CHAPTER 3

Stochastic Integration and Ito's Formula

In this chapter we discuss Itô's theory of stochastic integration. This is a vast subject. However, our goal is rather modest: we will develop this theory only generally enough for later applications. We will discuss stochastic integrals with respect to a Brownian motion and more generally with respect to a continuous local martingale. Instead of attempting to describe the largest possible class of integrand processes, we will only single out a class of integrand processes sufficiently large for our later applications. For example, in the case of a continuous local martingale as an integrator, we only restrict ourselves to continuous adapted integrand processes, a class of processes sufficiently large for most applications. This of course does not mean that integrands which are not continuous cannot be integrated with respect to a continuous martingale. It is usually the case that when dealing with a discontinuous integrand, we can quickly decide then and there whether the integral has a meaning. Since many excellent textbooks on stochastic integration are available (McKean [**8**], Ikeda and Watanabe [**6**], Chung and Williams [**3**], Oksendal [**10**], Karatzas and Shreve [**7**], to cite just a few), there is little motivation on the part of the author to go beyond what will be presented in this chapter.

1. Introduction

If A is a process of bounded variation and $f : \mathbb{R} \times \mathbb{R}$ is function such that $s \mapsto f(s, \omega)$ is Borel measurable function, then

$$\int_0^t f(s)\, dA_s$$

can be interpreted as a pathwise integral if proper integrability conditions are assumed and its value at $\omega \in \Omega$ is the usual Lebesques–Stieljes integral

$$\int_0^t f(s,\omega)\, dA_s(\omega).$$

In theoretical and applied problems, there is an obvious need to make sense out of an integral of the form

$$\int_0^t f(s)\, dB_s,$$

where B is a Brownian motion. As we know, Brownian motion sample paths are not functions of bounded variation (see REMARK 7.2). For this reason in general there is no easy and direct pathwise interpretation of the above integral. However, in some special situation, a simple interpretation is possible. For example, if $s \mapsto f(s, \omega)$ itself has bounded variation for each ω, we can define the above integral by integration by parts:

$$\int_0^t f(s)\,dB_s = f(t)B_t - \int_0^t B_s\,df(s).$$

Such stochastic integrals are rather limited in its scope of application. Itô's theory of stochastic integration greatly expands the class of integrand processes, thus making the theory into a powerful tool in pure and applied mathematics.

We first define the integration of a step (and deterministic) process with respect to a Brownian motion. Let

$$\Delta:\ 0 = t_0 < t_1 \cdots < t_n = t$$

be a partition of $[0, t]$. If f is a step process:

$$f(s) = f_{j-1}, \qquad t_{j-1} \le t < t_j,$$

then the natural definition of its stochastic integral with respect to Brownian motion B is

$$\int_0^t f(s)\,dB_s = \sum_{j=1}^{n-1} f_{j-1}\left[B_{t_j} - B_{t_{j-1}}\right].$$

Using the property of independent increments for Brownian motion, its second moment is given by

$$\begin{aligned}
\mathbb{E}\left[\int_0^t f(s)\,dB_s\right]^2 &= \sum_{i,j=1}^{n} f_{i1} f_{j-1} \mathbb{E}\left[\left(B_{t_i} - B_{t_{i-1}}\right)\left(B_{t_j} - B_{t_{j-1}}\right)\right] \\
&= \sum_{j=1}^{n} |f_{j-1}|^2 \mathbb{E}\left[\left(B_{t_j} - B_{t_{j-1}}\right)^2\right] \\
&= \sum_{j=1}^{n} |f_{j-1}|^2 \left(t_j - t_{j-1}\right).
\end{aligned}$$

This gives

$$\mathbb{E}\left|\int_0^1 f(s)\,dB_s\right|^2 = \sum_{j=1}^{n} |f_{j-1}|^2 (t_j - t_{j-1}) = \int_0^t |f(s)|^2\,ds. \tag{1.1}$$

If f and g are two step functions, then

$$\mathbb{E}|\int_0^t f(s)\,dB_s - \int_0^t g(s)\,dB_s|^2 = \int_0^t |f(s) - g(s)|^2\,ds = \|f - g\|_2^2.$$

For an arbitrary function $f : [0, t] \to \mathbb{R}$, it is now clear that as long as there is a sequence of step functions f_n such that $\|f_n - f\|_2 \to 0$, we can define the stochastic integral as the limit

$$\int_0^t f(s)\, dB_s = \lim_{n\to\infty} \int_0^t f_n(s)\, dB_s.$$

It is well know from real analysis that this approximation property is shared by all Borel measurable functions f such that $\|f\|_2 < \infty$. We have thus greatly enlarged the space of deterministic functions which can be integrated with respect to a Brownian motion.

The key observation in Itô's theory of stochastic integration is that we can pass to a random step process as long as $f_j \in \mathscr{F}_{t_j}$. Note that now the integrand has the form

$$f(s) = f_{j-1}, \quad t_{j-1} \le s < t_j, \quad f_{j-1} \in \mathscr{F}_{t_{j-1}},$$

Namely, in the time interval $[t_{j-1}, t_j]$, the integrand is measurable with respect to the σ-algebra at the *left* endpoint of the time interval. For this reason above argument still applies as long as we replace the equality (1.1) by the more general equality

$$\mathbb{E}\left|\int_0^1 f(s)\, dB_s\right|^2 = \mathbb{E}\sum_{j=1}^n |f_{j-1}|^2 (t_j - t_{j-1}) = \mathbb{E}\int_0^t |f(s)|^2\, ds.$$

The key step in establishing this equality is the vanishing of the off diagonal term

$$\mathbb{E}\left[f_{i-1} f_{j-1} (B_{t_i} - B_{t_{i1}})(B_{t_j} - B_{t_{j-1}})\right] = 0.$$

This holds because if $i < j$, then by conditioning on $\mathscr{F}_{t_{j-1}}$, we have

$$\begin{aligned} & \mathbb{E}\left[f_{i-1} f_{j-1} (B_{t_i} - B_{t_{i1}})(B_{t_j} - B_{t_{j-1}}) | \mathscr{F}_{t_{j-1}}\right] \\ = \; & f_{i-1} f_{j-1} (B_{t_i} - B_{t_{i1}}) \mathbb{E}\left[B_{t_j} - B_{t_{j-1}} | \mathscr{F}_{t_{j-1}}\right] \\ = \; & 0. \end{aligned}$$

For the diagonal term, by conditioning on $\mathscr{F}tj - 1$ again, we have

$$\mathbb{E}\left[|f_{i-1}|^2 (B_{t_i} - B_{t_{i-1}})^2\right] = \mathbb{E}|f_{i-1}|^2 (t_i - t_{i-1}).$$

An important property of stochastic integrals is that

$$M_t = \int_0^t f(s)\, dB_s$$

is a continuous martingale. Let us see why this is so if the integrand process is a step process

$$f(s) = f_{j-1}, \qquad t_{j-1} < s \le t_j, \quad f_{j-1} \in L^2(\Omega, \mathscr{F}_{t_{j-1}}, \mathbb{P}).$$

We can write

$$M_t = \int_0^t f(s)\, dB_s = \sum_n f_{j-1}\left(Bt_j \wedge t - B_{t_{j-1}\wedge t}\right).$$

Note that sum has only finitely many terms for each fixed t. The process M has continuous sample paths because Brownian motion does. Suppose that $s < t$. If s and t are not among the points t_j we may simply insert them into the sequence. Now s and t are among the sequence we have

$$M_t - M_s = \sum_{s \le t_{j-1} \le t} f_{j-1}\left(B_{t_j} - B_{t_{j-1}}\right).$$

The general term in the sum vanishes after we condition it on $\mathscr{F}_{t_{j-1}}$, hence it also vanishes after we condition on $\mathscr{F}_s$ because $\mathscr{F}_s \subset \mathscr{F}_{t_{j-1}}$. It follows that

$$\mathbb{E}\left[M_t - M_s | \mathscr{F}_s\right] = 0$$

and M_t is a continuous martingale.

