现代数学物理方法

第一章，特殊函数

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ζ-function regulation:

One-dimensional quantum oscillator is described by the Hamiltonian

$$H = \frac{p^2}{2\mu} + \frac{1}{2}\mu\omega^2 x^2$$

whose eigenvalue equation reads:

$$H\,|n\rangle = \left(n + \frac{1}{2}\right)\omega\,|n\rangle, \qquad (n = 0, 1, 2, \cdots)$$

In QFT, a quantized field can be viewed as a set of infinitely many quantum oscillators. The total energy of the field is:

$$E = \sum_{n=0}^{+\infty} \left(n + \frac{1}{2}\right)\omega \tag{1}$$

The sum is obviously divergent, which need to be regulated.

Question:

How to regulate the summation in Eq.(1) ?

ζ-function:

Consider the series

$$\zeta_R(s, a) = \sum_{n=0}^{+\infty} (n + a)^{-s} \tag{2}$$

where $\Re s > 1$ and $0 \leqslant a \leqslant 1$.Function $\zeta_R(s, a)$ is called the *generalized* Riemann ζ-function, or Hurwitz ζ-function.

The Riemann ζ-function is a special case of Eq.(2):

$$\zeta_R(s) = \zeta_R(s, 1) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \forall\ \Re s > 1 \tag{3}$$

In practical calculation, it is convenient to use an integral representation for $\zeta_R(s,a)$.

Note that for $\Re s > 0$, $\Re a > 0$ and $n \geqslant 0$, we have

$$\int_0^\infty dt t^{s-1} e^{-(n+a)t} = \frac{\Gamma(s)}{(n+a)^s}$$

Thereby,

$$\begin{aligned} \zeta_R(s,a) &= \sum_{n=0}^{\infty} (n+a)^{-s} = \sum_{n=0}^{\infty} \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{-(n+a)t} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{-at} \sum_{n=0}^{\infty} e^{-nt} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1} e^{-at}}{1-e^{-t}} \end{aligned}$$

That is to say,

$$\zeta_R(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1}\, \frac{e^{(1-a)t}}{e^t - 1} \tag{4}$$

Eq.(4) can be used to construct an analytical continuation of $\zeta_R(s,a)$ to the whole complex s-plane. In fact, $\zeta_R(s,a)$ has only one singularity $s = 1$ where it has a simple pole[1],

$$\zeta_R(s,a) \approx \frac{1-a}{s-1}$$

To study the analytical structure of $\zeta_R(s,a)$, we need the Bernoulli polynomials $B_n(x)$, which are defined as coefficients in the series:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi. \tag{5}$$

Hence,

$$B_n(x) = \left[\frac{\partial^n}{\partial t^n} \left(\frac{te^{xt}}{e^t - 1} \right) \right]_{t=0} \tag{6}$$

[1] The singularity appears at the lower limit $t \approx 0$ of integration in Eq.(4).

The first 5 Bernoulli polynomials are explicitly written down:

$$
\begin{aligned}
B_0(x) &= 1, & B_1(x) &= x - \frac{1}{2}, \\
B_2(x) &= x^2 - x + \tfrac{1}{6}, & B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \\
B_4(x) &= x^4 - 2x^3 + x^2 - \tfrac{1}{30}, & \cdots
\end{aligned}
\tag{7}
$$

The Bernoulli polynomials can be used to determine $\zeta_R(s, a)$ when s is a negative integer. To this aim let us represent Eq.(4) as

$$
\zeta_R(s, a) = \frac{1}{\Gamma(s)} \int_0^1 dt\, t^{s-1} \frac{e^{(1-a)t}}{e^t - 1} + I_1(s, a) \tag{8}
$$

where

$$
I_1(s, a) = \frac{1}{\Gamma(s)} \int_1^\infty dt\, t^{s-1} \frac{e^{(1-a)t}}{e^t - 1}
$$

Substitution of Eq.(5) into Eq.(8) gives,

$$\begin{aligned}\zeta_R(s,a) &= \frac{1}{\Gamma(s)}\int_0^1 dt\, t^{s-2}\sum_{n=0}^{\infty} B_n(1-a)\frac{t^n}{n!} + I_1(s,a) \\ &= \frac{1}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{B_n(1-a)}{n!}\int_0^1 dt\, t^{n+s-2} + I_1(s,a) \\ &= \frac{1}{\Gamma(s)}\sum_{n=0}^{N}\frac{B_n(1-a)}{n!}\int_0^1 dt\, t^{n+s-2} + I_1(s,a) + I_2(s,a)\end{aligned}$$

where N is a natural number such that $N + \Re s - 1 > 0$, and

$$I_2(s,a) = \frac{1}{\Gamma(s)}\sum_{n=N+1}^{\infty}\frac{B_n(1-a)}{n!}\int_0^1 dt\, t^{n+s-2}$$

Both $I_1(s,a)$ and $I_2(s,a)$ vanishes for $s = -k \in \mathbb{Z} \quad (N - k - 1 > 0)$ because

$$\Gamma(-k+\varepsilon) = \frac{(-1)^k}{k!}\left[\frac{1}{\varepsilon} - \gamma + \sum_{n=1}^{k}\frac{1}{n} + \mathscr{O}(\varepsilon)\right] \approx (-1)^k\frac{1}{k!\varepsilon}$$

but other factors *finite*.

