现代数学物理方法

第一章，特殊函数

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September 14, 2017

Feynman's integral formula:

We now begin the applications of Gamma function in QFT.

The first application is in generalizing the so-called *Feynman's integral formula*:

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[Ax + B(1-x)]^2} \tag{1}$$

Its generalization reads:

$$\frac{1}{\prod_{i=1}^n A_i^{\alpha_i}} = \frac{\Gamma(\sum_{i=1}^n \alpha_i)}{\prod_{i=1}^n \Gamma(\alpha_i)} \frac{1}{(n-1)!} \int dF_n \frac{\prod_{i=1}^n x_i^{\alpha_i - 1}}{\left(\sum_{i=1}^n x_i A_i\right)^{\sum_{i=1}^n \alpha_i}} \tag{2}$$

where $n \geqslant 2$, and the integration measure over the *Feynman parameters* x_i is,

$$\int dF_n = (n-1)! \int_0^1 dx_1 \cdots dx_n \delta\left(\sum_{i=1}^n x_i - 1\right) \tag{3}$$

Feyman's integral formula can be shown as follows:

$$\frac{1}{AB} = \frac{1}{A-B}\left(\frac{1}{B} - \frac{1}{A}\right) = \frac{1}{A-B}\left(-\frac{1}{\xi}\right)\Big|_B^A = \frac{1}{A-B}\int_B^A \frac{d\xi}{\xi^2}$$

Setting $\xi = B + (A-B)x$, we see that

$$\int_B^A \frac{d\xi}{\xi^2} = (A-B)\int_0^1 \frac{dx}{[B+(A-B)x]^2} = (A-B)\int_0^1 \frac{dx}{[Ax+B(1-x)]^2}$$

Therefore,

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[Ax+B(1-x)]^2}$$

Question:

How to prove Eq.(2) ?

Proof:

Firstly, we note that the measure is normalized so that,

$$\begin{aligned} \int dF_n 1 &= (n-1)! \int_0^1 dx_1 \cdots dx_n \delta(x_1 + x_2 + \cdots + x_n - 1) \\ &= (n-1)! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-x_2-\cdots-x_{n-2}} dx_{n-1} \end{aligned}$$

The reasoning is as follows.

- Non-vanishing of the first integral $\int_0^1 dx_n \delta(x_1 + \cdots + x_n - 1)$ requires $0 \leqslant x_n = 1 - x_1 - \cdots - x_{n-1} \leqslant 1$, implying that $0 \leqslant x_{n-1} \leqslant 1 - x_1 - \cdots - x_{n-2}$.
- The inequality $0 \leqslant 1 - x_1 - \cdots - x_{n-2}$ demands further $x_{n-2} \leqslant 1 - x_1 - \cdots - x_{n-3}$, so that the parameter x_{n-2} is subject to the inequalities $0 \leqslant x_{n-2} \leqslant 1 - x_1 - \cdots - x_{n-3}$.
- $\cdots$

Let us finish the integration case by case. When $n = 2$,

$$\int dF_2 1 = 1! \int_0^1 dx_1 = 1$$

When $n = 3$,

$$\int dF_3 1 = 2! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 = 2 \int_0^1 dx_1 (1 - x_1) = 2 \int_0^1 dx_1 x_1 = 1$$

In the next-to-last step we have made a replacement $1 - x_1 = y_1$, and then renamed y_1 as x_1.

When $n = 4$,

$$\begin{aligned} \int dF_4 1 &= 3! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \\ &= 3! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 (1 - x_1 - x_2) = 6 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 x_2 \\ &= 3 \int_0^1 dx_1 (1 - x_1)^2 = 3 \int_0^1 dx_1 x_1^2 = 1 \end{aligned}$$

In general,

$$\int dF_n 1 = (n-1)! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-2}} dx_{n-1}$$

After finishing the integration for x_{n-1}, it becomes,

$$= (n-1)! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-3}} dx_{n-2} \left(1 - \sum_{i=1}^{n-2} x_i\right)$$

$$= (n-1)! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-3}} dx_{n-2} x_{n-2}$$

After finishing the integration for x_{n-2}, it becomes,

$$= \frac{(n-1)!}{2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-4}} dx_{n-3} \left(1 - \sum_{i=1}^{n-3} x_i\right)^2$$

$$= \frac{(n-1)!}{2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-4}} dx_{n-3} x_{n-3}^2$$

After finishing the integration for x_{n-3}, it becomes,

$$
\begin{aligned}
&= \frac{(n-1)!}{3!} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-5}} dx_{n-4} \left(1 - \sum_{i=1}^{n-4} x_i\right)^3 \\
&= \frac{(n-1)!}{3!} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-5}} dx_{n-4} x_{n-4}^3
\end{aligned}
$$

And so forth. So we conclude that

$$\int dF_n 1 = 1 \tag{4}$$

Secondly, we recall that for $\alpha_i > 0$,

$$\frac{\Gamma(\alpha_i)}{A_i^{\alpha_i}} = \int_0^\infty dt t^{\alpha_i - 1} e^{-A_i t}$$

where $i = 1, 2, \cdots, n$.

