1.3 Least Upper Bound

MSP to LUB

(1)

Prove that Monotonic Sequence Property implies the Least Upper Bound Theorem. That is, $\forall S \subset \mathbb{R}$ that has an upper bound, $\exists! \operatorname{lub}(S)$ for S.
Proof.
Step 1: Construction of a monotone sequence.
Fix an upper bound m for S. $\forall$ fixed integer $n \geq 1$, consider the sequence:

$$a_k = m - \frac{k}{2^n},\ k = 1, 2, \cdots$$

Let k_n be the first integer s.t. a_k is the first element in the sequence that is not an upper bound, that is, $\forall a_1, a_2, \cdots, a_{k-1}$ are upper bounds of S. For this particular sequence $\{a_n\}$, for each fixed n, there is an associated k. Note that $\{a_k\}$ decreases linearly

Then, we construct another sequence by fixing the k obtained from $\{a_n\}$:

$$b_n = m - \frac{k_n}{2^n},\ n = 1, 2, \cdots$$

Observation: $\{b_n\}$ is increasing geometrically, b_b is not an upper bound, but $b_n + \frac{1}{2^n}$ is an upper bound.

Step 2: Applying MSP to $\{b_n\}$

i) bounded above because $\{b_n\}$ is bounded above by M.

ii) $\{b_n\}$ is increasing :

$$
\begin{aligned}
b_{n+1} - b_n &= M - \frac{k_{n+1}}{2^{n+1}} - M + \frac{k_n}{2^n} \\
&= \frac{k_n}{2^n} - \frac{k_{n+1}}{2^{n+1}} \\
&= \frac{2k_n - k_{n+1}}{2^{n+1}}
\end{aligned}
$$

Suppose that $b_{n+1} < b_n$, then:

$$
\begin{aligned}
\frac{2k_n - k_{n+1}}{2^{n+1}} &< 0 \\
2k_n - k_{n+1} &< -1
\end{aligned}
$$

And:

$$
\begin{aligned}
b_{n+1} - b_n = \frac{2k_n - k_{n+1}}{2^{n+1}} < \frac{2k_n - k_{n+1}}{2^n} &= -\frac{1}{2^{n+1}} \\
b_n &> b_{n+1} + \frac{1}{2^{n+1}}
\end{aligned}
$$

Recall that b_n is not an upper bound, but $b_{n+1} + \frac{1}{2^{n+1}}$ is an upper bound. Therefore, this is a contradiction, and $\{b_n\}$ should be an increasing sequence.

And by MSP, $\{b_n\}$ should converge, $\exists b$ such that $\forall \varepsilon > 0, \exists N$ such that $\forall n > N, |b_n - b| < \varepsilon$.

Step 3: Show that b is $\sup(S)$:

i). $\forall x \in S, b_n + \frac{1}{2^n}$ is an upper bound. Therefore,

$$
\begin{aligned}
x &\leq b_n + \frac{1}{2^n} \\
\lim_{n\to\infty} x &\leq \lim_{n\to\infty} \frac{1}{2^n} \\
x &\leq b + 0
\end{aligned}
$$

The last line implies that b is an upper bound of x.

ii). $b_n \to b$ meaning $\forall \varepsilon > 0, \exists N$ such that $\forall n > N, |b_n - b| < \varepsilon$. But b_n is not an upper bound. Therefore, $\exists x \in S$ such that

$$x \geq b_N$$
$$x \geq b - \epsilon$$

Therefore, checking through the property of least upper bound, b is the least upper bound.

QED

(2)

Construction of the Sequence:
Choose $x_0 \in S$ and M_0 as an arbitrary upper bound. Let $a_0 = (x_0 + M_0)/2$.

- If a_0 is an upper bound, then let $M_1 = a_0$ and $x_1 = x_0$.
- Otherwise, let $M_1 = M_0$ and $x_1 > a_0, x_1 \in S$.

Repeat, generating sequences x_n and M_n.
Proof.
Suppose that $\exists S \subset \mathbb{R}$ that is non-empty and is bounded above.

