现代数学物理方法

第二章，群论基础

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Conjugacy classes:

In a group G, the sets of elements $S = \{g_1, g_2, \cdots\}$ satisfying the condition

$$g^{-1}Sg = S, \quad \forall g \in G$$

are called to form the conjugacy classes of G.

Corollaries:

1. *A subgroup that is a union of conjugacy classes is a normal subgroup.*
2. *In an Abelean group, each group element forms an independent conjugacy class.*

Example:
Group S_3 has 3 conjugacy classes:

1. $C_1 = \{e\}$
2. $C_2 = \{a_1, a_2\}$
3. $C_3 = \{a_3, a_4, a_5\}$

Checking:

- The identity $\{e\}$ forms a conjugacy class itself, due to the fact that

$$g^{-1}eg = e, \quad \forall\, g \in S_3$$

- Moreover,

$$\begin{aligned}(a_3)^{-1}a_1a_3 &= a_3a_1a_3 &= a_4a_3 &= a_2\\ (a_4)^{-1}a_1a_4 &= a_4a_1a_4 &= a_5a_4 &= a_2\\ (a_5)^{-1}a_1a_5 &= a_5a_1a_5 &= a_3a_5 &= a_2\end{aligned}$$

The set $C_2 = \{a_1, a_2\}$ forms another conjugacy class of S_3.

- Similar calculations yield,

$$\begin{aligned}(a_1)^{-1}a_3a_1 &= a_2a_3a_1 &= a_4a_1 &= a_5\\ (a_2)^{-1}a_3a_2 &= a_1a_3a_2 &= a_5a_2 &= a_4\\ (a_4)^{-1}a_3a_4 &= a_4a_3a_4 &= a_2a_4 &= a_5\\ (a_5)^{-1}a_3a_5 &= a_5a_3a_5 &= a_1a_5 &= a_4\end{aligned}$$

Namely, $C_3 = \{a_3, a_4, a_5\}$ forms the 3rd conjugacy class of S_3.

Other concepts in group theory:

1. An isomorphism is a *one-to-one* mapping of group onto another group that preserves the multiplication law.

2. An automorphism is a *one-to-one* mapping of a group onto itself that preserve the multiplication law.

3. An inner automorphism is an automorphism that can be cast as the mapping

$$G \quad \to \quad G' = gGg^{-1}$$

   for a fixed group element $g \in G$.

4. An outer automorphism is an automorphism that can not be written as gGg^{-1} for any group element $g \in G$.

Schur's second lemma:

If

$$D_1(g)A = AD_2(g), \quad \forall g \in G$$

where D_1 and D_2 are inequivalent, irreducible representations of group G, then $A = 0$.

Proof:

The spaces and their dimensions of these two nonequivalent irreducible representations are denoted as $\mathcal{S}_1(d_1)$ and $\mathcal{S}_2(d_2)$ respectively, with $d_1 \geqslant d_2$.

Let A be an operator which maps from $\mathcal{S}_2$ into $\mathcal{S}_1$. When applied to $\mathcal{S}_2$, this A generates a subspace $\mathcal{S}_3$ of $\mathcal{S}_1$:

$$\mathcal{S}_3 = \{A\,|\Psi\rangle \in \mathcal{S}_1, \quad \forall\,|\Psi\rangle \in \mathcal{S}_2\}$$

with dimension $d_3 \leqslant d_2 \leqslant d_1$.

It follows from the proposed assumption that,

$$D_1(g)A|\Psi\rangle = AD_2(g)|\Psi\rangle = A[D_2(g)|\Psi\rangle] \equiv A|\Psi_g\rangle \quad \in \mathcal{S}_3$$

Because $|\Psi_g\rangle \equiv D_2(g)|\Psi\rangle \in \mathcal{S}_2$. Thus, $D_1(g)\mathcal{S}_3 = \mathcal{S}_3$. $\rightsquigarrow$ $\mathcal{S}_3$ is an invariant subspace of $\mathcal{S}_1$.

