现代数学物理方法

第三章，李群

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Rotation group $SO(3)$:

Consider a vector $\vec{r}$ in 3-dimensional space,

$$\vec{r} = \sum_{a=1}^{3} \vec{e}_a x_a \sim \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Rotation:

A linear transformation g

$$g : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = g \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

that leaves the bilinear form $\sum_{a=1}^{3} x_a x_a = x_1^2 + x_2^2 + x_3^2$ invariant is called a 3-dimensional rotation.

Because

$$x_1^2 + x_2^2 + x_3^2 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{aligned} x_1'^2 + x_2'^2 + x_3'^2 &= \begin{bmatrix} x_1' & x_2' & x_3' \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} g^T g \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

the 3-dimensional rotation transformations should be expressed as a set of 3×3 real orthogonal matrices,

$$g^T g = 1$$

Therefore,

$$1 = \det(g^T g) = \left[\det(g)\right]^2 \quad \rightsquigarrow \quad \det(g) = \pm 1$$

The determinant of every orthogonal matrix is either

$$\det(g) = +1$$

in which case the transformation describes pure rotation, or

$$\det(g) = -1$$

in which case it describes a rotation-reflection.

Orthogonal group $O(3)$:

The aggregate of all real orthogonal 3-dimensional matrices

$$g^T g = 1, \quad \det g = \pm 1$$

forms a Lie group, $O(3)$, the so-called 3-dimensional orthogonal group.

$SO(3)$:

Special orthogonal group $SO(3)$:

The aggregate of all pure 3-dimensional rotations

$$g^T g = 1, \quad \det(g) = 1$$

forms a Lie group, $SO(3)$, the 3-dimensional special orthogonal group.

Question:

What is the orthogonal matrix describing a pure rotation with an angle ψ about some direction

$$\vec{n} = \sin\theta\cos\phi\vec{e}_1 + \sin\theta\sin\phi\vec{e}_2 + \cos\theta\vec{e}_3 \sim \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} ?$$

$SO(3)$:

Solution:

In 3-dimensional Cartesian space, the other two *independent* unit vectors orthogonal to $\vec{n}$ read

$$\begin{aligned} \vec{t}_1 &= \cos\theta\cos\phi\vec{e}_1 + \cos\theta\sin\phi\vec{e}_2 - \sin\theta\vec{e}_3, \\ \vec{t}_2 &= -\sin\phi\vec{e}_1 + \cos\phi\vec{e}_2. \end{aligned}$$

From these three unit vectors we find the following *pure rotation* from $\vec{e}_3$ to $\vec{n}$:

$$h = \begin{bmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Evidently,

$$h: \quad \vec{e}_3 \sim \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \rightsquigarrow \quad h\vec{e}_3 \sim h \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} \sim \vec{n}$$

The expected orthogonal matrix describing the pure rotation with an angle ψ about the direction $\vec{n}$ is,

$$\begin{aligned}
g &= h \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} h^T \\
&= \begin{bmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&\quad \cdot \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix}
\end{aligned}$$

The explicit expressions for matrix elements, for example, read

$$\begin{aligned}
g_{11} &= c_\psi + s_\theta^2 c_\phi^2 (1 - c_\psi), \qquad g_{12} = s_\theta^2 c_\phi s_\phi (1 - c_\psi) - c_\theta s_\psi, \\
g_{13} &= s_\theta c_\theta c_\phi (1 - c_\psi) + s_\theta s_\phi s_\psi, \quad \cdots
\end{aligned}$$

where $c_\theta = \cos\theta$ and $s_\psi = \sin\psi$, eta.

In general,

$$\left[g(\theta,\phi,\psi)\right]_{ab} = \delta_{ab}c_\psi + n_a n_b(1-c_\psi) - \epsilon_{abc}n_c s_\psi$$

where indices $a,\ b$ and c take their values from 1 to 3, and $n_1 = s_\theta c_\phi$, $n_2 = s_\theta s_\phi$ and $n_3 = c_\theta$.

Generators of $SO(3)$:

In this definition representation, the generators of $SO(3)$ are defined by,

$$\left[X(\theta,\phi)\right]_{ab} = -i\partial_\psi\left[g(\theta,\phi,\psi)\right]_{ab}|_{\psi=0} = i\epsilon_{abc}n_c$$

Along the 3 axes of the Cartisian coordinate frame, we have:

$$(X_1)_{ab} = i\epsilon_{ab1} = i(\delta_{a2}\delta_{b3}-\delta_{a3}\delta_{b2}), \quad \rightsquigarrow \quad X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix}$$

$$(X_2)_{ab} = i\epsilon_{ab2} = i(\delta_{a3}\delta_{b1}-\delta_{a1}\delta_{b3}), \quad \rightsquigarrow \quad X_2 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}$$

$$(X_3)_{ab} = i\epsilon_{ab3} = i(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1}), \quad \rightsquigarrow \quad X_3 = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In short, in Cartisian coordinates, the generators of $SO(3)$ are as follows:

$$(X_a)_{mn} = i\epsilon_{mna}$$

Based on the famous mathematical identity

$$\epsilon_{ijk}\epsilon_{mnk} = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})$$

we get:

$$\begin{aligned} [X_a,\ X_b]_{mn} &= (X_a)_{mk}(X_b)_{kn} - (X_b)_{mk}(X_a)_{kn} \\ &= -\epsilon_{mka}\epsilon_{knb} + \epsilon_{mkb}\epsilon_{kna} = \epsilon_{amk}\epsilon_{bnk} - \epsilon_{bmk}\epsilon_{ank} \\ &= \delta_{ab}\delta_{mn} - \delta_{an}\delta_{mb} - \delta_{ba}\delta_{mn} + \delta_{bn}\delta_{ma} \\ &= \delta_{am}\delta_{bn} - \delta_{an}\delta_{bm} = \epsilon_{abc}\epsilon_{mnc} \\ &= -i\epsilon_{abc}(i\epsilon_{mnc}) = -i\epsilon_{abc}(X_c)_{mn} \end{aligned}$$

That is,

$$[X_a,\ X_b] = -i\epsilon_{abc}X_c$$

The structure constants of $SO(3)$ are components ϵ_{ijk} of the Levi-Civita antisymmetric tensor.

Relying on the fact,

$$-(X_a)_{bc} = -i\epsilon_{abc}$$

the definition representation of $SO(3)$ is just its adjoint representation.

Casimir operators:

Casimir operators of a Lie group are such operators that commute with all generators of the group.