We now calculate the quadratic variation process for the continuous martingale

$$M_t = \int_0^t f(s)\, dB_s.$$

We claim that

$$\langle M, M \rangle_t = \int_0^t |f(s)|^2\, ds.$$

We again verify this for a step process f. As before, we assume that $s < t$ are among the points t_j. We have

$$M_t^2 - M_s^2 = \sum f_{i-1} f_{j-1} (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}}),$$

where the sum is over those i and j such that $t_i \le t$, $t_j \le t$ and at least one of t_{i-1} and t_{j-1} is greater or equal o s. For an off-diagonal term, say, $i < j$, the term vanishes after conditioning on $\mathscr{F}_{t_{j-1}}$, hence also vanishes after conditioning on $\mathscr{F}_s$. For a diagonal term $i = j$, since $t_{j-1} \ge s$, by first conditioning on $\mathscr{F}_{t_{j-1}}$ and then conditioning on $\mathscr{F}_s$ we have

$$\mathbb{E}\left[|f_{j-1}|^2 (B_{t_j} - B_{t_{j-1}})^2 | \mathscr{F}_s\right] = \mathbb{E}\left[|f_{j-1}|^2 (t_j - t_{j-1}) | \mathscr{F}_s\right].$$

It follows that

$$\mathbb{E}\left[M_t^2 - M_s^2 | \mathscr{F}_s\right] = \mathbb{E}\left[\int_s^t |f(s)|^2\, ds \middle| \mathscr{F}_s\right].$$

This shows that

$$M_t^2 - \int_0^t |f(s)|^2\, ds$$

is a martingale.

2. Stochastic integrals with respect to Brownian motion

The general setting is as follow. We have a Brownian motion B with respect to a filtration $\mathscr{F}_*$ defined on a filtered probability space $(\Omega, \mathscr{F}_*, \mathbb{P})$. We assume that the filtration $\mathscr{F}_*$ satisfies the usual conditions.

DEFINITION 2.1. *A real-valued function $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is called a step process if there exists a nondecreasing sequence $0 = t_0 < t_1 < t_2 < \cdots$ increasing to infinity and a sequence of square-integrable random variables $f_{j-1} \in \mathscr{F}_{t_{j-1}}$ such that*

$$f_t = f_{j-1}, \qquad t_{j-1} \le t < t_j.$$

The space of step processes is denoted by $\mathscr{S}$. The space of step processes on $[0,t]$ is denoted by $\mathscr{S}_t$.

Note that on an interval $[t_{j-1}, t_j)$ where f is constant, it is measurable with respect to the σ-algebra on the left endpoint of the interval, i.e., $f = f_{j-1} \in \mathscr{F}_{t_{j-1}}$.

Let f be a step process as above. The stochastic integral of f with respect to Brownian motion B is the process

$$I(f)_t = \int_0^t f(s)\, dB_s = \sum_{j=1}^{\infty} f_{j-1} \left(B_{t_j \wedge t} - B_{t_{j-1} \wedge t} \right).$$

Of course this is a finite sum for each fixed t. If $t_i \le t < t_{i+1}$, then

$$I(f)_t = \sum_{j=1}^{i} f_{j-1}(B_{t_j} - B_{t_{j-1}}) + f_i(B_t - B_{t_i}).$$

It is easy to see that the definition of $I(f)_t$ is independent of the partition $\{t_j\}$. As we have shown in the last section $I(f)$ is a continuous martingale with the quadratic variation

$$\langle I(f), I(f) \rangle_t = \int_0^t |f(s)|^2 ds.$$

In particular we have

$$\mathbb{E}|I(f)_t|^2 = \mathbb{E}\left[\int_0^t |f(s)|^2\, ds \right].$$

This relation allows us to extend the definition of stochastic integrals to more general integrands by a limit procedure. Since the quadratic variation process of a Brownian motion corresponds to the Lebesgue measure on $\mathbb{R}_+$, a very wide class of processes can be used as integrand processes.

DEFINITION 2.2. *A function $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is called progressively measurable with respect to the filtration $\mathscr{F}_*$ if for each fixed t the restriction $f : [0,t] \times \Omega \to \mathbb{R}$ is measurable with respect to the product σ-algebra $\mathscr{B}[0,t] \times \mathscr{F}_t$.*

EXAMPLE 2.3. Here are some examples of progressively measurable processes:

(1) A step process is progressively measurable.

(2) A continuous and adapted process is progressively measurable.

(3) Let τ be a finite stopping time. The the process

$$f(t,\omega) = I_{[0,\tau(\omega)]}(s)$$

is progressively measurable. For a discrete τ, this can be verified directly. For a general τ, let

$$\tau_n = ([2^n\tau]+1)/2^n.$$

Then $\tau_n \downarrow \tau$ and $I_{[0,\tau_n(\omega)]}(s) \to I_{[0,\tau(\omega)]}(s)$ for all $(s,\omega) \in \mathbb{R}_+ \times \Omega$.

For a progressively measurable process f, we define for each fixed T,

$$\|f\|_{2,T}^2 = \mathbb{E}\left[\int_0^T |f(s)|^2 ds\right].$$

The progressive measurability assures that the integral on the right side has a meaning. We use $\mathscr{L}_T^2$ to denote the space of progressively measurable processes f on $[0,T]\times\Omega$ with $\|f\|_{2,T} < \infty$. The norm $\|\cdot\|_{2,T}$ makes $\mathscr{L}_T^2$ into a complete Hilbert space. We use $\mathscr{L}^2$ to denote the space of progressively measurable processes f such that $\|f\|_{2,T}$ is finite for all T. It can be made into a metric space by introducing the distance function

$$d(f,g) = \sum_{n=1}^{\infty} \frac{\|f-g\|_{2,n}}{1+\|f-g\|_{2,n}}.$$

Note that $\mathscr{S}_T \subset \mathscr{L}^2$ and $\mathscr{S} \subset \mathscr{L}^2$, and there is an obvious restriction map $\mathscr{L}^2 \to \mathscr{L}_t^2$.