Claim: Both $\{x_n\}$ and $\{M_k\}$ converge

For some arbitrary x_n, we observe the construct of $\{x_n\}$ and we obtain the following fact: $x_{n+1} = x_n$ or x_{n+1} is picked to be: $x_{n+1} > a_0 > x_n$. Therefore $\{x_n\}$ is monotonically increasing. It is also bounded above since M_0 is initiated to be an upper bound of S, therefore, M_0 is greater than or equal to all $x_n \in S$. So according to the monotonic sequence property, $\{x_n\}$ must converge. We suppose that $\exists x \in \mathbb{R}$ such that

$$\lim_{n \to \infty} x_n = x$$

For M, we first know that $\{M_k\}$ is constructed to be the upper bounds of the set S. Therefore, any element in S is smaller than equal to all elements in the sequence $\{M_k\}$, we know that $\{M_k\}$ is bounded below. Secondly, we know that M_{k+1} could be either obtained either by

$M_{k+1} = M_k$ if the generated a_n is not an upper bound of S, or by $M_{k+1} = a_n = \frac{x_n+M_k}{2}$ if a_n is an upper bound. From here,

$$\frac{x_n + M_k}{2} \leq \frac{M_k + M_k}{2} = M_k$$

as $x_n \leq M_k$ Therefore, $M_{k+1} \leq M_k\{M_k\}$ is a decreasing sequence bounded below. We know that $\{M_k\}$ converges. We suppose that $\exists M \in \mathbb{R}$ such that

$$\lim_{k\to\infty} M_k = M$$

Claim: $x = M$

We know that $M_{k+1} = a_n = \frac{x_n+M_k}{2}$. With sufficiently large choice of n and k, we obtain that:

$$\begin{aligned}
\lim_{k\to\infty} M_{k+1} &= \lim_{k\to\infty} \lim_{n\to\infty} \frac{x_n + M_k}{2} \\
M &= \frac{x + M}{2} \\
M &= x
\end{aligned}$$

Therefore, the two sequences converge to the same point.

Claim: $x = M = \sup(S)$
Previously, $\lim_{n\to\infty} x_n = x$, therefore, $\forall \varepsilon > 0, \exists N$ such that $\forall n \geq N, |x_n - x| < \varepsilon$. We can just pick the lowest index that satisfies this inequality, N.
Since $\{x_n\}$ is increasing, we know that $|x_N - x| = x - x_N$:

$$\begin{aligned}
|x_N - x| &< \varepsilon \\
x - x_N &< \varepsilon \\
x_N &> x - \varepsilon
\end{aligned}$$

Since $x = M$ thus is an upper bound for the set S. We have shown that $\forall \varepsilon, \exists N$ such that $x_N > x - \varepsilon$. Therefore, M is the least upper bound, $x = M = \sup(S)$ **QED**

LUB & GLB to MSP

Proof.

Claim: LUBP $\Rightarrow$ MSP

Suppose that $\{x_n\}$ is a monotonic increasing sequence bounded above. Therefore, the set $S = \{x_1, x_2, \cdots, x_n\}$ is bounded above and by the least upper bound property, $\exists x = \sup(S)$. By definition of the LUB, $x \geq x_n \ \forall x_n \in \{x_n\}$. $\forall \varepsilon > 0$, there exists some N such that $x_N > x - \epsilon$. We can obtain the inequality

$$x - x_N < \varepsilon$$

Recall $x \geq x_n \ \forall x_n \in \{x_n\}$, we know that $|x_N - x| = x - x_N$, therefore, fixing $\varepsilon > 0, \exists N$ such that:

$$|x_N - x| < \varepsilon$$

Also, given that $\{x_n\}$ monotonically increases, then, $x_n \geq x_N \ \forall n > N$. Therefore, $|x_n - x| \leq |x_N - x| < \varepsilon \ \forall n > N$. Eventually, by the definition of limit, $\{x_n\}$ converges. Least upper bound property implies the Monotonic Sequence Property.

