Probability and Statistical Inference

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13 Lebesgue Integration

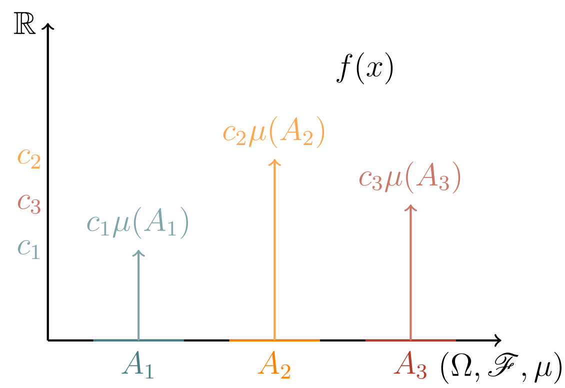
13.1 Set up:

- In $(\Omega, \mathscr{F}, \mu)$, let $f : \Omega \to \mathbb{R}$ be measurable.
- **Recall:** Indicator functions are measurable.
  - Integration of indicator functions equals the measure of the indicator function.

Definition 13.1.1 (Simple Functions).
f is said to be a simple function is f is a finite linear combinations of indicator functions:

$$f(x) = \sum_{i=1}^{n} c_i \cdot \mathbb{I}_{A_i}(x), \quad \text{for some measurable sets } \{A_i\} \text{ and constants } c_i \in \mathbb{R}.$$

Example 13.1.2 Simple functions



Definition 13.1.3 (Lebesgue Integral (for Simple Functions)).
Define the set

$$S^+ := \{f : \Omega \to \mathbb{R} \mid f \text{ is simple function, } f \geq 0\}.$$

For $f \in S^+$: define its Lebesgue integral w.r.t. its measure μ as:

$$\int_\Omega f \, d\mu := \sum_{i=1}^{n} c_i \cdot \mu(A_i), \quad \forall f \in S^+$$

Notation:

$$\int_\Omega f \, d\mu \quad \text{or} \quad \int_\Omega f(x) \, d\mu(x).$$

Proposition 13.1.4 Lebesgue Integral:

- **Linearity:** $\int_\Omega (\alpha f + \beta g) \, d\mu = \alpha \int_\Omega f \, d\mu + \beta \int_\Omega g \, d\mu, \quad \forall f, g \in S^+, \alpha, \beta \geq 0.$
- **Monotonicity:** If $f \leq g$, then $\int_\Omega f \, d\mu \leq \int_\Omega g \, d\mu$.

13.2 Lebesgue Integral

Definition 13.2.1 (Lebesgue Integral).

1. For any positive function $f : \Omega \to \mathbb{R}$, define the set $\mathcal{S}_f^+ = \{h \in S^+ \mid h < f\}$.
2. Compute all Lebesgue integral $I(h)$ according to the previous definition.

We define

$$\int_\Omega f d\mu = \sup_{h \in \mathcal{S}_{\{}^+} I(h)$$

And we note that f is μ-integrable (or just integrable if the context if clear) if $\int_\Omega f d\mu < \infty$.

13.3 Important results and theorems

Lemma 13.3.1 Fatou's Lemma:
Given a measure space $(\Omega, \mathscr{F}, \mu)$ and a sequence of measurable, non-negative functions $\{f_n\}$ each maps from $(\Omega, \mathscr{F}, \mu) \to (\mathbb{R}, \mathscr{B}, \cdot)$. Define a function $f \equiv \liminf_{n \to \infty} f_n(x)$, then f is measurable and we have the following inequality:

$$\int_\Omega f \, d\mu \leq \liminf_{n \to \infty} \int_\Omega f_n \, d\mu$$

Corollary 13.3.2 Beppo Levi's Monotone Convergence:

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and let $\{X_n\}$ be a sequence of non-negative measurable functions defined on Ω. Suppose:

$$0 \leq X_n(\omega) \leq X_{n+1}(\omega) \quad \forall \omega \in \Omega \quad \text{(monotonically increasing sequence of functions)}$$

Define the pointwise limit function:

$$X(\omega) = \lim_{n \to \infty} X_n(\omega), \quad \forall \omega \in \Omega$$

Then:

$$\lim_{n \to \infty} \int_\Omega X_n \, d\mu = \int_\Omega X \, d\mu$$

Theorem 13.3.3 Lebesgue's Dominated Convergence Theorem
Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of measurable functions mapping from $(\Omega, \mathscr{F}, \mu)$ to $(\mathbb{R}, \mathscr{B})$. Suppose:

- $f_n \to f$ pointwise almost everywhere on Ω, and
- there exists an integrable function $g \colon \Omega \to \mathbb{R}$ such that $|f_n(\omega)| \leq g(\omega)$ for all $\omega \in \Omega$ and all $n \in \mathbb{N}$.