Consequently,

$$\begin{aligned}
\zeta_R(-k+\varepsilon, a) &= \frac{1}{\Gamma(-k+\varepsilon)} \sum_{n=0}^{N} \frac{B_n(1-a)}{n!} \int_0^1 dt\, t^{n-k+\varepsilon-2} \\
&= \frac{1}{\Gamma(-k+\varepsilon)} \sum_{n=0}^{N} \frac{B_n(1-a)}{n!(n-k+\varepsilon-1)}\, t^{n-k+\varepsilon-1}\Big|_0^1
\end{aligned}$$

This integral is obviously divergent. The divergence appears at the lower limit $t = 0$. Rewrite the above formula as

$$\zeta_R(-k+\varepsilon, a) = (-1)^k k! \varepsilon \sum_{n=0}^{N} \frac{B_n(1-a)}{n!(n-k+\varepsilon-1)}\, t^{n-k+\varepsilon-1}\Big|_{\xi}^1$$

where $0 < \xi \ll 1$. The divergence behaviour of $\zeta_R(-k+\varepsilon, a)$ is then,

$$\zeta_R(-k+\varepsilon, a) \propto \frac{1}{\xi^{k+1}} \tag{9}$$

Omit such a divergence. The remaining part of $\zeta_R(-k+\varepsilon, a)$ reads,

$$
\begin{aligned}
\zeta_R(-k+\varepsilon, a) &= \frac{1}{\Gamma(-k+\varepsilon)} \sum_{n=0}^{N} \frac{B_n(1-a)}{n!(n-k+\varepsilon-2)} \\
&= (-1)^k k!\varepsilon \sum_{n=0}^{N} \frac{B_n(1-a)}{n!(n-k+\varepsilon-1)}
\end{aligned}
$$

All terms vanish in the summation except the term $n = k+1$. This is possible because $0 \leqslant n \leqslant N$ while $N > k+1$.

Therefore,

$$
\zeta_R(-k+\varepsilon, a) = (-1)^k \frac{B_{k+1}(1-a)}{k+1}
$$

It follow from Eq.(6) that $B_n(x) = (-1)^n B_n(1-x)$. We finally have:

$$\zeta_R(-k+\varepsilon, a) = -\frac{B_{k+1}(a)}{k+1} \tag{10}$$

As two corollaries of Eq.(9), we have[2]:

$$\sum_{n=1}^{\infty} n = \zeta_R(-1, 1) = -\frac{1}{2}B_2(1) = -\frac{1}{12}$$

and

$$\sum_{n=1}^{\infty}\left(n-\frac{1}{2}\right) = \sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right) = \zeta_R(-1, 1/2) = -\frac{1}{2}B_2(1/2) = \frac{1}{24}$$

These regulation formulae could alternatively be obtained in a simple manner. See below.

[2] $B_2(x) = x^2 - x + \frac{1}{6}$.

Consider the *regulation* of divergent summation

$$\sum_{n=1}^{\infty}(n-\theta)$$

for $0 < \theta < 1$. Notice that for $0 < \xi \ll 1$,

$$\sum_{n=1}^{\infty} e^{-(n-\theta)\xi} = e^{\theta\xi}\left(-1 + \frac{1}{1-e^{-\xi}}\right) = \frac{e^{\theta\xi}}{e^{\xi}-1}$$

Thereby,

$$\begin{aligned}\sum_{n=1}^{\infty}(n-\theta)e^{-(n-\theta)\xi} &= -\frac{d}{d\xi}\left(\sum_{n=1}^{\infty} e^{-(n-\theta)\xi}\right) = \frac{(1-\theta)e^{(1+\theta)\xi} + \theta e^{\theta\xi}}{(e^{\xi}-1)^2} \\ &\approx \left[(1-\theta)\left(1 + (1+\theta)\xi + \frac{1}{2}(1+\theta)^2\xi^2 + \cdots\right)\right. \\ &\quad \left. + \theta\left(1 + \theta\xi + \frac{1}{2}\theta^2\xi^2 + \cdots\right)\right] \\ &\quad \cdot \left(\xi + \frac{\xi^2}{2} + \frac{\xi^3}{6} + \cdots\right)^{-2}\end{aligned}$$

$$\approx \frac{1}{\xi^2}\left(1+\xi+\frac{1}{2}(1+\theta-\theta^2)\xi^2+\cdots\right)\left(1-\xi+\frac{5}{12}\xi^2+\cdots\right)$$

$$\approx \frac{1}{\xi^2}-\frac{1}{12}(6\theta^2-6\theta+1)+\mathscr{O}(\xi)$$

The *finite* part of this sum at the limit $\xi \to 0$ is,

$$\sum_{n=1}^{\infty}(n-\theta) = -\frac{1}{12}(6\theta^2-6\theta+1) \tag{11}$$

Important !

$$\sum_{n=1}^{\infty}\left(n-\frac{1}{2}\right)=\frac{1}{24}, \qquad \sum_{n=1}^{\infty} n = -\frac{1}{12}. \tag{12}$$

Eq.(12) is very useful in study *superstring* theory, where the first formula does also describe the regulation of total energy of a quantized field.