That $D_1(G)$ is an irreducible representation of G implies $\mathcal{S}_1$ *has no true invariant subspace.*

- Because $\mathcal{S}_3$ is an invariant subspace of $\mathcal{S}_1$, there is a contradiction unless $\mathcal{S}_3$ is either a null space ($A = 0$) or the full $\mathcal{S}_1$.
- The second possibility is excluded by the assumption that $D_1(G)$ and $D_2(G)$ are different (nonequivalent) representations[1].

Therefore, the single possibility $A = 0$ remains.

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[1] The second possibility happens when $d_3 = d_1 = d_2$. However, if $d_2 = d_1$, we could invert A so that the two representations would be equivalent,

$$D_1(g) = AD_2(g)A^{-1}, \quad \forall g \in G.$$

Schur's first lemma:

If

$$D(g)A = AD(g), \quad \forall g \in G$$

where D is a finite dimensional irreducible representation of group G, then[a], $A \propto I$.

[a] In other words, if a matrix A commutes with all elements of a finite dimensional irreducible representation, it must be proportional to the unit matrix I.

Proof:

The condition of a finite dimensional representation is important. Any finite dimensional matrix A has at least one eigenvalue,

$$A\,|\lambda\rangle = \lambda\,|\lambda\rangle \quad \rightsquigarrow \quad (A - \lambda I)\,|\lambda\rangle = 0.$$

This is because the characteristic equation

$$\det(A - \lambda I) = 0$$

has at least one root for finite dimensional A.

Proof (continued):

Let P be the projection operator of the corresponding eigenstate $|\lambda\rangle$,

$$(A - \lambda I)P = 0$$

The assumption $D(g)A = AD(g)$ for all $g \in G$ does then imply,

$$(A - \lambda I)D(g)P = D(g)(A - \lambda I)P = 0$$

This equation has two possible solutions:

1. either $D(g)P \propto P$
2. or $A = \lambda I$

The first possibility is excluded because $D(G)$ is assumed to be an irreducible representation of G.

Consequently,

$$A = \lambda I \propto I$$

Remark:

Schur's first lemma can be alternatively written as,

$$A^{-1}D(g)A = D(g),\ \forall g \in G \quad \rightsquigarrow \quad A \propto I$$

for any irreducible representation $D(G)$.

- The form of $D(G)$ is fixed and there is no further freedom to make nontrivial similarity transformations on the basis vectors.
- The basis vectors of an irreducible representation are essentially unique.
- The only unitary transformation you can make is to multiply all the states by the same phase factor.

Hilbert Space:

The orthonormal basis states of an QM object are of the form,

$$|a,\ j,\ x\rangle, \quad (1 \leqslant j \leqslant n_a)$$

where a labels the irreducible representation $D_a(G)$, j lables the states within $D_a(G)$ and x lables the other physical parameters. These states satisfy the relations:

$$\langle b,\ k,\ y|a,\ j,\ x\rangle = \delta_{ba}\delta_{kj}\delta_{yx}, \quad \sum_{a,j,x} |a,\ j,\ x\rangle\langle a,\ j,\ x| = I$$

Symmetry:

In QM, the symmetry is expressed as

$$\left[H, D(g)\right] = 0, \quad \forall g \in G$$

- Under the symmetry transformation, the states in Hilbert space transform like,

$$|\psi\rangle \to |\psi'\rangle = D(g)\,|\psi\rangle$$
$$\langle\psi| \to \langle\psi'| = \langle\psi|\,[D(g)]^\dagger$$

- The operators transform like

$$\mathscr{O} \to \mathscr{O}' = D(g)\mathscr{O}\,[D(g)]^\dagger$$

so that all matrix elements $\langle\phi|\,\mathscr{O}\,|\psi\rangle$ kept unchanged.

- An invariant observable satisfies,

$$\mathscr{O} \to \mathscr{O}' = D(g)\mathscr{O}\,[D(g)]^\dagger = \mathscr{O}$$

i.e.,

$$[\mathscr{O}, D(g)] = 0, \quad \forall\, g \in G$$

We have supposed that $D(G)$ forms a finite dimensional representation of group G.