Taking the product of these Gamma functions, we get

$$\frac{\prod_{i=1}^{n}\Gamma(\alpha_i)}{\prod_{i=1}^{n}A_i^{\alpha_i}} = \int_0^\infty dt_1 \cdots \int_0^\infty dt_n t_1^{\alpha_1 - 1} e^{-A_1 t_1} \ldots t_n^{\alpha_n - 1} e^{-A_n t_n}$$

$$= \int_0^\infty dt_1 \cdots \int_0^\infty dt_n \int_0^\infty ds \delta\left(s - \sum_{i=1}^{n} t_i\right) t_1^{\alpha_1 - 1} e^{-A_1 t_1} \ldots t_n^{\alpha_n - 1} e^{-A_n t_n}$$

Making the change of variables from $t_i \to x_i$ by setting

$$t_i = s x_i$$

we see that $0 \leqslant x_i \leqslant 1$, and

$$\delta\left(s - \sum_{i=1}^{n} t_i\right) = \frac{1}{s}\delta\left(1 - \sum_{i=1}^{n} x_i\right)$$

Therefore,

$$\frac{\prod_{i=1}^{n}\Gamma(\alpha_i)}{\prod_{i=1}^{n}A_i^{\alpha_i}} = \int_0^1 dx_1\cdots dx_n\delta\left(1-\sum_{i=1}^{n}x_i\right)x_1^{\alpha_1-1}\cdots x_n^{\alpha_n-1}$$

$$\cdot\int_0^\infty ds s^{\sum_i\alpha_i-1}\exp\left(-s\sum_{i=1}^{n}A_ix_i\right)$$

$$=\int_0^1 dx_1\cdots dx_n\delta\left(1-\sum_{i=1}^{n}x_i\right)\prod_{i=1}^{n}x_i^{\alpha_i-1}\frac{\Gamma\left(\sum_i\alpha_i\right)}{\left(\sum_i A_ix_i\right)^{\sum_i\alpha_i}}$$

$$=\frac{\Gamma\left(\sum_i\alpha_i\right)}{(n-1)!}\int dF_n\frac{\prod_{i=1}^{n}x_i^{\alpha_i-1}}{\left(\sum_i A_ix_i\right)^{\sum_i\alpha_i}}$$

It is very the generalized Feynman's integral formula.

The most popular case happens for $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$. In this case, the identity is called *Feynman formula*:

$$
\begin{aligned}
\frac{1}{A_1 A_2 \cdots A_n} &= \int dF_n \frac{1}{(x_1 A_1 + x_2 A_2 + \cdots + x_n A_n)^n} \\
&= (n-1)! \int_0^1 dx_1 \cdots dx_n \frac{\delta\left(1 - \sum_{i=1}^n x_i\right)}{(x_1 A_1 + \cdots + x_n A_n)^n} \\
&= (n-1)! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \\
&\qquad \cdot \int_0^{1-x_1-x_2\cdots-x_{n-2}} \frac{dx_{n-1}}{\left[\sum_{i=1}^{n-1} x_i A_i + \left(1 - \sum_{i=1}^{n-1} x_i\right) A_n\right]^n}
\end{aligned}
$$

If $n = 2$, we see that:

$$
\frac{1}{A_1 A_2} = \int_0^1 \frac{dx}{[xA_1 + (1-x)A_2]^2}
$$

This is just what we have expected.

A useful integral in d-dimensions:

In dimensional regulation scheme, there is an important integral defined in d-dimensional Euclidean space,

$$\int \frac{d^d x}{(2\pi)^d} \frac{(x^2)^a}{(x^2 + D)^b}$$

where $a > 0,\ b > 0$ and $d > 0$.

