Measure Theory
Study Note

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Contents

1 Preface

Welcome to my personal study notes on measure theory, with a particular emphasis on its applications in probability theory. To fully benefit from these notes, it is recommended that readers have a solid foundation in basic real analysis, including the following topics:

- Sequence and function convergence
- Point-set topology
- Integration analysis

This document aims to provide a structured and concise exploration of measure theory, blending rigorous mathematical concepts with practical insights to support further studies in probability theory and related fields.

– Tianqi Zhang
Dec 14th, 2024

2 Measure

2.1 Motivation and Definition of Outer Measure

Recall the definition of upper and lower Darboux sum in Riemann integral setting:

$$U(f) = \sum_{i=1}^{N} \sup_{x \in [x_i, x_{i+1}]} f(x) \cdot \underbrace{(x_{i+1} - x_i)}_{\text{length}}$$

Definition of Riemann non-integrable involves the upper integral and the lower integral as two limits do not agree, which likely boils down to some partitions $[x_i, x_{i+1}]$ not well-defined. Therefore, to propose a fix, we are motivated to properly define the "length" of any general subset of $\mathbb{R}$.

Definition 2.1.1 (Length). The length $\ell(I)$ of some open interval $I \subset \mathbb{R}$ is a function defined by

$$\ell(I) = \begin{cases} b - a, & I = (b - a), a < b, a, b \in \mathbb{R} \\ 0, & I = \varnothing, \\ \infty & I = (-\infty, a), a \in \mathbb{R} \\ \infty & I = (a, \infty), a \in \mathbb{R} \end{cases}$$

Then suppose $A \subset \mathbb{R}$. The size of A can at most be the **sum of lengths of a sequence of open intervals** I **whose union contains** A. Taking the infimum of such sums over all possible sequences of I, we obtain the outer measure of A, i.e.

Definition 2.1.2 (Outer Measure, $|A|$). For $A \subset \mathbb{R}$,

$$|A| \equiv \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \mid I_k \text{ open, } A \subset \bigcup_{k=1}^{\infty} I_k \right\}$$

Proposition 2.1.3 Finite sets have outer measure 0:

Proof: A.2.1

2.1.1 Properties of Outer Measure

Proposition 2.1.4 Countable subsets of $\mathbb{R}$ have outer measure 0:

Proof: A.2.2

Proposition 2.1.5 Order preserving of outer measure:

If $A \subset B \subset \mathbb{R}$, then $|A| \leq |B|$ Proof: A.2.3

Definition 2.1.6 (Translation). For any $A \subset \mathbb{R}, t \in \mathbb{R}$, the translation $t + A$ is defined by

$$t + A = \{t + a \mid a \in A\}$$

Note that the length function should be translation invariant. Therefore, we obtain the proposition that outer measure is translation invariant.

Proposition 2.1.7 Outer measure is translation invariant:

Suppose $t \in \mathbb{R}$ and $A \subset \mathbb{R}$, then $|t + A| = |A|$.

Proposition 2.1.8 Countable Sub-additivity of outer measure:

Suppose $A_1, A_2, \ldots, \subset \mathbb{R}$. Then

$$\left| \bigcup_{k=1}^{\infty} A_k \right| \leq \sum_{k=1}^{\infty} |A_k|$$

Note that this implies finite sub-additivity which could come handy in proof techniques:

$$|A_1 \cup \cdots \cup A_n| \leq |A_1| + \cdots + |A_n|$$

2.2 Outer Measure of Closed Bounded Interval

It is apparent for any closed interval $[a, b]$, we can construct a sequence of open cover $(a - \varepsilon, b + \varepsilon)$ and arbitrarily shrink ε. We obtain $|[a, b]| \leq b - a$. However, the other direction requires completeness of $\mathbb{R}$.

Proposition 2.2.1 $|[a, b]| = b - a$:

Suppose $a, b \in \mathbb{R}, a < b$. Then $|[a, b]| = b - a$. Proof: A.2.6

Proposition 2.2.2 Non-trivial intervals are uncountable:

Every **interval** in $\mathbb{R}$ that contains at least two distinct terms is uncountable.

Proposition 2.2.3 Non-additivity:

$\exists A, B \subset \mathbb{R}$ disjoint such that $|A \cup B| \neq |A| + |B|$.

2.3 Measurable Space and Functions

Countable additivity is desired for many scenario since we are mostly interested in proving theorems on limits. However, there is a fundamental problem of measure theory such as follows:

Proposition 2.3.1 Non-existence of extension of length to all subsets of $\mathbb{R}$:

There does not exist a function μ with all the following properties:
a). $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$
b). $\mu(I) = \ell(I) \forall$ open interval I on $\mathbb{R}$
c). Countable additivity: $\mu(\bigcup_k A_k) = \sum_k \mu(A_k)$ for all disjoint $A_k \subset \mathbb{R}$
d). Translation invariant: $\mu(t + A) = \mu(A)\ \forall A \subset \mathbb{R}$ and $t \in \mathbb{R}$.
The only relaxation we can make is a). Instead of defining the function on the entirety of power set of $\mathbb{R}$, we consider a more practice subset of the power set such that the subset is closed under complementation and countable unions. We make the following definition:

