# 现代数学物理方法

## 第三章，李群

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# $su(2)$ algebra:

Unitary group $SU(2)$ has 3 independent generators

$$J_a, \qquad a = 1, 2, 3$$

which satisfy the Lie brackets,

$$[J_a,\ J_b] = i\epsilon_{abc} J_c\ , \qquad (1 \leqslant a, b, c \leqslant 3)$$

This is known as $su(2)$ algebra.

**Remark:**

- The $SU(2)$ structure constants $\epsilon_{abc}$ is completely anti symmetric for exchanging any two indices. Therefore,

the adjoint representation of $SU(2)$ is irreducible.

**Question :**

What is the *adjoint* representation of $su(2)$ algebra ?

**Answer :**

The adjoint representation of $SU(2)$ is generated by the following traceless hermitian matrices,

$$(T_a)_{bc} = -i\epsilon_{abc}, \qquad (1 \leqslant a, b, c \leqslant 3)$$

It is 3-dimensional.

Obviously,

$$\begin{aligned}
[T_a,\ T_b]_{ij} &= (T_a)_{ik}(T_b)_{kj} - (T_b)_{ik}(T_a)_{kj} \\
&= -\epsilon_{aik}\,\epsilon_{bkj} + \epsilon_{bik}\,\epsilon_{akj} \\
&= -\delta_{aj}\,\delta_{bi} + \delta_{ai}\,\delta_{bj} \\
&= \epsilon_{abc}\,\epsilon_{ijc} \\
&= i\epsilon_{abc}\left[-i\epsilon_{cij}\right] = i\epsilon_{abc}\,(T_c)_{ij}
\end{aligned}$$

$\rightsquigarrow \quad [T_a,\ T_b] = i\epsilon_{abc}\,T_c$

The explicit matrices of the $SU(2)$ adjoint representation generators read,

$$T_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

**Our Goal** here is to find out all of the finite dimensional irreducible representations of $SU(2)$.

# $J_3$ eigenstates:

To conveniently find a finite-dimensional irreducible representations of a Lie algebra, we have to diagonalize as many of the generators in the algebra as possible.

$su(2)$ is a simple Lie algebra, in which the 3 generators don't commute with one another.

Consequently, we can only diagonalize one generator, say $J_3$,

$$J_3 = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & \ddots \end{bmatrix}$$

where $m_i$ is the eigenvalues of $J_3$,

$$J_3 \left| m_i \right\rangle = m_i \left| m_i \right\rangle$$

and $i = 1, 2, \cdots, N$.

**Discussions:**

1. In an irreducible representation with finite dimensions, the number of $J_3$'s eigenvalues is obviously finite, i.e.,

$$N \text{ takes a finite value,}$$

   among which exists the highest eigenvalue.

2. Call the highest eigenvalue of $J_3$ as $j$,

$$J_3 |j, \alpha\rangle = j |j, \alpha\rangle$$

   where $\alpha$ is another label necessary if there is more than one state of highest $J_3$.

3. The states of the representation space can be normalized so that

$$\langle j, \alpha | j, \beta\rangle = \delta_{\alpha\beta}$$

# $su(2)$'s adjoint representation :

Consider the adjoint representation of $su(2)$.

Let the eigenvalue equation of $T_3$ be

$$T_3 \left|\lambda\right\rangle = \lambda \left|\lambda\right\rangle$$

Recall that

$$T_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we see that the eigenvalues of $T_3$ obey an algebraic equation,

$$\begin{vmatrix} -\lambda & -i & 0 \\ i & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0 \qquad \rightsquigarrow \; -\lambda^3 + \lambda = 0,$$

Its solutions are:

$$\lambda = 0, \; \pm 1.$$

- The highest eigenvalue of $T_3$ is 1.
- Complete list of solutions to the eigenvalue problem of $T_3$ is:

$$|\lambda_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \qquad |\lambda_2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad |\lambda_3\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

$$\lambda_1 = 1 \qquad \lambda_2 = 0 \qquad \lambda_3 = -1$$

From these eigenvectors we can define a unitary matrix $U$:

$$U = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

Its inverse reads,

$$U^{-1} = U^{\dagger} = \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & i/\sqrt{2} & 0 \end{bmatrix}$$

The matrix $U$ enables us to diagonalize the $SU(2)$ adjoint representation generator $T_3$,

$$
\begin{aligned}
T_3^1 &= U^\dagger T_3 U \\
&= \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & i/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 0 & 0 \\ -1/\sqrt{2} & -i/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\end{aligned}
$$

The other two generators of $SU(2)$ in its adjoint representation become,

$$T_1^1 = U^\dagger T_1 U = -\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix},$$

$$T_2^1 = U^\dagger T_2 U = \frac{i}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

**Remark:**

- Among the 3 independent generators $T_a^1$ of $SU(2)$ adjoint representation, only is $T_3^1$ a diagonal matrix.