The result of this integral could also be expressed in terms of the Gamma functions:

$$\int \frac{d^d x}{(2\pi)^d} \frac{(x^2)^a}{(x^2 + D)^b} = \frac{\Gamma\left(b - a - \frac{d}{2}\right) \Gamma\left(a + \frac{d}{2}\right)}{(4\pi)^{d/2} \Gamma(b) \Gamma\left(\frac{d}{2}\right)} D^{-(b-a-d/2)} \tag{5}$$

Proof:

Becuase the integrand depends upon the Cartesian coordinates x^i only through $x^2 = (x^1)^2 + (x^2)^2 + \cdots + (x^d)^2$, the integration can easily be finished in spherical coordinates, where $\sqrt{x^2} = r$ and

$$d^d x = r^{d-1} dr d\Omega$$

The integrand depends only upon the radial coordinate r. The radial factor of the integral is,

$$\begin{aligned} I_1 &= \frac{1}{(2\pi)^d} \int_0^\infty \frac{r^{2a+d-1}}{(r^2+D)^b} dr = \frac{1}{(2\pi)^d} D^{-(b-a-d/2)} \int_0^\infty \frac{\rho^{2a+d-1}}{(\rho^2+1)^b} d\rho \\ &= \frac{1}{2(2\pi)^d} D^{-(b-a-d/2)} B\left(a+\frac{d}{2}, b-a-\frac{d}{2}\right) \\ &= \frac{1}{2(2\pi)^d} D^{-(b-a-d/2)} \frac{\Gamma\left(b-a-\frac{d}{2}\right)\Gamma\left(a+\frac{d}{2}\right)}{\Gamma(b)} \end{aligned} \tag{6}$$

where we have employed an integral expression of Beta function $B(a, b)$,

$$B(a, b) = 2\int_0^\infty \frac{\zeta^{2a-1}}{(1+\zeta^2)^{a+b}} d\zeta$$

The angular factor is $\Omega_d := \int d\Omega$, which can easily be finished by computing the Gaussian integral $\int d^d x e^{-x^2}$ in both Cartesian and spherical coordinates. In Cartesian coordinates,

$$\int d^d x e^{-x^2} = \prod_{i=1}^{d} \int_{-\infty}^{+\infty} dx_i e^{-x_i^2} = \pi^{d/2}$$

In spherical coordinates,

$$\int d^d x e^{-x^2} = \int_0^\infty r^{d-1} e^{-r^2} dr \int d\Omega = \frac{1}{2}\Omega_d \int_0^\infty \rho^{\frac{d}{2}-1} e^{-\rho} d\rho = \frac{\Gamma\left(\frac{d}{2}\right)}{2}\Omega_d$$

By comparison, we get

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{7}$$

Hence,

$$\int \frac{d^d x}{(2\pi)^d} \frac{(x^2)^a}{(x^2 + D)^b} = I_1 \Omega_d = \frac{\Gamma\left(b - a - \frac{d}{2}\right)\Gamma\left(a + \frac{d}{2}\right)}{(4\pi)^{d/2}\Gamma(b)\Gamma\left(\frac{d}{2}\right)} D^{-(b-a-d/2)}$$

This is as expected.

Loop corrections in QED:

As an application of Gamma function, now we compute *in detail* the one-loop corrections in spinor electrodynamics.

The free QED lagrangian reads,

$$\mathscr{L}_0 = i\bar{\Psi}\partial\!\!\!/\Psi - m\bar{\Psi}\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{8}$$

The renormalization scheme requires to add all possible terms whose coefficeints have positive or zero mass dimensions, and that respect the symmetries of the free lagrangian.

These symmetries include:

1. Lorentz symmetry
2. the $U(1)$ gauge symmetry
3. the discrete symmetries of parity, time reversal and the charge conjugation

In 4-dimensional spacetime, the mass dimensions of the relevant fields are

- $[A^\mu] = 1$
- $[\Psi] = 3/2$

Gauge invariance excludes the mass term $m_\gamma^2 A^\mu A_\mu$ of a photon, so A^μ apears only in the covariant derivative

$$D^\mu = \partial^\mu - ieA^\mu$$

Thus, the only possible term we could add to $\mathscr{L}_0$, that does not involve Ψ and that has non-negative mass dimension, is

$$\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

This term, however, is *odd* under parity and time reversal.

Similarly, except the mass term

$$m\bar{\Psi}\Psi$$

of the spinor field Ψ and the interacting lagrangian

$$e\bar{\Psi}\not{A}\Psi$$

which comes from replacing the common derivative ∂_μ in Eq.(8) with the covariant derivative D_μ, there are no terms meeting all the requirements that involve Ψ.

The only candidates are:[1]

1. $\bar{\Psi}\gamma_5\Psi$, forbidden by parity conservation.
2. $\Psi^T \mathcal{C}\Psi$, forbidden by $U(1)$ symmetry.