We now show how to define the stochastic integral

$$I(f)_t = \int_0^t f(s)\, dB_s$$

for every process in $\mathscr{L}^2$. We need the following approximation result.

LEMMA 2.4. *$\mathscr{S}_T$ is dense in $\mathscr{L}_T^2$. In other words, for any square integrable progressively measurable process $f \in \mathscr{L}_T^2$, there exists a sequence $\{f_n\}$ of step processes such that $\|f_n - f\|_{2,T} \to 0$.*

PROOF. First note that the set of uniformly bounded elements in $\mathscr{L}_T^2$ is dense. Suppose that f is a progressively measurable process uniformly bounded on $[0,T]\times\Omega$. Define

$$f_h(t,\omega) = \frac{1}{h}\int_{t-h}^t f(s,\omega)\, ds.$$

Each f_h is continuous and adapted. From real analysis we also know that

$$\lim_{h\to 0}\int_0^T |f_h(t,\omega) - f(t,\omega)|^2\, ds = 0.$$

It follows that $\|f_h - f\|_{2,T} \to 0$ as $h \to 0$. This shows that the set of uniformly bounded continuous and adapted processes is dense in $\mathscr{L}_T^2$.

Finally for a continuous, adapted, and uniformly bounded process f on $[0, T]$ we define

$$f_n(t) = f\left(\frac{j-1}{n}\right), \qquad \frac{j-1}{n} \le t < \frac{j}{n}.$$

Each $f_n \in \mathscr{S}_T$ and $f_n(t,\omega) \to f(t,\omega)$ for all $(t,\omega) \in [0,T] \times \Omega$ by continuity. Hence $\|f_n - f\|_{2,T} \to 0$ by the dominated convergence theorem. The lemma is proved. □

We are now ready to define the stochastic integral of a progressively measurable and square integrable process with respect to a Brownian motion. Suppose that $f \in \mathscr{L}^2$. By restricting to $[0,t]$ we can regard it as an element in $\mathscr{L}_t^2$ for any t. Let f_n be a sequence of step processes on $[0,T]$ such that $\|f_n - f\|_{2,T} \to 0$. The stochastic integral $I(f_n)_t$ is well defined for $0 \le t \le T$ as a Riemannan sum and is a square integrable martingale. We have

$$I(f_m)_t - I(f_n)_t = \int_0^t [f_m(s) - f_n(s)]\, dB_s.$$

By Doob's martingale inequality,

$$\begin{aligned} \mathbb{E}\left[\max_{0\le t\le T} |I(f_m)_t - I(f_n)_t|^2\right] &\le 4\mathbb{E}|I(f_m - f_n)_T|^2 \\ &= 4\mathbb{E}\left[\int_0^T |f_m(s) - f_n(s)|^2 ds\right] \\ &= 4\|f_m - f_n\|_{2,T}^2. \end{aligned}$$

This shows that $I(f_n)_t$ is a Cauchy sequence in $L^2(\Omega, \mathbb{P})$. By the completeness of $L^2(\Omega, \mathbb{P})$, the limit $I(f)_t = \lim_{n\to\infty} I(f_n)_t$ exists and we define

$$\int_0^t f(s)\, dB_s = \lim_{n\to\infty} \int_0^t f_n(s)\, dB_s.$$

Since $\|f_n - f\|_{2,T_1} \to 0$ implies $\|f_n - f\|_{2,T_2}$ for $T_2 \le T_1$, it is easy to see that $I(f)_t$ thus defined is independent of T and the choice of the approximating sequence $\{f_n\}$.

We have only defined $\{I(f)_t,\ t \ge 0\}$ as a collection of random variables. We can do more.

THEOREM 2.5. *Suppose that $f \in \mathscr{L}^2$ is a square integrable, progressively measurable process. Then the stochastic integral*

$$I(f)_t = \int_0^t f(s)\, dB_s$$

is a continuous martingale. Its quadratic variation process is

$$\langle I(f), I(f)\rangle_t = \int_0^t f(s)^2\, ds.$$

PROOF. The martingale property of $I(f)$ is inherited from the same property of $I(f_n)$ because this property is preserved when passing to the limit in $L^2(\Omega, \mathscr{F}*, \mathbb{P})$ as $n \to \infty$. We show that it has continuous sample paths. From

$$\mathbb{E}\left[\max_{0\le t\le T} |I(f_m)_t - I(f_n)_t|^2\right] \le 4\|f_m - f_n\|_{2,T}^2$$

we have by Chebyshev's inequality,

$$\mathbb{P}\left[\max_{0\le t\le T} |I(f_m)_t - I(f_n)_t| \ge \epsilon\right] \le \frac{4}{\epsilon^2}\|f_m - f_n\|_{2,t}^2.$$

Since $\|f_m - f_n\|_{2,T} \to 0$ as $m, n \to \infty$, by choosing a subsequence if necessary, we may assume that $\|f_n - f_{n+1}\|_{2,T} \le 1/n^3$, hence

$$\mathbb{P}\left\{\max_{0\le t\le T} |I(f_n)_t - I(f_{n+1})_t| \ge \frac{1}{n^2}\right\} \le \frac{4}{n^2}.$$

By the Borel–Cantelli lemma, there is a set Ω_0 with $\mathbb{P}\{\Omega_0\} = 1$ with the following property: for any $\omega \in \Omega_0$, there is an $n(\omega)$ such that

$$\max_{0\le t\le T} |I(f_n)_t(\omega) - I(f_{n+1})_t(\omega)| \le \frac{1}{n^2}$$

for all $n \ge n(\omega)$. It follows that $I(f_n)_t(\omega)$ converges uniformly on $[0, T]$. Since each $I(f_n)_t(\omega)$ is continuous in t, the limit, which necessarily coincide with $I(f)_t(\omega)$ with probability 1 for all t, must also be continuous. We therefore have shown that the stochastic integral $I(f)_t$ has a version with continuous sample paths.

We have shown before that for the step process f_n the process

$$I(f_n)_t^2 - \int_0^t f_n(s)^2\, ds$$

is a martingale. For each fixed t, as $n \to \infty$, the above random variable converges in $L^1(\Omega, \mathbb{P})$ to

$$I(f)_t^2 - \int_0^t f(s)^2\, ds.$$

Therefore the martingale property can be passed to the limiting process, which identifies the quadratic variation process of the stochastic integral as stated in the proposition. □

COROLLARY 2.6. *Suppose that $f, g \in \mathscr{L}^2$. Then*

$$\langle I(f), I(g)\rangle_t = \int_0^t f(s)g(s)\, ds.$$

PROOF. This follows from the equality

$$\langle M, N\rangle_t = \frac{\langle M+N, M+N\rangle_t - \langle M-N, M-N\rangle_t}{4}$$

for two continuous martingales M and N. □

If $Z \in \mathscr{F}_s$, we have

$$Z \int_s^t f_u \, dB_u = \int_s^t Z f_u \, dB_u.$$

This can be considered to be self-evident, even though it requires an approximation argument to verify. The same remark can be said of the identity

$$\int_s^t f_u \, dB_u = \int_0^t f_u \, dB_u - \int_0^s f_u \, dB_u.$$

Either we define the left side by the right side or we consider the left side as the stochastic integral with respect to $\{B_{s+u}, u \geq 0\}$, which is a Brownian motion with respect to the filtration $\mathscr{F}_{s+*} = \{\mathscr{F}_{s+u}, u \geq 0\}$, the end result is the same. However, if we replace s or t by a random time, the meaning of the two and other similar equalities may become ambiguous, especially when the random time is not a stopping time. In this respect, if we restrict ourselves to stopping times, the following results may help us clarify the situation.