Then all f_n and f integrable, and

$$\lim_{n \to \infty} \int_\Omega f_n d\mu = \int_\Omega \lim_{n \to \infty} f_n d\mu = \int_\Omega f d\mu$$

13.4 Expectation

Definition 13.4.1 (Expectation). Consider a measure space $(\Omega, \mathscr{F}, P)$ and a random variable $X : \Omega \to \mathbb{R}$. Its expectation is defined to be a Lebesgue integral

$$\mathbf{E}[X] = \int_\Omega X(\omega) dP(\omega)$$

Corollary 13.4.2 Jensen's Inequality:
If X is a random variable and $\varphi : \mathbb{R} \to \mathbb{R}$ is a convex function, then:

$$\varphi(\mathbb{E}[X]) \leq \mathbb{E}[\varphi(X)]$$

Special Case: For $\varphi(x) = |x|$, we obtain:

$$|\mathbb{E}[X]| \leq \mathbb{E}[|X|]$$

Intuition: The expectation of a convex function applied to X is at least as large as applying the function to the expectation of X . Convexity "pulls the curve upwards," leading to this inequality.
Corollary 13.4.3 Markov's Inequality:
For a non-negative random variable X and any $\alpha > 0$.

$$P(|X| > \alpha) \leq \frac{\mathbb{E}[|X|]}{\alpha}$$

Intuition: The probability that X exceeds some threshold α is bounded by the ratio of its expected value to α. It provides an upper bound on tail probabilities.

13.5 Chebyshev's Inequality

Statement: For $f : \mathbb{R}^d \to \mathbb{R}, f \geq 0, \alpha \in \mathbb{R}$, we have $m(\{f \geq \alpha\}) < \frac{1}{\alpha} \int f$.

Proof 1. By monotonicity of integral,

$$\int f \geq \int_{\{f \geq \alpha\}} f$$

Also observe that

$$\int_{\{f \geq \alpha\}} f \geq \alpha \cdot m(\{f \geq \alpha\}) \quad \text{Since } f \geq \alpha \text{ a.e. on the set}$$

Associating the two inequalities:

$$\frac{1}{\alpha} \int f \geq m(\{f \geq \alpha\})$$

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Two immediate lemmas follow:

13.5.1 Lemma 1

Statement: For $f : \mathbb{R}^d \to [0, \infty]$, if $\int f < \infty$, then $f < \infty$ a.e.

Proof 2. Fix any $n \in \mathbb{N}$, by Chebyshev's inequality,

$$m\{f \geq n\} < \frac{1}{n} \underbrace{\int f}_{<\infty}$$

And the sequence of sets: $\{f \geq n\}_{n\in\mathbb{N}}$ are nested and $\{f \geq n\} \searrow \{f > \infty\}$. Therefore by continuity of measure:

$$\lim_{n\to\infty} m(\{f \geq n\}) \leq \lim_{n\to\infty} \frac{1}{n} \int f$$
$$m(\{f > \infty\}) \leq 0$$

Therefore, f goes to infinity on a set of measure 0, which is equivalent as f is finite almost everywhere.

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13.5.2 Lemma 2

Statement: For $f : \mathbb{R}^d \to [0, \infty]$, if $\int f = 0$, then $f = 0$ a.e.

Proof 3. Similarly, fixing $n \in \mathbb{N}$, we have by Chebyshev's inequality:

$$m(\{f \geq 1/n\}) < n \int f = 0$$

Observe that the sequence of sets $\{f \geq 1/n\}_{n\in\mathbb{N}}$ is an increasing sequence of sets such that $\{f \geq 1/n\} \nearrow \{f > 0\}$. Therefore by continuity of measure:

$$\lim_{n\to\infty} m(\{f \geq 1/n\}) \leq n \int f$$
$$m(\{f > 0\}) \leq 0$$

Therefore, the set on which f is strictly larger than 0 has measure 0. This is equivalent as $f = 0$ almost everywhere.

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13.6 Fatou's Lemma

Statement Given a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$, $f_n \geq 0\ \forall n$. If $f_n \to f$ almost everywhere. Then $\int f \leq \liminf_n \int f_n$.

Proof 1. Same practice that we can always remove the subset from the domain on which f_n does not converge to f, therefore making $f_n \to f$ pointwise everywhere.

Consider a function g bounded with finite support: $m(\text{supp}(g)) < \infty$, and $0 \leq g \leq f$. Reminder that the construction of g is qualified for the bounded convergence theorem.

Then we set $g_n \equiv \min\{g, f_n\}$. Given $g < f$ and $f_n \to f$, we know that $g_n \to g$.

By bounded convergence theorem,

$$\int g = \lim_{n\to\infty} g_n$$

By monotonicity:

$$\forall n,\ g_n \leq f_n \Rightarrow \int g_n \leq \int f_n$$

We take the limit inferior on both ends (since we don't know yet if the limits exist):

$$\begin{aligned}
\underbrace{\liminf_n \int g_n}_{=\lim_n \int g_n} &\leq \liminf_n \int f_n \\
\int g &\leq \liminf_n \int f_n \quad \forall g \leq f \\
\underbrace{\sup_{g\leq f} \int g}_{\text{definition of } \int f} &\leq \liminf_n \int f_n \\
\int f &\leq \liminf_n \int f_n
\end{aligned}$$

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13.7 Monotone Convergence Theorem

Proof 1.

