Proof: Bolzano-Weierstrass Theorem
Real Analysis

Apr. 17th 2025

Theorem 0.1 (Bolzano-Weierstrass)

In a metric space $\mathbf{M}$, $\forall A \subset M$, A is compact if and only if A is sequentially compact

Proof. To begin with, we start by marking some necessary lemmas:

1 Lemmas

Lemma 1.1 (Let $A \subset M$ be a compact subset of a metric space M. Then A is closed):

We will show that the complement $A^C = M \setminus A$ is open.

Fix any point $x \in A^C$. For each $n \in \mathbb{N}$, define the open set

$$U_n = \left\{ y \in M : d(y, x) > \frac{1}{n} \right\}$$

Each U_n is open in M, and we claim that $\{U_n\}n = 1^\infty$ forms an open cover of A. To see this, take any point $a \in A$. Since $a \neq x$ (because $x \in A^C$), we have $d(a, x) = \delta > 0$. Then for any n such that $\frac{1}{n} < \delta$, it follows that $d(a, x) > \frac{1}{n}$, i.e., $a \in U_n$. Therefore, every point $a \in A$ lies in some U_n, and hence

$$A \subset \bigcup_{n=1}^{\infty} U_n$$

Since A is compact, there exists a finite subcover, say $A \subset \bigcup_{i=1}^{N} U_i$. But since $U_i \subset U_N$ for all $i \leq N$ (because $d(y, x) > 1/i$ implies $d(y, x) > 1/N$ when $i \leq N$), we have

$$A \subset U_N$$

Taking complements, we obtain

$$U_N^C \subset A^C$$

but $U_N^C = \left\{y \in M : d(y,x) \leq \frac{1}{N}\right\}$ is a closed ball centered at x, and in particular, it contains x. This shows that for every $x \in A^C$, there exists an open neighborhood $U = M \setminus U_N^C$ of x contained in A^C. Hence, A^C is open, so A is closed.

Lemma 1.2 (Let M be a compact metric space, and let $B \subset M$ be closed. Then B is compact.):

Let $U_i, i \in I$ be an open cover for B, that is

$$B \subset \bigcup_{i \in I} U_i$$

Since B is closed, the complement B^C is open. Then consider the union;

$$\bigcup_{i \in I} U_i \cup B^C$$

is also an open set and moreover an open cover for M. Since M is compact, there exists a finite subcover $\bigcup_{i=1}^N U_i \cup B^C \supset M$. Then we take B^C away on both sides since $B \cap B^C = \varnothing$ meaning B^C does not cover anything for B. That makes $\bigcup_{i=1}^N U_i$ a finite subcover for B. Therefore, B is compact.

2 Sufficiency (compact ⇒ sequentally compact)

Step 1: Set up and reduction

Suppose that A is compact, we need to show that for any sequence $\{x_k\}$, there exists a converging subsequence whose limit is in the set. Suppose to the contrary that A is not sequentially compact, i.e. x_k has no converging subsequence. Without loss of generality, assume all points are distinct in $\{x_k\}$.

Step 2: For all $k = 1, 2, \ldots, \exists r > 0$ such that $x_j \notin D(x_k, r)$ $\forall j \neq k$. Intuition: if points are not accumulating, then they have to be isolated.

Suppose otherwise that $\exists k$ such that $\forall \varepsilon > 0, \exists x_j \in D(x_k, \varepsilon)$. We fix $\varepsilon = \frac{1}{m}$ and obtain $\{x_{j_m}\}$ such that $x_{j_m} \in D(x_k, \frac{1}{m})$. Then $x_{j_m} \to x_k$ which is a contradiction to the assumption that x_k has no converging subsequence.

Step 3: Finish up

Consider $B = \{x_k | k = 1, 2, \ldots\} \subset A$. By Step 2, there should not be any accumulation point, therefore, $\mathrm{AC}(B) = \varnothing$. Therefore, B contains all its accumulation points trivially which implies that B is closed, and lemma 2 implies that B is compact.

Now consider the collection of open sets $\mathcal{U} = \{\{x_k\}\}_{k\in\mathbb{N}}$. Since each point x_k is isolated (by construction in Step 2), the singleton set $\{x_k\}$ is open in the subspace topology on B. Thus, $\mathcal{U}$ is an open cover of B in the subspace topology.