Definition 2.3.2 (Sigma algebra).
Suppose X is a set and $S \subset \mathcal{P}(X)$. Then S is a σ-algebra if the following conditions are satisfied:
1. $\varnothing \in S$
2. if $E \in S$, then $\overline{E} \in S$
3. if $\{E_k\}_k \subset S$, then $\bigcup_k E_k \in S$

A Proofs

A.2 Measure

A.2.1 Finite sets have outer measure 0

Proof. Suppose $A = \{a_1, a_2, \dots, a_N\} \subset \mathbb{R}$. It is equivalent to prove that $\forall \varepsilon$, $\exists \{I_k\}$ such that $A \subset \bigcup I_k$ and $\sum \ell(I_k) < \varepsilon$. Therefore, let

$$I_k = \begin{cases} \left(a_k - \frac{\varepsilon}{4N}, a_k + \frac{\varepsilon}{4N}\right), & k \leq N \\ \varnothing, & k > n \end{cases}$$

we have $A \subset \bigcup I_k$, and we have the sum $\sum_{k=1}^{\infty} I_k = \sum_{k=1}^{n} \frac{\varepsilon}{2n} = \frac{\varepsilon}{2} < \varepsilon$. Therefore, $|A| = 0$. □

A.2.2 Countable sets have outer measure 0

Proof. Suppose $A = \{a_1, a_2, \dots\} \subset \mathbb{R}$. With similar idea, for any ε, construct I_k to be

$$I_k = \left(a_k - \frac{\varepsilon}{2^{k+2}}, a_k + \frac{\varepsilon}{2^{k+2}}\right)$$

Then each $a_k \in I_k$, which implies $A \subset \bigcup I_k$. Now consider the infinite sum:

$$\begin{aligned} \sum_{k=1}^{\infty} I_k &= \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} \\ &= \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

Therefore, $|A| = 0$. □

A.2.3 Outer measure is order preserving

Proof. Since $A \subset B$, then any open cover for A would be an open cover for B. Taking infimum over all sequences of open covers of B, we have $|A| \leq |B|$. □

A.2.4 Outer Measure is translation invariant

Proof. Suppose $t \in \mathbb{R}, A \subset \mathbb{R}$ and suppose $\{I_k\}$ covers A. Then $\{t + I_k\}$ should be a set of cover for $t + A$, we obtain the following inequality:

$$|t + A| \leq \sum_k \ell(t + I_k)$$

Since ℓ is translational invariant, we have $\sum_k \ell(t + I_k) = \sum_k \ell(I_k)$. Taking the infimum on both sides:

$$|t + A| \leq |A|$$

Now consider $\{t + I_k\}$ to be a set of open cover for $t + A$ and $A = -t + (t + A)$, we have

$$\begin{aligned}
|A| &\leq \sum_k \ell(-t + (t + I_k)) = \sum_k \ell(t + I_k) \\
\inf_{t+I_k} |A| &\leq \inf_{t+I_k} \sum_k \ell(t + I_k) \\
|A| &\leq |t + A|
\end{aligned}$$

We have $|A| = |t + A|$. $\square$

A.2.5 Sub-additivity

Proof. Suppose for each A_k, we have a sequence of open cover $\{I_{ik}\}_i$ such that it is close enough above to the outer measure.

$$\sum_i \ell(I_{ik}) \leq \frac{\varepsilon}{2^k} + |A_k|$$

Then summing both sides over k, we have

$$\begin{aligned}
\sum_k \sum_i \ell(I_{ik}) &\leq \sum_k \frac{\varepsilon}{2^k} + |A_k| \\
\sum_k \sum_i \ell(I_{ik}) &\leq \varepsilon + \sum_k |A_k|
\end{aligned}$$

Observe that the left hand side $\cup_{i,k} I_{ik}$ is an open cover for the entire union $\cup_k A_k$, then we have

$$\left| \bigcup_k A_k \right| \leq \sum_k \sum_i \ell(I_{ik}) \leq \varepsilon + \sum_k |A_k|$$

$$\left|\bigcup_k A_k\right| < |A_k|$$

□

A.2.6 Closed Interval Length

Proof. One direction is trivially proven. Now for $|[a,b]| \leq b-a$, recall Heine-Borel theorem. $\exists\{I_k\}_{k=1}^K$ such that $[a,b] \subset \{I_k\}_{k=1}^K$, which implies that

$$|[a,b]| \leq \sum_{k=1}^{K} \ell(I_k)$$

We now need to show that the sum is bounded below by $b-a$, by induction. Note that the statement is true when $K=1$. Now assume that above holds for K, now consider a new set of open cover $\{I_k\}_{k=1}^{K+1}$ for $[a,b]$. Without loss of generality, we assume that $b \in I_{K+1} \equiv (c,d)$ with $c,d \in \mathbb{R}$. If $c<a$, I_{K+1} alone covers $[a,b]$ and the proof is finished. So consider $a<c<b<d$.

One immediate conclusion is that the sub-interval $[a,c]$ is covered by $\{I_k\}_{k=1}^K$, and by our induction hypothesis,

$$\begin{aligned}
\sum_{k=1}^{K} I_k &\geq c-a \\
\sum_{k=1}^{K+1} I_k &\geq c-a+\ell(I_{K+1}) \\
&\geq c-a+d-c = d-a \\
|[a,b]| \geq \sum_{k=1}^{K+1} I_k &\geq b-a
\end{aligned}$$

Therefore, we have both directions such that $|[a,b]| = b-a$. □

B Practices

B.2 Measure