Statement: GLBP $\Rightarrow$ MSP

Suppose that $\{x_n\}$ is a monotonic decreasing sequence bounded above. Therefore, the set $S = \{x_1, x_2, \cdots, x_n\}$ is bounded below and by the greatest lower bound property, $\exists x = \inf(S)$. By definition of the GLB, $x \leq x_n \ \forall x_n \in \{x_n\}$. $\forall \varepsilon > 0$, there exists some N such that $x_N < x + \epsilon$. We can obtain the inequality

$$x_N - x < \varepsilon$$

Recall $x \leq x_n \ \forall x_n \in \{x_n\}$, we know that $|x_N - x| = x_N - x$, therefore, fixing $\varepsilon > 0, \exists N$ such that:

$$|x_N - x| < \varepsilon$$

Also, given that $\{x_n\}$ monotonically decreasing, then, $x_n \leq x_N \ \forall n > N$. Therefore, $|x_n - x| \leq |x_N - x| < \varepsilon \ \forall n > N$. Eventually, by the definition of limit, $\{x_n\}$ converges to x. Greatest lower bound property implies the Monotonic Sequence Property. **QED**

1.4 Cauchy Sequence

Cauchy $\Rightarrow$ Convergence

Bounded Sequence has a Convergent Subsequence

Statement: $\forall x_n$ to be bounded, there is a subsequence that converges (Or: every bounded sequence has at least one cluster point, Theorem 1.4.3)
Proof.
Let x_n be bounded in $\mathbb{R}$. Then $\exists M \in \mathbb{R}$ such that $-M < x_n < M \ \forall n$.

We then divide the interval equally into two sub-intervals that **at least one of them contains infinitely many terms**, denote the new interval I_0 (ex. $I_0 = [0, M]$). We then choose $x_{n_0} \in I_0$.

We then divide I_0 into two sub-intervals that **at least one of them contains infinitely many terms**, denote the new interval I_1, choosing $x_{n_1} \in I_1$. We can perform this iteratively to obtain a subsequence x_{n_k}, and a sub-interval $I_k = [a_k, b_k]$.

Observation 1: $I_o \supset I_1 \supset \cdots \supset I_k$
Observation 2: $b_k - a_k = \frac{1}{2}(b_{k-1} - a_{k-1}) = \dfrac{M}{2k}$
Observation 3: $x_{n_k} \in I_k$

Lemma 1: a_k converges.
Note that the sequence $\{a_k\}$ monotonically increases and is bounded above by M, therefore by MSP, $\{a_k\}$ converges to some value. We name it a.

Lemma 2: x_{n_k} converges to a

$$
\begin{aligned}
|x_{n_k} - a| &= |x_{n_k} - a_k + a_k - a| \\
&\leq |x_{n_k} - a_k| + |a_k - a|
\end{aligned}
$$

Note that the sequence $|x_{n_k} - a_k|$ is bounded by $b_k - a_k$

$$|x_{n_k} - a_k| \leq |b_k - a_k| = \frac{M}{2^k} \to 0$$

Therefore,

$$|x_{n_k} - a| \leq |a_k - a| < \varepsilon$$

x_{n_k} converges to a

QED

Subsequence of a Cauchy Sequence Converges implies Cauchy Sequence Converges

1.5 liminf and limsup

Subsequence converge to liminf and limsup

3. Let x_n be a sequence with $\limsup x_n = b \in \mathbb{R}$ and $\liminf x_n = a \in \mathbb{R}$. Show that x_n has subsequences u_n and l_n with $u_n \to b$ and $l_n \to a$.

Proof.

Claim: $u_n \to b$.
Let $\varepsilon_1 = \frac{1}{1}$. By previous proposition (1): $\limsup x_n = b \in \mathbb{R}$ implies that $\forall \varepsilon > 0, \exists N$ such that $\forall n > N, x_n < b + \varepsilon$.

We start by choosing an $M = N_1$, and by proposition (2), we know that $\forall M, \exists n_1 > M$ such that $x_{n_1} > b - \varepsilon_1$. We select this x_{n_1} and name it u_1, then:

$$b - \varepsilon_1 < u_1 < b + \varepsilon_1$$

Constructing this sequence inductively, we let $\varepsilon_k = \frac{1}{k}$, then suppose that we have selected all terms from u_1 to u_k. Then we would show how to select u_{k+1}.

