Probability and Statistical Inference

Tianqi Zhang
Emory University

Apr 17th 2025

4 Lebesgue Measure

4.1 Motivation (Number Theory Version)

Focus on half closed interval. Random draws $\omega \in [0, 1]$: $\mathbb{P}(\omega \in [0, 0.47)) =?$

- Pick a number: $0.xy\ldots$
- Either $x \in \{0, 1, 2, 3\}$ or $x = 4 \cap y \in \{0, 1, \ldots, 0.6\}$.

$$\#\text{ possibilities} = (4 \cdot 10) + (1 \cdot 7) = 47 \text{ out of } (10 \cdot 10) \text{ possibilities.}$$

$$\mathbb{P}(w \in [0, 0.47]) = \frac{47}{100} = 0.47 \quad \text{Some shape of Uniform CDF.}$$

4.2 Construction

Definition 4.2.1 (Lebesgue Measure on $\mathbb{R} \cap [0, 1]$).
More Generally, for $a, b \in \mathbb{R} \cap [0, 1]$, $a < b$, Lebesgue Measure is a measure defined on the power set of $\mathbb{R} \cap [0, 1]$ (Warning! this is ideal but is not true. reason to be discussed in the next chapter.) onto $\mathbb{R}$ such that the mapping is non-negative and sigma-additive. It is defined by the following:

$$\mathbb{P}([0, a)) = a, \quad \mathbb{P}([0, b)) = b.$$

$$\mathbb{P}([a, b)) = \mathbb{P}([0, b)) - \mathbb{P}([0, a)) = b - a.$$

Proposition 4.2.2 Additional Features on Lebesgue Measure:

- Unity: $\mathbb{P}([0, 1]) = 1$

- Translational Invariant: $\mathbb{P}(x + A) = \mathbb{P}(A),\ \forall x \in \mathbb{R}$

Proposition 4.2.3 :

1. **Open interval:** $\mathbb{P}((a, b)) = \lim_{n\to 0} \mathbb{P}((a + \frac{1}{n}, b)) = \lim_{n\to 0} b - (a + \frac{1}{n}) = b - a.$
2. **Single element:** $\mathbb{P}(\{a\}) = \mathbb{P}([a, b]) - \mathbb{P}((a, b)) = b - a - (b - a) = 0.$
3. **Closed interval:** $\mathbb{P}([a, b]) = \mathbb{P}([a, b)) + \mathbb{P}(\{b\}) = \mathbb{P}([a, b]) = b - a.$

Remark 4.1 *Using Kolmogorov axioms for measures, $A \subset [0, 1]$, finite or countable, $\mathbb{P}(A) = 0$. As a result, $\mathbb{P}(\mathbb{Q} \cap [0, 1]) = 0$.*