- $SO(3)$ has one Casimir operator:

$$X^2 = \sum_{a=1}^{3} X_a X_a$$

Racah Theorem :

Here is a simple check:

$$
\begin{aligned}
[X^2,\ X_a] &= \sum_{b=1}^{3}[X_bX_b,\ X_a] = \sum_{b=1}^{3}\Big\{[X_b,\ X_a]X_b + X_b[X_b,\ X_a]\Big\} \\
&= \sum_{b,c=1}^{3}\left(-i\epsilon_{bac}X_cX_b - i\epsilon_{bac}X_bX_c\right) \\
&= i\sum_{b,c=1}^{3}\epsilon_{abc}(X_bX_c + X_cX_b)\ = 0.
\end{aligned}
$$

Racah theorem:

For any semi-simple Lie group G *of rank* l*, there exists a set of* l *Casimir operators,*

$$C_\lambda = C_\lambda(X_1,\ X_2,\ \cdots,X_N),\quad (1 \leqslant \lambda \leqslant l)$$

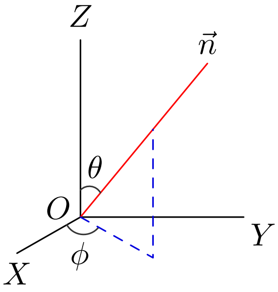
that commute with every generator of the group and therefore also amongst themselves, $[C_\lambda,\ C_\sigma] = 0$.

Group elements of $SO(3)$:

The general group elements of $SO(3)$, which describe the pure rotation with an angle ψ about the direction $\vec{n} = (s_\theta c_\phi, s_\theta s_\phi, c_\theta)$, read:[1]

$$\left[g(\theta,\phi,\psi)\right]_{ab} = \delta_{ab}\, c_\psi + n_a n_b\,(1 - c_\psi) - \epsilon_{abc} n_c\, s_\psi$$

where $n_1 = s_\theta c_\phi$, $n_2 = s_\theta s_\phi$ and $n_3 = c_\theta$.



[1] The ranges for the parameters take their values are $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi, \psi \leqslant 2\pi$.

In particular,

$$g\left(\frac{\pi}{2}, 0, \psi\right) \equiv R_x(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

Similarly,

$$g\left(\frac{\pi}{2}, \frac{\pi}{2}, \psi\right) \equiv R_y(\psi) = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix}$$

and

$$g(0, 0, \psi) \equiv R_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

With the previously defined generators,

$$X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

these special group elements of $SO(3)$ can be expressed as

$$R_x(\psi) = e^{i\psi X_1}, \quad R_y(\psi) = e^{i\psi X_2}, \quad R_z(\psi) = e^{i\psi X_3}$$

In general,

$$g(\theta, \phi, \psi) \equiv R_{\vec{n}}(\psi) = e^{i\psi \vec{n}\cdot\vec{X}} = e^{i\psi(s_\theta c_\phi X_1 + s_\theta s_\phi X_2 + c_\theta X_3)}$$

Our check is as follows:

$$(\vec{n} \cdot \vec{X})_{ij} = n_a (X_a)_{ij} = i\epsilon_{ija} n_a$$

$$
\begin{aligned}
\left[(\vec{n}\cdot\vec{X})^2\right]_{ij} &= (\vec{n}\cdot\vec{X})_{ik}(\vec{n}\cdot\vec{X})_{kj} \\
&= (i\epsilon_{ika}n_a)(i\epsilon_{kjb}n_b) \\
&= -\epsilon_{ika}\epsilon_{kjb}n_a n_b \\
&= \epsilon_{iak}\epsilon_{jbk}n_a n_b \\
&= (\delta_{ij}\delta_{ab} - \delta_{ib}\delta_{ja})n_a n_b \\
&= \delta_{ij}n_a n_a - n_i n_j \\
&= \delta_{ij} - n_i n_j
\end{aligned}
$$

In the last step, we have used the the condition $n_a n_a = 1$ for unit vector $\vec{n}$. Moreover,

$$
\begin{aligned}
\left[(\vec{n}\cdot\vec{X})^3\right]_{ij} &= \left[(\vec{n}\cdot\vec{X})^2\right]_{ik}(\vec{n}\cdot\vec{X})_{kj} \\
&= (\delta_{ik} - n_i n_k)(-i\epsilon_{kja}n_a) \\
&= -i\epsilon_{ija}n_a + i\epsilon_{akj}n_a n_k n_i \\
&= -i\epsilon_{ija}n_a = (\vec{n}\cdot\vec{X})_{ij}
\end{aligned}
$$

$$\left[(\vec{n}\cdot\vec{X})^4\right]_{ij} = [(\vec{n}\cdot\vec{X})^3]_{ik}(\vec{n}\cdot\vec{X})_{kj} = (\vec{n}\cdot\vec{X})_{ik}(\vec{n}\cdot\vec{X})_{kj} = [(\vec{n}\cdot\vec{X})^2]_{ij}$$

In general, for an arbitrary positive integer $m \in \mathbb{Z}^+$,

$$[(\vec{n}\cdot\vec{X})^{2m-1}]_{ij} = i\epsilon_{ija}n_a, \quad [(\vec{n}\cdot\vec{X})^{2m}]_{ij} = \delta_{ij} - n_i n_j \ .$$

Hence,

$$\begin{aligned}
\left[e^{i\psi(\vec{n}\cdot\vec{X})}\right]_{ij} &= \left[1 + i\psi(\vec{n}\cdot\vec{X}) + \frac{i^2\psi^2}{2!}(\vec{n}\cdot\vec{X})^2 + \frac{i^3\psi^3}{3!}(\vec{n}\cdot\vec{X})^3 + \cdots\right] \\
&= \delta_{ij} + i(\vec{n}\cdot\vec{X})_{ij}\left[\psi - \frac{\psi^3}{3!} + \cdots\right] \\
&\quad + [(\vec{n}\cdot\vec{X})^2]_{ij}\left[-\frac{\psi^2}{2!} + \frac{\psi^4}{4!} - \cdots\right] \\
&= \delta_{ij} + i(\vec{n}\cdot\vec{X})_{ij}s_\psi + [(\vec{n}\cdot\vec{X})^2]_{ij}(c_\psi - 1) \\
&= \delta_{ij} - \epsilon_{ija}n_a s_\psi + (\delta_{ij} - n_i n_j)(c_\psi - 1)
\end{aligned}$$

As expected,

$$\left[e^{i\psi(\vec{n}\cdot\vec{X})}\right]_{ij} = c_\psi \delta_{ij} + n_i n_j (1 - c_\psi) - \epsilon_{ijk} n_k s_\psi = \left[g(\theta, \phi, \psi)\right]_{ij}$$

In matrix form,
the group elements of $SO(3)$ in its adjoint representation are expressed as:

$$g(\theta, \phi, \psi) = e^{i\psi(\vec{n}\cdot\vec{X})} = e^{i\psi(s_\theta c_\phi X_1 + s_\theta s_\phi X_2 + c_\theta X_3)}$$

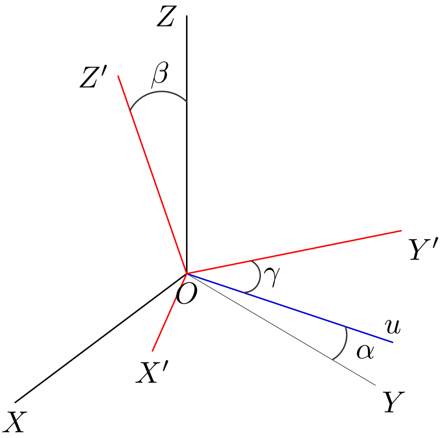
where $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi,\ \psi \leqslant 2\pi$.