For $\varepsilon = \varepsilon_{k+1} = \frac{1}{k+1}$, by prop (1) we choose a $N_{k+1} > k$ such that $\forall n > N_{k+1}$, $x_n < b + \varepsilon_{k+1}$.

Similarly by prop (2), we know that amongst all x_n that are smaller than $b + \varepsilon_{k+1}$, $\exists n_{k+1} > N_{k+1} > k$ such that $x_{n_{k+1}} > b - \varepsilon_{k+1}$. We then choose this element to be u_{k+1}. We have defined this subsequence from x_n inductively. Therefore, we observe that, $\forall k$

$$\begin{aligned} b - \varepsilon_n &< u_n < b + \varepsilon_n \\ b - \frac{1}{n} &< u_n < b + \frac{1}{n} \\ \lim_{n\to\infty} b - \frac{1}{n} &< \lim_{n\to\infty} u_n < \lim_{n\to\infty} b + \frac{1}{n} \end{aligned}$$

Both sides go to b, and by sandwich theorem, $u_n \to b$ as well.

Claim: $l_n \to a$
Suppose that $\liminf x_n = a$.

Again, we let $\varepsilon_1 = \frac{1}{1}$ and by previous proposition (1): $\liminf x_n = a \in \mathbb{R}$ implies that $\forall \varepsilon > 0, \exists N$ such that $\forall n > N, x_n > a - \varepsilon$.

We choose $\varepsilon = \varepsilon_1$ and an N_1 that satisfies this condition, and choose an $M = N_1$. By proposition (2), we know that $\forall M, \exists n_1 > M$ such that $x_{n_1} < a + \varepsilon_1$. We rename this x_{n_1} as l_1.

We then construct this sequence inductively. Suppose that we have chosen l_1 to l_k. Similarly, we choose $\varepsilon_{k+1} = \frac{1}{k+1}$, and by proposition (1), there exists some N_{k+1} such that $\forall n > N_{k+1}, x_n > a - \varepsilon_{k+1}$. We choose this $N_{k+1} > k$.

And by the second proposition, we know that amongst all $x_n > a - \varepsilon_{k+1}$, there exist one that is within the epsilon neighbourhood of a, i.e., for this fixed $N_{k+1}, \exists n_{k+1} > N_{k+1}$ such that $x_{n_{k+1}} < a + \varepsilon_{k+1}$. We choose this $x_{n_{k+1}}$ to be our l_{k+1}. This is how the sequence l_n is defined inductively.

Combining the results and applying to the sequence $\{l_n\}$:

$$\begin{gathered} a - \varepsilon_n < l_n < a + \varepsilon_n \\ a - \frac{1}{n} < l_n < a + \frac{1}{n} \\ \lim_{n\to\infty} a - \varepsilon_n < \lim_{n\to\infty} l_n < \lim_{n\to\infty} a + \varepsilon_n \end{gathered}$$

Both sides go to a. By sandwich theorem, $l_n \to a$.

QED

LUB & GLB to MSP

11. Show that i and ii of Theorem 1.3.4 both imply the completeness axiom for an ordered field.

(Theorem 1.3.4: Least upper bound property and Greatest lower bound property)

Proof.

Claim: LUBP $\Rightarrow$ MSP
Suppose an ordered field $\mathbb{F}$.
Suppose that $\{x_n\}$ is a monotonic increasing sequence bounded above. Therefore, the set $S = \{x_1, x_2, \cdots, x_n\}$ is bounded above and by the least upper bound property, $\exists x = \sup(S)$. By definition of the LUB, $x \geq x_n \ \forall x_n \in \{x_n\}$. $\forall \varepsilon > 0$, there exists some N such that

$x_N > x - \epsilon$. We can obtain the inequality

$$x - x_N < \varepsilon$$

Recall $x \geq x_n \ \forall x_n \in \{x_n\}$, we know that $|x_N - x| = x - x_N$, therefore, fixing $\varepsilon > 0, \exists N$ such that:

$$|x_N - x| < \varepsilon$$

Also, given that $\{x_n\}$ monotonically increases, then, $x_n \geq x_N \ \forall n > N$. Therefore, $|x_n - x| \leq |x_N - x| < \varepsilon \ \forall n > N$. Eventually, by the definition of limit, $\{x_n\}$ converges. Least upper bound property implies the Monotonic Sequence Property. The ordered field $\mathbb{F}$ with Least upper bound property is complete.