Hence, $D(G)$ can be equivalent to a unitary and completely reducible representation:

$$\langle a,\ j,\ x|\, D(g)\, |b,\ k,\ y\rangle = \delta_{ab}\delta_{xy}\left[D_a(g)\right]_{jk}$$

Consequently,

$$D(g) = \sum_{a,j,k,x} |a,\ j,\ x\rangle \left[D_a(g)\right]_{jk} \langle a,\ k,\ x|$$

In detail,

$$
\begin{aligned}
D(g) &= \left[\sum_{a,j,x} |a, j, x\rangle\langle a, j, x| \right] D(g) \left[|b, k, y\rangle\langle b, k, y| \right] \\
&= \sum_{a,j,x} \sum_{b,k,y} |a, j, x\rangle \left[\langle a, j, x| D(g) |b, k, y\rangle \right] \langle b, k, y| \\
&= \sum_{a,j,x} \sum_{b,k,y} |a, j, x\rangle \left\{ \delta_{ab} \delta_{xy} \left[D_a(g)\right]_{jk} \right\} \langle b, k, y| \\
&= \sum_{a,j,k,x} |a,\ j,\ x\rangle \left[D_a(g)\right]_{jk} \langle a,\ k,\ x|
\end{aligned}
$$

Wigner-Eckart Theorem:

For an invariant observable operator $\mathscr{O}$,

$$\left[\mathscr{O}, D(g)\right] = 0, \quad \forall\, g \in G$$

we get:

$$\begin{aligned} 0 &= \langle a, j, x|\,[\mathscr{O}, D(g)]\,|b, k, y\rangle \\ &= \sum_i \left\{ \langle a, j, x|\,\mathscr{O}\,|b, i, y\rangle\,[D_b(g)]_{ik} - [D_a(g)]_{ji}\,\langle a, i, x|\,\mathscr{O}\,|b, k, y\rangle \right\} \end{aligned}$$

The matrix element $\langle a, j, x|\,\mathscr{O}\,|b, k, y\rangle$ satisfies the hypotheses of Schur's Lemmas. Therefore, it either vanishes when $a \neq b$ or is proportional to identity δ_{jk} for $a = b$,

$$\langle a, j, x|\,\mathscr{O}\,|b, k, y\rangle = f_a(x, y)\delta_{ab}\delta_{jk}$$

This conclusion is called the Wigner-Eckart theorem.

Orthogonality relations:

Suppose that $D_a(G)$ and $D_b(G)$ are two finite dimensional irreducible representations of G. We define a linear operator:

$$A^{ab}_{jl} \equiv \sum_{g \in G} D_a(g^{-1}) \left| a, j \right\rangle \left\langle b, l \right| D_b(g)$$

Then,

$$\begin{aligned}
D_a(g_1) A^{ab}_{jl} &= \sum_{g \in G} D_a(g_1) D_a(g^{-1}) \left| a, j \right\rangle \left\langle b, l \right| D_b(g) \\
&= \sum_{g \in G} D_a(g_1 g^{-1}) \left| a, j \right\rangle \left\langle b, l \right| D_b(g) \\
&= \sum_{h \in G} D_a(h^{-1}) \left| a, j \right\rangle \left\langle b, l \right| D_b(h g_1) \\
&= \sum_{h \in G} D_a(h^{-1}) \left| a, j \right\rangle \left\langle b, l \right| D_b(h) D_b(g_1) \\
&= \left[\sum_{h \in G} D_a(h^{-1}) \left| a, j \right\rangle \left\langle b, l \right| D_b(h) \right] D_b(g_1) = A^{ab}_{jl} D_b(g_1)
\end{aligned}$$

Schur's lemmas indicate that,

$$A_{jl}^{ab} = \sum_{g \in G} D_a(g^{-1}) |a, j\rangle\langle b, l| D_b(g) = \delta_{ab}\lambda_{jl}^{a} I$$

