Probability and Statistical Inference

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4 Lebesgue Measure

4.1 Motivation: From Probability to Length (Number Theory Version)

Now suppose that we need to draw a real number at random from the interval [0, 1]. What is the probability that it lies in [0, 0.47)?

In a discrete setting, we count favorable outcomes. Here, outcomes are uncountably infinite — so what does "probability" mean?

We can break down the problem by **picking digits** one by one.

- Pick a number: $0.xy\ldots$
- Either $x \in \{0, 1, 2, 3\}$ or $x = 4 \cap y \in \{0, 1, \ldots, 0.6\}$.

$$\# \text{ possibilities} = (4 \cdot 10) + (1 \cdot 7) = 47 \text{ out of } (10 \cdot 10) \text{ possibilities.}$$

$$\mathbb{P}(w \in [0, 0.47]) = \frac{47}{100} = 0.47 \quad \text{Some shape of Uniform CDF.}$$

Note that this is not the cardinality of the set.

4.2 Defining Lebesgue Measure on Intervals

Definition 4.2.1 (Lebesgue Measure on $[0, 1]$). Let $a, b \in [0, 1]$ with $a < b$. The Lebesgue measure μ assigns:

$$\mu([a, b)) = b - a.$$

In a more mathematical sense, the "length" of an interval is a "size" to a subset of any general set. Therefore, the essence of a measure, or Lebesgue measure on $\mathbb{R}$, intends to describe the "size" of almost any general subset in $\mathbb{R}$ by extending this rule to finite unions of disjoint intervals and then, with more care, to broader collections of subsets.

This seems natural — the length of an interval. But to treat it as a true measure in probability context, we need to verify that it satisfies the Kolmogorov axioms and additional key properties.

4.3 Properties of Lebesgue Measure on $\mathbb{R}$

Proposition 4.3.1 Basic Properties:
Let μ denote Lebesgue measure on $\mathbb{R}$:

1. **Length:** $\mu([a, b)) = b - a$
2. **Normalization:** $\mu([0, 1]) = 1$
3. **Translation Invariance:** $\mu(x + A) = \mu(A), \ \forall x \in \mathbb{R}$

Proposition 4.3.2 Special Cases:

1. Open intervals: $\mu((a, b)) = \mu([a, b)) = \mu([a, b]) = b - a$
2. Single element set:$\mu(\{a\}) = 0$
3. Countable set: $\mu(\mathbb{Q} \cap [0, 1]) = 0$

Remark 4.1 *Lebesgue measure assigns zero measure to all finite or countable sets — even dense ones like* $\mathbb{Q} \cap [0, 1]$.

4.4 What's Missing?

Remark 4.2 *Why Care About Measurability? So far, we've assigned lengths to intervals and even countable unions. But it turns out — surprisingly — that not all subsets of* $\mathbb{R}$ *can be measured in this way, namely:*

Example 4.4.1 Banach–Tarski Paradox (Informal Version) In 3-dimensional space, there exists a decomposition of a solid ball into a finite number of disjoint pieces, which can be reassembled — using only rotations and translations — into two identical copies of the original ball.

This construction:

- Uses only finitely many pieces.
- Does not rely on scaling or stretching.

- Crucially depends on the Axiom of Choice.

But it cannot be carried out with measurable pieces.

Implication: Any measure that is translation invariant and countably additive cannot be defined on **all** subsets of $\mathbb{R}^3$ if we hold onto the axiom of choice.

Remark 4.3 *To avoid these paradoxes, we restrict our attention to a special collection of subsets called **measurable sets**. In the next chapter, we will construct Lebesgue measure more carefully, using the concept of outer measure and the **Carathéodory criterion** to define which sets can be measured consistently.*