LECTURE 5: LIE GROUPS AND THEIR LIE ALGEBRAS

1. Lie groups

Definition 1.1. A *Lie group* G is a smooth manifold equipped with a group structure so that the group multiplication

$$\mu : G \times G \to G, \quad (g_1, g_2) \mapsto g_1 \cdot g_2$$

is a smooth map.

Example. Here are some basic examples:

- $\mathbb{R}^n$, considered as a group under addition.
- $\mathbb{R}^* = \mathbb{R} - \{0\}$, considered as a group under multiplication.
- S^1, Considered as a group under multiplication.
- Linear Lie groups $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$, $O(n)$ etc.
- If M and N are Lie groups, so is their product $M \times N$.

Remarks. (1) The famous Hilbert's fifth problem, solved by Gleason and Montgomery-Zippin in the 1950's, confirms that any locally Euclidean group (=topological group whose underlying space is a topological manifold) is "actually" a Lie group, and moreover, The underlying space of any Lie group is in fact an analytic manifold, and the group operations are analytic. *analytic*. One can define analytic maps between analytic manifolds using local charts as in the smooth case.)

(2) Not every smooth manifold admits a Lie group structure. For example, the only spheres that admit a Lie group structure are S^0, S^1 and S^3; among all the compact 2 dimensional surfaces the only one admits a Lie group structure is $T^2 = S^1 \times S^1$. There are many constraints for a manifold to be a Lie group. For example, a Lie group must be analytic manifold, and the tangent bundle of a Lie group is always trivial: $TG \simeq G \times \mathbb{R}^n$. In particular, any Lie group is orientable.

Now suppose G is a Lie group. For any elements $a, b \in G$, there are two natural maps, the left multiplication

$$L_a : G \to G, \quad g \mapsto a \cdot g$$

and the right multiplication

$$R_b : G \to G, \quad g \mapsto g \cdot b.$$

It is obviously that $L_a^{-1} = L_{a^{-1}}$ and $R_b^{-1} = R_{b^{-1}}$. So both L_a and R_b are diffeomorphisms. Moreover, L_a and R_b commutes with each other: $L_a R_b = R_b L_a$.

Lemma 1.2. *The differential of the multiplication map* $\mu : G \times G \to G$ *is given by*

$$d\mu_{a,b}(X_a, Y_b) = (dR_b)_a(X_a) + (dL_a)_b(Y_b)$$

for any $(X_a, Y_b) \in T_aG \times T_bG \simeq T_{(a,b)}(G \times G)$.

Proof. Notice that as a function of a, $\mu(a,b) = R_b(a)$, and as a function of y, $\mu(a,b) = L_a(b)$. Thus for any function $f \in C^\infty(G)$,

$$\begin{aligned}(d\mu_{a,b}(X_a, Y_b))(f) &= (X_a, Y_b)(f \circ \mu(a,b)) \\ &= X_a(f \circ R_b(a)) + Y_b(f \circ L_a(b)) \\ &= (dR_b)_a(X_a)(f) + (dL_a)_b(Y_b)(f)\end{aligned}$$

□

As an application, we can prove

Proposition 1.3. *For any Lie group* G*, the group inversion map*

$$i : G \to G, \quad g \mapsto g^{-1}$$

is smooth.

Proof. Consider the map

$$f : G \times G \to G \times G, (a,b) \mapsto (a, ab).$$

It is obviously a bijective smooth map. According to lemma above, the derivative of f is

$$df_{(a,b)} : T_aG \times T_bG \to T_aG \times T_{ab}G, \quad (X_a, Y_b) \mapsto (X_a, (dR_b)_a(X_a) + (dL_a)_b(Y_b)).$$

This is a bijective linear map since dR_b, dL_a are. It follows by inverse function theorem that f is locally a diffeomorphism near each pair (a,b). However, since f is globally bijective, it must be a globally diffeomorphism. We conclude that its inverse,

$$f^{-1} : G \times G \to G \times G, \quad (a,c) \mapsto (a, a^{-1}c)$$

is a diffeomorphism. Thus the inversion map i, as the composition

$$\begin{array}{ccccccc} G & \hookrightarrow & G \times G & \xrightarrow{f^{-1}} & G \times G & \xrightarrow{\pi_2} & G \\ g & \longmapsto & (g,e) & \longmapsto & (g, g^{-1}) & \longmapsto & g^{-1} \end{array}$$

is smooth. □

Definition 1.4. A *Lie group homomorphism* between two Lie groups is a smooth map which is also a homomorphism of groups.

For example, the map $\varphi : \mathbb{R} \to \mathbb{S}^1, t \mapsto e^{it}$ is a Lie group homomorphism.

Similarly one can define Lie group isomorphisms.

2. Lie algebras associated to Lie groups

Suppose G is a Lie group. From the left translations L_a one can, for any vector $X_e \in T_eG$, define a vector field X on G by

$$X_a = (dL_a)(X_e).$$

It is not surprising that X is invariant under any left translation:

$$(dL_a)(X_b) = dL_a \circ dL_b(X_e) = dL_{ab}(X_e) = X_{ab}.$$

Definition 2.1. A *left invariant vector field* on a Lie group G is a smooth vector field X on G which satisfies $(dL_a)(X_b) = X_{ab}$.