By computing the trace of the above equation in the sub-Hilbert space of dimension n_a,

$$\begin{aligned}
\delta_{ab}\lambda_{jl}^{a}\, n_a &= \delta_{ab}\lambda_{jl}^{a}\, \mathrm{Tr} I = \mathrm{Tr} A_{jl}^{ab} \\
&= \mathrm{Tr}\left[\sum_{h \in G} D_a(h^{-1}) |a, j\rangle\langle b, l| D_b(h)\right] \\
&= \delta_{ab}\mathrm{Tr}\left[\sum_{h \in G} \langle a, l| D_a(h) D_a(h^{-1}) |a, j\rangle\right] \\
&= \delta_{ab}\mathrm{Tr}\left[\sum_{h \in G} \langle a, l| D_a(h h^{-1}) |a, j\rangle\right] \\
&= \delta_{ab}\mathrm{Tr}\sum_{h \in G} \langle a, l|a, j\rangle = N\delta_{ab}\delta_{jl} \quad \rightsquigarrow \quad \lambda_{jl}^{a} = \frac{N}{n_a}\delta_{jl}
\end{aligned}$$

Therefore,

$$\sum_{g\in G} D_a(g^{-1})\,|a,j\rangle\langle b,l|\,D_b(g) = \frac{N}{n_a}\delta_{ab}\delta_{jl} I$$

Orthogonality relations:

The matrix element of the above equation between the states $|a,k\rangle$ and $|b,m\rangle$ reads,

$$\begin{aligned}
\frac{N}{n_a}\delta_{ab}\delta_{jl}\delta_{km} &= \frac{N}{n_a}\delta_{ab}\delta_{jl}\,\langle a,k|a,m\rangle \\
&= \langle a,k|\left[\frac{N}{n_a}\delta_{ab}\delta_{jl} I\right]|b,m\rangle \\
&= \langle a,k|\left[\sum_{g\in G} D_a(g^{-1})\,|a,j\rangle\langle b,l|\,D_b(g)\right]|b,m\rangle \\
&= \sum_{g\in G}\langle a,k|\,D_a(g^{-1})\,|a,j\rangle\langle b,l|\,D_b(g)\,|b,m\rangle
\end{aligned}$$

These equations are known as the *orthogonality relations* for the matrix elements of irreducible representations. They can be rewritten as:

$$\sum_{g\in G} \frac{n_a}{N} \left[D_a(g^{-1})\right]_{kj} \left[D_b(g)\right]_{lm} = \delta_{ab}\delta_{jl}\delta_{km}$$

Notice:

- The matrix elements $\left[D_a(g)\right]_{jk}$ are linearly independent of one another.
- The whole set of $\left[D_a(g)\right]_{jk}$ are complete. An arbitrary function of g can be expanded in them.

For the unitary irreducible representations, the orthogonality can be recast as,

$$\sum_{g\in G} \frac{n_a}{N} [D_a(g)]^*_{jk} [D_b(g)]_{lm} = \delta_{ab}\delta_{jl}\delta_{km}$$

With proper normalization,

$$\Phi^a_{jk}(g) \equiv \sqrt{\frac{n_a}{N}}\ [D_a(g)]_{jk}$$

the matrix elements of unitary irreducible representations become the orthonormal functions of the group elements $\{g\}$:

$$\sum_{g\in G} \left[\Phi^a_{jk}(g)\right]^*\ \Phi^b_{lm}(g) = \delta_{ab}\delta_{jl}\delta_{km}$$

Definition:

The characters $\chi_D(g)$ of a group representation $D(G)$ are the traces of the matrices $\{D(g)\}$ in the representation,

$$\chi_D(g) = \mathrm{Tr}\,[D(g)] = \sum_i [D(g)]_{ii}$$

Orthogonality:

The characters of non-equivalent irreducible representations are different from each other. In fact, they satisfy the so-called orthogonality relations,

$$\frac{1}{N}\sum_{g\in G}\chi^*_{D_a}(g)\chi_{D_b}(g) = \delta_{ab}$$

Therefore, the characters of different irreducible representations are different.