PROPOSITION 2.7. *Suppose that $f \in \mathscr{L}_T^2$ and $\tau \leq T$ a stopping time.*
(1) We have

$$\int_0^\tau f_s \, dB_s = \int_0^T I_{[0,\tau]}(s) f_s \, dB_s.$$

(2) If $Z \in \mathscr{F}_\tau$ is a bounded bounded random variable, then

$$Z \int_\tau^T f \, dB_s = \int_0^T Z I_{[\tau,\infty)}(s) f_s \, dB_s.$$

PROOF. We will verify the equalities for discrete stopping times and step processes. In general we can approximate the stopping times σ and τ by discrete stopping times from above and the integrand process f by step processes in the $\| \cdot \|_{2,T}$ norm. The equalities are clearly preserved when passing to the limit under such approximations.

Let $0 = t_0 < t_1 < \cdots < t_n = T$ be a partition such that $f = f_{j-1} \in \mathscr{F}_{t_{j-1}}$ for $t_{j-1} \leq t < t_j$ and τ takes values only in the sequence $\{t_j\}$.

(1) Note that $I_{\{s<\tau\}}$ is a step process. We have

$$\begin{aligned}
I(f)_\tau &= \sum_{i=1}^{n} I_{\{\tau=t_i\}} \int_0^{t_i} f_s\, dB_s \\
&= \sum_{i=1}^{n} I_{\left\{\tau=t_j\right\}} \sum_{j=1}^{i} f_{j-1}(B_{t_j} - B_{t_{j-1}}) \\
&= \sum_{j=1}^{n} f_{j-1}(B_{t_j} - B_{t_{j-1}}) \sum_{i=j}^{n} I_{\{\tau=t_i\}} \\
&= \sum_{j=1}^{n} I_{\left\{t_{j-1}<\tau\right\}} f_{j-1}(B_{t_j} - B_{t_{j-1}}) \\
&= \int_0^T I_{[0,\tau)}(s) f_s\, dB_s.
\end{aligned}$$

(2) Note that $ZI_{\{\tau\le s\}}$ is a step process and belongs to $\mathscr{F}_s$ for fixed s. We have

$$\begin{aligned}
Z\int_\sigma^T f_s\, dB_s Z &\sum_{i=1}^{n} ZI_{\{\tau=t_i\}} \int_0^{t_i} f_s dB_s \\
&= \sum_{i=1}^{n} I_{\{\tau=t_i\}} \sum_{j=i+1}^{n} Zf_{j-1}(B_{t_j} - B_{t_{j-1}}) \\
&= \sum_{j=1}^{n} Zf_{j-1}(B_{t_j} - B_{t_{j-1}}) \sum_{i=1}^{j-1} I_{\{\tau=t_i\}} \\
&= \sum_{j=1}^{n} ZI_{\left\{\tau\le t_{j-1}\right\}} f_{j-1}(B_{t_j} - B_{t_{j-1}}) \\
&= \int_0^T ZI_{[\tau,\infty)}(s) f_s\, dB_s.
\end{aligned}$$

□

EXAMPLE 2.8. Consider the stochastic integral

$$\int_0^t B_s\, dB_s = \lim \sum_{i=1}^{n} B_{t_{j-1}}(B_{t_j} - B_{t_{j-1}}).$$

Using the identity $2a(b-a) = b^2 - a^2 + (b-a)^2$ we have

$$2\sum_{i=1}^{n} B_{t_{j-1}}(B_{t_j} - B_{t_{j-1}}) = B_t^2 + \sum_{i=1}^{n}(B_{t_j} - B_{t_{j-1}})^2.$$

The last sum converges to the quadratic variation $\langle B, B\rangle_t = t$. Hence we have

$$2\int_0^t B_s\, dB_s = B_t^2 - t,$$

which is indeed a martingale.

EXAMPLE 2.9. One may wonder what will happen if in the approximating sum of a stochastic integral we do not take the value of the integrand at the left endpoint of a partition interval. For example, what is the sum

$$\sum_{j=1}^{n} B_{t_j}(B_{t_j} - B_{t_{j-1}})?$$

This is not hard to figure out. The difference between this sum and the usual sum taking the left endpoint value of the integrand is just the quadratic variation along the partition

$$\sum_{j=1}^{n}(B_{t_j} - B_{t_{j-1}})^2,$$

hence we ave

$$\lim_{|\Delta|\to 0}\sum_{j=1}^{n} B_{t_j}(B_{t_j} - B_{t_{j-1}}) = \int_0^t B_s\, dB_s + t.$$

This example shows where to take the value of the integrand on each partition interval really matters.

3. Extension to more general integrands

We have defined stochastic integral with respect to a Brownian motion for integrand process $f \in \mathscr{L}^2$. These are progressively measurable processes f such that

$$\mathbb{E}\left[\int_0^t f_s^2\, ds\right] < \infty$$

for all t. This integrability is too restrictive. Without too much effort we can define a much wider class of integrand processes.

DEFINITION 3.1. *We use $\mathscr{L}^2$ to denote the space of progressively measurable processes f such that*

$$\mathbb{P}\left[\int_0^t f_s^2\, ds < \infty\right] = 1$$

for all t.

This is a class of process wider than $\mathscr{L}^2$ and is sufficient for our applications. For example, all adapted continuous processes belong to $\mathscr{L}^2_{loc}$. We will extend the definition of stochastic integrals to processes in $\mathscr{L}^2_{loc}$. The resulting stochastic integral process is no longer a square integrable martingale but a local martingale.

Recall the definition of a local martingale.