Theta functions:

Theta functions are necessary mathematical tools for studying string theory.

Let us begin with solving the one-dimensional Schrödinger equation[3] of a free particle confined in the interval $0 \leqslant \nu \leqslant 1/2$,

$$i\partial_\tau \psi(\tau,\nu) = \frac{1}{4\pi}\partial_\nu^2 \psi(\tau,\nu) \tag{13}$$

under the boundary conditions $\psi(\tau,0) = \psi(\tau,1/2) = 0$.
We make the separation ansatz,

$$\psi(\tau,\nu) = T(\tau)\Phi(\nu)$$

Substitution of this trial solution into Eq.(13) leads to,

$$\frac{i}{T}\frac{dT}{d\tau} = \frac{1}{4\pi\Phi}\frac{d^2\Phi}{d\nu^2} = -n^2\pi$$

[3] More accurate, Eq.(13) is the time reversal image of Schrödinger equation.

Hence,

$$T(\tau) = C\, e^{in^2\pi\tau} \tag{14}$$

$$\frac{d^2\Phi}{d\nu^2} + 4\pi^2 n^2 \Phi = 0 \tag{15}$$

When $n^2 \geqslant 0$, the solution of Eq.(15) is

$$\Phi(\nu) = A\sin(2\pi n\nu) + B\cos(2\pi n\nu)$$

The boundary conditions $\Phi(0) = \Phi(1/2) = 0$ require $B = 0$ and $n = 0, \pm 1, \pm 2, \cdots$.

The general solution to Schrödinger Eq.(13) reads,

$$\begin{aligned} \psi(\tau,\nu) &= \sum_{n=-\infty}^{+\infty} A_n \sin(2\pi n\nu) e^{in^2\pi\tau} \\ &= \frac{1}{2i} \sum_{n=-\infty}^{+\infty} A_n \Big[e^{\pi i n^2\tau + 2\pi i n\nu} - e^{\pi i n^2\tau - 2\pi i n\nu} \Big] \\ &= \frac{1}{2i} \sum_{n=-\infty}^{+\infty} (A_n - A_{-n}) \exp(\pi i n^2\tau + 2\pi i n\nu) \end{aligned} \tag{16}$$

If by some physics we have $(A_n - A_{-n}) = 2i$ for all possible n, we get,

$$\psi(\tau, \nu) = \sum_{n=-\infty}^{+\infty} \exp(\pi i n^2 \tau + 2\pi i n \nu)$$

This is just the so-called basic *theta function*.

According to Polchinski,[4] the basic theta function is defined as:

$$\vartheta(\nu, \tau) = \sum_{n=-\infty}^{+\infty} \exp(\pi i n^2 \tau + 2\pi i n \nu) \tag{17}$$

[4] J. Polchinski, String Theory, CUP, 2003, Vol1.

Manifestly, it has the periodicity properties,

$$\begin{aligned}\vartheta(\nu+1,\tau) &= \sum_{n=-\infty}^{+\infty} \exp[\pi i n^2\tau + 2\pi i n(\nu+1)] \\ &= \sum_{n=-\infty}^{+\infty} e^{2\pi i n} \exp(\pi i n^2\tau + 2\pi i n\nu) \\ &= \vartheta(\nu,\tau) \end{aligned} \tag{18}$$

$$\begin{aligned}\vartheta(\nu+\tau,\tau) &= \sum_{n=-\infty}^{+\infty} \exp[\pi i n^2\tau + 2\pi i n(\nu+\tau)] \\ &= e^{-\pi i\tau} \sum_{n=-\infty}^{+\infty} \exp[\pi i(n+1)^2\tau + 2\pi i n\nu] \\ &= e^{-\pi i\tau - 2\pi i\nu}\vartheta(\nu,\tau) \end{aligned} \tag{19}$$

The theta function has a unique zero, up to the periodicity properties (18) and (19), at $\nu = (1+\tau)/2$.