  Consequently,

  *The adjoint representation of $\mathfrak{su}(2)$ algebra is irreducible.*

# $J_\pm$ :

The $su(2)$ algebra can alternatively be formulated as:

$$[J_3,\ J_\pm] = \pm J_\pm, \quad [J_+,\ J_-] = J_3$$

if we introduce the so-called *raising* and *lowering* operators

$$J_\pm = \frac{1}{\sqrt{2}}\left[J_1 \pm iJ_2\right]$$

- $J_\pm$ are not hermitian. The meaning of $J_\pm$ can be revealed by the comparison of eigenvalue equation

$$J_3 \left|m\right\rangle = m \left|m\right\rangle$$

and its inference,

$$\begin{aligned} J_3 J_\pm \left|m\right\rangle &= \Big\{[J_3,\ J_\pm] + J_\pm J_3\Big\} \left|m\right\rangle \\ &= \Big\{ \pm J_\pm + J_\pm m\Big\} \left|m\right\rangle = (m \pm 1) J_\pm \left|m\right\rangle \end{aligned}$$

We now try to build the finite dimensional irreducible representations of $su(2)$. The key idea is to use the *raising* and *lowering* operators $J_{\pm}$.

**Step 1.**

Because we have assumed that $j$ is the highest value of $J_3$, there is no state with $J_3 = j + 1$. Therefore,

$$J_+ \left|j, \alpha\right\rangle = 0, \;\; \forall\, \alpha$$

Of course, the states $J_- \left|j, \alpha\right\rangle$ with different $\alpha$ are orthogonal

$$\langle j, \alpha | j, \beta \rangle = \delta_{\alpha\beta}$$

On the other hand,

$$J_- \left|j, \alpha\right\rangle = N_j(\alpha) \left|j - 1, \alpha\right\rangle$$

with $N_j(\alpha)$ the normalization coefficient.

Notice that

$$\left(J_\pm\right)^\dagger = J_\mp, \quad \left(|\psi\rangle\right)^\dagger = \langle\psi|$$

and

$$\langle j-1,\alpha|j-1,\beta\rangle = \delta_{\alpha\beta}$$

we have:

$$\begin{aligned}
N_j(\beta)^* N_j(\alpha)\delta_{\alpha\beta} &= N_j(\beta)^* N_j(\alpha)\,\langle j-1,\beta|j-1,\alpha\rangle \\
&= \langle j,\beta|\, J_+ J_- \,|j,\alpha\rangle \\
&= \langle j,\beta|\, [J_+,\ J_-] \,|j,\alpha\rangle \\
&= \langle j,\beta|\, J_3 \,|j,\alpha\rangle = \langle j,\beta|\, j \,|j,\alpha\rangle \\
&= j\,\langle j,\beta|j,\alpha\rangle \\
&= j\delta_{\alpha\beta} \qquad \rightsquigarrow \qquad N_j(\alpha) = \sqrt{j} \equiv N_j
\end{aligned}$$

Hence,

$$J_-\,|j,\alpha\rangle = N_j\,|j-1,\alpha\rangle, \qquad \rightsquigarrow \qquad |j-1,\alpha\rangle = \frac{1}{N_j} J_-\,|j,\alpha\rangle$$

The last equation further implies that,

$$\begin{aligned} J_+ |j-1,\alpha\rangle &= \frac{1}{N_j} J_+ J_- |j,\alpha\rangle \\ &= \frac{1}{N_j} [J_+,\ J_-] |j,\alpha\rangle \\ &= \frac{1}{N_j} J_3 |j,\alpha\rangle \\ &= \frac{j}{N_j} |j,\alpha\rangle = N_j |j,\alpha\rangle \end{aligned} \qquad \left\{ \text{Reminder: } N_j = \sqrt{j}\ . \right\}$$

So far we have achieved the following conclusion:

$$J_- |j,\alpha\rangle = N_j |j-1,\alpha\rangle, \qquad J_+ |j-1,\alpha\rangle = N_j |j,\alpha\rangle .$$

**Step 2:**

Focus on the states $J_- |j-1,\alpha\rangle$.

