CHAPTER 5

Stochastic Differential Equations

1. Simplest stochastic differential equations

In this section we discuss a stochastic differential equation of a very simple type.

Let M be a martingale in and A a process of bounded variation. Let a and b be two real-valued functions and consider the following stochastic differential equation

$$dX_t = a(X_t)\, dM_t + b(X_t)\, dA_t, \quad X_0 = x.$$

The meaning of this equation is the following: we are looking for a semimartingale X such that

$$X_t = x + \int_0^t a(X_s)\, dM_s + \int_0^t b(X_s)\, dAs.$$

Thus the equation we want to solve is a stochastic differential equation question only in name; it is more appropriate to be called a stochastic integral equation, but nobody does that. Note that diffusion coefficient $a(X_s)$ and drift coefficient $b(X_s)$ are functions of the current position X_s of the solution rather than functions of the path X. This type of stochastic equation is often called an Itô type stochastic differential equation. In the most popular case M is a Brownian motion and A is the time itself, i.e.,

$$dX_t = a(X_t)\, dB_t + b(X_t)\, dt, \qquad X_0 = x.$$

This is a stochastic differential equation driven by the Brownian motion B with starting point x. The handling of the general form of equations we formulated above is only slightly more complicated (by a time change) than that of this special case.

The main theorem below concerns the existence and uniqueness of Itô type stochastic differential equation under global Lipschtiz condition on the coefficients.

THEOREM 1.1. *Assume that $a(x), b(x)$ satisfy the global Lipschitz condition: there exists a constant K such that*

$$|a(x) - a(y)| + |b(x) - b(y)| \leq K|x - y|$$

for all x, y. Then the stochastic differential equation

$$dX_t = a(X_t)\, dM_t + b(X_t)\, dA_t, \quad X_0 = x$$

has a unique solution.

PROOF. The stochastic differential equation looks very much like an ordinary differential equation: $dx_t = b(x_t)\,dt$. In fact this is a special case of the general stochastic differential equation formulated above. Recall that ordinary differential equations of this type can be solved by Picard's iteration. The same method can be used to solve the stochastic differential equation.

As will be seen below, it will be a great convenience if the quadratic variation $\langle M, M\rangle$ and the total variation $|A|$ are absolutely continuous with respect to a fixed non-random increasing function. This is the case if M is a Brownian motion and A is the time. In general we can achieve this by a time change. Let

$$\phi(t) = \langle M, M\rangle_t + |A|_t + t.$$

The random function ϕ is continuous and strictly increasing, the latter property being one of the reasons we have added t in the definition of $\phi(t)$. We denote its (pathwise) inverse function by τ. It is easy to see that $\tau(t)$ is a stopping time for each fixed t. Furthermore, $t \mapsto \tau(t)$ is absolutely continuous with respect to t, which is the second reason we have added t in the definition of $\phi(t)$. We can write this property as $d\tau(t) \le dt$. We make a time change to all the processes involved; namely,

$$\tilde{M}_t = M_{\tau(t)}, \quad \tilde{A}_t = A_{\tau(t)}, \quad \tilde{X}_t = X_{\tau(t)}.$$

Then we have

$$d\langle \tilde{M}, \tilde{M}\rangle_t \le dt \quad \text{and} \quad d|A|_t \le dt.$$

The time changed equation is

$$d\tilde{X}_t = a(\tilde{X}_t)\,d\tilde{M}_t + b(\tilde{X}_t)\,d\tilde{A}_t, \quad \tilde{X}_0 = x.$$

If the existence and uniqueness hold for this equation for the semimartingale $\tilde{X}$ (with respect to the time changed filtration), then they also hold for the original equation of for X, and vice versa. Thus without loss of generality we may assume that

$$d\langle M, M\rangle_t \le dt, \quad \text{and} \quad d|A|_t \le dt \tag{1.1}$$

for the original equation. The advantage of these assumptions will be clear in the course of the proof.

We will prove the uniqueness first. Suppose that that Y is another solution,

$$dY_t = a(Y_t)\,dM_t + b(Y_t)\,dA_t, \quad Y_0 = x.$$

Subtracting the two equations, squaring the result, using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ on the right side, and taking the expectation, we have

$$\begin{aligned}
E|Y_t - X_t|^2 \leq & 2\mathbb{E}\left(\int_0^t \{a(Y_s) - a(X_s)\}\, dM_s\right)^2 \\
& + 2\mathbb{E}\left(\int_0^t \{b(Y_s) - b(X_s)\}\, dA_s\right)^2 \\
\leq & 2\mathbb{E}\int_0^t |a(Y_s) - a(X_s)|^2 d\langle M, M\rangle_s \\
& + 2\mathbb{E}\left(|A|_t \int_0^t |b(Y_s) - b(X_s)|^2\, d|A|_s\right) \\
\leq & 2K^2\mathbb{E}\int_0^t |Y_s - X_s|^2\, dt + 2K^2 t\mathbb{E}\int_0^t |Y_s - X_s|^2\, ds.
\end{aligned}$$

Note that we have taken the advantage of the assumptions (1.1) and the fact that $|A|_t \leq t$. Now let $c(t) = \mathbb{E}\left[|Y_t - X_t|^2\right]$. For $t \leq T$, we have

$$c(t) \leq C_T \int_0^t c(s)\, ds.$$

with $C_T = 2K^2(1+T)$. From this inequality it follows immediately that $c(t) = 0$, which proves the uniqueness.