Evidently,
3 parameters are required to describe an arbitrary 3-dimensional rotation. *They may be related to the rotation axis*[2] *and the angle ψ of rotation.*

[2]The axis $\vec{n}$ is described by 2 parameters θ and ϕ. Since $g(\vec{n}, \psi) = g(-\vec{n}, 2\pi - \psi)$, the space of $SO(3)$ group parameters is a sphere of radius π, i.e., $0 \leqslant \phi \leqslant 2\pi$ and $0 \leqslant \theta,\ \psi \leqslant \pi$, if the one-to-one correspondence exists between the parameters and the $SO(3)$ group elements.

Euler angles

Alternatively, the 3 parameters may be chosen as Euler angles, defined as the *three successive angles of rotation* by the sequent rotations from the fixed system of axes $Oxyz$:



1. Rotate through angle α about axis Oz, carrying Oy into Ou;
2. Rotate through angle β about axis Ou, carrying Oz into Oz';
3. Rotate through angle γ about axis Oz', carrying Ou into Oy';

At the end of this process Ox will have been carried into Ox'. The range of these Euler angles is $0 \leqslant \alpha, \gamma \leqslant 2\pi$ and $0 \leqslant \beta \leqslant \pi$.

Euler angle representation:

The net rotation is described by the orthogonal matrix,

$$R(\alpha,\ \beta,\ \gamma) = e^{i\gamma X_{z'}} e^{i\beta X_u} e^{i\alpha X_z} = R_{z'}(\gamma) R_u(\beta) R_z(\alpha)$$

Because the factor rotation $R_z(\alpha) = e^{i\alpha X_z}$ carries axis Oy into ou,

$$X_u = R_z(\alpha) X_y R_z(-\alpha) = e^{i\alpha X_z} X_y e^{-i\alpha X_z}$$

Hence,

$$R_u(\beta) = e^{i\beta X_u} = e^{i\alpha X_z} e^{i\beta X_y} e^{-i\alpha X_z}$$

Similarly, because $R_u(\beta)$ carries axis Oz into Oz', we have,

$$R_{z'}(\gamma) = e^{i\gamma X_{z'}} = e^{i\beta X_u} e^{i\gamma X_z} e^{-i\beta X_u}$$

Consequently,

$$\begin{aligned}
R(\alpha,\ \beta,\ \gamma) \quad &= R_{z'}(\gamma) R_u(\beta) R_z(\alpha) \\
&= \left[e^{i\beta X_u} e^{i\gamma X_z} e^{-i\beta X_u} \right] e^{i\beta X_u}\ R_z(\alpha) \\
&= e^{i\beta X_u}\ e^{i\gamma X_z}\ R_z(\alpha) \\
&= \left[e^{i\alpha X_z}\ e^{i\beta X_y}\ e^{-i\alpha X_z} \right]\ e^{i\gamma X_z}\ e^{i\alpha X_z} \\
&= e^{i\alpha X_z} e^{i\beta X_y} e^{i\gamma X_z}
\end{aligned}$$

In conclusion, an arbitrary pure rotation in 3-dimensional Cartesian space can be recast as

$$R(\alpha,\ \beta,\ \gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma) = e^{i\alpha X_z} e^{i\beta X_y} e^{i\gamma X_z}$$

in terms of Euler angles α, β and γ in the original fixed coordinate system $Oxyz$.

The range of Euler angles:

It follows from the explicit orthogonal matrices $R_y(\beta)$ and $R_z(\alpha)$ that,

$$R_z(\gamma)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 \\ s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$R_y(\beta)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} s_\beta \\ 0 \\ c_\beta \end{bmatrix}$$

$$R_z(\alpha)\begin{bmatrix} s_\beta \\ 0 \\ c_\beta \end{bmatrix} = \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} s_\beta \\ 0 \\ c_\beta \end{bmatrix} = \begin{bmatrix} s_\beta c_\alpha \\ s_\beta s_\alpha \\ c_\beta \end{bmatrix}$$

It implies,

$$R(\alpha,\ \beta,\ \gamma)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = R_z(\alpha)R_y(\beta)R_z(\gamma)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} s_\beta c_\alpha \\ s_\beta s_\alpha \\ c_\beta \end{bmatrix}$$

Namely,

$$R(\alpha,\ \beta,\ \gamma)\vec{e}_3 = \vec{n} = s_\beta c_\alpha \vec{e}_1 + s_\beta s_\alpha \vec{e}_2 + c_\beta \vec{e}_3$$

Hence $0 \leqslant \alpha \leqslant 2\pi$ and $0 \leqslant \beta \leqslant \pi$.

Similarly,

$$[R_z(\alpha)]^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_\alpha & s_\alpha & 0 \\ -s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$[R_y(\beta)]^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_\beta & 0 & -s_\beta \\ 0 & 1 & 0 \\ s_\beta & 0 & c_\beta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -s_\beta \\ 0 \\ c_\beta \end{bmatrix}$$

$$[R_z(\gamma)]^T \begin{bmatrix} -s_\beta \\ 0 \\ c_\beta \end{bmatrix} = \begin{bmatrix} c_\gamma & c_\gamma & 0 \\ -s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s_\beta \\ 0 \\ c_\beta \end{bmatrix} = \begin{bmatrix} -s_\beta c_\gamma \\ s_\beta s_\gamma \\ c_\beta \end{bmatrix}$$

These formulae yield,

$$
\begin{aligned}
[R(\alpha,\ \beta,\ \gamma)]^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} &= [R_z(\gamma)]^T [R_y(\beta)]^T [R_z(\alpha)]^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -s_\beta c_\gamma \\ s_\beta s_\gamma \\ c_\beta \end{bmatrix}
\end{aligned}
$$

That is to say,

$$
\begin{aligned}
[R(\alpha,\ \beta,\ \gamma)]^T \vec{e}_3 &= \vec{n}' \\
&= -s_\beta c_\gamma \vec{e}_1 + s_\beta s_\gamma \vec{e}_2 + c_\beta \vec{e}_3 \\
&= s_\beta c_{(\pi-\gamma)} \vec{e}_1 + s_\beta s_{(\pi-\gamma)} \vec{e}_2 + c_\beta \vec{e}_3
\end{aligned}
$$

Hence $0 \leqslant (\pi - \gamma) \leqslant 2\pi$ or equivalently $-\pi \leqslant \gamma \leqslant \pi$.