To confirm this conclusion *conveniently*, we rewrite $\vartheta(\nu,\tau)$ as an infinite product,

$$\vartheta(\nu,\tau) = \prod_{m=1}^{+\infty}(1-q^m)(1+zq^{m-1/2})(1+z^{-1}q^{m-1/2}) \tag{20}$$

where,

$$q = \exp(2\pi i\tau), \quad z = \exp(2\pi i\nu). \tag{21}$$

In terms of Eq.(20), we have:

$$\begin{aligned}
\vartheta(\nu,\tau)\big|_{\nu=(1+\tau)/2} &= \prod_{m=1}^{+\infty}(1-e^{2\pi i m\tau})(1+e^{2\pi i\nu+2\pi i m\tau-\pi i\tau}) \\
&\quad \cdot (1+e^{-2\pi i\nu+2\pi i m\tau-\pi i\tau})\Big|_{\nu=(1+\tau)/2} \\
&= \prod_{m=1}^{+\infty}(1-e^{2\pi i m\tau})(1-e^{2\pi i m\tau})(1-e^{2\pi i(m-1)\tau}) \\
&= 0
\end{aligned}$$

Question:

Why $\vartheta(\nu, \tau)$ can equivalently be written as Eq.(20) ?

The equivalence is based on the so-called Jacobi triple product identity. Assume $q, w \in \mathbb{C}$, $|q| < 1$ and $w \neq 0$. Jacobi triple product identity reads:

$$\prod_{n=1}^{+\infty}(1 - q^{2n})(1 + q^{2n-1}w)(1 + q^{2n-1}w^{-1}) = \sum_{n=-\infty}^{+\infty} q^{n^2} w^n \qquad (22)$$

To show Eq.(22), we have to show the *Euler's lemma* at first.

1. For $q, w \in \mathbb{C}$ with $|q| < 1$, we have:

$$\prod_{n=0}^{+\infty}(1 + q^n w) = \sum_{m=0}^{+\infty}\left[\frac{q^{m(m-1)/2}w^m}{\prod_{k=1}^{m}(1 - q^k)}\right] \tag{23}$$

2. If also $|w| < 1$, then,

$$\prod_{n=0}^{+\infty}\frac{1}{(1 + q^n w)} = \sum_{m=0}^{+\infty}\left[\frac{(-1)^m w^m}{\prod_{k=1}^{m}(1 - q^k)}\right] \tag{24}$$

Focus on deriving the first formula in Eq.(23). Let

$$f(q, w) = \prod_{n=0}^{+\infty}(1 + q^n w) \tag{25}$$

We see that

$$f(q, qw) = \prod_{n=0}^{+\infty}(1+q^{n+1}w) = \prod_{m=1}^{+\infty}(1+q^m w) = \frac{\prod\limits_{m=0}^{+\infty}(1 + q^m w)}{1 + w} = \frac{f(q, w)}{1 + w}$$

i.e.,

$$f(q, w) = (1 + w)f(q, qw) \tag{26}$$

Let[5]

$$f(q, w) = \sum_{n=0}^{+\infty} a_n(q)w^n \tag{27}$$

we see from Eq.(25) that $a_0(q) = 1$. The functional equation in Eq.(26) yields,

$$\sum_{n=0}^{+\infty} a_n(q)w^n = (1 + w) \sum_{n=0}^{+\infty} a_n(q)q^n w^n$$

This implies that

$$a_n(q) = a_n(q)q^n + a_{n-1}(q)q^{n-1}, \quad \forall\ n \geqslant 1$$

Equivalently,

$$a_n(q) = \frac{q^{n-1}}{1 - q^n} a_{n-1}(q), \qquad (n \geqslant 1). \tag{28}$$

[5] The function $f(q, w)$ defined in Eq.(25) converges absolutely for $|q| < 1$ and any $w \in \mathbb{C}$ because of the convergence of the series $\sum_{n=0}^{\infty} |q^n w|$.

Since $a_0(q) = 1$, we get from Eq.(28) that,

$$\begin{aligned} a_n(q) &= \frac{q^{n-1}}{1-q^n} a_{n-1}(q) \\ &= \frac{q^{n-1}}{1-q^n} \frac{q^{n-2}}{1-q^{n-1}} a_{n-2}(q) \\ &= \frac{q^{n-1}}{1-q^n} \frac{q^{n-2}}{1-q^{n-1}} \frac{q^{n-3}}{1-q^{n-2}} a_{n-3}(q) \\ &= \cdots \\ &= \frac{q^{(n-1)+(n-2)+\cdots+1}}{(1-q^n)(1-q^{n-1})\cdots(1-q)} a_0(q) = \frac{q^{n(n-1)/2}}{\prod_{k=1}^{n}(1-q^k)} \end{aligned}$$

Substitution of these coefficients into Eq.(27) gives the Euler's first lemma (23):

$$\prod_{n=0}^{+\infty}(1+q^n w) = f(q,w) = \sum_{n=0}^{+\infty} a_n(q) w^n = \sum_{m=0}^{+\infty}\left[\frac{q^{m(m-1)/2} w^m}{\prod_{k=1}^{m}(1-q^k)}\right]$$