However, $\mathcal{U}$ does not have a finite subcover for B. Therefore, this is a contradiction, and $\exists$ a converging subsequence for x_k, which implies that A is sequentially compact.

3 Necessity (sequentially compact ⇒ compact)

Suppose that $A \subset M$ is sequentially compact. Show that A is compact. Let $\{U_i\}_{i\in I}$ be an open cover for A.

Claim 3.1 $\exists r > 0$ such that $\forall y \in A, B(y, r) \subset U_i$ for some i. Intuition: there's a uniform ball size that fits entirely within at least one set in the open cover at every point.

Suppose not. Then for every $n \in \mathbb{N}$, let $r_n = \frac{1}{n}$. By assumption, for each r_n, there exists a point $y_n \in A$ such that for all $i \in I$,

$$B(y_n, r_n) \not\subseteq U_i.$$

Since A is sequentially compact, the sequence $\{y_n\} \subset A$ has a convergent subsequence $y_{n_k} \to z \in A$. Since $\{U_i\}i \in I$ is an open cover of A, there exists some $i_0 \in I$ such that $z \in Ui_0$.

Because U_{i_0} is open, there exists $\varepsilon > 0$ such that $B(z, \varepsilon) \subset U_{i_0}$. By convergence, there exists $K \in \mathbb{N}$ such that for all $k \geq K$, we have

$$d(y_{n_k}, z) < \frac{\varepsilon}{2}, \quad \text{and} \quad r_{n_k} = \frac{1}{n_k} < \frac{\varepsilon}{2}$$

Then for such k, we have

$$B(y_{n_k}, r_{n_k}) \subset B(z, \varepsilon) \subset U_{i_0}$$

which contradicts our assumption that $B(y_n, r_n) \not\subseteq U_i$ for any $i \in I$. Therefore, the claim is proven.

Claim 3.2 A is totally bounded.

Suppose not, then $\exists\varepsilon > 0$ such that A cannot be covered by finite number of balls radius ε. Then we can choose $y_1 \in A, y_2 \in A\backslash B(y_1, \varepsilon), \cdots, y_{k+1} \in A\backslash \bigcup_{i=1}^{k} B(y_i, \varepsilon)$.

Observe that $d(y_i, y_j) \geq \varepsilon \; \forall i \neq j$ for this chosen ε. Then y_n is not Cauchy which implies that y_n does not have a converging subsequence. This is a contradiction to the previous assumption that A is sequentially compact. The claim is proven.

To wrap up the proof, by Claim 3.1, $\exists r$ such that $\forall y \in A, B(y, r) \subset U_i$ for some i. By Claim 3.2, $\exists\{y_1, y_2, \cdots, y_m\} \in A$ such that $A \subset \bigcup_{i=1}^{m} B(y_i, r)$. Also by Claim 3.1, $B(y_{i_j}, r) \subset U_{i_j}$ for some i_j. Therefore, $\{U_{i_j}\}_{j=1}^{m}$is a finite subcover for A. A is compact.

Now apply total boundedness with $\varepsilon = r$. There exist points $\{x_1, \ldots, x_m\} \subset A$ such that

$$A \subset \bigcup_{j=1}^{m} B(x_j, r)$$

By claim 3.1, each ball $B(x_j, r) \subset U_{i_j}$ for some $i_j \in I$. Then

$$A \subset \bigcup_{j=1}^{m} B(x_j, r) \subset \bigcup_{j=1}^{m} U_{i_j}$$

Thus, $\{U_{i_j}\}_{j=1}^{m}$ is a finite subcover of A. Therefore, A is compact.

□

Remark 3.1 *B-W Theorem:*
The Bolzano–Weierstrass theorem is a foundational result in real analysis and topology to characterize compactness in metric spaces.

This highlights a central philosophy in analysis: Compactness enables global control using only local data.

One intuitive interpretation is that a sequentially compact set "does not let sequences escape": every sequence has a cluster point that remains within the set. This is particularly important in function spaces later, where we often work with approximating sequences and need to ensure their limits stay within a prescribed domain.