We now prove the existence by iteration. Let $X_t^0 = x$ and define

$$X_t^n = x + \int_0^t a(X_s^{n-1})\, dM_s + \int_0^t b(X_s^{n-1})\, dA_s. \tag{1.2}$$

We prove that X^n converges to a process which is a solution of the equation. Consider the difference $X_t^{n+1} - X_t^n$. We have as before

$$\begin{aligned}
\max_{0\leq s\leq t} |X_s^{n+1} - X_s^n| \leq & 2\max_{0\leq s\leq t}\left(\int_0^s \left\{a(X_u^n) - a(X_u^{n-1})\right\}\, dM_u\right)^2 \\
& + 2TK^2 \int_0^t \max_{0\leq u\leq s} |X_u^n - X_u^{n-1}|^2\, ds.
\end{aligned}$$

It follows from Doob's martingale moment inequality we have for $t \leq T$

$$\eta_{n+1}(t) \leq D_T \int_0^t \eta_n(s)\, ds,$$

where

$$\eta_n(t) = \mathbb{E}\left[\max_{0\leq s\leq t} |X_s^n - X_s^{n-1}|^2\right]$$

and $D_T = 2K^2(4+T)$. Let $C = \eta_1(T)$. We have by the recursive inequality for $\eta_n(t)$,

$$\eta_n(T) \leq \frac{C(TD_T)^{n-1}}{(n-1)!}.$$

From this we have by the Markov inequality,

$$\sum_{n=1}^{\infty} \mathbb{P}\left[\max_{0\le s\le T} |X_s^n - X_s^{n-1}| \ge 2^{-n}\right] < \infty.$$

By the easy part of the Borel–Cantelli lemma, we conclude that the limit $\lim_{n\to\infty} X_t^n = X_t$ exists and the limiting process X_t is continuous. Since the convergence is also in L^2 sense, we may take the limit in the recursive relation (1.2) and obtain

$$X_t = x + \int_0^t a(X_s)\, dM_s + \int_0^t b(X_s)\, dA_s.$$

Thus the limit process is a continuous semimartingale which satisfies the equation. The existence has been proved. □

Although we have proved the existence and uniqueness for one dimensional Itô type stochastic differential equations only, the theorem and the proof itself can be easily generalized to multi-dimensional setting. We point out that the dimensions of the driving process and the solution semimartingale are not necessarily the same. A general Itô type stochastic differential equation has the form

$$dX_t = \sigma(X_t)\, dZ_t,$$

where $\sigma : \mathbb{R}^n \to \mathscr{M}(n, m)$ (the space of $(n \times m)$ matrices, Z an $\mathbb{R}^m$-valued semimartingale, and the solution is an $\mathbb{R}^n$-valued semimartingale. If σ is globally Lipschitz, then the equation has a unique solution. For example, if we take $Z = (M, A)^T$ and $\sigma = (\sigma_1, b)$, then this general form reduces to the one dimensional case considered above. Note that for the proof, we need to separate the martingale and bounded variation part of the semimartingale Z and treat them differently, as we have done in the proof of the above theorem.

2. Locally Lipschitz coefficients. Explosion.

In applications the coefficients of a stochastic differential equation may only be locally Lipschitz. A function $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ from a euclidean space to another is called locally Lipschitz if for any positive integer N there is a constant C_N such that

$$|\sigma(x) - \sigma(y)| \le C_N |x - y|$$

for all $|x| \le N$ and $|y| \le N$. A globally Lipschitz function grows at most linearly, but this is not so for a locally Lipschitz function. For this reason, we have to allow the possibility of explosion of a stochastic differential equation.

For a continuous function $x : [0, e) \to \mathbb{R}^d$ we say that e is the explosion time for x if either $e = \infty$ or $\lim_{t\uparrow e} |x_t| = \infty$. The explosion time of a path x is denoted by $e(x)$. Let τ be a stopping time and X a continuous process defined on the time interval $[0, \tau)$. If there exists a sequence of stopping times $\tau_n \uparrow \tau$ such that each stopped process $X_t^{\tau_n} = X_{t\wedge\tau_n}$ is a semimartingale for each n, then X is called a semimartingale up to time τ, or a semimartingale defined on $[0, \tau)$. Consider a stochastic differential equation

$$dX_t = \sigma(X_t)\, dZ_t,$$

where the semimartingale Z is defined up to a stopping time τ. A semimartingale X up to a stopping time τ is a solution of the stochastic differential equation if

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dZ_s, \qquad t < \tau.$$

Note that if X is a semimartingale defined up to time τ, then the stochastic integral on the right side makes sense for all $t < \tau$. Equivalently, X is a solution if there exists a sequence of stopping times $\tau_n \uparrow \tau$ such that for all $t \geq 0$ and all n,

$$X_{t\wedge\tau_n} = X_0 + \int_0^{t\wedge\tau_n} \sigma(X_s) dZ_s.$$

It now makes sense to speak of a solution X of an Itô type stochastic differential equation up to its explosion time. The main theorem is as follows.

THEOREM 2.1. *Suppose that that σ is locally Lipschitz and Z is a semimartingale (defined for all time). Then there is a unique solution to the stochastic differential equation*

$$dX_t = \sigma(X_t)\, dZ_t$$

up to its explosion time.

PROOF. We divide the proof into several steps. We first assume that X_0 is uniformly bounded, say, by N_0. In the last step we remove this restriction.

(a) *Construction of a solution.* For a fixed positive integer $N \geq N_0$, let $\sigma^N : \mathbb{R}^d \to \mathbb{R}^{d\times l}$ be globally Lipschitz such that $\sigma^N(z) = \sigma(z)$ for $z \in B(0; N)$, the ball of radius N centered at the origin. Consider the equation

$$X_t^N = X_0 + \int_0^t \sigma^N\left(X_s^N\right) dZ_s. \tag{2.1}$$

This equation has a unique solution. We show that X^N and X^{N+1} coincides before either one of them wanders more than a distance of N from the origin. Let

$$\tau_N = \inf\left\{t \geq 0 : |X_t^N| \text{ or } |X_t^{N+1}| = N\right\}.$$

Since $\sigma^N = \sigma^{N+1}$ on $B(0; N)$, both X^{N,τ_N} and X^{N+1,τ_N} (the two processes X^N and X^{N+1} stopped at τ_N) are solutions of the equation

$$Y_t = X_0 + \int_0^t \sigma^N(Y_s)\, dZ_s^{\tau_N}. \tag{2.2}$$

By uniqueness we have $X^{N,\tau_N} = X^{N+1,\tau_N}$, hence $X_t^N = X_t^{N+1}$ for $0 \leq t \leq< \tau_N$, and τ_N is the first time the common process reaches a distance of N from the origin. It is clear that $\tau_N \leq \tau_{N+1}$.