By an similar procedure, we can find out a set of orthonormal states $|j-2,\alpha\rangle$ which satisfy,

$$\langle j-2,\alpha|j-2,\beta\rangle = \delta_{\alpha\beta}$$

and

$$J_- \left| j-1, \alpha \right\rangle = N_{j-1} \left| j-2, \alpha \right\rangle, \qquad J_+ \left| j-2, \alpha \right\rangle = N_{j-1} \left| j-1, \alpha \right\rangle.$$

**Question :**

What is the coefficient $N_{j-1}$ equal to ? $\quad N_{j-1} \stackrel{?}{=} \sqrt{j-1}$

**Step 3:**

By continuing the procedure, we can easily build a series of orthonormal states $|j-k, \alpha\rangle$,

$$\langle j-k, \alpha | j-k, \beta \rangle = \delta_{\alpha\beta}, \qquad k = 0,\ 1,\ 2,\ \cdots$$

such that

$$\begin{cases} J_- \left| j-k, \alpha \right\rangle = N_{j-k} \left| j-k-1, \alpha \right\rangle, \\ J_+ \left| j-k-1, \alpha \right\rangle = N_{j-k} \left| j-k, \alpha \right\rangle. \end{cases}$$

**Explanation :**

In general, we should express the action of $J_\pm$ as follows:

$$\begin{cases} J_- \left| j-k, \alpha \right\rangle = N_{j-k} \left| j-k-1, \alpha \right\rangle, \\ J_+ \left| j-k-1, \alpha \right\rangle = \widetilde{N}_{j-k} \left| j-k, \alpha \right\rangle. \end{cases}$$

Notice that,

$$\begin{aligned} N_{j-k} &= N_{j-k} \langle j-k-1, \alpha | j-k-1, \alpha \rangle \\ &= \langle j-k-1, \alpha | \, J_- \left| j-k, \alpha \right\rangle \end{aligned}$$

Because we have assumed that $N_{j-k}$ is real, we have:

$$\begin{aligned} N_{j-k} &= N^*_{j-k} \\ &= \langle j-k, \alpha | \, J_+ \left| j-k-1, \alpha \right\rangle \\ &= \widetilde{N}_{j-k} \langle j-k, \alpha | \left| j-k, \alpha \right\rangle \end{aligned}$$

That is,

$$N_{j-k} = \widetilde{N}_{j-k}$$

Hence, it is not necessary to distinguish $N_{j-k}$ and $\widetilde{N}_{j-k}$.

The normalization coefficients $N_{j-k}$ are generally chosen to be real, and determined by a *recursion* relation. Because,

$$\begin{aligned}
\left(N_{j-k}\right)^2 &= \left(N_{j-k}\right)^2 \langle j-k-1, \alpha | j-k-1, \alpha \rangle \\
&= \langle j-k, \alpha | \, J_+ J_- \, | j-k, \alpha \rangle \\
&= \langle j-k, \alpha | \left\{ [J_+, \; J_-] + J_- J_+ \right\} | j-k, \alpha \rangle \\
&= \langle j-k, \alpha | \, J_3 \, | j-k, \alpha \rangle + \langle j-k, \alpha | \, J_- J_+ \, | j-k, \alpha \rangle \\
&= (j-k) + \left(N_{j-k+1}\right)^2
\end{aligned}$$

the expected recursion relation is,

$$\left(N_{j-k}\right)^2 - \left(N_{j-k+1}\right)^2 = j-k, \qquad k = 0, \; 1, \; 2, \; \cdots$$

- Setting $k = 1$ in the recursion relation gives,

$$\left(N_{j-1}\right)^2 = \left(N_j\right)^2 + (j-1) \;= j + (j-1) \;= 2j - 1$$

$$\rightsquigarrow \quad N_{j-1} = \sqrt{2j-1} \neq \sqrt{j-1}\,.$$

It follows from the above recursion relation that,

$$
\begin{aligned}
(N_j)^2 &= j \\
(N_{j-1})^2 - (N_j)^2 &= j-1 \\
(N_{j-2})^2 - (N_{j-1})^2 &= j-2 \\
(N_{j-3})^2 - (N_{j-2})^2 &= j-3 \\
\cdots & \quad \cdots \\
(N_{j-k})^2 - (N_{j-k+1})^2 &= j-k
\end{aligned}
$$