[1] The charge conjugation matrix $\mathcal{C}$ has the properties,

$$\mathcal{C}^T = \mathcal{C}^\dagger = \mathcal{C}^{-1} = -\mathcal{C}, \quad \mathcal{C}^{-1}\gamma^\mu \mathcal{C} = -(\gamma^\mu)^T.$$

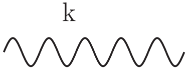
Our starting point to calculate *one-loop corrections* in QED is the lagrangian

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1$$

where

$$\begin{aligned} \mathscr{L}_1 \quad &= Z_1 e \bar{\Psi} \not{A} \Psi + i(Z_2 - 1)\bar{\Psi}\not{\partial}\Psi - (Z_m - 1) m \bar{\Psi}\Psi \\ &- \frac{1}{4}(Z_3 - 1) F^{\mu\nu} F_{\mu\nu} \end{aligned} \tag{9}$$

We begin with the *photon propagator* in momentum space. The free photon propagator in so-called R_ξ gauge[2] reads:



$$\tilde{\Delta}_{\mu\nu}(k) = \frac{1}{k^2 - i\epsilon}\left[\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right] \quad (10)$$

Two special gauges are as follows:

- $\xi = 1, \quad \rightsquigarrow$ Feynman gauge
- $\xi = 0, \quad \rightsquigarrow$ Lorenz gauge (or Landau gauge)

Taking the electron-photon interaction into account, $\tilde{\Delta}_{\mu\nu}(k)$ should be replaced by the so-called *exact* photon propagator:

$$\tilde{\Delta}^{(exact)}_{\mu\nu}(k) = \tilde{\Delta}_{\mu\nu}(k) + \tilde{\Delta}_{\mu\rho}(k)\Pi^{\rho\sigma}(k)\tilde{\Delta}_{\sigma\nu}(k) + \cdots \quad (11)$$

where $i\Pi^{\mu\nu}(k)$ is the one-loop correction to the photon propagator.

[2] *R* stands for *renormalizable* and ξ a gauge parameter.

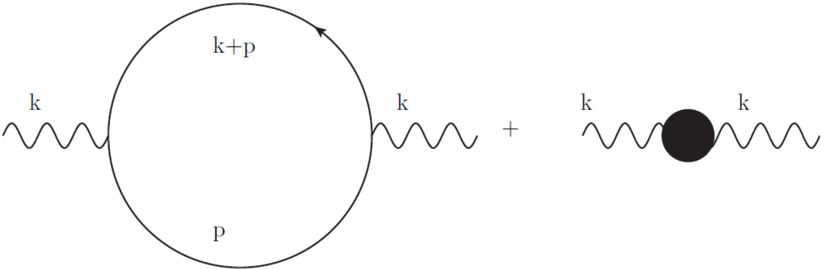


Figure : One-loop correction and counterterm to the photon propagator

Observable squared amplitudes should not depend on the choice of gauge parameter ξ. This suggests that $\Pi^{\mu\nu}(k)$ should be transverse,

$$k_\mu \Pi^{\mu\nu}(k) = \Pi^{\mu\nu}(k) k_\nu = 0 \tag{12}$$

$\Pi^{\mu\nu}(k)$ can be written into,

$$\Pi^{\mu\nu}(k) = k^2 \Pi(k) \left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \tag{13}$$

or $\Pi^{\mu\nu}(k) = k^2 \Pi(k^2) P^{\mu\nu}$, where $\Pi(k^2)$ is a scalar function of k^2, and

$$P^{\mu\nu} = \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \tag{14}$$

is the projection operator in momentum space:

$$P^{\mu\nu} P_{\nu\rho} = \left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \left(\eta_{\nu\rho} - \frac{k_\nu k_\rho}{k^2} \right) = \delta^\mu_\rho - \frac{k^\mu k_\rho}{k^2} = P^\mu{}_\rho$$

The free photon propagator can be expressed in terms of $P^{\mu\nu}$,

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{1}{k^2 - i\epsilon}\left(P^{\mu\nu} + \xi\frac{k_\mu k_\nu}{k^2}\right) \tag{15}$$

Consequently, Eq.(11) has a closed form:

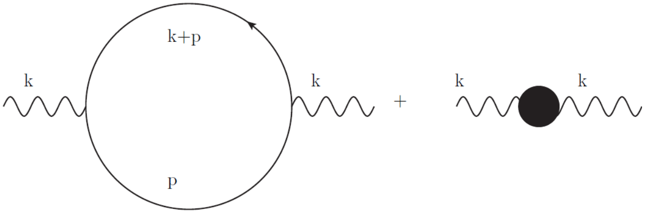
$$\begin{aligned}
\tilde{\Delta}^{(exact)}_{\mu\nu} &= \frac{P_{\mu\nu}}{k^2 - i\epsilon} + \xi\frac{k_\mu k_\nu/k^2}{k^2 - i\epsilon} + \frac{P_{\mu\nu}}{k^2 - i\epsilon}\frac{k^2\Pi(k^2)}{k^2 - i\epsilon} \\
&\quad + \frac{P_{\mu\nu}}{k^2 - i\epsilon}\left[\frac{k^2\Pi(k^2)}{k^2 - i\epsilon}\right]^2 + \cdots \\
&= \frac{P_{\mu\nu}}{k^2[1 - \Pi(k^2)] - i\epsilon} + \xi\frac{k_\mu k_\nu/k^2}{k^2 - i\epsilon}
\end{aligned} \tag{16}$$

The ξ-dependent term should be physically irrelevant[3]. The remaining term has a pole at $k^2 = 0$ with residue

$$\frac{P_{\mu\nu}}{1 - \Pi(0)}$$

[3]It can be set to zero by the gauge choice $\xi = 0$, corresponding to Lorentz gauge.