Let $e = \lim_{N\uparrow\infty} \tau_N$, and define a semimartingale X on $[0, e)$ by $X_t = X_t^N$ for $0 \le t < \tau_N$. We have from (2.1)

$$X_t = X_0 + \int_0^t \sigma^N (X_s)\, dZ_s, \qquad 0 \le t < \tau_N.$$

Therefore X is a solution of the stochastic differential equation up to time e.

(b) *e is the explosion time for X.* This is the key step of the proof. This means that almost surely, either $e = \infty$ or $e < \infty$ and $\lim_{t\uparrow e} |X_t| = \infty$. Equivalently, if $e < \infty$, then for each fixed positive $R \ge N_0$, then there exists a $t_R < e$ such that $|X_t| \ge R$ for all $t_R \le t < e$.

The idea of the proof is as follows. Because on the ball $B(0; R+1)$ the coefficients of the equation are bounded, X needs to spend at least a fixed amount of time (in an appropriate probabilistic sense) when it crosses from $\partial B(0; R)$ to $\partial B(0; R+1)$. If $e < \infty$, this can happen only finitely many times. Thus after some time, X either never returns to $B(0; R)$ or stays inside $B(0; R+1)$ forever; but the second possibility contradicts the facts that $|X_{\tau_N}| = N$ and $\tau_N \uparrow e$ as $N \uparrow \infty$.

To proceed rigorously, define two sequences $\{\eta_n\}$ and $\{\zeta_n\}$ of stopping times inductively by

$$\begin{aligned} \zeta_0 &= 0, \\ \eta_n &= \inf\{t > \zeta_{n-1} : |X_t| = R\}, \\ \zeta_n &= \inf\{t > \eta_n : |X_t| = R+1\}, \end{aligned}$$

with the convention that $\inf \emptyset = e$. If $\zeta_n < e$, the difference $\zeta_n - \eta_n$ is the time X takes to cross from $\partial B(0; R)$ to $\partial B(0; R+1)$ for the nth time. By Itô's formula applied to the function $f(x) = |x|^2$ we have for $t < e$,

$$|X_t|^2 = |X_0|^2 + N_t + \int_0^t \Psi_s dQ_s, \tag{2.3}$$

where

$$\begin{aligned} N_t &= 2\int_0^t \sigma_\alpha^i(X_s) X_s^i \, dM_s^\alpha \\ \Psi_s &= 2\sigma_\alpha^i(X_s) X_s^i \frac{dA_s^\alpha}{dQ_s} + \sigma_\alpha^i(X_s)\sigma_\beta^i(X_s)\frac{d\langle M^\alpha, M^\beta\rangle_s}{dQ_s} \\ \langle N, N\rangle_t &= \int_0^t \Phi_s dQ_s \\ \Phi_s &= 4\sigma_\alpha^i(X_s)\sigma_\beta^j(X_s) X_s^i X_s^j \frac{d\langle M^\alpha, M^\beta\rangle_s}{dQ_s}. \end{aligned}$$

Here we have assumed that $Z = M + A$ is the Dooby-Meyer decomposition of the semimartingale Z and By Lévy's criterion, there exists a one-dimensional Brownian motion W such that

$$N_{s+\eta_n} - N_{\eta_n} = W_{\langle N,N\rangle_{s+\eta_n} - \langle N,N\rangle_{\eta_n}}.$$

When $\eta_n \le s \le \zeta_n$ we have $|X_s| \le R+1$, hence there is a constant C depending on R such that $|\Psi_s| \le C$ and $|\Phi_s| \le C$. From (2.3) we have

$$\begin{aligned} 1 &\le |X_{\zeta_n}|^2 - |X_{\eta_n}|^2 \\ &= W_{\langle N,N\rangle_{\zeta_n} - \langle N,N\rangle_{\eta_n}} + \int_{\eta_n}^{\zeta_n} \Psi_s dQ_s \\ &\le W^*_{\langle N,N\rangle_{\zeta_n} - \langle N,N\rangle_{\eta_n}} + C\left(Q_{\zeta_n} - Q_{\eta_n}\right), \end{aligned}$$

where $W_t^* = \max_{0\le s\le t}|W_s|$, and

$$\langle N,N\rangle_{\zeta_n} - \langle N,N\rangle_{\eta_n} \le C(Q_{\zeta_n} - Q_{\eta_n}).$$

Now it is clear that $\zeta_n < e$ and $Q_{\zeta_n} - Q_{\eta_n} \le (Cn)^{-1}$ imply that

$$W^*_{1/n} \ge 1 - \frac{1}{n} \ge \frac{1}{2},$$

and

$$\begin{aligned} P\left\{\zeta_n < e, Q_{\zeta_n} - Q_{\eta_n} \le \frac{1}{Cn}\right\} &\le P\left\{W^*_{1/n} \ge 1/2\right\} \\ &= \sqrt{\frac{2n}{\pi}} \int_{1/2}^{\infty} e^{-nu^2/2} du \le \sqrt{\frac{8}{\pi n}} e^{-n^2/8}. \end{aligned}$$

By the Borel-Cantelli lemma, we have either $\zeta_n = e$ for some n or $\zeta_n < e$ and $Q_{\zeta_n} - Q_{\eta_n} \ge (Cn)^{-1}$ for all sufficiently large n. The second possibility implies that

$$Q_{\zeta_n} \ge \sum_{m=1}^{n-1} (Q_{\zeta_m} - Q_{\eta_m}) \to \infty.$$

This implies in turn that $\zeta_n \uparrow \infty$ and $e > \zeta_n \to \infty$. Thus if $e < \infty$, we must have $\zeta_n = e$ for some n. Let ζ_{n_0} be the last ζ_n strictly less than e. Then X never returns to $B(0;R)$ for $\zeta_{n_0} \le t < e$. This shows that e is indeed the explosion time of X.