DEFINITION 3.2. *We say that $M = \{M_t, t \geq 0\}$ is a local martingale if there exists a sequence of stopping times $\tau_n \uparrow \infty$ such that the stopped processes $M^\tau = \{M_{t\wedge\tau_n}, t \geq 0\}$ are martingales.*

For continuous local martingale M with $M_0 = 0$, we can always take

$$\tau_n = \inf \{t \geq 0 : |M_t| \geq n\}.$$

Suppose that $f \in \mathscr{L}^2_{loc}$. Define the stopping time

$$\tau_n = \inf \left\{ t > 0 : \int_0^t |f(s)|^2 ds \geq n \right\}.$$

the assumption that $f \in \mathscr{L}^2_{loc}$ implies that $\tau_n \uparrow \infty$ (with probability one, if one wants to be very precise). Now consider the process

$$f_n(s) = f(s) I_{\{s \leq \tau_n\}}.$$

It is clear that $f \in \mathscr{L}^2$, in fact $\|f_n\|_{2,T} \leq n$ for all T. The stochastic integral $I(f_n)$ is well-defined and is a square–integrable continuous martingale. We now define the stochastic integral $I(f)_t$ as follows:

$$I(f)_t = I(f_n)_t, \qquad t \leq \tau_n.$$

In order that this definition make sense, we have to verify the consistency: if $m \leq n$, then

$$I(f_n)_t = I(f_m)_t, \qquad t \leq \tau_m.$$

This is immediate. Indeed,

$$I(f_n)_{t \wedge \tau_m} = \int_0^t f_n I_{\{s \leq \tau_m\}} \, dB_s = \int_0^t f_m dB_s = I(f_m)_t.$$

From definition we have

$$I(f)_{t \wedge \tau_n} = I(f_n)_t,$$

which shows that $I(f)$ is a continuous local martingale.

EXAMPLE 3.3. The process $Be^{B^2} \notin \mathscr{L}^2$ because $\mathbb{E}\left[|B_t|^2 e^{B_t^2}\right]$ is infinite for $t \geq 1/2$. However, since it is continuous it belongs to $\mathscr{L}^2_{loc}$ and its stochastic integral with respect to Brownian motion is well defined. In fact, Itô's formula will show that

$$2\int_0^t B_s e^{B_s^2} \, dB_s = e^{B_t^2} - \int_0^t \left[1 + 2B_s^2\right] e^{B_s^2} \, ds.$$

4. Stochastic integration with respect to continuous local martingales

We have discussed stochastic integration with respect to Brownian motion so that only the martingale properties of Brownian motion is used. For this reason not much needs to be changed if we replace Brownian motion by a general continuous martingale. All we need to do is to replace the quadratic variation process of Brownian motion $\langle B, B\rangle_t = t$ by $\langle M.M\rangle_t$, which is a continuous increasing process. More specifically, the theory of stochastic integration with respect to Brownian motion is based on several properties of Brownian motion:

(1) B is a continuous square integrable martingale;
(2) The quadratic variation process is $\langle B, B\rangle_t = t$.

(3) If f is a step process, then

$$\mathbb{E}\left[\left(\int_0^t f\,dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t |f_s|^2\,ds\right].$$

(4) The space of step processes $\mathscr{S}_T$ is dense in the space of progressively measurable and square integrable processes $\mathscr{L}^2_T$.

For a theory of stochastic integration with respect to a general continuous martingale M, we make the following observations:

(1) By the usual stopping time argument, we can restrict ourselves initially to uniformly bounded continuous martingales.

(2) The quadratic variation process $\langle M, M\rangle_t$ is continuous, adapted, and increasing. Therefore integration of a progressively measurable process with respect to $\langle M, M\rangle$ is well defined.

(3) For a bounded step process f we have

$$\mathbb{E}\left|\int_0^t f_s\,dM_s\right|^2 = \mathbb{E}\left[\int_0^t f_s^2\,d\langle M, M\rangle_s\right].$$

(4) We can introduce the norm

$$\|f\|_{2,T;M} = \sqrt{\mathbb{E}\left[\int_0^t f_s^2\,d\langle M, M\rangle_s\right]}.$$

There does not seem to be a good description of the closure of the space of $\mathscr{S}_T$ under this norm that is good for all M, but at least all progressively measurable and continuous processes with finite norm can be approximated by step processes.

It is also possible develop a theory of stochastic integration with respect to a general, not necessarily continuous martingale. Since we will mostly only deal with continuous martingales, we will restrict ourselves to continuous local martingales.

Suppose that M is a continuous local martingale and $f \in \mathscr{S}$ is a step process. The stochastic integral of f with respect to the martingale M is

$$I(f)_t = \int_0^t f\,dM_s = \sum_j f_{j-1}\left(M_{t_{j-1}\wedge t} - M_{t_j\wedge t}\right).$$

Just as in the case of Brownian motion we have

$$\mathbb{E}\left|\int_0^t f_s\,dM_s\right|^2 = \mathbb{E}\left[\int_0^t |f_s|^2 d\langle M, M\rangle\right].$$

Let $\mathscr{L}^2(M)$ be the space of progressively measurable processes f such that

$$\|f\|^2_{2,T;M} = \mathbb{E}\left[\int_0^T |f_s|^2 d\langle M, M\rangle_s\right] < \infty$$

for all $t \geq 0$. It can be shown that $\mathscr{S}_T$ is dense in $\mathscr{L}^2(M)_T$ (see Karatzas and Shreve **[7]**). Thus by the usual approximation argument, we see that the stochastic integral

$$I(f)_t = \int_0^t f_s \, dM_s$$

is well defined for all $f \in \mathscr{L}^2(M)$ and the quadratic variation process is

$$\langle I(f), I(f) \rangle_t = \int_0^t |f_s|^2 \, d\langle M, M \rangle_s.$$

Although it is not easy to find a good description of $\mathscr{L}^2(M)$ that fits all continuous martingales, it does contain all continuous adapted processes such that $\|f\|_{2,T;M}$ is finite for all T.

By the standard stopping time argument we have used in the case of Brownian motion, the stochastic integral

$$I(f)_t = \int_0^t f_s \, dM_s$$

can be defined for a continuous local martingale M and a progressively measurable process f such that

$$\mathbb{P}\left[\int_0^t |f_s|^2 \, d\langle M, M \rangle_s < \infty \right] = 1$$

for all t. Under these conditions, the stochastic integral $I(f)$ is a continuous local martingale.

EXAMPLE 4.1. Let M be a uniformly bounded continuous martingale. As in the case of Brownian motion we have

$$M_t^2 - M_0^2 = 2 \sum_{i=1}^n M_{t_{j-1}} (M_{t_j} - M_{t_{j-1}}) + \sum_{i=1}^n (M_{t_j} - M_{t_{j-1}})^2.$$

The sum after the equal sign converges to the stochastic integral $\displaystyle\int_0^t M_s \, dM_s$. Therefore the second sum must also converge. It is clear that the limit of the second sum is an continuous, increasing, and adapted process. Hence by the Doob-Meyer decomposition theorem, the limit must be the quadratic variation process, i.e.,

$$\lim_{|\Delta| \to 0} \sum_{i=1}^n (M_{t_j} - M_{t_{j-1}})^2 = \langle M, M \rangle_t$$

and

$$M_t^2 = M_0^2 + 2 \int_0^t M_s \, dM_s + \langle M, M \rangle_t.$$

The convergence of the quadratic variation process takes place in $L^2(\Omega, \mathbb{P})$. For a general local martingale, it can be shown by routine argument that the convergence takes place at least in probability.