We now turn to the calculation of $\Pi(k^2)$. The one-loop correction is shown in the attached Feynman diagram.



It follows from Feynman's rule that,

$$i\Pi^{\mu\nu}(k) = (-1)(iZ_1e)^2(-i)^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\tilde{S}(\not{p}+\not{k})\gamma^\mu \tilde{S}(\not{p})\gamma^\nu\right] - i(Z_3-1)k^2 P^{\mu\nu} + \mathcal{O}(e^4) \tag{17}$$

The explanation is as follows:

1. For each interaction vertex, write $(iZ_1 e\gamma^\mu)$, where γ^μ stands for the famous Dirac's Gamma matrices, satisfying $\{\gamma^\mu,\ \gamma^\nu\} = -2\eta^{\mu\nu}$. Anticipating that $Z_1 = 1 + \mathcal{O}(e^2)$, we will set $Z_1 = 1$ in the following.
2. For each spinor propagator (internal line) of momentum p, write $[-i\tilde{S}(\not{p})]$, which is the free fermion propagator in momentum space, with
$$\tilde{S}(\not{p}) := \frac{1}{\not{p} + m} = \frac{(-\not{p} + m)}{p^2 + m^2 - i\epsilon} \tag{18}$$
where we have used the identity $\not{p}\not{p} = -p^2$.
3. Integrate over internal momentum.
4. There is an extra factor of *minums one* and a trace operation for the closed fermionic loop.
5. For the counterterm of the photon propagator, write $[-i(Z_3 - 1)k^2 P^{\mu\nu}]$.

We now simplify the trace appearing in Eq.(17),

$$\mathrm{Tr}\left[\tilde{S}(\not{p}+\not{k})\gamma^\mu\tilde{S}(\not{p})\gamma^\nu\right] = \frac{\mathrm{Tr}[(-\not{p}-\not{k}+m)\gamma^\mu(-\not{p}+m)\gamma^\nu]}{[(p+k)^2+m^2-i\epsilon](p^2+m^2-i\epsilon)}$$
$$= \frac{4N^{\mu\nu}}{[(p+k)^2+m^2-i\epsilon](p^2+m^2-i\epsilon)}$$

In the numerator, $N^{\mu\nu}$ is defined by,

$$N^{\mu\nu} = \frac{1}{4}\mathrm{Tr}[(-\not{p}-\not{k}+m)\gamma^\mu(-\not{p}+m)\gamma^\nu] \quad (19)$$

Gamma matrices obey identities:

$$\mathrm{Tr}(\not{a}\not{b}) = -4(a\cdot b) \quad (20)$$
$$\mathrm{Tr}(\not{a}\not{b}\not{c}) = 0 \quad (21)$$
$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4\big[(a\cdot d)(b\cdot c)-(a\cdot c)(b\cdot d)+(a\cdot b)(c\cdot d)\big] \quad (22)$$

where $(a\cdot b) = a^\mu b_\mu$.

Employment of these identities leads to:

$$
\begin{aligned}
a_\mu N^{\mu\nu} b_\nu &= \frac{1}{4}\mathrm{Tr}\Big[(-\not{p} - \not{k} + m)\not{a}(-\not{p} + m)\not{b}\Big] \\
&= \frac{1}{4}\mathrm{Tr}[\not{p}\not{a}\not{p}\not{b} - m\not{p}\not{a}\not{b} + \not{k}\not{a}\not{p}\not{b} - m\not{k}\not{a}\not{b} - m\not{a}\not{p}\not{b} + m^2\not{a}\not{b}] \\
&= \frac{1}{4}\mathrm{Tr}[\not{p}\not{a}\not{p}\not{b} + \not{k}\not{a}\not{p}\not{b} + m^2\not{a}\not{b}] \\
&= 2(a \cdot p)(p \cdot b) - (a \cdot b)p^2 + (a \cdot k)(p \cdot b) - (a \cdot b)(k \cdot p) \\
&\qquad + (a \cdot p)(k \cdot b) - m^2(a \cdot b) \\
&= a_\mu\Big[2p^\mu p^\nu - p^2\eta^{\mu\nu} + k^\mu p^\nu + p^\mu k^\nu - (k \cdot p)\eta^{\mu\nu} - m^2\eta^{\mu\nu}\Big] b_\nu
\end{aligned}
$$

where a_μ and b_ν are two arbitrary non-zero 4-vector.