(c) *Uniqueness.* Suppose that Y is another solution up to its explosion time. Let τ_N be the first time either $|X_t|$ or $|Y_t|$ is equal to N. Then X and Y stopped at time τ_N are solutions of the equation (see (2.2)). By uniqueness we have $X_t = Y_t$ for $0 \le t < \tau_N$ and τ_N is the time the common process first reaches a distance N from the origin. Hence

$$e(X) = e(Y) = \lim_{N\uparrow\infty} \tau_N$$

and $X_t = Y_t$ for all $0 \le t < e(X)$.

(d) *General initial condition.* For a general X_0 let $\Omega_N = \{|X_0| \le N\}$ and X^N the solution of the equation with the initial condition $X_0 I_{\Omega_N}$). Define a new probability measure by $\mathbb{Q}^N(C) = \mathbb{P}(C\cap\Omega_N)/\mathbb{P}(\Omega_N)$. Since $\Omega_N \in \mathscr{F}_0$, both X^N and X^{N+1} are solutions to the same equation under Q^N but now also with the same initial condition $X_0 I_{\Omega_N}$. Hence by Part (c) they must coincide, i.e., $X^N = X^{N+1}$ and $e(X^N) = e(X^{N+1})$ on Ω_N. In view of the fact that $\mathbb{P}(\Omega_N) \uparrow 1$ as $N \uparrow \infty$, we can define $X = X^N$ and $e = e(X^N)$ on Ω_N. Then it is clear that X is a semimartingale and satisfies the SDE up to its explosion time. The uniqueness follows from the observation that if Y is another solution, then it must be a solution to the SDE with the initial condition $X_0 I_{\Omega_N}$ under $\mathbb{Q}^N$. Thus it must coincide with X on the set Ω_N for all N. $\square$

For future reference we need the following slightly more general form of uniqueness. We will leave its proof to the reader.

PROPOSITION 2.2. *Suppose that σ is locally Lipschitz. Let X and Y be two solutions of the stochastic differential equation*

$$dX_t = \sigma(X_t)\, dZ_t$$

up to stopping times τ and η, respectively, with the same initial condition. Then $X_t = Y_t$ for $0 \leq t < \tau \wedge \eta$. In particular, if X is a solution up to its explosion time $\tau = e(X)$, then $\eta \leq e(X)$ and $X_t = Y_t$ for $0 \leq t < \eta$.

The following result gives a well known sufficient condition for non-explosion.

PROPOSITION 2.3. *If σ is locally Lipschitz and there is a constant C such that $|\sigma(x)| \leq C(1 + |x|)$, then solutions of the stochastic differential equation of the type*

$$dX_t = \sigma(X_t)\, dZ_t$$

do not explode.

PROOF. We may assume that X_0 is uniformly bounded (see Part (d)) of the proof of THEOREM 2.1). Let η_t be defined as before and let $\tau_N = \inf\{t > 0 : |X_t| = N\}$. We have

$$X_t = X_0 + \int_0^t \sigma(X_s) dZ_s$$

we have

$$\begin{aligned}
\mathbb{E}|X|^2_{\infty,\eta_T \wedge \tau_N} &\leq 2\mathbb{E}|X_0|^2 + C\mathbb{E}\int_0^{\eta_T \wedge \tau_N} \left\{1 + |X_s|^2\right\} dQ_s \\
&\leq 2\mathbb{E}|X_0|^2 + CT + C\int_0^T \mathbb{E}|X|^2_{\infty,\eta_s \wedge \tau_N} ds,
\end{aligned}$$

Hence by Gronwall's inequality

$$\mathbb{E}|X|^2_{\infty,\eta_T \wedge \tau_N} \leq \left(2\mathbb{E}|X_0|^2 + CT\right) e^{CT}.$$

Letting $N \uparrow \infty$ we see that $|X|_{\infty,\eta_T \wedge e(X)} < \infty$, a.s. This implies $\eta_T \wedge e(X) < e(X)$, hence $\eta_T < e(X)$. Now $e(X) = \infty$ follows from the fact that $\eta_T \uparrow \infty$ as $T \uparrow \infty$. $\square$

EXAMPLE 2.4. *Accelerated Brownian motion.* According to PROPOSITION 2.3, if σ has at most linear growth, then no explosion is possible for any driving semimartingales Z and initial condition X_0. However, explosion or non-explosion is not a property of σ alone. It may happen that if Z is intrinsically slow, there will be no explosion no matter how fast σ grows. To see this consider the equation

$$dX_t = \sigma(X_t)dB_t, \quad X_0 = 0, \tag{2.4}$$

where B be a d-dimensional Brownian motion and σ a positive, locally Lipschitz function on $\mathbb{R}^d$. When $d = 1$ or 2, there will be no explosion regardless what σ is. The reason is, of course, that Brownian motion itself is recurrent for in these dimensions. To see this more clearly let

$$\phi_t = \int_0^t \sigma(X_s)^2 ds, \tag{2.5}$$

and $\tau : [0, \phi_\infty) \to [0, e(X))$ the inverse function of $\phi : [0, e(X)) \to [0, \phi_\infty)$. Then $W_t = X_{\tau_t}$ is a Brownian motion. If the original process $X_t = W_{\phi_t}$ explodes, then $|X_t| \to \infty$ as $t \uparrow e(X) < \infty$ and this cannot happen in dimensions 1 and 2 because W is recurrent in these dimensions. In dimensions 3 and higher, W is transient: $|W_t| \to \infty$ as $t \to \infty$. Thus X explodes if and only if ϕ explodes, and this happens only when τ_∞ is finite. By the law of iterated logarithm, $|W_t| \leq C\sqrt{t \log\log t}$ for large t. Thus if σ grows at most linearly, we have

$$\tau_t = \int_0^t \frac{ds}{a(W_s)^2} \geq C \int_{t_0}^t \frac{ds}{s \log\log s} \to \infty, \quad \text{as } t \to \infty$$

and X does not explode, in agreement with PROPOSITION 2.3.