5. Itô's formula

Itô's formula is the fundamental theorem for stochastic calculus. Let us recall that the fundamental theorem of calculus states that if F is a continuously differentiable function then

$$F(t) - F(0) = \int_0^t F'(s)\, ds.$$

Let $0 = t_0 < t_1 < \cdots < t_n = t$ be a partition of the interval $[a, b]$. We have

$$F(t) - F(0) = \sum_{i=1}^{n} \left[F(t_j) - F(t_{j-1}) \right].$$

Since F' is continuous, by the mean value theorem, there is a point $\xi_i \in [t_{j-1}, t_j]$ such that

$$F(t_j) - F(t_{j-1}) = F(\xi_j)(t_j - t_{j-1}).$$

Hence, by the definition of Riemann integrals we have as $|\Delta| \to 0$,

$$F(b) - F(a) = \sum_{j=1}^{n} F'(\xi_j)(t_j - t_{j-1}) \to \int_0^t F'(s)\, ds.$$

What happens to this proof if we replace t by B_t. In this case ξ_j will be a point between $B_{t_{j-1}}$ and B_{t_j}. We have to deal with the sum

$$\sum_{i=1}^{n} F'(\xi_j)(B_{t_j} - B_{t_{j-1}}).$$

Here the argument has to depart from what we have done above. As EXAMPLE 2.9 shows, the place where we take the value of the integrand on each partition interval can change the limit of the Riemann sum. Therefore we should not expect that the above sum will converges to the stochastic integral $\int_0^t F'(B_s)\, dB_s$, for which the left endpoint value of the integrand is used in each partition interval. The method to remedy this situation is to take one more term of the Taylor expansion

$$F(B_{t_j}) - F(B_{t_{j-1}}) = F'(B_{t_{j-1}})(B_{t_j} - B_{t_{j-1}}) + \frac{1}{2} F''(\xi_j)(B_{t_j} - B_{t_{j-1}})^2.$$

The sum of the first term on the right side will converge to the stochastic integral as usual. The crucial observation is that because Brownian motion has finite quadratic variation, the sum of the second term on the right side

$$\sum_{i=1}^{n} F''(\xi_j)(B_{t_j} - B_{t_{j-1}})^2 \to \int_0^t F''(B_s)\, ds$$

as $|\Delta| \to 0$. The final result is the well know Itô's formula

$$F(B_t) = F(B_0) + \int_0^t F'(B_s)\, dB_s + \frac{1}{2} \int_0^t F''(B_s)\, ds.$$

On the right side, the first term is a continuous martingale, and the second term is a process of bounded variation. Thus if F is continuous and uniformly bounded together with its first and second derivatives, then the composition $F(B_t)$ is a semimartingale and Itô's formula gives an explicit expression for its Doob-Meyer decomposition.

The proof of Itô's formula for for a general semimartingale Z will not be a lot more difficult than the case of Brownian motion. For this reason, in this section we will prove it in this generality.

THEOREM 5.1. *Suppose that $F \in C^2(\mathbb{R})$ and Z is a semimartingale. Then*

$$F(Z_t) = F(Z_0) + \int_0^t F'(Z_s)\, dZ_s + \frac{1}{2}\int_0^t F''(Z_s)\, d\langle Z, Z\rangle_s. \tag{5.1}$$

Recall that a semimartingale has the form

$$Z_t = M_t + A_t,$$

where M is a local martingale and A a process of bounded variation. We first use a stopping time argument to reduce the proof to the case where Z_0, M, $\langle M, M\rangle$, and A are all uniformly bounded and F together with its first and second derivatives are uniformly bounded and uniformly continuous. First of all, let $\Omega_N = \{|Z_0| \le N\}$ and define

$$\mathbb{P}_N(C) = \frac{\mathbb{P}(C \cap \Omega_N)}{\mathbb{P}(\Omega_0)}.$$

It is clear that under the $\mathbb{P}_N$, the process Z is still a semimartingale with the same decomposition $Z = M + A$ and $\mathbb{P}_N\{|Z_0| \le N\} = 1$, i.e., Z_0 is bounded. Next, define the stopping time

$$\tau_N = \inf\{t : \, |M_t - M_0| + \langle M, M\rangle_t + |A|_t \ge N\}.$$

The stopped processes $Z^{\tau_N} = M^{\tau_N} + A^{\tau_N}$ and $\langle M^{\tau_N}, M^{\tau_N}\rangle_t = \langle M, M\rangle_t^{\tau_N}$ have the bounded properties we desired. Define a function $F_N \in C^2(\mathbb{R})$ such that it coincides with F on $[-2N, 2N]$ and vanishes outside $[-3N, 3N]$. On the probability space $(\Omega_N, \mathscr{F}_* \cap \Omega_N, \mathscr{P}_N)$ suppose that we have the formula

$$F_N(Z^{\tau_N}) = F_N(Z^{\tau_N}) + \int_0^t F_N'(Z_s^{\tau_N})\, dZ_s + \frac{1}{2}\int_0^t F_N''(Z_s^{\tau_N})\, d\langle Z^{\tau_N}, Z^{\tau_N}\rangle_s.$$

Replacing t there by $t \wedge \tau_N$ we see that

$$F(Z_{t\wedge\tau_N}) = F(Z_0) + \int_0^{t\wedge\tau_N} F'(Z_s)\, dZ_s + \frac{1}{2}\int_0^{t\wedge\tau_N} F''(Z_s)\, d\langle Z, Z\rangle_s.$$

Here we have used the definition of stochastic integral with respect to a local martingale. This shows that the formula (5.1) holds on Ω_N and $t < \tau_N$. Finally from $\mathbb{P}\{\Omega_N\} \uparrow 1$ and $\mathbb{P}\{\tau_N \uparrow \infty\} = 1$, we see that Itô's formula holds without any extra conditions.

After these reductions, we start the proof of Itô's formula itself. By Taylor's formula with remainder we have

$$F(Z_{t_j}) - F(Z_{t_{j-1}}) = F'(Z_{t_{j-1}})(Z_{t_j} - Z_{t_{j-1}}) + \frac{1}{2}F''(\xi_i)(Z_{t_j} - Z_{t_{j-1}})^2,$$

where ξ is a point between $Z_{t_{j-1}}$ and Z_{t_j}. For simplicity let us denote

$$\Delta Z_j = Z_{t_j} - Z_{t_{j-1}}, \quad \Delta M_j = M_{t_j} - M_{t_{j-1}}, \quad \Delta\langle M\rangle_j = \langle M\rangle_{t_j} - \langle M\rangle_{t_{j-1}}.$$

With these notations we can write

$$F(Z_t) - F(Z_0) = \sum_{i=1}^n F'(Z_{t_{j-1}})\Delta Z_j + \frac{1}{2}\sum_{i=1}^n F''(\xi_i)(\Delta Z_j)^2.$$

The first sum converges to the stochastic integral in $L^2(\Omega, \mathbb{P})$:

$$\sum_{i=1}^n F'(Z_{t_{j-1}})\Delta Z_j \to \int_0^t F'(Z_s)\, dZ_s.$$

The real work starts with the proof of

$$\sum_{i=1}^n F''(\xi_i)(\Delta Z_j)^2 \to \int_0^t F''(Z_s)\, d\langle Z, Z\rangle_s. \tag{5.2}$$

We will prove this through a series of three replacements

$$F''(\xi_i)(\Delta Z_j)^2 \Rightarrow F''(\xi_i)(\Delta M_j)^2 \Rightarrow F''(Z_{t_{j-1}})(\Delta M_j)^2 \Rightarrow F''(Z_{t_{j-1}})\Delta\langle M\rangle_j.$$

We will show that each replacement produces an error which will vanish in probability as the mesh of the partition $\Delta| \to 0$. After these replacements we will have

$$\sum_{i=1}^n F''(Z_{t_{j-1}})\Delta\langle M\rangle_j \to \int_0^t F''(Z_s)\, d\langle M, M\rangle_s,$$

which will complete the proof.