Therefore,

$$
N^{\mu\nu} = 2p^\mu p^\nu - p^2\eta^{\mu\nu} + k^\mu p^\nu + p^\mu k^\nu - (k \cdot p)\eta^{\mu\nu} - m^2\eta^{\mu\nu} \quad (23)
$$

Besides, the Feynman's formula

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[xA + (1-x)B]^2}$$

tells us that,

$$\frac{1}{[(p+k)^2 + m^2 - i\epsilon](p^2 + m^2 - i\epsilon)} = \int_0^1 \frac{dx}{[x(p+k)^2 + (1-x)p^2 + m^2 - i\epsilon]^2}$$

Therefore,

$$\mathrm{Tr}\left[\tilde{S}(\not{p} + \not{k})\gamma^\mu \tilde{S}(\not{p})\gamma^\nu\right] = 4\int_0^1 dx \frac{2p^\mu p^\nu - p^2\eta^{\mu\nu} + k^\mu p^\nu + p^\mu k^\nu - (k\cdot p)\eta^{\mu\nu} - m^2\eta^{\mu\nu}}{[x(p+k)^2 + (1-x)p^2 + m^2 - i\epsilon]^2} \tag{24}$$

We change the variable p to $q^\mu = p^\mu + xk^\mu$ to remove the cross term $q \cdot k$ in the denominator.

The square-root of the denominator in Eq.(24) becomes,

$$\begin{aligned}\sqrt{\text{denominator}} &= x(p+k)^2 + (1-x)p^2 + m^2 - i\epsilon \\ &= x[q + (1-x)k]^2 + (1-x)(q - xk)^2 + m^2 - i\epsilon \\ &= q^2 + x(1-x)k^2 + m^2 - i\epsilon \\ &= q^2 + D\end{aligned}$$

where,

$$D = x(1-x)k^2 + m^2 - i\epsilon \tag{25}$$

So,

$$\mathrm{Tr}\left[\tilde{S}(\not{p} + \not{k})\gamma^\mu \tilde{S}(\not{p})\gamma^\nu\right] = 4\int_0^1 dx \frac{N^{\mu\nu}}{(q^2 + D)^2} \tag{26}$$

where,

$$
\begin{aligned}
N^{\mu\nu} &= 2(q-xk)^\mu (q-xk)^\nu - (q-xk)^2\eta^{\mu\nu} + k^\mu (q-xk)^\nu \\
&\quad + (q-xk)^\mu k^\nu - [k\cdot(q-xk)]\eta^{\mu\nu} - m^2\eta^{\mu\nu} \\
&= 2q^\mu q^\nu - 2x(1-x)k^\mu k^\nu - [q^2 - x(1-x)k^2 + m^2]\eta^{\mu\nu} \\
&\quad + (1-2x)(q^\mu k^\nu + k^\mu q^\nu) - (1-2x)(q\cdot k)\eta^{\mu\nu} \qquad (27)
\end{aligned}
$$

Notice that,

$$
\int d^d q q^\mu f(q^2) = 0 \qquad (28)
$$

This is because the integrand is odd under $q \to -q$, and so vanished when integratged. Based on Eqs.(28) and (17), the terms linear in q in Eq.(27) can be safely discarded.

As a result,

$$
N^{\mu\nu} = 2q^\mu q^\nu - 2x(1-x)k^\mu k^\nu - [q^2 - x(1-x)k^2 + m^2]\eta^{\mu\nu}
$$

There is another identity for integration in d-dimensional momentum space,

$$\int d^d q q^\mu q^\nu f(q^2) = \frac{1}{d}\eta^{\mu\nu} \int d^d q q^2 f(q^2) \tag{29}$$

The reasoning is as follows. The LHS of Eq.(29) is two index constant symmetric 4-tensor, and so must equal to $\eta^{\mu\nu} A$ by Lorentz invariance,

$$\int d^d q q^\mu q^\nu f(q^2) = \eta^{\mu\nu} A$$

where A is a Lorentz scalar. To determine A, we contract both sides of the above formula with $\eta_{\mu\nu}$,

$$\int d^d q q^2 f(q^2) = \eta_{\mu\nu}\eta^{\mu\nu} A = dA \quad \rightsquigarrow \quad A = \frac{1}{d}\int d^d q q^2 f(q^2)$$

So Eq.(29) is proved.