We conclude that the range of Euler angles in $R(\alpha, \beta, \gamma)$ are:

$$0 \leqslant \alpha,\ \gamma \leqslant 2\pi, \quad 0 \leqslant \beta \leqslant \pi.$$

$SO(3)$ rotation in Hilbert space:

Scalar wave function :

Scalar wave-function has one-component $\psi(\vec{x})$. Under a rotation of position coordinates, $\vec{x} \rightsquigarrow \vec{x}' = R\vec{x}$, it remains invariant,

$$\psi(\vec{x}) \rightsquigarrow \psi'(\vec{x}') = \psi(\vec{x})$$

As a result,

$$\psi'(\vec{x}) = \psi(R^{-1}\vec{x})$$

Here R^{-1} is the inverse of a 3×3 coordinate rotation matrix R.

Let us introduce the operator $\mathcal{R}$ in Hilbert space to describe *the rotation of the wave functions themselves,*

$$\begin{aligned} \vec{x} \quad &\rightsquigarrow \quad \vec{x}' = R\vec{x}, \\ \psi(\vec{x}) \quad &\rightsquigarrow \quad \psi'(\vec{x}) = \mathcal{R}\psi(\vec{x}) \end{aligned}$$

Therefore,

$$\mathcal{R}\psi(\vec{x}) = \psi(R^{-1}\vec{x})$$

The complete set of operators $\{\mathcal{R}\}$ defines a representation of $SO(3)$, called *the rotation group in Hilbert space.*

Proof:

The unit element in $\{\mathcal{R}\}$ does trivially exist. Moreover, under two successive coordinate rotations,

$$\vec{x} \rightsquigarrow \vec{x}' = R_1 \vec{x} \rightsquigarrow \vec{x}'' = R_2 \vec{x}' = R_2 R_1 \vec{x}$$

the scalar wave function $\psi(\vec{x})$ transforms into:

$$\psi(\vec{x}) \rightsquigarrow \psi'(\vec{x}') = \psi(\vec{x}) \rightsquigarrow \psi''(\vec{x}'') = \psi'(\vec{x}') = \psi(\vec{x})$$

Namely,

$$\psi''(\vec{x}) = \psi((R_2 R_1)^{-1} \vec{x})$$

On the other hand, $\mathcal{R}_1 \psi(\vec{x}) = \psi'(\vec{x})$ and $\mathcal{R}_2 \psi'(\vec{x}) = \psi''(\vec{x})$. Hence,

$$\psi''(\vec{x}) = \mathcal{R}_2 \psi'(\vec{x}) = \mathcal{R}_2 \mathcal{R}_1 \psi(\vec{x})$$

By comparison, we get

$$\mathcal{R}_2\mathcal{R}_1\psi(\vec{x}) = \psi((R_2R_1)^{-1}\vec{x})$$

This justifies that the rule

$$\mathcal{R}\psi(\vec{x}) = \psi(R^{-1}\vec{x})$$

is kept by the successive transformations, as expected. So $\{\mathcal{R}\}$ forms a representation of $SO(3)$ in Hilbert space.

- Recall that the rotation matrices in coordinate space are expressed as $R_{\vec{n}}(\psi) = e^{i\psi(\vec{n}\cdot\vec{X})}$, whose infinitesimal form reads,

$$[R_{\vec{n}}(\varphi)]_{ij} \approx \delta_{ij} + i\varphi(\vec{n}\cdot\vec{X})_{ij} = \delta_{ij} - \varphi\epsilon_{ijk}n_k$$

Hence, the infinitesimal rotation in Hilbert space should satisfy,

$$\begin{aligned}
\mathcal{R}_{\vec{n}}(\varphi)\psi(\vec{x}) \quad &= \psi(R_{\vec{n}}^{-1}(\varphi)\vec{x}) = \psi([R_{\vec{n}}^{-1}(\varphi)]_{ij}x_j) \\
&= \psi(x_i + \varphi\epsilon_{ijk}x_jn_k) \\
&= \psi(\vec{x}) + \varphi\epsilon_{ijk}x_jn_k\partial_{x_i}\psi(\vec{x}) + \cdots
\end{aligned}$$

Namely,

$$\mathcal{R}_{\vec{n}}(\varphi)\psi(\vec{x}) \approx \psi(\vec{x}) - \varphi n_i \epsilon_{ijk} x_j \partial_k \psi(\vec{x})$$

Generators:

Define the generators L_i $(i = 1, 2, 3)$ of $SO(3)$ in Hilbert space by

$$\mathcal{R}_{\vec{n}}(\varphi) \approx 1 - i\varphi(\vec{n} \cdot \vec{L})$$

- These generators turn out to be the orbital angular momentum operators:

$$L_i = -i\epsilon_{ijk} x_j \partial_k$$

- It is easy to check that

$$[L_i,\ L_j] = i\epsilon_{ijk} L_k$$

Multicomponent wave functions :

Under a 3-dimensional rotation $\vec{x} \rightsquigarrow \vec{x}' = R\vec{x}$ in coordinate space, the components of a multicomponent wave function

$$\begin{bmatrix} \psi_1(\vec{x}) \\ \psi_2(\vec{x}) \\ \vdots \\ \psi_N(\vec{x}) \end{bmatrix}$$

transform as,

$$\mathcal{R}\psi_a(\vec{x}) = D_{ab}\psi_b(R^{-1}\vec{x}), \qquad (a, b = 1, 2, \cdots, N)$$

In addition to the coordinate transformation $R^{-1}\vec{x}$, *a $N \times N$ matrix D has to act on the internal degrees of freedom so that a linear combination of the wave function components forms.*

Hence,

$$\mathcal{R}_{\vec{n}}(\varphi) = e^{-i\varphi(\vec{n}\cdot\vec{L})} D_{\vec{n}}(\varphi)$$

The matrix D must be unitary and so it can be written as:

$$D_{\vec{n}}(\varphi) = e^{-i\varphi(\vec{n}\cdot\vec{S})}$$

with the $N \times N$ hermitian matrices $\vec{S}$ obeying Lie brackets

$$[S_i,\ S_j] = i\epsilon_{ijk}S_k$$

and

$$[S_i,\ L_j] = 0$$

Such a $\vec{S}$ is called the spin angular momentum of the particle described by the given multi-component wave function. *e.g.*,

1. $N = 1$, scalar.
2. $N = 2$, spinor.
3. $N = 3$, vector.
4. $N = 4$, double-spinor ?

$SO(N)$:

$O(N)$:

The orthogonal group $O(N)$ is formed by the set of all $N \times N$ real orthogonal matrices

$$R^T R = 1, \quad R^* = R$$

under the matrix multiplications.