So far we have not consider the possibility that the driving semimartingale Z may be defined only up to a stopping time τ.

THEOREM 2.5. *Let Z be a semimartingale defined up to a stopping time τ. Then the stochastic differential equation*

$$dX_t = \sigma(X_t)\, dZ_t$$

has a unique solution X up to the stopping time $e(X) \wedge \tau$. If Y is another solution up to a stopping time $\eta \leq \tau$, then $\eta \leq e(X) \wedge \tau$ and $X_t = Y_t$ for $0 \leq t < e(X) \wedge \eta$.

3. Weak solution and weak uniqueness

For simplicity, we will denote a general stochastic differential equation

$$dX_t = \sigma(X_t)\, dZ_t$$

with initial condition X_0 by $SDE(\sigma, Z, X_0)$. Stochastic differential equations of this type makes sense at least for continuous coefficients and can be solved for coefficients much more general than Lipschitz ones if they are interpreted appropriately. Uniqueness for such equations is in general a complicated question. In contrast to the pathwise uniqueness , the weak uniqueness (also called uniqueness in law) asserts roughly that if (Z, X_0) and $(\hat{Z}, \hat{X}_0)$ have the same law, then the solutions X and $\hat{X}$ of $SDE(\sigma, Z, X_0)$ and $SDE(\sigma, \hat{Z}, \hat{X}_0)$ (not necessarily on the same probability space or with the same filtration) also have the same law. This concept is most useful when we deal with classical stochastic differential equations of Itô type, those driven by multi-dimensional Brownian motion: $Z = (W, t), \sigma = (\tilde{\sigma}, \tilde{b})$, where W is an euclidean Brownian motion. In this case the equation can be written in the form

$$X_t = X_0 + \int_0^t \sigma(X_s) dW_s + \int_0^t b(X_s) ds. \tag{3.1}$$

We will need the weak uniqueness for this type of equations when we discuss the uniqueness of diffusion measures generated by second order elliptic operators. In the following $W(\mathbb{R}^d)$ denotes the space of continuous functions $x : [0, e(x)) \to \mathbb{R}^d$ such that $\lim_{t \uparrow e(x)} |x_t| = \infty$. Our main result is that for such equations weak uniqueness holds.

PROPOSITION 3.1. *Suppose that σ is locally Lipschitz. Then the weak uniqueness holds for the stochastic differential equation (3.1). More precisely, let $\hat{X}$ be the solution of*

$$\hat{X}_t = \hat{X}_0 + \int_0^t \sigma(\hat{X}_s) d\hat{W}_s + \int_0^t b(\hat{X}_s) ds, \tag{3.2}$$

where $\hat{W}$ is a Brownian motion, possibly on a different filtered probability space $(\hat{\Omega}, \hat{\mathscr{F}}_*, \tilde{\mathbb{P}})$ and $\hat{X}_0$ and X_0 have the same distribution, then $\hat{X}$ and X have the same law.

PROOF. The idea is to pass to the product probability space

$$(\mathbb{R}^d \times W(\mathbb{R}^l), \mathscr{B}(\mathbb{R}^d) \times \mathscr{B}(W(\mathbb{R}^l)), \mathscr{B}(\mathbb{R}^d) \times \mathscr{B}(W(\mathbb{R}^l))_*, \mu_0 \times \mu^W),$$

where μ_0 is the common distribution of X_0 and $\hat{X}_0$, and μ^W is the law of l-dimensional Brownian motion. A point in this space is denoted by (x_0, w). The stochastic differential equation on this space

$$\phi_t = x_0 + \int_0^t \sigma(\phi_s) dw_s + \int_0^t b(\phi_s) ds \tag{3.3}$$

has a unique solution $\phi = \phi(x_0, w)$ up to its explosion time. Now $\phi : \mathbb{R}^d \times W(\mathbb{R}^l) \to W(\mathbb{R}^d)$ is a measurable map defined $(\mu_0 \times \mu^W)$-almost everywhere. Since the law of (X_0, B) is $\mu_0 \times \mu^W$, the composition $\phi(X_0, W) : \Omega \to W(\mathbb{R}^d)$ is well defined. By (3.3), $\phi(X_0, W)$ as a stochastic process satisfies (3.1). It should be pointed out that although the stochastic integral on the right side is not defined path by path, the above passage from (3.3) to (3.1) is valid because the stochastic integral is the limit (in probability) of an appropriate sequence of discrete approximations, e.g.,

$$\sum_{j=0}^{m} \sigma(\phi_{jt/m}) \left\{ w_{(j+1)t/m} - w_{jt/m} \right\} \to \int_0^t \sigma(\phi_s) dw_s.$$

Now by the pathwise uniqueness for (3.1) we have $X = \phi(X_0, W)$. Likewise we have $\hat{X} = \phi(\hat{X}_0, \hat{W})$. Because (X_0, W) and $(\hat{X}_0, \hat{W})$ have the same law, we conclude that X and $\hat{X}$ must also have the same law. □

4. Stratonovich formulation

We now turn to the Stratonovich formulation of stochastic differential equations. The advantage of this formulation is that Itô's formula appears in the same form as the fundamental theorem of calculus and stochastic calculus in this formulation takes a more familiar form (compare (4.2)). This is a very convenient feature when we study stochastic differential equations under different coordinate systems (e.g., on a differentiable manifold). However, it often happens that useful probabilistic and geometric information reveals itself only after we separate martingale and bounded variation components of the equation under consideration and its solutions.