Statement For a sequence of function $f_k : \mathbb{R}^d \to \mathbb{R}$, $f_k \geq 0$. If f_k monotonically converges to some limit function f, then

$$\lim_{k\to\infty}\int f_k = \int f$$

By monotonicity of integral, $\{f_k\}$ being a monotone sequence implies that $\left\{\int f_k\right\}$ is also a monotone sequence bounded by $\int f$. Therefore, $\lim_{k\to\infty}\int f_k$ exists and is smaller than $\int f$.

The other side of the inequality is given by Fatou's lemma:

$$\int f \leq \liminf_k \int f_k = \lim_k \int f_k$$

Therefore, we have $\int f = \lim_k \int f_k$.

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13.8 Dominated Convergence Theorem

Statement For a sequence of functions $f_k : \mathbb{R}^d \to \mathbb{R}$ and $f_k \to f$ a.e.. If $\exists g$ integrable such that $|f_k| \leq g\ \forall k$, then $\lim_{k\to\infty}\int |f_k - f| = 0$.

Proof 1. We focus on the function $2g - |f_k - f|$, note that $|f_k| \leq g$ implies that $|f| \leq g$, and

$$|f_k - f| \leq |f_k| + |f| \leq 2g$$

Therefore $2g - |f_k - f| \geq 0$.

By Fatou's lemma,

$$\begin{aligned}
\liminf_k \int 2g - |f_k - f| \geq \int 2g - \lim_k |f_k - f| = \int 2g \\
\int 2g - \liminf_k \int |f_k - f| \geq \int 2g \\
\limsup_k \int |f_k - f| - \int 2g \leq -\int 2g \\
\limsup_k \int |f_k - f| \leq 0
\end{aligned}$$

Given that $\int |f_k - f| \geq 0$ by definition, we conclude that $\lim_{k\to\infty}\int |f_k - f| = 0$.

Remark: This conclusion, in particular, states that $\int f_k \to \int f$.

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13.9 Other Propositions to Integrability

13.9.1 Proposition I

Statement For $f : \mathbb{R}^d \to \mathbb{R}$ integrable, then $\forall \varepsilon > 0, \exists E \subset \mathbb{R}^d, m(E) < \infty, and \int_E |f| < \varepsilon$.

Proof 1. We define $E_N \equiv \{|x| < N\}$.
Observe the sequence of functions $|f| \cdot \chi_{E_N}$ are measurable and converges pointwise monotonically to $|f|$. We can then apply monotonic convergence theorem to conclude that

$$\lim_{N\to\infty} \int |f|\chi_{E_N} = \int |f|$$

Then by this convergence, $\exists N$ such that the difference $E \equiv \mathbb{R}^d \backslash E_N$ such that

$$\int_E |f| = \int_{\mathbb{R}^d} |f| - \int_{E_n} |f| < \varepsilon$$

Remark: The setup $|f|\chi_{E_N}$ is also dominated by f so DCT also works for this setup.

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13.9.2 Proposition II: Absolute Continuity

Statement For $f : \mathbb{R}^d \to \mathbb{R}$ integrable, then $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall E \subset \mathbb{R}^d$ measurable with $m(E) < \delta$, $\int_E |f| < \varepsilon$.

Proof 2. We will prove this by contradiction: Suppose this is not true.

Then $\exists \varepsilon > 0$ such that we cannot find a δ, that is, $\forall n$, there are some exceptions on some sets E_n, we set it up by letting $m(E_n) < 1/2^n$ and $\int_E |f| \geq \varepsilon$.

Then notice that the measure of E_n is designed to be summable and the infinite sum converges - then we have Borel-Cantelli Lemma:

$$\sum_n m(E_n) < \infty \Rightarrow m\left(\limsup_n E_n\right) = 0$$

Recall $\limsup_n E_n = \cap_{n\in\mathbb{N}} \cup_{k\geq n} E_n$

We define the limsup set to be E. Since $m(E) = 0$, $\int_E |f| = 0$. Observe the set of union

$\cup_{k\geq n} E_n$, the sequence of functions $|f| \cdot \chi_{\bigcup_{k\geq n} E_n}$ are all bounded above by $|f|$, then by DCT:

$$\lim_{n\to\infty} \int_{\bigcup_{k\geq n} E_n} |f| = \int_E |f| = 0$$

However, $\int_{E_n} |f| \geq \varepsilon > 0 \ \forall n$. This is a contradiction to the previous conclusion that the limit of the integral goes to 0. We therefore proved the proposition of absolute continuity.