- Obviously,

$$\det R = \pm 1$$

- The condition $R^T R = 1$ stands for $N(N+1)/2$ independent constraints

$$R_{ij} R_{ik} = \delta_{jk}$$

  Hence, the number of independent real parameters for describing an $O(N)$ group element is:

$$g = N^2 - \frac{1}{2} N(N+1) = \frac{1}{2} N(N-1)$$

$SO(N)$:

$SO(N)$ is the normal subgroup of $O(N)$ consisting of the $N \times N$ real orthogonal matrices with unit determinant,

$$\det R = 1$$

Remarks:

- The total number of real independent parameters for describing a $SO(N)$ group element is $N(N-1)/2$.
- These real parameters can be written as

$$\omega_{ab}, \qquad (a, b = 1, 2, \cdots, N)$$

with antisymmetry,

$$\omega_{ab} = -\omega_{ba}$$

Consequently, an arbitrary $SO(N)$ group element is expressed as,

$$R = \exp\left[-i \sum_{b>a} \sum_{a=1}^{N-1} \omega_{ab} \, T_{ab}\right]$$

where T_{ab} with symmetry $T_{ab} = -T_{ba}$ are $N(N-1)/2$ generators of $SO(N)$.

Discussions:

- Because R is real and unitary, *each generator T_{ab} is purely imaginary and antisymmetric hermitian matrix.*
- $\det R = 1$ requires that *all T_{ab} are traceless.*

$so(N)$ Algebra

We choose the generators of $SO(N)$ in its definition representation as

$$(T_{ab})_{jk} = -i(\delta_{aj}\delta_{bk} - \delta_{ak}\delta_{bj})$$

where indices a, b label the name of the generator T_{ab}, while indices j, k specify the matrix element of T_{ab}.

Obviously,

1. T_{ab} are purely imaginary.
2. $(T_{ab})_{jk} = -(T_{ab})_{kj}$
3. $\mathrm{Tr}(T_{ab}) = (T_{ab})_{jj} = -i(\delta_{aj}\delta_{bj} - \delta_{aj}\delta_{bj}) = -i(\delta_{ab} - \delta_{ab}) = 0$

$so(N)$ algebra is,

$$\begin{aligned}
[T_{ab},\ T_{cd}]_{ij} &= (T_{ab})_{ik}(T_{cd})_{kj} - (T_{cd})_{ik}(T_{ab})_{kj} \\
&= -(\delta_{ai}\delta_{bk} - \delta_{ak}\delta_{bi})(\delta_{ck}\delta_{dj} - \delta_{cj}\delta_{dk}) \\
&\qquad +(\delta_{ci}\delta_{dk} - \delta_{ck}\delta_{di})(\delta_{ak}\delta_{bj} - \delta_{aj}\delta_{bk}) \\
&= -i\delta_{bc}(T_{ad})_{ij} + i\delta_{bd}(T_{ac})_{ij} + i\delta_{ac}(T_{bd})_{ij} - i\delta_{ad}(T_{bc})_{ij}
\end{aligned}$$

Namely,

$$[T_{ab},\ T_{cd}] = -i(\delta_{ad}T_{bc} + \delta_{bc}T_{ad} - \delta_{ac}T_{bd} - \delta_{bd}T_{ac})$$

Equivalently,

$$[T_{ab},\ T_{cd}] = i f_{ab,cd,ij} T_{ij}$$

where the structure constants

$$\begin{aligned}
f_{ab,cd,ij} &= \frac{1}{2}\Big[\delta_{ad}\delta_{ci}\delta_{bj} - \delta_{ad}\delta_{bi}\delta_{cj} + \delta_{bc}\delta_{di}\delta_{aj} - \delta_{bc}\delta_{ai}\delta_{dj} \\
&\qquad - \delta_{ac}\delta_{di}\delta_{bj} + \delta_{ac}\delta_{bi}\delta_{dj} - \delta_{bd}\delta_{ci}\delta_{aj} + \delta_{bd}\delta_{ai}\delta_{cj}\Big]
\end{aligned}$$

are completely antisymmetric for exchanging any two groups of indices.

Note:

- The definition representation of $SO(N)$ is just its *adjoint* representation.
- For $SO(2M)$ and $SO(2M+1)$, the mutually commuting generators are:

$$H_a = T_{(2a-1)(2a)}, \quad (1 \leqslant a \leqslant M)$$

The normalization conditions of the $SO(N)$ generators read,

$$\begin{aligned}
\mathrm{Tr}(T_{ab}T_{cd}) &= (T_{ab})_{ij}(T_{cd})_{ji} \\
&= -(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi})(\delta_{cj}\delta_{di} - \delta_{ci}\delta_{dj}) \\
&= 2(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})
\end{aligned}$$

Definition Rep. of $SU(N)$:

The aggregate of all $N \times N$ unitary matrices $\{u\}$ with unit determinant provides the group $SU(N)$,

$$u^\dagger u = u u^\dagger = 1, \quad \det u = 1$$

Number of the real parameters :

- The unitary condition can be written as

  $$\delta_{ij} = (u^\dagger)_{ik} u_{kj} = u^*_{ki} u_{kj}$$

  It gives N real constraints when $i = j$ while $N(N-1)/2$ complex constraints or equivalently $N(N-1)$ real constraints when $i \neq j$.

- $\det u = 1$ gives an additional constraint.

Totally, the number of real independent parameters for describing an arbitrary $SU(N)$ group element should be,

$$g = 2N^2 - N - N(N-1) - 1 = N^2 - 1$$

These $N^2 - 1$ real parameters could be chosen to be

$$\left\{ \begin{array}{l} \omega_{ab}^{(1)} \\ \omega_{ab}^{(2)} \\ \omega_{c}^{(3)} \end{array} \right. \qquad a = 1, 2, \cdots, N-1; \quad a < b; \quad b, c = 2, 3, \cdots, N$$

with properties

$$\omega_{ab}^{(1)} = \omega_{ba}^{(1)}, \;\; \omega_{ab}^{(2)} = -\omega_{ba}^{(2)}.$$

Generators:

The (N^2-1) *traceless hermitian* generators of the definition Rep. of unitary group $SU(N)$ could be chosen as follows:

1. $N(N-1)/2$ *hermitian* $T^{(1)}_{ab}$ $(a<b)$ *with* $T^{(1)}_{ab}=T^{(1)}_{ba}$
2. $N(N-1)/2$ *hermitian* $T^{(2)}_{ab}$ $(a<b)$ *with* $T^{(2)}_{ab}=-T^{(2)}_{ba}$
3. $(N-1)$ *diagonal hermitian* $T^{(3)}_{c}$

so that

$$u=\exp\left[\sum_{a<b}\sum_{b=2}^{N}\left(\omega^{(1)}_{ab}T^{(1)}_{ab}+\omega^{(2)}_{ab}T^{(2)}_{ab}\right)+\sum_{c=2}^{N}\omega^{(3)}_{c}T^{(3)}_{c}\right]$$

The matrix elements of these traceless hermitian generators can explicitly be defined as,

$$(T^{(1)}_{ab})_{ij}=\frac{1}{2}\left(\delta_{ai}\delta_{bj}+\delta_{aj}\delta_{bi}\right)$$

$$(T^{(2)}_{ab})_{ij} = -\frac{i}{2}\Big(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}\Big)$$

and

$$(T^{(3)}_{c})_{ij} = \begin{cases} \delta_{ij}\frac{1}{\sqrt{2c(c-1)}}, & \text{if } \ i < c\,; \\ -\delta_{ij}\sqrt{\frac{(c-1)}{2c}}, & \text{if } \ i = c\,; \\ 0, & \text{if } \ i > c. \end{cases}$$

For $SU(2)$, they are simply related to the famous Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Obviously,

$$T^{(1)}_{12} = \sigma_1/2, \quad T^{(2)}_{12} = \sigma_2/2, \quad T^{(3)}_{2} = \sigma_3/2.$$

$SU(2)$:

Remainder:

The aggregate of all unitary matrices of order 2 and determinant unity forms the group $SU(2)$.