The summation of these equations yields:

$$
\left(N_{j-k}\right)^2 = \sum_{n=0}^{k}(j-n) = j(k+1) - \frac{k(k+1)}{2} = \frac{1}{2}(k+1)(2j-k)
$$

i.e.,

$$
N_m = \frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}
$$

Consequently,

$$
\begin{aligned}
J_- \left|m, \alpha\right\rangle &= \tfrac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}\left|m-1, \alpha\right\rangle \\
J_+ \left|m-1, \alpha\right\rangle &= \tfrac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}\left|m, \alpha\right\rangle \qquad \forall \;\; m \leqslant j
\end{aligned}
$$

**Step 4:**

The representations under consideration are assumed to have finite dimensions. Therefore, *there must be some maximum number of the lowering operators, $p$,* that we can apply to $|j,\alpha\rangle$

$$(J_-)^p\,|j,\alpha\rangle \;\propto\; |j-p,\alpha\rangle$$

so that

$$J_-\,|j-p,\alpha\rangle = 0\,.$$

Since,

$$J_-\,|j-k,\alpha\rangle = N_{j-k}\,|j-k,\alpha\rangle \;=\sqrt{\frac{(2j-k)(k+1)}{2}}\,|j-k-1,\alpha\rangle$$

we have:

$$N_{j-p} = \sqrt{\frac{(2j-p)(p+1)}{2}} = 0, \qquad \rightsquigarrow \quad j = \frac{p}{2}$$

*$p$ is obviously a non-negative integer.* As a result,

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$$

## Discussions:

1. The lowest value of $m$ (the eigenvalue of $J_3$) is,

$$m_{\text{min}} = j - p = j - 2j = -j$$

2. The operator $J_3$ has $(2j+1)$ possible eigenvalues in total,

$$J_3 \left| m, \alpha \right\rangle = m \left| m, \alpha \right\rangle , \qquad -j \leqslant m \leqslant j \ .$$

## Remark :

The parameter $\alpha$ for denoting the states $\left| m, \alpha \right\rangle$ is in fact unwanted.

- All of the $SU(2)$ generators do not change $\alpha$. *The representation space breaks into subspaces that are invariant under $su(2)$, one for each value of $\alpha$.*
- Due to the assumption of *irreducibility*, there must be only one $\alpha$ value. So we can drop the parameter $\alpha$ entirely.

In standard notation, we label the states of the *irreducible representations of* $su(2)$ by 2 parameters

$$|jm\rangle$$

where,

1. $j$ is the highest eigenvalue of $J_3$ in the considered representation.
2. $m$ is the eigenvalue of $J_3$ in a concrete state in the representation.

In short, the spin-$j$ representation of $su(2)$ is defined by

$$\begin{cases} J_3 \, |jm\rangle = m \, |jm\rangle \\ J_\pm \, |jm\rangle = \frac{1}{\sqrt{2}} \sqrt{(j \mp m)(j \pm m + 1)} \, |j, m \pm 1\rangle \end{cases}$$

where

$$j = 0, \; \frac{1}{2}, \; 1, \; \frac{3}{2}, \; 2, \cdots$$

and

$$-j \leqslant m \leqslant j$$

The spin-$j$ representation of $su(2)$ has dimensions of $(2j + 1)$.

In spin-$j$ representation, the matrix elements of the $SU(2)$ generators are given by,

$$
\begin{aligned}
(J_3^j)_{m'm} &= \langle jm'|\, J_3 \,|jm\rangle &= m\,\delta_{m'm} \\
(J_+^j)_{m'm} &= \langle jm'|\, J_+ \,|jm\rangle &= \sqrt{(j-m)(j+m+1)/2}\,\delta_{m',m+1} \\
(J_-^j)_{m'm} &= \langle jm'|\, J_- \,|jm\rangle &= \sqrt{(j+m)(j-m+1)/2}\,\delta_{m',m-1}
\end{aligned}
$$