Suppose that $V_\alpha, \alpha = 1, \ldots, l$ are smooth vector fields on $\mathbb{R}^d$. Each V_α can be regarded as a smooth function $V_\alpha : \mathbb{R}^d \to \mathbb{R}^d$ so that $V = (V_1, \ldots, V_l)$ is an $\mathscr{M}(d, 1)$-valued function on $\mathbb{R}^d$. Let Z be a semimartingale and consider a Stratonovich stochastic differential equation

$$X_t = X_0 + \int_0^t V(X_s) \circ dZ_s,$$

where the stochastic integral is in the Stratonovich sense. To emphasize the fact that V is a set of l vector fields, we rewrite the equation as

$$X_t = X_0 + \int_0^t V_\alpha(X_s) \circ dZ_s^\alpha. \tag{4.1}$$

Converting the Stratonovich integral into the equivalent Itô integral, we have

$$X_t = X_0 + \int_0^t V_\alpha(X_s)\, dZ_s^\alpha + \frac{1}{2} \int_0^t \nabla_{V_\beta} V_\alpha(X_s)\, d\langle Z^\alpha, Z^\beta \rangle_s.$$

Here $\nabla_\beta V_\alpha$ is the derivative of V_α along V_β. This is an Itô type stochastic differential equation we have studied before and is equivalent to the original equation (4.1). Itô's formula in this setting becomes the following. We leave its proof as an exercise.

PROPOSITION 4.1. *Let X be a solution to the SDE (4.1) and $f \in C^2(\mathbb{R}^d)$. Then*

$$f(X_t) = f(X_0) + \int_0^t V_\alpha f(X_s) \circ dZ_s, \qquad 0 \le s < e(X). \tag{4.2}$$

5. Stochastic differential equation with reflecting boundary condition

When the space where a solution of a stochastic differential equation lives in has a boundary, we in general have to specify its behavior at the boundary. Normal reflection is the most typical boundary condition. In this section we will consider the simplest case where the space is the half line $\mathbb{R}_+ = [0, \infty)$. A stochastic differential equation with normal reflection has the following form

$$dX_t = a(X_t)\, dM_t + b(X_t)\, dA_t + d\phi_t \quad X_0 = x \geq 0.$$

The meaning of the equation is as follows. As before, a is the diffusion coefficient and b is the drift coefficient. Given a continuous local martingale M and a process of bounded variation A, we look for a semimartingale X such that

$$X_t = x + \int_0^t \sigma(X_s)\, dM_s + \int_0^t b(X_s)\, dA_s + \phi_t,$$

where ϕ is an increasing process which increases only when $X_t = 0$. More precisely, (X_t, ϕ_t) is the solution of the Skorokhod equation for the process

$$\int_0^t a(X_s)\, dM_s + \int_0^t b(X_s)\, dA_s.$$

Note that it is boundary local time term ϕ that keeps the process in the positive half line. Also note that we are seeking not only the semimartingale X but also the boundary local time ϕ at the same time. The solution is the pair (X, ϕ), not just X alone. When $a = 1$, $b = 0$, and M is a Brownian motion, we have the Skorokhod problem discussed in SECTION 7 and the solution (X, ϕ) consists of a reflecting Brownian motion X and its local time ϕ.

The fundamental result of stochastic differential equations with reflection is as follows.

THEOREM 5.1. *Suppose that $a(x), b(x)$ satisfy a global Lipschitz condition, i.e., there exists a constant K such that for all x and y,*

$$|a(x) - a(y)| + |b(x) - b(y)| \leq K|x - y|.$$

Then the stochastic differential equation with reflecting boundary condition

$$dX_t = a(X_t)\, dM_t + b(X_t)\, dA_t + \phi_t, \quad X_0 = x \geq 0$$

has a unique solution (X, ϕ).

For the existence of the solution, we need the explicit solution of the Skorokhod problem. Recall that for a continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$, the Skorokhod equation is

$$x_t = f_t - \min_{0 \leq s \leq t} f_s \wedge 0.$$

Define the Skorokhod map γ from the space $W(\mathbb{R})$ of continuous paths on $\mathbb{R}^1$ to the space $W(\mathbb{R}_+)$ of continuous paths on $\mathbb{R}_+$ by

$$\gamma(f)_t = f_t - \min_{0 \leq s \leq t} f_s \wedge 0.$$

The map γ is Lipschitz continuous in the following sense

$$\max_{0\le s\le t}|\gamma(f)_s-\gamma(g)_s|\le\max_{0\le s\le t}|f_s-g_s|.$$

Consider the following stochastic differential equation:

$$dY_t=\sigma(\gamma(Y)_t)\,dM_t+b(\gamma(Y)_t)\,dA_t,\quad Y_0=x. \tag{5.1}$$

Note that the boundary local time term has been removed from the original equation and the diffusion and drift coefficients are no longer functions of Y_t, the solution at time t. Instead, they depend on the whole past history of the solution. Thus we are not dealing with an Itô type stochastic differential equation. Nevertheless, because of the Lipschitz continuity of the map γ mentioned above, the equation can be solved in exactly the same way as we have done in SECTION 1, using the Lipschitz continuity

$$\max_{0\le s\le t}|\sigma(\gamma(f)_s)-\sigma(\gamma(g)_s)|\le K\max_{0\le s\le t}|f_s-g_s|$$

and

$$\max_{0\le s\le t}|b(\gamma(f)_s)-b(\gamma(g)_s)|\le K\max_{0\le s\le t}|f_s-g_s|.$$

The equation (5.1) can be solved uniquely.

The obvious thing to do now is to set

$$\phi_t=-\min_{0\le s\le t}Y_s\wedge 0\quad\text{and}\quad X_t=Y_t+\phi_t$$

and show that the pair (X,ϕ) is a solution to the original stochastic differential equation with reflection. This is a triviality, for $X_t=\gamma(Y)_t$ and

$$X_t=Y_t+\phi_t=x+\int_0^t\sigma(X_s)\,dM_s+\int_0^t b(X_s)\,dA_s+\phi_t.$$

We have therefore completed the proof of the existence part of THEOREM 5.1.