An arbitrary $SU(2)$ group element has the form,

$$u(\omega) = e^{i\left[\omega_{12}^{(1)} T_{12}^{(1)} + \omega_{12}^{(2)} T_{12}^{(2)} + \omega_{2}^{(3)} T_{2}^{(3)}\right]}$$

Equivalently,

$$u(\vec{n}, \psi) = e^{i\psi(\vec{n}\cdot\vec{\sigma})/2}$$

where

$$\vec{n} = c_\theta \vec{e}_3 + s_\theta c_\phi \vec{e}_1 + s_\theta s_\phi \vec{e}_2$$

is a two-parameter unit vector in the 3-dimensional parameter space $(\theta,\ \phi,\ \psi)$.

The Pauli matrices satisfy relation

$$\sigma_a\sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c.$$

Hence,

$$(\vec{n}\cdot\vec{\sigma})^2 = n_a n_b \sigma_a\sigma_b = n_a n_b(\delta_{ab} + i\epsilon_{abc}\sigma_c) = n_a n_a = 1$$

The $SU(2)$ group element becomes,

$$\begin{aligned}
u(\vec{n},\psi) &= e^{i\psi(\vec{n}\cdot\vec{\sigma})/2} \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!}(\psi/2)^n(\vec{n}\cdot\vec{\sigma})^n \\
&= \cos(\psi/2) + i\sin(\psi/2)(\vec{n}\cdot\vec{\sigma}) \\
&= \cos(\psi/2) + i\sin(\psi/2)\begin{bmatrix} n_3 & n_1 - in_2 \\ n_1 + in_2 & -n_3 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\psi/2) + i\sin(\psi/2)c_\theta & i\sin(\psi/2)s_\theta e^{-i\phi} \\ i\sin(\psi/2)s_\theta e^{i\phi} & \cos(\psi/2) - i\sin(\psi/2)c_\theta \end{bmatrix}
\end{aligned}$$

It follows from

$$u(\vec{n},\psi) = \begin{bmatrix} \cos(\psi/2) + i\sin(\psi/2)c_\theta & i\sin(\psi/2)s_\theta e^{-i\phi} \\ i\sin(\psi/2)s_\theta e^{i\phi} & \cos(\psi/2) - i\sin(\psi/2)c_\theta \end{bmatrix}$$

that:

1. $\det u = \cos^2(\psi/2) + \sin^2(\psi/2)c_\theta^2 + \sin^2(\psi/2)s_\theta^2 = 1.$
2. $u(\vec{n},\psi)$ is indeed unitary, $u^\dagger(\vec{n},\psi) = u^{-1}(\vec{n},\psi)$, with

$$u^\dagger(\vec{n},\psi) = \begin{bmatrix} \cos(\psi/2) - i\sin(\psi/2)c_\theta & -i\sin(\psi/2)s_\theta e^{-i\phi} \\ -i\sin(\psi/2)s_\theta e^{i\phi} & \cos(\psi/2) + i\sin(\psi/2)c_\theta \end{bmatrix}$$

3. $u(\vec{n},2\pi) = -1$ while $u(\vec{n},\psi) = -u(-\vec{n},2\pi-\psi)$. Therefore, the range for these 3 real parameters taking their values could be,

$$0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \phi \leqslant 2\pi, \quad 0 \leqslant \psi \leqslant 2\pi.$$

4. There is a Homomorphism between the groups $SO(3)$ and $SU(2)$,

$$u^\dagger(\vec{n},\psi)\sigma_b u(\vec{n},\psi) = \sum_{a=1}^{3} \sigma_a \left[R(\vec{n},\psi)\right]_{ab}$$

Homomorphism between $SO(3)$ and $SU(2)$:

So, two $SU(2)$ matrices, $u(\vec{n}, \psi)$ and $u(-\vec{n}, 2\pi - \psi)$, correspond to the same $SO(3)$ rotation $R(\vec{n}, \psi)$.

Proof:
Consider an arbitrary vector $\vec{r}$ in the $SU(2)$ parameter space,

$$\vec{r} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Because

$$u(\vec{n}, \psi) = e^{i\psi(\vec{n}\cdot\vec{\sigma})/2} = \cos(\psi/2) + i\sin(\psi/2)(\vec{n}\cdot\vec{\sigma})$$

we have

$$\begin{aligned} & u^\dagger(\vec{n}, \psi)(\vec{r}\cdot\vec{\sigma})u(\vec{n}, \psi) \\ & \quad = \Big[\cos(\psi/2) - i\sin(\psi/2)(\vec{n}\cdot\vec{\sigma})\Big](\vec{r}\cdot\vec{\sigma}) \\ & \qquad \cdot\Big[\cos(\psi/2) + i\sin(\psi/2)(\vec{n}\cdot\vec{\sigma})\Big] \end{aligned}$$

$$= \cos^2(\psi/2)(\vec{r}\cdot\vec{\sigma}) - i\sin(\psi/2)\cos(\psi/2)\left[(\vec{n}\cdot\vec{\sigma}),\ (\vec{r}\cdot\vec{\sigma})\right] + \sin^2(\psi/2)(\vec{n}\cdot\vec{\sigma})(\vec{r}\cdot\vec{\sigma})(\vec{n}\cdot\vec{\sigma})$$

Employment of identity $\sigma_a\sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c$ yields,

$$\left[(\vec{n}\cdot\vec{\sigma}),\ (\vec{r}\cdot\vec{\sigma})\right] = n_a x_b[\sigma_a, \sigma_b] = 2i n_a x_b \epsilon_{abc}\sigma_c = 2i(\vec{n}\times\vec{r}\,)\cdot\vec{\sigma}$$

and

$$\begin{aligned}
(\vec{n}\cdot\vec{\sigma})(\vec{r}\cdot\vec{\sigma})(\vec{n}\cdot\vec{\sigma}) &= n_a n_b x_c \sigma_a\sigma_c\sigma_b \\
&= n_a n_b x_c(\delta_{ac} + i\epsilon_{acd}\sigma_d)\sigma_b \\
&= (\vec{n}\cdot\vec{r})(\vec{n}\cdot\vec{\sigma}) + i n_a n_b x_c \epsilon_{acd}(\delta_{db} + i\epsilon_{dbe}\sigma_e) \\
&= (\vec{n}\cdot\vec{r})(\vec{n}\cdot\vec{\sigma}) - i n_a n_b x_c \epsilon_{abc} - n_a n_b x_c(\epsilon_{acd}\epsilon_{bed})\sigma_e \\
&= (\vec{n}\cdot\vec{r})(\vec{n}\cdot\vec{\sigma}) - n_a n_b x_c(\delta_{ab}\delta_{ce} - \delta_{ae}\delta_{cb})\sigma_e \\
&= (\vec{n}\cdot\vec{r})(\vec{n}\cdot\vec{\sigma}) - (\vec{r}\cdot\vec{\sigma}) + (\vec{n}\cdot\vec{r})(\vec{n}\cdot\vec{\sigma}) \\
&= 2(\vec{n}\cdot\vec{r})(\vec{n}\cdot\vec{\sigma}) - (\vec{r}\cdot\vec{\sigma})
\end{aligned}$$