The last two equations can be recast as

$$
\left(J_1^j\right)_{m'm} = \frac{1}{2}\left[\sqrt{(j-m)(j+m+1)}\,\delta_{m',m+1} \right. \\
\left. +\sqrt{(j+m)(j-m+1)}\,\delta_{m',m-1}\right]
$$

$$
\left(J_2^j\right)_{m'm} = \frac{1}{2i}\left[\sqrt{(j-m)(j+m+1)}\,\delta_{m',m+1} \right. \\
\left. -\sqrt{(j+m)(j-m+1)}\,\delta_{m',m-1}\right]
$$

**Examples :**

- Spin-1/2 Representation of $su(2)$.

$$j = 1/2 \qquad \Rightarrow \qquad m = \pm 1/2$$

Hence,

$$J_3^{1/2} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_3/2, \qquad J_1^{1/2} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_1/2,$$

$$J_2^{1/2} = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \sigma_2/2.$$

Exponentiating the above generators yields the general elements of group $SU(2)$ in spin-1/2 representation:

$$g = e^{\frac{i}{2}\vec{\alpha}\cdot\vec{\sigma}} = \sum_{n=0}^{\infty} \frac{(i/2)^n}{n!} \left(\vec{\alpha}\cdot\vec{\sigma}\right)^n$$

Since,

$$\begin{aligned}(\vec{\alpha}\cdot\vec{\sigma})^2 &= \alpha_a\alpha_b(\sigma_a\sigma_b) = \alpha_a\alpha_b(\delta_{ab}+i\epsilon_{abc}\sigma_c)\\ &= \alpha_a\alpha_b\delta_{ab} = \alpha_a\alpha_a \equiv \alpha^2\end{aligned}$$

we have:

$$\begin{cases}(\vec{\alpha}\cdot\vec{\sigma})^{2n} = \alpha^{2n}\\ (\vec{\alpha}\cdot\vec{\sigma})^{2n+1} = \alpha^{2n}(\vec{\alpha}\cdot\vec{\sigma})\end{cases}$$

where $n$ is an arbitrary *non-negative* integer. Therefore,

$$\begin{aligned}e^{\frac{i}{2}\vec{\alpha}\cdot\vec{\sigma}} &= \cos(\alpha/2)+i(\vec{n}\cdot\vec{\sigma})\sin(\alpha/2)\\ &= \begin{bmatrix}\cos(\alpha/2)+in_3\sin(\alpha/2) & (in_1+n_2)\sin(\alpha/2)\\ (in_1-n_2)\sin(\alpha/2) & \cos(\alpha/2)-in_3\sin(\alpha/2)\end{bmatrix}\end{aligned}$$

where $\alpha = \sqrt{\alpha_a\alpha_a}$ and $n_a$ are the Cartesian components of the unit vector

$$\vec{n} = \vec{\alpha}/\alpha = \vec{e}_3 c_\theta + \vec{e}_1 s_\theta c_\phi + \vec{e}_2 s_\theta s_\phi$$

This is obviously a unitary matrix with unity determinant.

- Spin-1 Representation of $su(2)$.

$$j = 1 \qquad \Rightarrow \quad m = 0, \ \pm 1.$$

Hence,

$$J_3^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad J_1^1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$J_2^1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}.$$

The corresponding 3-d irreducible representation of group $SU(2)$ is given by,

$$e^{i\vec{\alpha}\cdot\vec{J}^1} = e^{i(\alpha_1 J_1^1 + \alpha_2 J_2^1 + \alpha_3 J_3^1)}$$

- Spin-3/2 Representation of $su(2)$.

$$j = 3/2 \quad \Rightarrow \quad m = \pm 3/2,\ \pm 1/2.$$

Hence,

$$J_3^{3/2} = \begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix},$$

$$J_1^{3/2} = \begin{bmatrix} 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ \sqrt{\frac{3}{2}} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{\frac{3}{2}} \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 \end{bmatrix},$$

and

$$
J_2^{3/2} = \begin{bmatrix} 0 & -i\sqrt{\frac{3}{2}} & 0 & 0 \\ i\sqrt{\frac{3}{2}} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{\frac{3}{2}} \\ 0 & 0 & i\sqrt{\frac{3}{2}} & 0 \end{bmatrix}.
$$

The corresponding 4-d irreducible representation of group $SU(2)$ is given by,

$$
e^{i\vec{\alpha}\cdot\vec{J}^{3/2}} = e^{i(\alpha_1 J_1^{3/2} + \alpha_2 J_2^{3/2} + \alpha_3 J_3^{3/2})}
$$

Let us now consider the homomorphism between $SU(2)$ and $SO(3)$.