The proof of uniqueness is quite long, but it follows by and large the idea in the proof of the uniqueness for stochastic differential equation without boundary; see THEOREM 1.1. Let (Y,ψ) be another solution. As was done in SECTION 1, by a time change we may assume that

$$d\langle M,M\rangle_t\le t\quad\text{and}\quad d|A|_t\le dt.$$

We subtract the equation for Y from that for X. The result can be written as

$$X_t-Y_t=Z_t+\phi_t-\psi_t, \tag{5.2}$$

where

$$Z_t=\int_0^t\{a(X_s)-a(Y_s)\}\,dM_s+\int_0^t\{b(X_s)-b(Y_s)\}\,dA_s. \tag{5.3}$$

The first term in (5.2) can be estimated as in the case of stochastic differential equation with boundary condition, but the second local time term needs a slightly different treatment. Instead of squaring the difference, we take the fourth power; the reason for doing so will be clear in later in the proof. We use the inequality

$$(x+y)^4 \leq 2^4 x^4 + 2^4 y^4,$$

take the maximum over the time interval $[0, t]$, and then take the expectation. After these steps we have

$$\mathbb{E} \max_{0 \leq s \leq t} |X_s - Y_s|^4 \leq 2^4 \mathbb{E} \max_{0 \leq s \leq t} |Z_s|^4 + 2^4 \mathbb{E} \max_{0 \leq s \leq t} |\phi_s - \psi_s|^4.$$

Now, separating the martingale part

$$N_t = \int_0^t \{a(X_s) - a(Y_s)\} \, dM_s$$

of Z from its boundary variation part, we have

$$\mathbb{E} \max_{0 \leq s \leq t} |Z_s|^4 \leq 2^4 \mathbb{E} \max_{0 \leq s \leq t} |N_s|^4 + 2^4 \mathbb{E} \left(\int_0^t |b(X_s) - b(Y_s)| \, d|A|_s \right)^4.$$

The first term can be bounded by the martingale moment inequality and we obtain

$$\begin{aligned}
\mathbb{E} \max_{0 \leq s \leq t} |N_s|^4 \leq & 4\mathbb{E} \langle N, N \rangle_t^2 \\
= & 4\mathbb{E} \left(\int_0^t |a(X_s) - a(Y_s)|^2 \, d\langle M, M \rangle_s \right)^2 \\
\leq & 4K^4 \mathbb{E} \left(\int_0^t |X_s - Y_s|^2 \, ds \right)^2 \\
\leq & 4K^4 T \mathbb{E} \int_0^t |X_s - Y_s|^4 \, ds.
\end{aligned}$$

The second term of $\mathbb{E} \max_{0 \leq s \leq t} |Z_s|^4$ can be bounded easily by Hölder's inequality and we have

$$\mathbb{E} \left(\int_0^t |b(X_s) - b(Y_s)| \, d|A|_s \right)^4 \leq K^4 T^3 \mathbb{E} \int_0^t |X_s - Y_s|^4 \, ds.$$

Combining the two parts, we have

$$\mathbb{E} \max_{0 \leq s \leq t} |Z_s|^4 \leq C_T E \left[\int_0^t |X_s - Y_s|^4 \, ds \right].$$

Note that we are assuming that $t \leq T$ and C_T is the notation for a generic constant depending on T, whose value may vary from one appearance to another. So far we have proved the following inequality

$$\mathbb{E} \max_{0 \leq s \leq t} |X_s - Y_s|^4 \leq C_T \mathbb{E} \int_0^t |X_s - Y_s|^4 \, ds + 2^4 \mathbb{E} \max_{0 \leq s \leq t} |\phi_s - \psi_s|^4. \tag{5.4}$$

We now bound the second term involving $\phi - \psi$. Since it is continuous with bounded variation, we have

$$2^{-1}|\phi_t - \psi_t|^2 = \int_0^t (\phi_s - \psi_s)\, d(\phi_s - \psi_s).$$

Using

$$\phi_s - \psi_s = X_s - Y_s - Z_s$$

in the integrand, we have

$$2^{-1}|\phi_t - \psi_t|^2 = \int_0^t (X_s - Y_s)\, d(\phi_s - \psi_s) - \int_0^t Z_s\, d(\phi_s - \psi_s). \tag{5.5}$$

The crucial step in this proof is that the first term on the right is negative. In order to see this, we write

$$\int_0^t (X_s - Y_s)\, d(\phi_s - \psi_s) = \int_0^t (X_s - Y_s)\, d\phi_s - \int_0^t (X_s - Y_s)\, d\psi_s.$$

Since ϕ increases only when $X_s = 0$ and Y_s is always nonnegative, we see that ϕ increases on when $X_s - Y_s \leq 0$, hence the first term on the right side is always negative; likewise the second term is always positive. It follows that

$$\int_0^t (X_s - Y_s)\, d(\phi_s - \psi_s) \leq 0.$$

We are allowed to integrate by parts in the second term in (5.5) and write have by Itô's formula

$$\int_0^t Z_s\, d(\phi_s - \psi_s) = Z_t(\phi_t - \psi_t) - \int_0^t (\phi_s - \psi_s)\, dZ_s$$

Splitting the integrating process Z in the last term according to (5.3) and using the inequality

$$(x + y + z)^2 \leq 2^3(x^2 + y^2 + z^2)$$

we have

$$2^{-5}\mathbb{E} \max_{0 \leq s \leq t} |\phi_s - \psi_s|^4 = J_1 + J_2 + J_3 \tag{5.6}$$

where

$$J_1 = \mathbb{E} \max_{0 \leq s \leq t} |\phi_s - \psi_s|^2 \max_{0 \leq s \leq t} |Z_s|^2$$

$$J_2 = \mathbb{E} \max_{0 \leq s \leq t} \left| \int_0^t (\phi_s - \psi_s)[a(X_s) - a(Y_s)]\, dM_s \right|^2$$

$$J_3 = \mathbb{E} \left| \int_0^t (\phi_s - \psi_s)\, dA_s \right|^2.$$

It should be clear now why we have taken the fourth power at the beginning of the proof. If we took the second power, we would be at a loss for a way to estimate the expected value of the maximum of the absolute value of a martingale in J_2. Now Using the inequality

$$x^2 y^2 \le \epsilon x^4 + \frac{1}{4\epsilon} y^4$$

we have ($\epsilon = 2^{-7}$)

$$\begin{aligned} J_1 \le & 2^{-7} \mathbb{E} \max_{0 \le s \le t} |\phi_s - \psi_s|^4 + 2^5 \mathbb{E} \max_{0 \le s \le t} |Z_s|^4 \\ \le & 2^{-7} \mathbb{E} \max_{0 \le s \le t} |\phi_s - \psi_s|^4 + C_T \mathbb{E} \int_0^t |X_s - Y_s|^4 \, ds. \end{aligned}$$