Therefore,

$$
\begin{aligned}
u^\dagger(\vec{n},\psi)(\vec{r}\cdot\vec{\sigma})u(\vec{n},\psi) &= \Big[\cos^2(\psi/2)-\sin^2(\psi/2)\Big](\vec{r}\cdot\vec{\sigma}) \\
&\quad + 2\sin(\psi/2)\cos(\psi/2)(\vec{n}\times\vec{r}\,)\cdot\vec{\sigma} \\
&\quad + 2\sin^2(\psi/2)(\vec{n}\cdot\vec{r})(\vec{n}\cdot\vec{\sigma}) \\
&= \cos\psi(\vec{r}\cdot\vec{\sigma}) + \sin\psi(\vec{n}\times\vec{r}\,)\cdot\vec{\sigma} + (1-\cos\psi)(\vec{n}\cdot\vec{r})(\vec{n}\cdot\vec{\sigma}) \\
&= \cos\psi\,\sigma_a x_a + \sin\psi\,\sigma_a\epsilon_{acb}n_c x_b + (1-\cos\psi)n_b x_b n_a\sigma_a \\
&= \sigma_a\Big[\delta_{ab}\cos\psi + n_a n_b(1-\cos\psi) - \epsilon_{abc}n_c\sin\psi\Big]x_b
\end{aligned}
$$

Recall that the $SO(3)$ group element

$$
R(\vec{n},\psi) \equiv g(\theta,\phi,\psi) = e^{i\psi(\vec{n}\cdot\vec{X})}
$$

can explicitly be expressed as

$$
\big[R(\vec{n},\psi)\big]_{ab} = \delta_{ab}\cos\psi + n_a n_b(1-\cos\psi) - \epsilon_{abc}n_c\sin\psi
$$

Therefore,

$$u^\dagger(\vec{n},\psi)(\vec{r}\cdot\vec{\sigma})u(\vec{n},\psi) = \sigma_a\big[R(\vec{n},\psi)\big]_{ab}x_b$$

It implies that the unitary group $SU(2)$ is homomorphic to the orthogonal group $SO(3)$,

$$u^\dagger(\vec{n},\psi)\,\sigma_b\,u(\vec{n},\psi) = \sigma_a\big[R(\vec{n},\psi)\big]_{ab}$$

Recall that

$$R(-\vec{n},\ 2\pi-\psi) = R(\vec{n},\psi)$$

we have also,

$$\begin{aligned} u^\dagger(-\vec{n},2\pi-\psi)\,\sigma_b\,u(-\vec{n},2\pi-\psi) &= \sigma_a\big[R(-\vec{n},2\pi-\psi)\big]_{ab} \\ &= \sigma_a\big[R(\vec{n},\psi)\big]_{ab} \end{aligned}$$

Therefore, two unitary matrices of $SU(2)$:

$$u(\vec{n},\psi),\quad u(-\vec{n},2\pi-\psi) = -u(\vec{n},\psi)$$

are mapped to the same rotation matrix $R(\vec{n},\psi)$ in $SO(3)$.

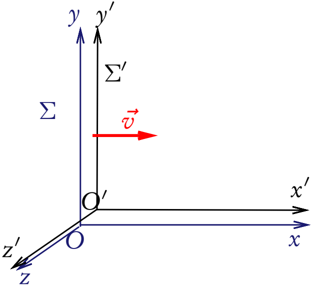
Lorentz group $SO(3,1)$:

The genuine Lorentz transformations (LTs), called **boost**, are those connecting two inertial frames moving with a relative speed v.

If the relative motion ia along the common x_1-direction, boost is:

$$
\begin{aligned}
x_1' &= \gamma(x_1 - \beta ct) \\
x_2' &= x_2 \\
x_3' &= x_3 \\
ct' &= \gamma(ct - \beta x_1)
\end{aligned}
$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.



Introduce the so-called boost parameter ζ by setting,

$$\gamma = \cosh\zeta\,, \qquad \gamma\beta = -\sinh\zeta\,.$$

Genuine LTs can be viewed as pseudo-orthogonal transformations in 4-dimensional Minkowski space $\mathbb{M}_4$,

$$\begin{bmatrix} ct' \\ x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

As expected,

$$\cosh^2\zeta - \sinh^2\zeta = \gamma^2 - \gamma^2\beta^2 = \left[\frac{1}{\sqrt{1-\beta^2}}\right]^2 (1-\beta^2) = 1$$

- The characteristic of Lorentz transformations is that they preserve the invariance of the **interval:**

$$S^2 = x_1^2 + x_2^2 + x_3^2 - c^2t^2 = x_1'^2 + x_2'^2 + x_3'^2 - c^2t'^2$$

The boost matrix

$$B = \begin{bmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are not orthogonal matrices, $BB^T \neq 1$. However, by introducing the metric matrix η in $\mathbb{M}_4$,

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we have:

$$B^{-1} = \eta B^T \eta = \begin{bmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let

$$X = \begin{bmatrix} ct \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

the boosts and the **interval** can be expressed as

$$X' = BX, \quad S^2 = X^T \eta X$$

The interval invariance under the boosts is then manifest,

$$\begin{aligned} S'^2 &= X'^T \eta X' = X^T B^T \eta B X \\ &= X^T \eta (\eta B^T \eta) B X = X^T \eta B^{-1} B X = X^T \eta X = S^2 \end{aligned}$$

The general form of boosts reads,

$$\begin{cases} ct' &= \gamma(ct - \vec{\beta}\cdot\vec{x}) \\ \vec{x}' &= -\gamma\vec{\beta}ct + \vec{x} + \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{x}) \end{cases}$$

Thereby,

$$B = \begin{bmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & 1+\frac{\gamma^2\beta_1^2}{\gamma+1} & \frac{\gamma^2\beta_1\beta_2}{\gamma+1} & \frac{\gamma^2\beta_1\beta_3}{\gamma+1} \\ -\gamma\beta_2 & \frac{\gamma^2\beta_2\beta_1}{\gamma+1} & 1+\frac{\gamma^2\beta_2^2}{\gamma+1} & \frac{\gamma^2\beta_2\beta_3}{\gamma+1} \\ -\gamma\beta_3 & \frac{\gamma^2\beta_3\beta_1}{\gamma+1} & \frac{\gamma^2\beta_3\beta_2}{\gamma+1} & 1+\frac{\gamma^2\beta_3^2}{\gamma+1} \end{bmatrix}$$

- Describing an arbitrary boost requires 3 real independent parameters.
- These parameters can be chosen as β_a $(a = 1, 2, 3)$.