Proof:

Notice that $n_a = \sum_i \delta_{ii}$ is the dimension of $D_a(G)$. It follows from the orthogonality relations

$$\sum_{g\in G} \frac{n_a}{N} \left[D_a(g^{-1})\right]_{kj} \left[D_b(g)\right]_{lm} = \delta_{ab}\delta_{jl}\delta_{km}$$

that

$$\begin{aligned}
\delta_{ab} n_a &= \delta_{ab} \sum_j \delta_{jj} = \sum_j \sum_l \delta_{ab}\delta_{jl}\delta_{jl} \\
&= \sum_j \sum_l \left\{ \sum_{g\in G} \frac{n_a}{N} \left[D_a(g^{-1})\right]_{jj} \left[D_b(g)\right]_{ll} \right\} \\
&= \sum_{g\in G} \frac{n_a}{N} \left\{ \sum_j \left[D_a(g)\right]^*_{jj} \right\} \left\{ \sum_l \left[D_b(g)\right]_{ll} \right\} \\
&= \frac{n_a}{N} \sum_{g\in G} \chi^*_{D_a}(g) \chi_{D_b}(g) \quad \rightsquigarrow \quad \frac{1}{N} \sum_{g\in G} \chi^*_{D_a}(g)\chi_{D_b}(g) = \delta_{ab}
\end{aligned}$$

Properties of $\chi_D(G)$:

- The characters are constants on conjugacy classes.

$$\begin{aligned}
\chi_D(g) &= \mathrm{Tr} D(g) = \mathrm{Tr}\left[D(h)^{-1} D(g) D(h)\right] \\
&= \mathrm{Tr}\left[D(h^{-1}) D(g) D(h)\right] \\
&= \mathrm{Tr} D(h^{-1} g h) \\
&= \chi_D(h^{-1} g h)
\end{aligned}$$

- By labeling the conjugacy classes in integers α and letting κ_α be the number of elements in $\mathcal{C}_\alpha$, we can rewrite the previous orthogonality relations of the characters as,

$$\frac{1}{N} \sum_\alpha \kappa_\alpha \chi^*_{D_a}(g_\alpha) \chi_{D_b}(g_\alpha) = \delta_{ab}$$

From this we get,

$$
\begin{aligned}
\chi_{D_b}(g_\beta) &= \sum_a \left[\delta_{ab}\chi_{D_a}(g_\beta)\right] \\
&= \sum_a \left[\chi_{D_a}(g_\beta)\frac{1}{N}\sum_\alpha \kappa_\alpha \chi^*_{D_a}(g_\alpha)\chi_{D_b}(g_\alpha)\right] \\
&= \frac{1}{N}\sum_\alpha \kappa_\alpha \left[\sum_a \chi^*_{D_a}(g_\alpha)\chi_{D_a}(g_\beta)\right]\chi_{D_b}(g_\alpha)
\end{aligned}
$$

Therefore,

$$
\sum_a \chi^*_{D_a}(g_\alpha)\chi_{D_a}(g_\beta) = \frac{N}{\kappa_\alpha}\delta_{\alpha\beta}
$$

Corollaries:

- The finite dimensional representation $D(G)$ of group G is irreducible iff

$$\frac{1}{N}\sum_{\alpha}\kappa_{\alpha}|\chi_{D}(g_{\alpha})|^{2}=1$$

- There is a relation between the order of group G and the dimensions of its irreducible representations

$$N=\sum_{a}n_{a}^{2}$$

Remark:

The formula $N = \sum_a n_a^2$ is shown below.

