Probability and Statistical Inference

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6 Measurable maps

6.1 Measurable maps

Definition 6.1.1 (Measurable map).
Let $(\Omega_1, \mathscr{F}_1)$, $(\Omega_2, \mathscr{F}_2)$ be measurable spaces.
A map $f : \Omega_1 \to \Omega_2$ is measurable with respect to $(\mathscr{F}_1, \mathscr{F}_2)$ if:

$$f^{-1}(A) \in \mathscr{F}_1, \ \forall A \in \mathscr{F}_2.$$

iff Pre-image of measurable sets are measurable.

Example 6.1.2 Indicator Function
Consider $\chi_A : (\Omega, \mathscr{F}) \to (\mathbb{R}, \mathscr{B})$, where

$$\chi_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{else.} \end{cases}$$

To show χ_A is measurable, check all pre-images:

$$\chi_A^{-1}(\varnothing) = \varnothing, \quad \chi_A^{-1}(\mathbb{R}) = \Omega, \quad \chi_A^{-1}(\{1\}) = A, \quad \chi_A^{-1}(\{0\}) = A^c.$$

Since all are in $\mathscr{F}$, χ_A is measurable.

Example 6.1.3 Suppose $f : (\Omega_1, \mathscr{F}_1) \to (\Omega_2, \mathscr{F}_2)$ and $g : (\Omega_2, \mathscr{F}_2) \to (\Omega_3, \mathscr{F}_3)$. If f and g are measurable, then $g \circ f$ is measurable.
For any $A \in \mathscr{F}_3$,

$$(g \circ f)^{-1}(A) = \underbrace{f^{-1}\underbrace{\left(g^{-1}(A)\right)}_{\in \mathscr{F}_2}}_{\in \mathscr{F}_1}.$$

Since $g^{-1}(A) \in \mathscr{F}_2$ and f^{-1} preserves measurability, $(g \circ f)^{-1}(A) \in \mathscr{F}_1$.

Proposition 6.1.4 :
Let $(\Omega, \mathscr{F})$, $(\mathbb{R}, \mathscr{B})$, and $f, g : \Omega \to \mathbb{R}$ be measurable. Then:

1. $f + g, f - g$ are measurable,
2. $|f|$ is measurable.

Proposition 6.1.5 Measurable and Continuity:
Let $f : X \to \mathbb{R}$ be a continuous function. Then f is measurable with respect to the Borel sigma algebra.

6.2 Probability measure to a random variable (Push-forward)

Definition 6.2.1 (Push-forward measure). Push-forward measure P_X is the measure induced on $\mathbb{R}^d$ via the function X, such that:

$$\forall A \in \mathscr{B}^d,\ P_X(A) = P(X^{-1}(A)).$$

For a probability (measure) space $(\Omega, \mathscr{F}, P)$ and a function (random variable)

$$X : (\Omega, \mathscr{F}, P) \to (\mathbb{R}^d, \mathscr{B}^d, \cdot),$$

There are generally two ways to make X measurable map:

1. Fix $\mathscr{B}^d$ and $\mathscr{F}$, ask if X is measurable.
2. Alter $\mathscr{F}$ according to X such that X is measurable.

6.2.1 Method 1: Validation

Ask whether X is measurable with respect to the fixed σ-algebras:

$$\forall A \in \mathscr{B}^d, \quad X^{-1}(A) \in \mathscr{F}.$$

This directly checks the measurability condition. If satisfied, we say then X as a function is measurable.

- *Non-Measurability* shows by the *same set having 2 different sizes (measures)*

Criterion Check (1):

- For $\Omega \xrightarrow{X} \mathbb{R}^d$, $\forall A \in \mathscr{B}^d$, $X^{-1}(A) \in \mathscr{F}$, then X is measurable.
- BUT: Too hard to validate $\forall A \in \mathscr{B}^d$.

6.2.2 Method 2: Generation

Fix $\mathscr{B}^d$, then construct a σ-algebra $\mathscr{F}_X$ (generated by X) that ensures measurability:

$$X^{-1}(\mathscr{B}^d) := \{X^{-1}(A) \mid A \in \mathscr{B}^d\}.$$

Define:

$$\mathscr{F}_X \equiv \sigma(X) \quad \text{(the } \sigma\text{-algebra generated by } X).$$

This is the smallest σ-algebra that makes X measurable:

- *Actually, $\sigma(X)$ is the smallest σ-algebra for X to be measurable.*
- $\sigma(X)$ contains all pre-images of $\mathscr{B}^d$ under X.

Example 6.2.2

Let $X(\omega) = \mathbb{I}_A(\omega)$ for some $A \subseteq \Omega$, where $\mathbb{I}_A$ is the indicator function of A. Then the σ-algebra generated by X is:

$$\sigma(X) = \{\varnothing, A, A^c, \Omega\}.$$

This σ-algebra usually represents *information sets* in practice.

Theorem 6.2.3 Measurability Validation

Solution: It is sufficient to check the generators B of $\mathscr{B}$ such that $X^{-1}(B) \subset \mathscr{F}$.

Example 6.2.4
Let $\mathcal{L} = \left\{\prod_{i=1}^{d}(-\infty, x_i] \mid x_i \in \mathbb{R}\right\}$ be a generator of $\mathscr{B}^d$. We have $X^{-1}(\mathcal{L}) \subseteq \mathscr{F}$, then we generate the smallest σ-algebra:

$$\sigma(X^{-1}(\mathcal{L}))$$

We need to show that

$$\sigma(X) \subseteq \sigma(X^{-1}(\mathcal{L})) \subseteq \mathscr{F}.$$

which makes X measurable.