Next, we consider

$$g(q,w) = \prod_{n=0}^{+\infty} \frac{1}{1+q^n w} \tag{29}$$

For $|q| < 1$ and $|w| < 1$ this product converges absolutely because of the convergence of

$$\sum_{n=0}^{+\infty} \left|1 - \frac{1}{1+q^n w}\right| = \sum_{n=0}^{+\infty} \left|\frac{q^n w}{1+q^n w}\right| \leqslant \left(\frac{1}{1-|w|}\right) \sum_{n=0}^{+\infty} |q^n w|$$

Therefore, for any q with $|q| < 1$, $g(q,w)$ is an analytical function of w with a power expansion

$$g(q,w) = \sum_{n=0}^{+\infty} b_n(q) w^n \tag{30}$$

which is valid for $|w| < 1$, and $b_0(q) = 1$.

Eq.(29) implies that,

$$g(q,qw) = \prod_{n=0}^{+\infty} \frac{1}{1+q^{n+1}w} = \prod_{m=1}^{+\infty} \frac{1}{1+q^m w} = \prod_{m=0}^{+\infty} \frac{(1+w)}{1+q^m w} = (1+w)g(q,w) \tag{31}$$

By combining Eqs.(30) and (31),

$$b_m(q) = -\frac{b_{m-1}(q)}{1-q^m}, \quad \forall\, m \geqslant 1$$

Recall that $b_0(q) = 1$, we have:

$$b_m(q) = \frac{(-1)^m}{\prod_{k=1}^m (1-q^k)} \tag{32}$$

Hence, the Euler's second lemma given in Eq.(24) results in:

$$\prod_{n=0}^{+\infty} \frac{1}{1+q^n w} = g(q,w) = \sum_{n=0}^{+\infty} b_n(q) w^n = \sum_{m=0}^{+\infty} \left[\frac{(-1)^m w^m}{\prod_{k=1}^m (1-q^k)} \right]$$

It is time to show Jacobi's triple product identity (22). Assume $|q| < 1$ and $w \in \mathbb{C}$. From Eq.(23) we have

$$\begin{aligned}\prod_{n=0}^{+\infty}(1+q^{2n+1}w) &= \prod_{n=0}^{+\infty}[1+(q^2)^n(qw)] \\ &= \sum_{m=0}^{+\infty}\left[\frac{(q^2)^{m(m-1)/2}(qw)^m}{\prod_{k=1}^{m}(1-q^{2k})}\right] \\ &= \sum_{m=0}^{+\infty}\left[\frac{q^{m^2}w^m}{\prod_{k=1}^{+\infty}(1-q^{2k})}\cdot\prod_{k=m+1}^{+\infty}(1-q^{2k})\right] \\ &= \frac{1}{\prod_{k=1}^{+\infty}(1-q^{2k})}\cdot\sum_{m=0}^{+\infty}q^{m^2}w^m\cdot\prod_{n=0}^{+\infty}(1-q^{2n+2m+2})\end{aligned} \tag{33}$$

The summation over m in the RHS of Eq.(33) can be extended from $0 \leqslant m < +\infty$ to $-\infty < m < +\infty$[6].

[6]This is because for $m < 0$ the product inside the infinite summation vanishes identically because of the factor with $n = -m-1$.

Therefore,

$$\prod_{n=0}^{+\infty}(1+q^{2n+1}w) = \frac{1}{\prod_{k=1}^{+\infty}(1-q^{2k})} \cdot \sum_{m=-\infty}^{+\infty} q^{m^2} w^m \cdot \prod_{n=0}^{+\infty}(1-q^{2n+2m+2}) \tag{34}$$

Applying (23) once more, we get

$$\prod_{n=0}^{+\infty}(1-q^{2n+2m+2}) = \prod_{n=0}^{+\infty}[1+(q^2)^n(-q^{2m+2})] = \sum_{i=0}^{+\infty} \frac{(q^2)^{i(i-1)/2}(-q^{2m+2})^i}{\prod_{j=1}^{i}(1-q^{2j})}$$

Combined with Eq.(34), it yields:

$$\prod_{n=0}^{+\infty}(1+q^{2n+1}w) = \prod_{k=1}^{+\infty} \frac{1}{(1-q^{2k})} \sum_{m=-\infty}^{+\infty} \sum_{i=0}^{+\infty} \frac{(-1)^i q^{m^2+i^2+2im+i} w^m}{\prod_{j=1}^{i}(1-q^{2j})} \tag{35}$$