Based on Eq.(29), we can simplify $N^{\mu\nu}$ into

$$N^{\mu\nu} = -2x(1-x)k^\mu k^\nu + \left[\left(\frac{2}{d} - 1\right)q^2 + x(1-x)k^2 - m^2\right]\eta^{\mu\nu} \tag{30}$$

Substitution of Eqs.(26) and (30) into Eq.(17) gives,

$$\begin{aligned}
\Pi^{\mu\nu}(k) &= -4e^2 \int_0^1 dx \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2 + D)^2} \Big[- x(1-x)(2k^\mu k^\nu - k^2\eta^{\mu\nu}) \\
&\quad + \left(\frac{2}{d} - 1\right)\bar{q}^2\eta^{\mu\nu} - m^2\eta^{\mu\nu} \Big] \\
&\quad - (Z_3 - 1)k^2 P^{\mu\nu} + \mathscr{O}(e^4) \\
&= 4e^2(2k^\mu k^\nu - k^2\eta^{\mu\nu}) \int_0^1 dx x(1-x) \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2 + D)^2} \\
&\quad - \frac{4(2-d)e^2}{d}\eta^{\mu\nu} \int_0^1 dx \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{\bar{q}^2}{(\bar{q}^2 + D)^2} \\
&\quad + 4m^2 e^2 \eta^{\mu\nu} \int_0^1 dx \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2 + D)^2} \\
&\quad - (Z_3 - 1)k^2 P^{\mu\nu} + \mathscr{O}(e^4)
\end{aligned} \tag{31}$$

where the spacetime dimensions have been prolonged from $4 \to d$, so has a Wick rotation: $q^0 = i\bar{q}_d, \;\; q^i = \bar{q}_i \;\; (i = 1, 2, \cdots, d-1)$.

From Eq.(5) we see that,

$$\int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2 + D)^2} = \frac{\Gamma\left(2 - \frac{d}{2}\right)}{(4\pi)^{d/2}} D^{-2+d/2}$$

$$\int \frac{d^d\bar{q}}{(2\pi)^d} \frac{\bar{q}^2}{(\bar{q}^2 + D)^2} = \frac{\frac{d}{2}\Gamma\left(1 - \frac{d}{2}\right) D^{-1+d/2}}{(4\pi)^{d/2}} = \frac{\Gamma\left(2 - \frac{d}{2}\right) D^{-1+d/2}}{\left(\frac{2}{d} - 1\right)(4\pi)^{d/2}}$$

Both integrals diverge when $d \to 4$, due to the fact $\Gamma(0) \to \infty$.

To analyze their divergence behaviour, we set

$$d = 4 - 2\varepsilon, \quad 0 < \varepsilon \ll 1$$

We also replace the eletron's charge e with $e\tilde{\mu}^{\varepsilon}$ so that e remains dimensionless for any dimensions d.

The first term in RHS of Eq.(31) becomes,

$$
\begin{aligned}
\text{Term-1} &= 4e^2\tilde{\mu}^{2\varepsilon}(2k^\mu k^\nu - k^2\eta^{\mu\nu})\int_0^1 dxx(1-x)\frac{\Gamma(\varepsilon)}{16\pi^2(4\pi)^{-\varepsilon}}D^{-\varepsilon} \\
&= \frac{e^2}{4\pi^2}(2k^\mu k^\nu - k^2\eta^{\mu\nu})\int_0^1 dxx(1-x)\,\Gamma(\varepsilon)\left(\frac{4\pi\tilde{\mu}^2}{D}\right)^\varepsilon
\end{aligned}
\tag{32}
$$

Because

$$
\begin{aligned}
\Gamma(\varepsilon) &= \varepsilon^{-1} - \gamma + \mathcal{O}(\varepsilon) \\
\left(\frac{4\pi\tilde{\mu}^2}{D}\right)^\varepsilon &= \exp\left[\varepsilon\ln\left(4\pi\tilde{\mu}^2/D\right)\right] = 1 + \varepsilon\ln\left(4\pi\tilde{\mu}^2/D\right) + \mathcal{O}(\varepsilon^2)
\end{aligned}
$$

we see that:

$$
\Gamma(\varepsilon)\left(\frac{4\pi\tilde{\mu}^2}{D}\right)^\varepsilon = \frac{1}{\varepsilon} - \gamma + \ln\left(4\pi\tilde{\mu}^2/D\right) + \mathcal{O}(\varepsilon) = \frac{1}{\varepsilon} - \ln\left(D/\mu^2\right) + \mathcal{O}(\varepsilon)
$$

where $\mu^2 = 4\pi e^{-\gamma}\tilde{\mu}^2$.