Suppose that G has a finite dimensional reducible representation $D(G)$, which can be expressed as the direct sum of a set of irreducible representations,

$$D(g) \sim \oplus_{a=1}^{M} c_a D_a(g), \quad \forall\, g \in G$$

This implies $\chi_D(g) = \sum_{a=1}^{M} c_a \chi_{D_a}(g)$. Therefore,

$$\begin{aligned}
\frac{1}{N}\sum_{g\in G}\chi_{D_a}^*(g)\chi_D(g) &= \sum_{b=1}^{M} c_b\left[\frac{1}{N}\sum_{g\in G}\chi_{D_a}^*(g)\chi_{D_b}(g)\right] \\
&= \sum_{b=1}^{M} c_b\delta_{ab} \\
&= c_a \quad \rightsquigarrow \quad c_a = \frac{1}{N}\sum_{g\in G}\chi_{D_a}^*(g)\chi_D(g)
\end{aligned}$$

Consider the regular representation $D_{\text{reg}}(G)$, where

$$\chi_{\text{reg}}(e) = \text{Tr} D_{\text{reg}}(e) = N,$$
$$\chi_{\text{reg}}(g) = \text{Tr} D_{\text{reg}}(g) = 0, \quad \forall\, g \neq e$$

Hence,

$$c_a = \frac{1}{N} \sum_{g \in G} \chi^*_{D_a}(g) \chi_{\text{reg}}(g) = \chi^*_{D_a}(e) = n_a$$

and

$$N = \chi_{\text{reg}}(e) = \sum_{a=1}^{M} c_a \chi_{D_a}(e) = \sum_{a=1}^{M} n_a^2$$

Corollary:

The number of non-equivalent irreducible representations of a finite group is equal to the number of its conjugacy classes.

Explanation:

Let $F(g_1)$ be a function of group element g_1 that is some constant on each conjugacy class,

$$F(g_1) = F(h^{-1} g_1 h)$$

The full set of $[D_a(g)]_{jk}$ of the irreducible representations are complete. Thereby, $F(g_1)$ can be expanded in terms of these matrix elements,

$$F(g_1) = \sum_{a,j,k} c^a_{jk} [D_a(g_1)]_{jk}$$

That $F(g_1)$ is some constant on each conjugacy class further suggests:

$$F(g_1) = \sum_a \left[\sum_j \left(\frac{c^a_{jj}}{n_a} \right) \right] \chi_{D_a}(g_1)$$

In detail,

$$
\begin{aligned}
F(g_1) &= \frac{1}{N}\sum_{g\in G} F(g^{-1}g_1 g) = \frac{1}{N}\sum_{g\in G}\sum_{a,j,k} c^a_{jk}\left[D_a(g^{-1}g_1 g)\right]_{jk} \\
&= \frac{1}{N}\sum_{g\in G}\sum_{a,j,k} c^a_{jk}\left\{\left[D_a(g^{-1})\right]_{jl}\left[D_a(g_1)\right]_{lm}\left[D_a(g)\right]_{mk}\right\} \\
&= \frac{1}{N}\sum_{a,j,k} c^a_{jk}\left\{\sum_{g\in G}\left[D_a(g^{-1})\right]_{jl}\left[D_a(g)\right]_{mk}\right\}\cdot\left[D_a(g_1)\right]_{lm} \\
&= \frac{1}{N}\sum_{a,j,k} c^a_{jk}\left\{\frac{N}{n_a}\delta_{lm}\delta_{jk}\right\}\cdot\left[D_a(g_1)\right]_{lm} \\
&= \sum_a\left[\sum_j\left(\frac{c^a_{jj}}{n_a}\right)\right]\left[D_a(g_1)\right]_{ll} \\
&= \sum_a\left[\sum_j\left(\frac{c^a_{jj}}{n_a}\right)\right]\chi_{D_a}(g_1)
\end{aligned}
$$

This formula

$$F(g_1) = \sum_a \left[\sum_j \left(\frac{c^a_{jj}}{n_a} \right) \right] \chi_{D_a}(g_1)$$

for functions that are constants on the conjugacy classes implies that the characters of the independent irreducible representations form a complete, orthonormal set of basis vectors in "Class Space".

Therefore,

the number of irreducible representations of a group G equals to the number of its conjugacy classes.