Claim: GLBP $\Rightarrow$ MSP

Suppose that $\{x_n\}$ is a monotonic decreasing sequence bounded above. Therefore, the set $S = \{x_1, x_2, \cdots, x_n\}$ is bounded below and by the greatest lower bound property, $\exists x = \inf(S)$. By definition of the GLB, $x \leq x_n \ \forall x_n \in \{x_n\}$. $\forall \varepsilon > 0$, there exists some N such that $x_N < x + \epsilon$. We can obtain the inequality

$$x_N - x < \varepsilon$$

Recall $x \leq x_n \ \forall x_n \in \{x_n\}$, we know that $|x_N - x| = x_N - x$, therefore, fixing $\varepsilon > 0, \exists N$ such that:

$$|x_N - x| < \varepsilon$$

Also, given that $\{x_n\}$ monotonically decreasing, then, $x_n \leq x_N \ \forall n > N$. Therefore, $|x_n - x| \leq |x_N - x| < \varepsilon \ \forall n > N$. Eventually, by the definition of limit, $\{x_n\}$ converges to x. Greatest lower bound property implies the Monotonic Sequence Property. The ordered field $\mathbb{F}$ with Greatest lower bound property is complete. **QED**

Inequality of limsup and liminf

22. **a.** If x_n and y_n are bounded sequences in $\mathbb{R}$, prove that

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n.$$

b. Is the product rule true for lim sups?

a.

Proof.
Suppose that x_n, y_n are bounded sequence in $\mathbb{R}$. By definition of limsup, $\limsup(x_n) = \lim_{N\to\infty} \sup(\{x_k | k \geq N\})$. Now fixing some $N \in \mathbb{N}$, and suppose some x_n with $n > N$, then since $\{x_n\}$ are bounded, $\sup(\{x_k | k \geq N\}) \in \mathbb{R}$ and we know that $x_n \leq \sup(\{x_k | k \geq N\})$ $\forall n \geq k \geq N$

Similarly for $\{y_n\}$, we know that $\sup(\{y_k | k \geq N\}) \in \mathbb{R}$ and for some $y_n, n > N$, $y_n \leq \sup(\{y_k | k \geq N\})$
Therefore,

$$x_n + y_n \leq \sup(\{x_k | k \geq N\}) + \sup(\{y_k | k \geq N\})$$

The right hand side is constant and unique for all fixed N. If this holds $\forall n > N$, then

$$\begin{aligned}
\sup(\{x_n + y_n | n > N\}) &\leq \sup(\{x_k | k \geq N\}) + \sup(\{y_k | k \geq N\}) \\
\lim_{N\to\infty} \sup(\{x_n + y_n | n > N\}) &\leq \lim_{N\to\infty} [\sup(\{x_k | k \geq N\}) + \sup(\{y_k | k \geq N\})] \\
\lim_{N\to\infty} \sup(\{x_n + y_n | n > N\}) &\leq \lim_{N\to\infty} \sup(\{x_k | k \geq N\}) + \lim_{N\to\infty} \sup(\{y_k | k \geq N\}) \\
\limsup(x_n + y_n) &\leq \limsup(x_n) + \limsup(y_n)
\end{aligned}$$

QED

Finite limsup/liminf $\to$ existence of subsequence convergence

Claim: If $\exists \limsup x_n = b \in \mathbb{R}$, then $\exists x_{n_k} \to b$
First, reiterating the two propositions related: Suppose that $\exists \limsup x_n = b$, then $\forall \varepsilon > 0$,

1. $\exists N$ such that $\forall n \geq N$, $x_n < b + \varepsilon$

2. and $\forall M$, $\exists n > M$ such that $x_n > b - \varepsilon$

Proof.