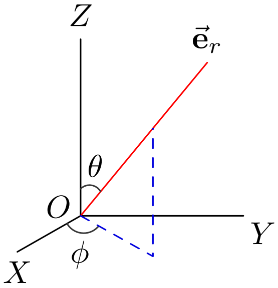
## Question:

Why the magnetic quantum number $m$ of orbital angular momentum $\vec{L}$ of an object must be an integer ?

The angular momentum operator is defined as $\vec{L} = \vec{r} \times \vec{p}$. In coordinate representation,

$$\vec{L} = -i\hbar\vec{r} \times \vec{\nabla}$$

To solve the eigenvalue problem of $\vec{L}$, we generally employ the spherical coordinates $(r,\ \theta,\ \phi)$.



So $\vec{r} = r\vec{e}_r$,

$$\vec{e}_r = \vec{e}_3 c_\theta + \vec{e}_1 s_\theta c_\phi + \vec{e}_2 s_\theta s_\phi,$$

and

$$\begin{aligned} \vec{e}_\theta &= \partial_\theta \vec{e}_r \\ &= -\vec{e}_3 s_\theta + \vec{e}_1 c_\theta c_\phi + \vec{e}_2 c_\theta s_\phi, \end{aligned}$$

$$\begin{aligned} \vec{e}_\phi &= \frac{1}{s_\theta} \partial_\phi \vec{e}_r \\ &= -\vec{e}_1 s_\phi + \vec{e}_2 c_\phi. \end{aligned}$$

In spherical coordinates, the gradient operator $\vec{\nabla}$ becomes:

$$\vec{\nabla} = \vec{e}_r \ \partial_r + \frac{1}{r} \vec{e}_\theta \partial_\theta + \frac{1}{r s_\theta} \vec{e}_\phi \partial_\phi$$

Hence,

$$\vec{L} = -i\hbar (r\vec{e}_r) \times \vec{\nabla} \ = -i\hbar \left[ \vec{e}_\phi \partial_\theta - \vec{e}_\theta \frac{1}{s_\theta} \partial_\phi \right]$$

Equivalently,

$$\vec{L} = -i\left[(-\vec{e}_1 s_\phi + \vec{e}_2 c_\phi)\partial_\theta - (-\vec{e}_3 s_\theta + \vec{e}_1 c_\theta c_\phi + \vec{e}_2 c_\theta s_\phi)\frac{1}{s_\theta}\partial_\phi\right]$$

Consequently, the Cartesian components of orbital angular momentum $\vec{L}$ can be expressed as

$$\begin{aligned} L_1 &= i\left[s_\phi \partial_\theta + \cot\theta c_\phi \partial_\phi\right] \\ L_2 &= -i\left[c_\phi \partial_\theta - \cot\theta s_\phi \partial_\phi\right] \\ L_3 &= -i\partial_\phi \end{aligned}$$

in terms of the spherical coordinates $(\theta,\ \phi)$.

## Casimir operator $L^2$ of $SO(3)$ :

Notice that $\vec{e}_\phi \cdot \vec{e}_\phi = \vec{e}_\theta \cdot \vec{e}_\theta = 1$ and $\vec{e}_\phi \cdot \vec{e}_\theta = 0$. The derivatives of the first two orthonormal conditions with respect to the angles $\theta$ and $\phi$ give,

$$\vec{e}_\phi \cdot \partial_\theta \vec{e}_\phi = \vec{e}_\phi \cdot \partial_\phi \vec{e}_\phi = 0, \qquad \vec{e}_\theta \cdot \partial_\theta \vec{e}_\theta = \vec{e}_\theta \cdot \partial_\phi \vec{e}_\theta = 0.$$