So any tangent vector $X_e \in T_eG$ determines a left invariant vector field on G. Conversely, any left invariant vector field X is uniquely determined by its "value" X_e at $e \in G$, since for any $a \in G$, $X(a) = (dL_a)X_e$. We will denote the set of all left invariant vector fields on Lie group G by $\mathfrak{g}$, i.e.

$$\mathfrak{g} = \{X \in \Gamma^\infty(TG) \mid X \text{ is left invariant}\}.$$

Obviously $\mathfrak{g}$ is a vector subspace of $\Gamma^\infty(TG)$. Moreover, as a vector space, $\mathfrak{g}$ is isomorphic to T_eG. In particular, $\dim \mathfrak{g} = \dim G$.

We have already seen that $\Gamma^\infty(TG)$ has a Lie bracket operation $[\cdot,\cdot]$ which makes $\Gamma^\infty(TG)$ into an infinitely dimensional Lie algebra.

Proposition 2.2. *If $X, Y \in \mathfrak{g}$, so is their Lie bracket $[X, Y]$.*

Proof. We only need to show that $[X, Y]$ is left-invariant if X and Y are. We first notice

$$Y(f \circ L_a)(b) = Y_b(f \circ L_a) = (dL_a)_b(Y_b)f = Y_{ab}f = (Yf)(L_a b) = (Yf) \circ L_a(b)$$

for any smooth function $f \in C^\infty(G)$. Thus

$$X_{ab}(Yf) = (dL_a)_b(X_b)(Yf) = X_b((Yf) \circ L_a) = X_b Y(f \circ L_a).$$

Similarly $Y_{ab}Xf = Y_b X(f \circ L_a)$. Thus

$$dL_a([X, Y]_b)f = X_b Y(f \circ L_a) - Y_b X(f \circ L_a) = X_{ab}(Yf) - Y_{ab}Xf = [X, Y]_{ab}(f).$$

□

It follows that the space $\mathfrak{g}$ of all left invariant vector fields on G together with the Lie bracket operation $[\cdot,\cdot]$ is an n-dimensional *Lie subalgebra* of the Lie algebra of all smooth vector fields $\Gamma^\infty(TG)$.

Definition 2.3. $\mathfrak{g}$ of is called *the Lie algebra of G*.

As before we can define Lie algebra homomorphism.

Definition 2.4. A *Lie algebra homomorphism* between two Lie algebras is a linear map that preserves the Lie algebra structures.

Now suppose $\varphi : G \to H$ is a Lie group homomorphism, then its differential at e gives a linear map from T_eG to T_eH. Under the identification of T_eG with $\mathfrak{g}$ and T_eH with $\mathfrak{h}$, we get an induced map, still denoted by $d\varphi$, from $\mathfrak{g}$ to $\mathfrak{h}$.

Theorem 2.5. *If $\varphi : G \to H$ is a Lie group homomorphism, then the induced map $d\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

Proof. We need to show that $d\varphi$ preserves the Lie bracket. Since φ is a group homomorphism, we have $\varphi \circ L_a = L_{\varphi(a)} \circ \varphi$. Let X, Y be left invariant vector field on G. We denote $d\varphi(X)$ and $d\varphi(Y)$ to be the left invariant vector fields on H that corresponding to $d\varphi_e(X_e)$ and $d\varphi_e(Y_e)$. We need to check $d\varphi_e([X,Y]_e) = [d\varphi(X), d\varphi(Y)]_e$.

For any $f \in C^\infty(H)$, we have

$$d\varphi_e([X,Y]_e)f = [X,Y]_e(f \circ \varphi) = X_e(Y(f \circ \varphi)) - Y_e(X(f \circ \varphi))$$

and

$$[d\varphi(X), d\varphi(Y)]_e f = d\varphi_e(X_e)(d\varphi(Y)f) - d\varphi_e(Y_e)(d\varphi(X)f) = X_e(d\varphi(Y)f \circ \varphi) - Y_e(d\varphi(X)f \circ \varphi).$$

So it is enough to check that as functions on G, $Y(f \circ \varphi) = d\varphi(Y)f \circ \varphi$. In fact, for any $a \in G$, we have

$$Y(f \circ \varphi)(a) = Y_a(f \circ \varphi) = dL_a(Y_e)(f \circ \varphi) = Y_e(f \circ \varphi \circ L_a) = Y_e(f \circ L_{\varphi(a)} \circ \varphi),$$

while

$$d\varphi(Y)f \circ \varphi(a) = d\varphi(Y)(f)(\varphi(a)) = dL_{\varphi(a)} \circ d\varphi_e(Y_e)(f) = Y_e(f \circ L_{\varphi(a)} \circ \varphi).$$

This completes the proof. $\square$

Example. The Euclidean group $\mathbb{R}^n$.

This is obviously a Lie group, since the group operation

$$\mu((x_1, \cdots, x_n), (y_1, \cdots, y_n)) := (x_1 + y_1, \cdots, x_n + y_n).$$

is smooth.

Moreover, for any $a \in \mathbb{R}^n$, the left translation L_a is just the usual translation map on $\mathbb{R}^n$. So dL_a *is* the identity map, as long as we identify $T_a\mathbb{R}^n$ with $\mathbb{R}^n$ in the usual way. It follows that any left invariant vector field is in fact a constant vector field, i.e.

$$X_v = v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n}$$

for $\vec{v} = (v_1, \cdots, v_n) \in T_0\mathbb{R}^n$. Since $\frac{\partial}{\partial x_i}$ commutes with $\frac{\partial}{\partial x_j}$ for any pair (i, j), we conclude that the Lie bracket of any two left invariant vector fields vanishes. In other words, the Lie algebra of $G = \mathbb{R}^n$ is $\mathfrak{g} = \mathbb{R}^n$ with vanishing Lie bracket.