LIE GROUPS 2013 FALL
PROBLEM SET 3

(1) Suppose G acts on M smoothly.
  (a) Prove: The infinitesimal action $d\tau : \mathfrak{g} \to \Gamma^\infty(M)$ defined in lecture 13 is linear.
  (b) Suppose the action is *effective*, i.e. $\cap_{m \in M} G_m = \{e\}$. Prove: $d\tau$ is injective.
  (c) Suppose the action is transitive. Prove: the linear map
$$(\mathrm{dev}_m)_e : \mathfrak{g} \to T_m M, \qquad X \mapsto X_M(m)$$
  is onto for all m. In particular, $\dim G \geq \dim M$.

(2) Let H be a closed Lie subgroup of G. Consider the three actions of H on G via
$$\tau_h(g) = hg, \quad \rho_h(g) = gh^{-1}, \quad \pi_h(g) = hgh^{-1}.$$
Justify whether these actions are proper or free or transitive or effecitve.

(3) Suppose G acts on M smoothly.
  (a) Prove: The formula
$$(g \cdot \varphi)(x) := \varphi(g^{-1} \cdot x)$$
  defines a linear action of G on $C^\infty(M)$.
A fixed point of the above induced action of G on $C^\infty(M)$ is called an *invariant* of the G-action on M. The set of invariants is thus denoted by $C^\infty(M)^G$.
  (b) Prove: If G is connected, then $f \in C^\infty(M)^G$ if and only if $X_M f = 0$ for all $X \in \mathfrak{g}$.
  (c) Suppose G acts on M properly and freely. Prove: $C^\infty(M)^G \simeq C^\infty(M/G)$.
  (d) What are the invariants for the standard $\mathrm{O}(n)$ action on $\mathbb{R}^n$?
  (e) What are the invariants for the adjoint action (c.f. lecture 13) of $\mathrm{GL}(n, \mathbb{C})$ on $\mathfrak{gl}(n, \mathbb{C})$?

(4) Recall that $\mathrm{SO}(3)$ is a 3-dimensional Lie group whose Lie algebra $\mathfrak{so}(3)$ consists of all 3×3 real anti-symmetric matrices. We identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ via
$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \quad \mapsto \quad \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$
Prove: Under this identification, the adjoint action of $\mathrm{SO}(3)$ on $\mathfrak{so}(3)$ is just the usual $\mathrm{SO}(3)$ action on $\mathbb{R}^3$. What are the orbits of this action? What is the orbit space?

(5) Let $\mathcal{H}$ be the vector space of all $n \times n$ hermitian matrices. For each $\lambda = (\lambda_1, \cdots, \lambda_n)$, where $\lambda_i \in \mathbb{R}$ and $\lambda_1 \leq \cdots \leq \lambda_n$, let $\mathcal{H}_\lambda$ be the set of all $n \times n$ hermitian matrices whose eigenvalues are $\lambda_1, \cdots, \lambda_n$. Note that $\mathcal{H}$ can be identified with the Lie algebra of the unitary group $\mathrm{U}(n)$, and thus $\mathrm{U}(n)$ acts on $\mathcal{H}$ the adjoint action:
$$\text{For } A \in \mathrm{U}(n),\ \xi \in \mathcal{H}: \qquad A \cdot \xi := A\xi A^{-1}$$
  (a) Prove: The orbits of the $\mathrm{U}(n)$-action are H_λ's. (In particular, each $\mathcal{H}_\lambda$ is a smooth compact manifold!)
  (b) Describe the infinitesimal action of $\mathfrak{u}(n)$ associated to this action.

(c) For each $\xi \in \mathcal{H}$, describe the stabilizer $\mathrm{U}(n)_\xi$ and its Lie algebra $\mathfrak{u}(n)_\xi$.

(6) Consider the action of $\mathrm{SL}(2,\mathbb{R})$ on the upper half plane $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\}$ by the Möbius transformation:

$$A \cdot z := \frac{az+b}{cz+d} \qquad \text{for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$

(a) Show that the action is transitive.
(b) What is the stabilizer of $\sqrt{-1} \in H$?
(c) Show that the action above is equivalent to the adjoint action of $\mathrm{SL}(2,\mathbb{R})$ on the adjoint orbit $\mathcal{O}(B)$ of $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
(d) Find explicitly an $\mathrm{SL}(2,\mathbb{R})$-equivariant diffeomorphism $f : H \to \mathcal{O}(B)$.

(7) Let $G = \mathrm{SL}(2,\mathbb{C})$. Let $M = P_n$ be the linear space of homogeneous polynomials of degree n of two variables z_1 and z_2.
(a) Prove: The formula

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f)(z_1, z_2) := f(az_1 + cz_2, bz_1 + dz_2)$$

defines a smooth action of $\mathrm{SL}(2,\mathbb{R})$ on M.
(b) Prove:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form a basis of $\mathfrak{sl}(2,\mathbb{R})$, and the corresponding infinitesimal action are

$$X_M(f) = z_1 \frac{\partial f}{\partial z_2}, \quad Y_M(f) = z_2 \frac{\partial f}{\partial z_1}, \quad H_M(f) = z_1 \frac{\partial f}{\partial z_1} - z_2 \frac{\partial f}{\partial z_2}.$$