(1) From $(\Delta Z_j)^2 - (\Delta M_j)^2 = \Delta A_j(\Delta Z_j + \Delta M_j)$ we have

$$\left|\sum_{i=1}^n F''(\xi_i)\left\{(\Delta Z_j)^2 - (\Delta M_j)^2\right\}\right| \le \|F''\|_\infty |A|_t \max_{1\le i\le n} |\Delta Z_j + \Delta M_j|.$$

By sample path continuity we have

$$\lim_{|\Delta|\to 0} \max_{1\le i\le n} |\Delta Z_j + \Delta M_j| = 0$$

almost surely. Hence the error produced by the first replacement vanishes in probability as $|\Delta| \to 0$.

(2) We have

$$\left|\sum_{i=1}^n \left[F''(\xi_i) - F''(Z_{t_{j-1}})\right](\Delta M_j)^2\right| \le \max_{1\le i\le n} |F''(\xi_i) - F''(Z_{t_{j-1}})| \sum_{i=1}^n (\Delta M_j)^2.$$

Again by sample path continuity we have

$$\max_{1\le i\le n} |F''(\xi_i) - F''(Z_{t_{j-1}})| \to 0$$

almost surely. On the other hand, by EXAMPLE 4.1,

$$\lim_{|\Delta|\to 0} \sum_{i=1}^{n} (\Delta M_j)^2 = \langle M, M\rangle_t$$

in $L^2(\Omega, \mathbb{P})$. Therefore the error produced by this replacement vanishes in probability as $|\Delta| \to 0$.

(3) This is the most delicate replacement of the three. The error is

$$E_n = \sum_{i=1}^{n} F''(Z_{t_{j-1}}) \left\{ (\Delta M_j)^2 - \Delta\langle M\rangle_j \right\}.$$

The square $|E_n|^2$ is a sum of n^2 terms. From EXAMPLE 4.1

$$(\Delta M_j)^2 - \Delta\langle M\rangle_j = 2\int_{t_{j-1}}^{t_j} (M_s - M_{t_{j-1}})\, dM_s.$$

This is a stochastic integral with respect to a martingale, hence the conditional expectation of this expression with respect to $\mathscr{F}_{t_{j-1}}$ is clearly zero. For an off-diagonal term, say $i < j$, in $|E_n|^2$, its conditional expectation with respect to $\mathscr{F}_{t_{j-1}}$ is therefore zero. Hence all off-diagonal term has expectation value zero. For the diagonal term we have

$$\mathbb{E}\left[\left(\int_{t_{j-1}}^{t_j} (M_s - M_{t_{j-1}})\, dM_s \right)^2 \middle| \mathscr{F}_{t_{j-1}} \right] = \mathbb{E}\left[\int_{t_{j-1}}^{t_j} (M_s - M_{t_{j-1}})^2\, d\langle M, M\rangle_s \right].$$

Therefore the expected value of this off-diagonal term is bounded by

$$\|F''\|_\infty^2\, \mathbb{E}\left[\Delta\langle M\rangle_j \max_{t_{j-1}\le s\le t_j} |M_s - M_{t_{j-1}}| \right].$$

Adding the diagonal terms together we obtain

$$\mathbb{E}|E_n|^2 \le \|F''\|_\infty^2\, \mathbb{E}\left[\langle M\rangle_t \max_{1\le j\le n} \max_{t_{j-1}\le s\le t_j} |M_s - M_{t_{j-1}}| \right].$$

Note that under our assumption, the expression under the expectation is uniformly bounded. This expectation converges to zero by sample path continuity and the dominated convergence theorem.

With this final replacement, we have completed the proof of (5.2) and also the proof of Itô's formula.

We end this section with the statement of Itô's formula for multidimensional semimartingales. The basic ingredients of its proof have already explained in great detail in the proof of the one dimensional case. Since nothing will be learned from its tedious proof, we will justifiably omit it.

THEOREM 5.2. *Suppose that $Z = (Z^1, \dots, Z^n)$ be an $\mathbb{R}^n$ valued semimartingale. Then for any $F \in C^2(\mathbb{R}^n)$.*

$$F(Z_t) = F(Z_0) + \sum_{i=1}^n \int_0^t F_{x_i}(Z_s)\, dZ_s^i + \frac{1}{2} \sum_{i,j=1}^n \int_0^t F_{x_i x_j}(Z_s)\, d\langle Z^i, Z^j\rangle_s.$$

6. Differential notation and Stratonovich integrals

This section does not contain new results. We introduce differential notation for stochastic calculus and Stratonovich integrals, a form of stochastic integral more restrictive than Itô integrals.

We often work with the following classes of processes:

$\mathscr{M}$ = the space of continuous local martingales;

$\mathscr{A}$ = the space of continuous processes of bounded variation;

$\mathscr{I}$ = the space of increasing processes;

$\mathscr{Q}$= the space of semimartingales

$\mathscr{H}$ = the space of integrand processes.

Every process of bounded variation is the difference of two increasing processes: $\mathscr{A} = \mathscr{I} - \mathscr{I}$. A semimartingale is the sum of a local continuous martingale and a process of bounded variation: $\mathscr{Q} = \mathscr{M} + \mathscr{A}$

It is often convenient to write equalities among these processes in a differential form. We have defined the meaning of an expression such as $\int_0^t H_s dX_s$ for $H \in \mathscr{H}$ and $X \in \mathscr{Q}$. What exactly is the meaning a differential expression such as dX? We should consider dX as a symbol for the equivalence class of semimartingales Y such that

$$X_t - X_s = Y_t - Y_s$$

for all $s \le t$. The equivalent classes of semimartingale differentials is denoted by $d\mathscr{Q}$. We can carry out some purely symbolic calculations. To start with we can set

$$\int_s^t dX_u = X_t - X_s.$$

We can define a multiplication in $\mathscr{Q}$ by setting

$$dX \cdot dY = d\langle X, Y\rangle.$$

We also define the multiplication of an element from $\mathscr{Q}$ by an element from $\mathscr{H}$: HdX is the equivalence class containing the semimartingale $\int_0^t H_s\, dX_s^t$.