For J_2 we have ($\epsilon = 2^{-9} K^{-2}$)

$$\begin{aligned} J_2 \le & 4\mathbb{E} \int_0^t |\phi_s - \psi_s|^2 |a(X_s) - a(Y_s)|^2 \, d\langle M, M\rangle_s \\ \le & 4K^2 \mathbb{E} \max_{0 \le s \le t} |\phi_s - \psi_s|^2 \cdot \int_0^t |X_s - Y_s|^2 \, ds \\ \le & 2^{-7} \mathbb{E} \max_{0 \le s \le t} |\phi_s - \psi_s|^4 + C_T \mathbb{E} \int_0^t |X_s - Y_s|^4 \, ds. \end{aligned}$$

For J_3 we have ($\epsilon = 2^{-7}$)

$$J_3 \le 2^{-7} \mathbb{E} \max_{0 \le s \le t} |\phi_s - \psi_s|^4 + C_T \mathbb{E} \int_0^t |X_s - Y_s|^4 \, ds.$$

These estimates for J_1, J_2, and J_3 are to be used in (5.6). The coefficient of $\mathbb{E} \max_{0 \le s \le t} |\phi_s - \psi_s|^4$ on the right side is $3 \cdot 2^{-7}$, which is less than the coefficient 2^{-5} on the left side, hence we have

$$\mathbb{E} \max_{0 \le s \le t} |\phi_s - \psi_s|^4 \le C_T \mathbb{E} \int_0^t |X_s - Y_s|^4 \, ds$$

for some constant C_T. Using this inequality in (5.4), we obtain the following inequality for all $t \le T$:

$$\mathbb{E}|X_s - Y_s|^4 \le C_T \int_0^t \mathbb{E}|X_s - Y_s|^4 \, ds.$$

It follows that $X_t = Y_t$ for all t, which is the uniqueness, and with this, we have completed the proof of THEOREM 5.1.

6. Fifth assignment

EXERCISE 5.1. Let X be the solution of the stochastic differential equation

$$dX_t^x = \sigma(X_t^x) \, dB_t + b(X_t^x) \, dt, \qquad X_0^x = x.$$

We assume that a, b are continuous and uniformly bounded. If f is a twice continuously differential function and f, f', and f'' are uniformly bounded. Show that

$$F(x, t) = \mathbb{E} f\left(X_t^x\right)$$

satisfies the parabolic partial differential equation

$$\frac{\partial F}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2} + b\frac{\partial F}{\partial x}, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+$$

and the initial condition $\lim_{t\to 0} F(x,t) = f(x)$.

EXERCISE 5.2. Prove the Itô's formula in the Stratonovich form stated in PROPOSITION 4.1.

EXERCISE 5.3. Consider the stochastic differential equation

$$dX_t = \operatorname{sgn}(X_t)\, dB_t, \qquad X_0 = 0.$$

Show that the uniqueness does not hold for this equation. Note that by convention $\operatorname{sgn} 0 = 0$.

EXERCISE 5.4. Let $B = (B^1, B^2, \cdots, B^n)$ be an n-dimensional Brownian motion. Derive an Itô type stochastic differential equation for its radial process $r = |B|$.

EXERCISE 5.5. Show that a two-dimensional Brownian motion B can be written in the form $B_t = r_t e^{i\Theta_t}$, where the angular process Θ is a time-changed one-dimensional Brownian motion.

EXERCISE 5.6. One dimensional stochastic differential equations are uniquely solvable for a Hölder continuous diffusion coefficient with exponent $1/2$. Consider the equation

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt.$$

If there is a constant C such that

$$|\sigma(x) - \sigma(y)|^2 + |b(x) - b(y)| \le K|x - y|,$$

then the equation has a unique solution.

EXERCISE 5.7. Consider the stochastic differential equation

$$dX_t = \sqrt{X_t \vee 0}\, dB_t + \alpha dt, \qquad X_0 = x \ge 0$$

with $\alpha > 0$. Show that it has a unique positive solution.

EXERCISE 5.8. Consider the stochastic differential equation

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt.$$

Suppose that the diffusion and drift coefficients are even functions are even. Show that $Y_t = |X_t|$ is the solution of a stochastic differential equation with reflection.

EXERCISE 5.9. The equation

$$dX_t = X_t^2\, dB_t$$

does not explode, whereas the equation

$$dX_t = X_t^2\, dB_t + X_t^2\, dt$$

explodes.

EXERCISE 5.10. Consider the stochastic differential equation

$$dX_t = dB_t - X_t\, dt, \qquad X_0 = x.$$

Its solution is called an Ornstein-Uhlenbeck process. Show that it is a Gaussian process and

$$\mathbb{E} f(X_t^x) = \int_{\mathbb{R}} f\left(e^{-t/2}x + \sqrt{1-e^{-t}}y\right) \mu(dy),$$

where μ is the standard Gaussian measure on $\mathbb{R}$. This is called Mehler's formula

EXERCISE 5.11. Consider the stochastic differential equation for a Brownian bridge

$$dX_t = dB_t - \frac{X_t}{1-t}\, dt, \qquad X_0 = 0.$$

Show that $\lim_{t\uparrow 1} X_t = 0$ almost surely.