We want to interchange the summation order in the double sum, and for this purpose we need absolute convergence. We have convergence for all $w \in \mathbb{C}$. But an estimate of the double sum in reversed order of summation shows that absolute convergence does only hold if $|q| < 1$ and $|w| > |q|$. Under this assumption we get

$$
\begin{aligned}
&\prod_{n=0}^{+\infty}(1+q^{2n+1}w) \\
&\qquad = \prod_{k=1}^{+\infty}\frac{1}{(1-q^{2k})}\sum_{i=0}^{+\infty}\frac{(-1)^i q^i}{\prod_{j=1}^{i}(1-q^{2j})}\sum_{m=-\infty}^{+\infty} q^{(m+i)^2}w^m \\
&\qquad = \left(\sum_{m=-\infty}^{+\infty} q^{m^2}w^m\right)\prod_{k=1}^{+\infty}\frac{1}{(1-q^{2k})}\sum_{i=0}^{+\infty}\frac{(-1)^i (q/w)^i}{\prod_{j=1}^{i}(1-q^{2j})}
\end{aligned}
\tag{36}
$$

Since $|q/w| < 1$, the Euler's second lemma (24) is valid,

$$
\sum_{i=0}^{+\infty}\frac{(-1)^i (q/w)^i}{\prod_{j=1}^{i}(1-q^{2j})} = \prod_{n=0}^{+\infty}\frac{1}{1+q^{2n+1}w^{-1}}
$$

This yields the expected Triple Product Identity:

$$\sum_{m=-\infty}^{+\infty} q^{m^2} w^m = \prod_{n=1}^{+\infty} (1 - q^{2n})(1 + q^{2n-1} w)(1 + q^{2n-1} w^{-1}) \tag{37}$$

under the assumptions that $|q| < 1$ and $|q| < |w|$.

By the principle of analytic continuation it holds for $|q| \leqslant 1$ and all $w \neq 0$. In Eq.(37), if we replace q with $\sqrt{q}$ and w with z, we have

$$\sum_{m=-\infty}^{+\infty} q^{m^2/2} z^m = \prod_{n=1}^{+\infty} (1 - q^n)(1 + q^{n-1/2} z)(1 + q^{n-1/2} z^{-1}) \tag{38}$$

Let $q = \exp(2\pi i \tau)$ and $z = \exp(2\pi i \nu)$ in Eq.(38). It becomes

$$\sum_{m=-\infty}^{+\infty} e^{\pi i m^2 \tau + 2\pi i m \nu} = \prod_{n=1}^{+\infty} (1 - e^{2\pi i n \tau})(1 + e^{\pi i (2n-1)\tau + 2\pi i \nu})(1 + e^{\pi i (2n-1)\tau - 2\pi i \nu}) \tag{39}$$

The LHS and RHS of Eq.(39) do just correspond to two equivalent expressions of $\vartheta(\nu, \tau)$, respectively.

Modular transformations:

In $\vartheta(\nu,\tau)$, τ is called the *modular* parameter.
Modular transformations are generated by,

$$\tau \to \tau + 1, \qquad \tau \to -1/\tau \tag{40}$$

under which the basic theta function $\vartheta(\nu,\tau)$ transforms as follows:

$$\vartheta(\nu,\tau+1) = \vartheta(\nu+1/2,\tau) \tag{41}$$

$$\vartheta(\nu/\tau,-1/\tau) = \sqrt{-i\tau}\,\exp(\pi i\nu^2/\tau)\vartheta(\nu,\tau) \tag{42}$$

Eq.(41) can easily be verified by using definition,

$$\begin{aligned}
\vartheta(\nu,\tau+1) &= \sum_{n=-\infty}^{+\infty} \exp[\pi i n^2(\tau+1) + 2\pi i n\nu] \\
&= \sum_{n=-\infty}^{+\infty} \exp[\pi i n^2\tau + 2\pi i n(\nu+1/2) + \pi i n(n-1)] \\
&= \sum_{n=-\infty}^{+\infty} \exp[\pi i n^2\tau + 2\pi i n(\nu+1/2)] = \vartheta(\nu+1/2,\tau)
\end{aligned}$$

Next consider the proof of Eq.(42). Let $f(x)$ be a continuous function of x defined for $-\infty < x < +\infty$, From which we define a periodic function

$$g(x) = \sum_{n=-\infty}^{+\infty} f(x+n) \tag{43}$$

so that $g(x) = g(x+m)$ for any $m \in \mathbb{Z}$. A periodic function is always expanded as a Fourier series in one period,

$$g(x) = \sum_{n=-\infty}^{+\infty} a_n e^{2\pi inx}, \quad 0 \leqslant x \leqslant 1. \tag{44}$$

Coefficients a_n are found to be

$$\begin{aligned} a_n &= \int_0^1 g(x) e^{-2\pi inx} dx = \int_0^1 \left[\sum_{k=-\infty}^{+\infty} f(x+k) \right] e^{-2\pi inx} dx \\ &= \sum_{k=-\infty}^{+\infty} \int_0^1 f(x+k) e^{-2\pi inx} dx = \sum_{k=-\infty}^{+\infty} \int_k^{k+1} f(x) e^{-2\pi inx} dx \\ &= \int_{-\infty}^{+\infty} f(x) e^{-2\pi inx} dx \end{aligned} \tag{45}$$

Consequently, we have the following identity:

$$\sum_{n=-\infty}^{+\infty} f(x+n) = \sum_{n=-\infty}^{+\infty} e^{2\pi inx} \int_{-\infty}^{+\infty} f(y) e^{-2\pi iny} dy \tag{46}$$

The special case $x = 0$ of Eq.(46) is called Poisson resummation formula.

