x)dx = F(b) - F(a)$

$F(b) = \int_a^b f(t)dt$, $F(a) = \int_a^a f(t)dt = 0$. Then

$$
F(b) - F(a) = \int_a^b f(t)dt
$$

**QED**

## Proof on the Properties of Integration

### Monotonicity

**Statement:** If $f \leq g$ on $[a, b]$ and are both integrable, then $\int_a^b f dx \leq \int_a^b g dx$.

**Statement:** In particular, $\forall f$, we have the following

$$
-|f| \leq f \leq |f|
$$

Therefore

$$\int_a^b -|f|dx \le \int_a^b f dx \le \int_a^b |f|dx$$
$$\left|\int_a^b f dx\right| \le \int_a^b |f|dx$$

## Riemann Integrable

**7.** Let $f : [0,1] \to \mathbb{R}$, $f(x) = 1$ if $x = 1/n$, $n$ an integer, and $f(x) = 0$ otherwise.

**a.** Prove that $f$ is integrable.

**b.** Show that $\int_0^1 f(x)\,dx = 0$.

**Proof.**
It is equivalent as showing that $\forall \varepsilon, \exists$ partition $P$ over $[0,1]$ such that

$$U(f,P) - L(f,P) < \varepsilon$$

Fix $\varepsilon_0 > 0$. We know that the sequence $\{1/n\} \to 0$ as $n \to \infty$. Therefore, $\exists N$ such that $\forall n > N, x_n < \frac{\varepsilon_0}{N}$. Consider a partition:

$$\mathcal{P} = \{x_1, x_2^1, x_2^2, \cdots, x_i^1, x_i^2, \cdots x_N^1, x_N^2, x_0\}$$

With $x_1 = 1$ and $x_0 = 0$ and $x_i^1 = \frac{1}{i} + \frac{\varepsilon_0}{2N}$ and $x_i^2 = \frac{1}{i} - \frac{\varepsilon_0}{2N}$, $i \in \mathbb{N} \cap [2, N]$.

There are four types of sub-intervals in this partition:

1. $[x_0, x_N^2]$, equivalent as $[0, x_N^2]$. This interval contains infinitely many $x = 1/n$. Therefore,
$$\sup\{f(x)|x \in [x_0, x_N^2]\} = 1$$
But the interval length is bounded since $x_N^2 = \frac{1}{N} - \frac{\varepsilon_0}{2N} < \frac{1}{N} < \frac{\varepsilon_0}{N}$.

2. $[x_i^2, x_i^1]$ with $i \in \mathbb{N} \cap [2, N]$, equivalent as $[\frac{1}{i} - \frac{\varepsilon_0}{2N}, \frac{1}{i} + \frac{\varepsilon_0}{2N}]$. Observe that $\forall i, \exists!\, x = \frac{1}{i} \in$

$[x_i^2, x_i^1]$. Therefore,

$$\sup\{f(x)|x \in [x_i^2, x_i^1]\} = 1$$

The interval length as the width is fixed to be $\frac{\varepsilon_0}{N}$.

3. $[x_i^1, x_{i+1}^2]$. The interval lengths vary but all such interval does not contain any $x = 1/n$. Therefore,

$$\sup\{f(x)|x \in [x_i^1, x_{i+1}^2]\} = 0$$

4. $[x_2^1, x_1]$, equivalent as $[\frac{1}{2} + \frac{\varepsilon_0}{2N}, 1]$. This interval does not contain $x = 1/n$ therefore,

$$\sup\{f(x)|x \in [x_2^1, x_1]\} = 0$$

To compute the upper sum:

$$\begin{aligned}
U(f, \mathcal{P}) = &\underbrace{\sup\{f(x)|x \in [x_0, x_N^2]\} \cdot (x_N^2 - x_0)}_{\text{First interval, has } \frac{1}{n}} \\
&+ \underbrace{\sum_{i=2}^{N} \sup\{f(x)|x \in [x_i^2, x_i^1]\} \cdot (x_i^1 - x_i^2)}_{\text{those contain } \frac{1}{n}} \\
&+ \underbrace{\sum_{i=2}^{N} \sup\{f(x)|x \in [x_i^1, x_{i+1}^2]\} \cdot (x_{i+1}^2 - x_i^1)}_{\text{those do not contain } \frac{1}{n}} \\
&+ \underbrace{\sup\{f(x)|x \in [x_2^1, x_1]\} \cdot (x_1 - x_2^1)}_{\text{Last interval}} \\
=&1 \cdot x_N^2 + (N-1) \cdot 1 \cdot \frac{\varepsilon_0}{N} + 0 + 0 \cdot \left(1 - \frac{1}{2} - \frac{\varepsilon_0}{2N}\right) \\
<&\frac{\varepsilon_0}{N} + \frac{N-1}{N}\varepsilon_0 = \varepsilon_0
\end{aligned}$$

**Claim:** $L(f, P) = 0 \ \forall P$.
To show this, fix any partition $P = \{y_1, \cdots y_K\}$. For any sub-interval $[y_k, y_{k+1}]$.
As for the lower sum, since $\mathbb{R}\backslash\mathbb{Q}$ is dense on $\mathbb{R} \cap [0, 1]$. Therefore at least one irrational $x$

exists in all sub-intervals. The infimum of $f(x)$ is always 0. Therefore,

$$L(f, \mathcal{P}) = 0$$

And the difference

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon_0 - 0 = \varepsilon_0$$

Therefore, I have shown that given $\varepsilon_0, \exists \mathcal{P}$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon_0$. $f$ is Riemann integrable. **QED**

## 2nd Mean Value Theorem - Integration

**Statement:** Let $f, g : [a, b] \to \mathbb{R}, f \geq 0, f$ integrable, $g$ continuous. Then $\exists x_0 \in (a, b)$ such that

$$\int_a^b f(x)g(x)dx = g(x_0) \int_a^b f(x)dx$$

**Proof.**

We know that $g$ is continuous and the domain is compact. Therefore, by the min/max theorem, $\exists x_1, x_2$ such that $g(x_1) = \min\{g(x)\}$ and $g(x_2) = \max\{g(x)\}$. Therefore, we have the following inequality:

$$\begin{gathered} g(x_1)f(x) \leq g(x)f(x) \leq g(x_1)f(x) \\ g(x_1) \int_a^b f(x)dx \leq \int_a^b f(x)g(x)dx \leq g(x_2) \int_a^b f(x)dx \\ g(x_1) \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b f(x)dx} \leq g(x_2) \end{gathered}$$

Since $g$ is continuous, by the intermediate value theorem, $\exists x_0$ such that $g(x_0) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b f(x)dx}$. Therefore, we have

$$g(x_0) \int_a^b f(x)dx = \int_a^b f(x)g(x)dx$$

**QED**