Probability and Statistical Inference

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> **Theorem 0.0.1 Binomial converges in distribution to Poisson**
> Let $\{X_n\}_{n\in\mathbb{N}^+}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{B}, P)$ and suppose that
> $$X_n \sim \text{Binom}\left(n, \frac{\lambda}{n}\right)$$
> Then $X_n \xrightarrow{d} X$ such that $X \sim \text{Poisson}(\lambda)$.

Proof. To show the convergence in distribution, we need to show that the probability densities converge to a limit density, that is,

$$\lim_{n\to\infty} P(X_n = k) = P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

Fix $k \in \mathbb{N}^+$, given each $X_n \sim \text{Binom}\big(n, \frac{\lambda}{n}\big)$, we have

$$\begin{aligned}
\lim_{n\to\infty} P(X_n = k) &= \lim_{n\to\infty} \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} \\
&= \lim_{n\to\infty} \frac{n!}{k!(n-k)!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k} && \text{definition of binomial coefficient} \\
&= \underbrace{\frac{\lambda^k}{k!}}_{\text{Poisson-ish}} \lim_{n\to\infty} \frac{n!}{(n-k)!}\left(1-\frac{\lambda}{n}\right)^n \frac{1}{n^k\left(1-\frac{\lambda}{n}\right)^k} && \text{grouping terms and factoring} \\
&= \frac{\lambda^k}{k!} \lim_{n\to\infty} \{n(n-1)\cdots(n-k+1)\}\left(1-\frac{\lambda}{n}\right)^n \frac{1}{(n-\lambda)^k} \\
&= \frac{\lambda^k}{k!} \lim_{n\to\infty} \underbrace{\frac{\{n(n-1)\cdots(n-k+1)\}}{(n-\lambda)^k}}_{(1)} \underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{\to e^{-\lambda}\text{ by def}}
\end{aligned}$$

Notice that term (1) expands to:

$$\begin{aligned}
\lim_{n\to\infty} \frac{n(n-1)\cdots(n-k+1)}{(n-\lambda)^k} &= \lim_{n\to\infty} \frac{n^k}{n^k\left(1-\frac{\lambda}{n}\right)^k} \prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right) \\
&= \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{-k} \cdot \prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right) \\
&= 1
\end{aligned}$$

The first term and each term in the product go to 1. Therefore, $\lim_{n\to\infty}(1) = 1$.

We have shown that $\exists X$ s.t. $\lim_{n\to\infty} X_n = X$ and $X \sim \text{Poisson}(\lambda)$. $\square$

A more standard approach in proving weak convergence (convergence in distribution) is to look at the pointwise convergence in the characteristic functions. For $X_n \sim \text{Binomial}\left(n, \frac{\lambda}{n}\right)$,

Claim

$$\varphi_{X_n}(t) = \mathbb{E}\left[e^{i\,t\,X_n}\right] = \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n}\,e^{i\,t}\right)^n$$

Then the convergence becomes more apparent as

$$\begin{aligned}
\lim_{n\to\infty} \varphi_{X_n}(t) &= \lim_{n\to\infty} \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n}\,e^{i\,t}\right)^n \\
&= \lim_{n\to\infty} \left(1 + \frac{\lambda(e^{i\,t}-1)}{n}\right)^n \\
&= \exp\left\{\lambda\left(e^{i\,t}-1\right)\right\}
\end{aligned}$$

The result is precisely the characteristic function of a Poisson-distributed random variable.