Recall that $N = \sum_a n_a^2$.

- All of the irreducible representations of a finite Abelian group are 1-dimensional.

An example:

Question:

Determine the characters of all independent irreducible representations of permutation group S_3.

Solution:

There are 3 independent conjugacy classes in S_3. Hence S_3 has 3 non-equivalent irreducible representations D_0, D_1 and D_2 in total.

D_0 is the trivial 1-dimensional irreducible representation,

$$D_0(g) = 1, \quad \forall\, g \in S_3$$

It means $\chi_0(g) = 1$, $\forall g \in S_3$. The constraint $N = \sum_a n_a^2$ further indicates:

$$6 = 1 + n_1^2 + n_2^2$$

Hence, $n_1 = 1$ and $n_2 = 2$. $\rightsquigarrow$ Besides D_0, S_3 has a 1d irreducible representation D_1 and a 2d irreducible representation D_2.

The elements of the Factor Group $S_3/Z_3 = Z_2$ form the cosets of subgroup Z_3,

$$Z_3 = \{e,\ a_1,\ a_2\}, \quad Z_3 a_3 = \{a_3,\ a_4,\ a_5\}$$

We can identify D_1 as this $Z_2 = \{1,\ -1\}$:

$$\begin{cases} D_1(e) = D_1(a_1) = D_1(a_2) = 1, \\ D_1(a_3) = D_1(a_4) = D_1(a_5) = -1. \end{cases}$$

The corresponding characters read,

$$\begin{cases} \chi_1(e) = \chi_1(a_1) = \chi_1(a_2) = 1, \\ \chi_1(a_3) = \chi_1(a_4) = \chi_1(a_5) = -1. \end{cases}$$

So far we have got an unfinished Characters table for S_3:

	$\{e\}$	$\{a_1, a_2\}$	$\{a_3, a_4, a_5\}$
χ_0	1	1	1
χ_1	1	1	-1
χ_2	2	?	?

We can fill the remaining 2 entries by using orthogonality relations of the characters,

$$\sum_{\alpha} \kappa_\alpha \chi^*_{D_a}(g_\alpha) \chi_{D_b}(g_\alpha) = \delta_{ab}$$

Concretely,

$$\begin{aligned}
6 &= |\chi_2(e)|^2 + 2|\chi_2(a_1)|^2 + 3|\chi_2(a_3)|^2 \\
&= 4 + 2|\chi_2(a_1)|^2 + 3|\chi_2(a_3)|^2 \\
0 &= \chi_1^*(e)\chi_2(e) + 2\chi_1^*(a_1)\chi_2(a_1) + 3\chi_1^*(a_3)\chi_2(a_3) \\
&= 2 + 2\chi_2(a_1) - 3\chi_2(a_3) \\
0 &= \chi_0^*(e)\chi_2(e) + 2\chi_0^*(a_1)\chi_2(a_1) + 3\chi_0^*(a_3)\chi_2(a_3) \\
&= 2 + 2\chi_2(a_1) + 3\chi_2(a_3)
\end{aligned}$$

Therefore,

$$\chi_2(a_1) = -1, \;\; \chi_2(a_3) = 0.$$

Exercise (optional):

Show these results by checking the alternative orthogonality relations

$$\sum_a \chi^*_{D_a}(g_\alpha)\chi_{D_a}(g_\beta) = \frac{N}{\kappa_\alpha}\delta_{\alpha\beta}$$

The *finished* Characters Table of S_3 is,

	$\{e\}$	$\{a_1, a_2\}$	$\{a_3, a_4, a_5\}$
χ_0	1	1	1
χ_1	1	1	-1
χ_2	2	-1	0

Homework:

1. Suppose that D_1 and D_2 are equivalent, irreducible representations of a finite group G such that

$$D_2(g) = SD_1(g)S^{-1}, \quad \forall g \in G.$$

What can you say about an operator A that satisfies

$$AD_1(g) = D_2(g)A, \quad \forall g \in G \ ?$$