Here are some properties of these operations:

(1) $d\mathscr{Q} \cdot d\mathscr{Q} \subset d\mathscr{A}$;

(2) $d\mathscr{Q} \cdot d\mathscr{A} = 0$;

(3) $H(dX \cdot dY) = (H \cdot dX) \cdot dY$;

(4) $H_1(H_2 \cdot dX) = (H_1 H_2) dX$.

With these formal symbolic notations, the multi-dimensional Itô's formula can be written as

$$d\left\{f(Z_t)\right\} = \sum_{j=1}^{n} f_{x_i}(Z_t)\, dZ_t^i + \frac{1}{2}\sum_{i,j=1}^{n} f_{x_i x_j}(Z_t)\, dX^i \cdot dX^j.$$

If we are willing to adopt Einstein's summation convention, we may even drop the two summation signs.

Suppose that X and Y are two semimartingales. We define a new semimartingale differential by

$$X \circ dY = X \cdot dY + \frac{1}{2} dX \cdot dY.$$

This means that

$$\int_0^t X_s \circ dY_s = \int_0^t X_s\, dY_s + \frac{1}{2}\langle X, Y\rangle_t.$$

This is called the Stratonovich integral. Unlike Itô integral, the Stratonovich integral requires that the integrand process is also a continuous semimartingale, for in its definition we need the quadratic covariation of X and Y. In this sense Stratonovich integral is a weaker form of stochastic integral and its use is much more limited than Itô integrals. The main advantage of S-tratonovich integrals is that Itô's formula takes a form similar to that of the fundamental theorem of calculus.

PROPOSITION 6.1. *Let $F \in C^3(\mathbb{R}^n)$ and $Z = (Z^1, \ldots, Z^n)$ an n-dimensional semimartingale. Then*

$$d\left\{f(Z_t)\right\} = \sum_{i=1}^{n} F_{x_i}(Z_t) \circ dZ_t^i.$$

PROOF. The right side is equal to

$$F_{x_i}(Z_t) \cdot dZ_t^j + \frac{1}{2} F_{x_i x_j}(Z_t) \cdot dZ_t^j. \tag{6.1}$$

Using It&o's formula we have

$$dF_{x_i}(Z_t) = F_{x_i x_j}(Z_t) \cdot dZ_t^i + \frac{1}{2} F_{x_i x_j x_k}(Z_t) \cdot dZ^j \cdot dZ^k.$$

Thus the last term in (6.1) becomes

$$\frac{1}{2} F_{x_i x_j}(Z_t) \cdot dZ_t^i \cdot dZ_t^j + \frac{1}{4} F_{x_i x_j x_k}(Z_t) \cdot dZ_t^i \cdot dZ_t^j \cdot dZ_t^k.$$

The last triple sum is equal to zero because $dZ_t^i \cdot dZ_t^j \cdot dZ_t^k = 0$. Therefore the equality we wanted to prove is just Itô's formula. $\square$

Finally we mention that Stratonovich integral can also be approximated by a Riemann sums. It is instructive to compare this Riemann sum with the one that approximates the corresponding Itô integral. The difference is subtle but crucial in making Itô integration such a successful theory.

THEOREM 6.2. *Let X and Y be continuous semimartingales. Then*

$$\int_0^t X_s \circ dY_s = \lim_{|\Delta|\to 0} \sum_{j=1}^n \frac{X_{t_{j-1}} + X_{t_j}}{2} \left[Y_{t_j} - Y_{t_{j-1}}\right],$$

where the convergence is in probability.

PROOF. Rewrite the summation as

$$\sum_{j=1}^n X_{t_{j-1}} \left[Y_{t_j} - Y_{t_{j-1}}\right] + \frac{1}{2}\left[X_{t_j} - X_{t_{j-1}}\right]\left[Y_{t_j} - Y_{t_{j-1}}\right].$$

The limit of the first term is the Itô integral $\int_0^t X_s dY_s$, that of the second term is exactly the quadratic covariation$\langle X, Y\rangle_t/2$. □

7. Third Assignment

EXERCISE 3.1. Let f be a (nonrandom) function of bound variation. Show that

$$\int_0^t f_s\, dB_s = f_t B_t - \int_0^t B_s\, df_s,$$

where the last integral is understood to be a Lebesgue-Stieljes integral.

EXERCISE 3.2. Let F be an entire function in the complex plane $\mathbb{C}$ and $Z = X + iY$ be the complex Brownian motion (meaning that X and Y are independent standard Brownian motion). We have

$$\int_0^t F'(Z_s)\, dZ_s = \int_0^t F'(Z_s) \circ dZ_s = F(Z_t) - F(Z_0).$$

EXERCISE 3.3. The price S_t of a stock with average rate of return μ and volatility σ is usually described by the stochastic differential equation

$$dS_t = S_t\left(\mu\, dt + \sigma\, dB_t\right).$$

Using Itô's formula to show that the solution of the above stochastic differential equation is

$$S_t = S_0 \exp\left[\sigma B_t + \left(\mu - \frac{\sigma^2}{2}\right) t\right].$$

EXERCISE 3.4. Let $B = (B^1, B^2, B^3)$ be a 3-dimensional Brownian motion which does not start from zero. Using Itô's formula to show that $1/|B_t|$ is a local martingale.

EXERCISE 3.5. Let B be an n-dimensional Brownian motion starting from zero and

$$X_t = |B_t| = \sqrt{\sum_{i=1}^n |B_t^i|^2}$$

be the radial process. Show that X satisfies the following Itô type stochastic differential equation:

$$X_t = W_t + \frac{n-1}{2}\int_0^t \frac{ds}{X_s},$$

where W is a one-dimensional Brownian motion.

EXERCISE 3.6. Let M and N be two continuous local martingales. Show that

$$|\langle M, N\rangle_t| \leq \sqrt{\langle M\rangle_t \langle N\rangle_t}.$$

EXERCISE 3.7. Suppose that X, Y and Z be three semimartingales. Are the following relations true?
(1) $X(Y\,dZ) = (XY)\,dZ$;
(2) $X \circ (Y \circ Z) = (XY) \circ dZ$.

EXERCISE 3.8. A Brownian bridge is defined by

$$dX_t = dB_t - \frac{X_t\,dt}{1-t}, \qquad X_0 = 0.$$

Show that X is a Gaussian process with mean zero and

$$\mathbb{E}\left[X_s X_t\right] = \min\{s,t\} - st.$$

Show that $\{X_t,\, 0 \leq t \leq 1\}$ and the reversed process $\{X_{1-t},\, 0 \leq t \leq 1\}$ have the same law.

EXERCISE 3.9. Suppose that M and N are two bounded continuous martingales which are independent. Show that MN is a continuous martingale with respect to the filtration $\mathscr{F}_*^{M,N} = \sigma\{M_s,\, N_s;\, s \leq t\}$ generated by M and N.

EXERCISE 3.10. Suppose that M is a strictly positive local martingale. Show that there is a local martingale N such that

$$M_t = M_0 \exp\left[N_t - \frac{1}{2}\langle N, N\rangle_t\right].$$