Construct a subsequence out from x_n inductively, Suppose that $\epsilon_1 = \frac{1}{1}$, then by prop 1, $\exists N_1$ such that $\forall n \geq N_1$, $x_n < b + \varepsilon_1$. Then we set $M = N_1$ and by prop 2, $\exists n_1 \geq M$ such that $x_{n_1} > b - \varepsilon_1$, we pick this x_{n_1} to be our first term of our subsequence that suffice the following inequality:

$$b - \varepsilon_1 < x_{n_1} < b + \varepsilon_1$$

Then, suppose that we have chosen $x_{n_1}, x_{n_2} \cdots x_{n_k}$, then we show how to choose $x_{n_{k+1}}$:

Define $\varepsilon_{k+1} = \frac{1}{k+1}$, then by prop 1, $\exists N_{k+1}$, particularly that it is greater than n_k, such that $\forall n \geq N_{k+1}, x_n < b + \varepsilon_{k+1}$, then we define $M = N_{k+1}$ and by prop 2, $\exists n_{k+1} > M$ such that $x_{n_{k+1}} > b - \varepsilon_{k+1}$, we have chosen our $x_{n_{k+1}}$ to be:

$$b - \varepsilon_{k+1} < x_{n_{k+1}} < b + \varepsilon_{k+1}$$

Observing the sequence x_{n_k}, we have

$$\begin{aligned}
b - \varepsilon_k <& x_{n_k} < b + \varepsilon_k \\
\lim_{k\to\infty} b - \varepsilon_k <& \lim_{k\to\infty} x_{n_k} < \lim_{k\to\infty} b + \varepsilon_k \\
b <& \lim_{k\to\infty} x_{n_k} < b
\end{aligned}$$

By sandwich theorem, $\lim_{k\to\infty} x_{n_k} = b$. **QED**

1.6 Euclidean Space

Cauchy-Schwarz in any IPS

Claim: $\forall (V, \langle | \rangle)$ as inner product space, we have $\forall v, w \in V$,

$$|\langle v|w\rangle| \leq \langle v|v\rangle^{\frac{1}{2}} \cdot \langle w|w\rangle^{\frac{1}{2}}$$

Proof.

First, we eliminate the two trivial cases: that is, when v or w are zero vector, this inequality holds automatically.

Therefore, we start by assuming that both are non-zero.

By bilinearity and positive definiteness, we know that $\forall \alpha \in \mathbb{R}$ we have,

$$\begin{aligned} \langle \alpha v + w | \alpha v + w \rangle &\geq 0 \\ \alpha^2 \langle v|v \rangle + 2\alpha \langle v|w \rangle + \langle w|w \rangle &\geq 0 \end{aligned}$$

We define

$$\begin{cases} \langle v|v \rangle = a \\ \langle v|w \rangle = b \\ \langle w|w \rangle = c \end{cases}$$

And we define a function $f(\alpha) : \mathbb{R} \to \mathbb{R}$ such that

$$f(\alpha) = a \cdot \alpha^2 + 2b\alpha + c$$

Since a, b, c are non-negative, $f(\alpha)$ is a non-negative function. Also, α has a min at

$$\begin{aligned} f'(\alpha) &= 0 \\ 2a\alpha + 2b &= 0 \\ \alpha &= -\frac{b}{a} \end{aligned}$$

Therefore, $f(-\frac{b}{a}) \geq 0$,

$$\begin{aligned} a\left(-\frac{b}{a}\right)^2 + 2b\left(-\frac{b}{a}\right) + c &\geq 0 \\ -\frac{b^2}{a} + c &\geq 0 \\ b^2 &\leq ac \end{aligned}$$

Substitute the values back, we obtain:

$$\langle v|w\rangle^2 \leq \langle v|v\rangle \cdot \langle w|w\rangle$$
$$|\langle v|w\rangle| \leq \langle v|v\rangle^{\frac{1}{2}} \cdot \langle w|w\rangle^{\frac{1}{2}}$$

QED