Consequently,

$$\text{Term-1} = \frac{e^2}{4\pi^2}(2k^\mu k^\nu - k^2\eta^{\mu\nu}) \int_0^1 dx x(1-x)\left[\frac{1}{\varepsilon} - \ln\left(D/\mu^2\right) + \mathscr{O}(\varepsilon)\right] \tag{33}$$

The second term in RHS of Eq.(31) becomes,

$$\begin{aligned}
\text{Term-2} &= -\frac{4(2-d)e^2\tilde{\mu}^{2\varepsilon}}{d}\eta^{\mu\nu}\int_0^1 dx \frac{d\,\Gamma(\varepsilon)}{16\pi^2(4\pi)^{-\varepsilon}(2-d)}D^{1-\varepsilon} \\
&= -\frac{e^2}{4\pi^2}\eta^{\mu\nu}\int_0^1 dx[x(1-x)k^2 + m^2 - i\epsilon]\Gamma(\varepsilon)\left(\frac{4\pi\tilde{\mu}^2}{D}\right)^{\varepsilon} \\
&= -\frac{e^2}{4\pi^2}k^2\eta^{\mu\nu}\int_0^1 dx x(1-x)\left[\frac{1}{\varepsilon} - \ln\left(D/\mu^2\right) + \mathscr{O}(\varepsilon)\right] \\
&\quad - \frac{e^2}{4\pi^2}m^2\eta^{\mu\nu}\int_0^1 dx\left[\frac{1}{\varepsilon} - \ln\left(D/\mu^2\right) + \mathscr{O}(\varepsilon)\right]
\end{aligned} \tag{34}$$

The third term becomes,

$$\begin{aligned}
\text{Term-3} &= 4m^2e^2\tilde{\mu}^2\eta^{\mu\nu}\int_0^1 dx\frac{\Gamma(\varepsilon)}{16\pi^2(4\pi)^{-\varepsilon}}D^{-\varepsilon} \\
&= \frac{m^2e^2}{4\pi^2}\eta^{\mu\nu}\int_0^1 dx\Gamma(\varepsilon)\left(\frac{4\pi\tilde{\mu}^2}{D}\right)^{\varepsilon} \\
&= \frac{m^2e^2}{4\pi^2}\eta^{\mu\nu}\int_0^1 dx\left[\frac{1}{\varepsilon}-\ln\left(D/\mu^2\right)+\mathscr{O}(\varepsilon)\right] \qquad (35)
\end{aligned}$$

Summation of these three terms leads to,

$$\begin{aligned}
\Pi^{\mu\nu}(k) &= -\frac{e^2}{2\pi^2}k^2P^{\mu\nu}\int_0^1 dxx(1-x)\left[\frac{1}{\varepsilon}-\ln\left(D/\mu^2\right)+\mathscr{O}(\varepsilon)\right] \\
&\quad -(Z_3-1)k^2P^{\mu\nu}+\mathscr{O}(e^4)
\end{aligned} \qquad (36)$$

i.e.,

$$\Pi(k^2) = -\frac{e^2}{2\pi^2}\int_0^1 dxx(1-x)\left[\frac{1}{\varepsilon}-\ln\left(D/\mu^2\right)+\mathscr{O}(\varepsilon)\right]-(Z_3-1)+\mathscr{O}(e^4) \qquad (37)$$

In the so-called *on-shell* renormalization scheme,

$$\Pi(0) = 0 \tag{38}$$

The condition $\Pi(0) = 0$ can be used to fix the renormalization constant Z_3. Recall that $D = x(1-x)k^2 + m^2 - i\epsilon$, we have:

$$\begin{aligned} Z_3 &= 1 - \frac{e^2}{2\pi^2}\int_0^1 dx x(1-x)\left[\frac{1}{\varepsilon} - \ln\left(m^2/\mu^2\right) + \mathcal{O}(\varepsilon)\right] + \mathcal{O}(e^4) \\ &= 1 - \frac{e^2}{12\pi^2}\left[\frac{1}{\varepsilon} - 2\ln(m/\mu) + \mathcal{O}(\varepsilon)\right] + \mathcal{O}(e^4) \end{aligned} \tag{39}$$

Therefore, a finite one-loop correction to the photon's propagator results in

$$\Pi(k^2) = \frac{e^2}{2\pi^2}\int_0^1 dx x(1-x)\ln(D/m^2) + \mathcal{O}(e^4) \tag{40}$$

Of course, this is just what we expect from renormalization scheme.

Homework:

1. Let $A_i, (i = 1, 2, \cdots, n)$ be positive real numbers. Start from the obvious identity
$$\frac{1}{A_i} = \int_0^\infty dse^{-sA_i}$$
and prove the Feynman's integral formula.
2. Simplify
$$\int d^dx\, x^\mu x^\nu x^\rho x^\sigma f(x^2)$$
3. Calculate the integral:
$$\int \frac{d^dk}{(2\pi)^d} \frac{(p+k)_\mu k_\nu}{(p+k)^2 k^2}$$