Poisson resummation formula:

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(y) e^{-2\pi iny} dy \tag{47}$$

Now we try to verify Eq.(42) by using Eq.(47). Taking

$$f(n) = \exp\left[-\frac{\pi in^2}{\tau} + 2\pi in\frac{\nu}{\tau} \right]$$

Defining an auxiliary integral:

$$\begin{aligned}
I_n &= \int_{-\infty}^{+\infty} f(y) e^{-2\pi i n y} dy \\
&= \int_{-\infty}^{+\infty} \exp\left[-\frac{\pi i y^2}{\tau} + 2\pi i y\left(\frac{\nu}{\tau} - n\right)\right] dy \\
&= \exp\left[\frac{\pi i}{\tau}(\nu - n\tau)^2\right] \int_{-\infty}^{+\infty} dy \exp\left[-\frac{\pi i}{\tau}(y - \nu + n\tau)^2\right] \\
&= e^{\frac{\pi i \nu^2}{\tau}} \exp(\pi i n^2 \tau - 2\pi i n \nu) \int_{-\infty}^{+\infty} d\xi \exp\left[-\frac{\pi i}{\tau}\xi^2\right] \\
&= \sqrt{-i\tau}\, e^{\frac{\pi i \nu^2}{\tau}} \exp(\pi i n^2 \tau - 2\pi i n \nu)
\end{aligned}$$

In the last step, we have used the Fresnel integral formula,

$$\int_{-\infty}^{+\infty} e^{itx^2} dx = \int_{-\infty}^{+\infty} e^{-(-it)x^2} dx = \sqrt{\frac{\pi}{-it}} = \sqrt{i\pi/t} \qquad (48)$$

Employment of Eq.(47) leads to,

$$
\begin{aligned}
\vartheta(\nu/\tau, -1/\tau) &= \sum_{n=-\infty}^{+\infty} f(n) \\
&= \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(y) e^{-2\pi i n y} dy \\
&= \sum_{n=-\infty}^{+\infty} I_n \\
&= \sqrt{-i\tau} e^{\frac{\pi i \nu^2}{\tau}} \sum_{n=-\infty}^{+\infty} \exp(\pi i n^2 \tau - 2\pi i n \nu) \\
&= \sqrt{-i\tau} e^{\frac{\pi i \nu^2}{\tau}} \sum_{n=-\infty}^{+\infty} \exp(\pi i n^2 \tau + 2\pi i n \nu) \\
&= \sqrt{-i\tau} e^{\frac{\pi i \nu^2}{\tau}} \vartheta(\nu, \tau)
\end{aligned}
$$

This is very the content of modular property in Eq.(42).

It is also necessary to define the *theta functions with characteristics* in superstring theory,

$$\begin{aligned}
&\vartheta\left[\begin{array}{c} a \\ b \end{array}\right](\nu,\tau) \\
&\quad = \exp\left[\pi i a^2\tau + 2\pi i a(\nu+b)\right]\vartheta(\nu + a\tau + b, \tau) \\
&\quad = \sum_{n=-\infty}^{+\infty} \exp\left[\pi i (n+a)^2\tau + 2\pi i (n+a)(\nu+b)\right]
\end{aligned} \tag{49}$$

where the parameters a and b take their values of either 0 or $1/2$.

Other common notations are as follows:

$$
\begin{aligned}
\vartheta_{00}(\nu,\tau) &= \vartheta_3(\nu|\tau) = \vartheta\left[\begin{array}{c} 0 \\ 0 \end{array}\right](\nu,\tau) = \sum_{n=-\infty}^{+\infty} q^{n^2/2} z^n \\
\vartheta_{01}(\nu,\tau) &= \vartheta_4(\nu|\tau) = \vartheta\left[\begin{array}{c} 0 \\ 1/2 \end{array}\right](\nu,\tau) = \sum_{n=-\infty}^{+\infty} (-1)^n q^{n^2/2} z^n \\
\vartheta_{10}(\nu,\tau) &= \vartheta_2(\nu|\tau) = \vartheta\left[\begin{array}{c} 1/2 \\ 0 \end{array}\right](\nu,\tau) \\
&= \sum_{n=-\infty}^{+\infty} q^{(n-1/2)^2/2} z^{n-1/2} \\
\vartheta_{11}(\nu,\tau) &= -\vartheta_1(\nu|\tau) = \vartheta\left[\begin{array}{c} 1/2 \\ 1/2 \end{array}\right](\nu,\tau) \\
&= -i \sum_{n=-\infty}^{+\infty} (-1)^n q^{(n-1/2)^2/2} z^{n-1/2} \qquad (50)
\end{aligned}
$$

where $q = \exp(2\pi i\tau)$ and $z = \exp(2\pi i\nu)$.