Using these parameters, the infinitesimal Lorentz boosts can be cast as,

$$B \approx 1 + \beta_a \frac{\partial B}{\partial \beta_a}|_{\vec{\beta}=0} = 1 + i\beta_a K_a$$

The generators for Lorentz boost are then:

$$K_a = -i \frac{\partial B}{\partial \beta_a}|_{\vec{\beta}=0}, \qquad (a = 1, 2, 3).$$

Recall $\gamma = 1/\sqrt{1-\beta^2}$. We have,

$$\frac{\partial \gamma}{\partial \beta_a} = -\gamma^3 \beta_a$$

This formula enables us to find out the explicit matrices of the boost generators:

$$K_1 = -i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K_2 = -i \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_3 = -i \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Obviously, these generators are not hermitian matrices:

$$K_a^\dagger = -K_a.$$

In terms of matrix elements, these boost generators have the form:

$$(K_a)_{\mu\nu} = -i(\delta_{\mu 0}\delta_{\nu a} + \delta_{\mu a}\delta_{\nu 0}), \qquad (a = 1, 2, 3).$$

Therefore,

$$\begin{aligned}
[K_a,\ K_b]_{\mu\nu} &= (K_a)_{\mu\rho}(K_b)_{\rho\nu} - (K_b)_{\mu\rho}(K_a)_{\rho\nu} \\
&= -(\delta_{\mu 0}\delta_{\rho a} + \delta_{\mu a}\delta_{\rho 0})(\delta_{\rho 0}\delta_{\nu b} + \delta_{\rho b}\delta_{\nu 0}) \\
&\qquad +(\delta_{\mu 0}\delta_{\rho b} + \delta_{\mu b}\delta_{\rho 0})(\delta_{\rho 0}\delta_{\nu a} + \delta_{\rho a}\delta_{\nu 0}) \\
&= -(\delta_{a\mu}\delta_{b\nu} - \delta_{a\nu}\delta_{b\mu})
\end{aligned}$$

Namely,

$$\begin{aligned}
[K_a,\ K_b]_{\mu 0} &= 0, \\
[K_a,\ K_b]_{0\nu} &= 0, \\
[K_a,\ K_b]_{de} &= -(\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}) = -\epsilon_{abc}\epsilon_{cde}
\end{aligned}$$

Introducing 4×4 matrices $(J_a)_{\mu\nu}$ $(a = 1, 2, 3)$ by,

$$(J_a)_{\mu 0} = (J_a)_{0\nu} = 0, \qquad (J_a)_{bc} = -i\epsilon_{abc}$$

then,

$$[K_a,\ K_b]_{\mu\nu} = -i\epsilon_{abc}(J_c)_{\mu\nu} \quad \rightsquigarrow \quad [K_a,\ K_b] = -i\epsilon_{abc}J_c$$

We see that the genuine Lorentz boosts do not form a group.

$so(3,1)$ algebra :

The above matrix J_a $(a = 1, 2, 3)$ can be written into compact forms,

$$(J_a)_{\mu\nu} = -\frac{i}{2}\epsilon_{abc}\left[\delta_{b\mu}\delta_{c\nu} - \delta_{b\nu}\delta_{c\mu}\right]$$

- Each J_a is purely imaginary and antisymmetric. So, all three J_a's are hermitian matrices.
- In fact, J_a are generators of 3-d rotations in 4-dimensional Minkowski space.

Together with the boost generators K_a $(a = 1, 2, 3)$, these six traceless matrices form a closed algebra under Lie brackets,

$$\left\{ \begin{array}{l} [K_a,\ K_b] = -i\epsilon_{abc}J_c \\ [K_a,\ J_b] = i\epsilon_{abc}K_c \\ [J_a,\ K_b] = i\epsilon_{abc}K_c \\ [J_a,\ J_b] = i\epsilon_{abc}J_c \end{array} \right.$$

It is called Lorentz algebra or $so(3,1)$ algebra.

$so(3,1) \sim su(2) \times su(2)$:

We can redefine the hermitian generators of Lorentz group $SO(3,1)$ as follows:

$$J_a^{\pm} = \frac{1}{2}\Big[J_a \pm iK_a\Big] \qquad (a = 1, 2, 3).$$

Evidently,

$$(J_a^{\pm})^{\dagger} = \frac{1}{2}\Big[J_a^{\dagger} \mp iK_a^{\dagger}\Big] = \frac{1}{2}\Big[J_a \pm iK_a\Big] = J_a^{\pm}$$

With these hermitian generators, $so(3,1)$ algebra becomes,

$$\begin{aligned} [J_a^{+},\ J_b^{+}] &= i\epsilon_{abc}J_c^{+} \\ [J_a^{-},\ J_b^{-}] &= i\epsilon_{abc}J_c^{-} \\ [J_a^{+},\ J_b^{-}] &= 0 \end{aligned}$$

This shows that $\{J_a^{+}\}$ and $\{J_a^{-}\}$ each generate a group $SU(2)$, and the two groups commute.

Hence the Lorentz algebra $so(3,1)$ is equivalent to two copies of $su(2)$,

$$so(3,1) \sim su(2) \times su(2)$$

$SO(3,1)$ group elements:

In terms of the *exponential* parameterization, the group elements of Lorentz group $SO(3,1)$ are expressed as:

$$D(\boldsymbol{\theta}, \boldsymbol{\lambda}) = \exp\left[-i\sum_{a=1}^{3}(\theta_a J_a + \lambda_a K_a)\right]$$

in some finite-dimensional representations. Surprisingly, each of them is a direct product of two $SU(2)$ group elements in their non-unitary representations:

$$D(\boldsymbol{\theta}, \boldsymbol{\lambda}) = e^{-i(\theta_a - i\lambda_a)J_a^+}\, e^{-i(\theta_a + i\lambda_a)J_a^-}$$

Homework:

1. The generators of Lorentz group $SO(3,1)$ are

$$(K_a)_{\mu\nu} = -i\Big[\delta_{\mu 0}\delta_{\nu a} + \delta_{\mu a}\delta_{\nu 0}\Big]$$
$$(J_a)_{\mu\nu} = -\frac{i}{2}\epsilon_{abc}\Big[\delta_{b\mu}\delta_{c\nu} - \delta_{b\nu}\delta_{c\mu}\Big]$$

   where $a, b, c = 1, 2, 3$ but $\mu, \nu = 0, 1, 2, 3$.
   Please check the $so(3,1)$ algebra by computing all possible Lie brackets.