They have also the following infinite product representations:

$$
\begin{aligned}
\vartheta_{00}(\nu,\tau) &= \prod_{m=1}^{+\infty}(1-q^m)(1+zq^{m-1/2})(1+z^{-1}q^{m-1/2}) \\
\vartheta_{01}(\nu,\tau) &= \prod_{m=1}^{+\infty}(1-q^m)(1-zq^{m-1/2})(1-z^{-1}q^{m-1/2}) \\
\vartheta_{10}(\nu,\tau) &= 2e^{\pi i\tau/4}\cos(\pi\nu)\prod_{m=1}^{+\infty}(1-q^m)(1+zq^m)(1+z^{-1}q^m) \\
\vartheta_{11}(\nu,\tau) &= -2e^{\pi i\tau/4}\sin(\pi\nu)\prod_{m=1}^{+\infty}(1-q^m)(1-zq^m)(1-z^{-1}q^m)
\end{aligned}
\tag{51}
$$

It follows obviously from the last formula in Eq.(51) that,

$$
\vartheta_{11}(0,\tau) = 0. \tag{52}
$$

The modular transformations of these theta functions read,

$$\begin{aligned}
\vartheta_{00}(\nu,\tau+1) &= \vartheta_{01}(\nu,\tau)\\
\vartheta_{01}(\nu,\tau+1) &= \vartheta_{00}(\nu,\tau)\\
\vartheta_{10}(\nu,\tau+1) &= \exp(\pi i/4)\vartheta_{10}(\nu,\tau)\\
\vartheta_{11}(\nu,\tau+1) &= \exp(\pi i/4)\vartheta_{11}(\nu,\tau)
\end{aligned} \tag{53}$$

and

$$\begin{aligned}
\vartheta_{00}(\nu/\tau,-1/\tau) &= (-i\tau)^{1/2}\exp(\pi i\nu^2/\tau)\vartheta_{00}(\nu,\tau)\\
\vartheta_{01}(\nu/\tau,-1/\tau) &= (-i\tau)^{1/2}\exp(\pi i\nu^2/\tau)\vartheta_{10}(\nu,\tau)\\
\vartheta_{10}(\nu/\tau,-1/\tau) &= (-i\tau)^{1/2}\exp(\pi i\nu^2/\tau)\vartheta_{01}(\nu,\tau)\\
\vartheta_{11}(\nu/\tau,-1/\tau) &= -i(-i\tau)^{1/2}\exp(\pi i\nu^2/\tau)\vartheta_{11}(\nu,\tau)
\end{aligned} \tag{54}$$

Besides, The theta functions satisfy Jacobi's *abstruse* identity,

$$\vartheta_{00}^4(0,\tau)-\vartheta_{01}^4(0,\tau)-\vartheta_{10}^4(0,\tau)=0 \tag{55}$$

Finally, the Dedekind eta function is defined as:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{+\infty} (1 - q^n) = \left[-\frac{1}{2\pi} \partial_\nu \vartheta_{11}(0, \tau) \right]^{\frac{1}{3}} \tag{56}$$

Its modular properties are:

$$\begin{aligned} \eta(\tau + 1) &= e^{i\pi/12}\, \eta(\tau) \\ \eta(-1/\tau) &= \sqrt{-i\tau}\, \eta(\tau) \end{aligned} \tag{57}$$

Homework:

1. Let $B_n(x)$ be the Bernoulli polynomials defined by
$$B_n(x) = \left[\frac{\partial^n}{\partial t^n} \left(\frac{te^{xt}}{e^t - 1} \right) \right]_{t=0}$$
Show that $B_n(x) = (-1)^n B_n(1-x)$.
2. Regulate the divergent summation
$$\sum_{n=0}^{+\infty} (n + 1/3)$$
3. Start from the Jacobi's triple product identity to show the equivalence between two expressions of the basic theta function.
4. Check the following modular properties of the theta functions:
$$\vartheta_{00}(\nu/\tau, -1/\tau) = (-i\tau)^{1/2} \exp(\pi i \nu^2/\tau) \vartheta_{00}(\nu, \tau)$$
$$\vartheta_{01}(\nu/\tau, -1/\tau) = (-i\tau)^{1/2} \exp(\pi i \nu^2/\tau) \vartheta_{10}(\nu, \tau)$$
$$\vartheta_{10}(\nu/\tau, -1/\tau) = (-i\tau)^{1/2} \exp(\pi i \nu^2/\tau) \vartheta_{01}(\nu, \tau)$$
$$\vartheta_{11}(\nu/\tau, -1/\tau) = -i(-i\tau)^{1/2} \exp(\pi i \nu^2/\tau) \vartheta_{11}(\nu, \tau)$$