Therefore,

$$\begin{aligned} L^2 &= \vec{L}\cdot\vec{L} \\ &= -\left[\vec{e}_\phi\partial_\theta - \vec{e}_\theta\frac{1}{s_\theta}\partial_\phi\right]\cdot\left[\vec{e}_\phi\partial_\theta - \vec{e}_\theta\frac{1}{s_\theta}\partial_\phi\right] \\ &= -\partial_\theta^2 + (\vec{e}_\phi\cdot\partial_\theta\vec{e}_\theta)\frac{1}{s_\theta}\partial_\phi + (\vec{e}_\theta\cdot\partial_\phi\vec{e}_\phi)\frac{1}{s_\theta}\partial_\theta - \frac{1}{s_\theta^2}\partial_\phi^2 \end{aligned}$$

Recall the transformation of basis vectors between the Cartesian and spherical coordinate systems

$$\begin{aligned} \vec{e}_r &= \vec{e}_3 c_\theta + \vec{e}_1 s_\theta c_\phi + \vec{e}_2 s_\theta s_\phi \\ \vec{e}_\theta &= -\vec{e}_3 s_\theta + \vec{e}_1 c_\theta c_\phi + \vec{e}_2 c_\theta s_\phi \\ \vec{e}_\phi &= -\vec{e}_1 s_\phi + \vec{e}_2 c_\phi \end{aligned}$$

we see that: $\vec{e}_r s_\theta + \vec{e}_\theta c_\theta = \vec{e}_1 c_\phi + \vec{e}_2 s_\phi$. Therefore,

$$\begin{aligned} \partial_\theta\vec{e}_\theta &= -\vec{e}_3 c_\theta - \vec{e}_1 s_\theta c_\phi - \vec{e}_2 s_\theta s_\phi \; = -\vec{e}_r \\ \partial_\phi\vec{e}_\phi &= -\vec{e}_1 c_\phi - \vec{e}_2 s_\phi \; = -\vec{e}_r s_\theta - \vec{e}_\theta c_\theta \end{aligned}$$

Hence,

$$(\vec{e}_\phi \cdot \partial_\theta \vec{e}_\theta) = 0, \qquad (\vec{e}_\theta \cdot \partial_\phi \vec{e}_\phi) = -c_\theta .$$

Substitution of these results into the previous formula yields,

$$L^2 = -\partial_\theta^2 - \cot\theta\, \partial_\theta - \frac{1}{s_\theta^2} \partial_\phi^2$$

In QM textbooks, $L^2$ is commonly recast as:

$$L^2 == -\left[ \frac{1}{s_\theta} \partial_\theta (s_\theta \partial_\theta) + \frac{1}{s_\theta^2} \partial_\phi^2 \right]$$

- $L^2$ is called the **Casimir** operator of $so(3)$. Its crucial property is,

$$[L^2,\ L_a] = 0, \quad a = 1,\ 2,\ 3.$$

Thereby, $L^2$ and $L_3$ can have common eigenvectors.

- The eigenvalue problem

$$L_3 \left| lm \right\rangle = m \left| lm \right\rangle, \quad L^2 \left| lm \right\rangle = l(l+1) \left| lm \right\rangle$$

in spherical coordinates becomes,

$$\begin{cases} \partial_\phi Y = imY, \\ s_\theta \partial_\theta (s_\theta \partial_\theta) Y + \left[ s_\theta^2 l(l+1) - m^2 \right] Y = 0. \end{cases}$$

- The common eigenfunction $Y(\theta, \phi)$ of $L_3$ and $L^2$ can be factorized into

$$Y(\theta, \phi) = \Theta(\theta) e^{im\phi}$$

**Insight:**

If $Y(\theta, \phi)$ is single-valued under rotation: $Y(\theta, \phi + 2\pi) = Y(\theta, \phi)$ , the magnetic quantum number $m$ has to be some integers: $m \in Z$.

**Question :**

Why should $Y(\theta, \phi)$ be single-valued under rotation ?

**Remarks :**

- In QM, physical significance is attached, not to wavefunction $Y$ itself, but to its bilinear functions, e.g., $|Y|^2$.
- These bilinear functions are unchanged by a $2\pi$ rotation *even if $m$ is a half-integer* and $Y$ changes sign.

For $l = m = 1/2$, the common eigenfunction of Casimir operator $L^2$ and $L_3$ becomes:

$$Y = \Theta(\theta) e^{\frac{i}{2}\phi}$$

where the factor function $\Theta$ obeys,

$$s_\theta \partial_\theta (s_\theta \partial_\theta)\Theta + \frac{1}{4}[3s_\theta^2 - 1]\Theta = 0$$

A special solution to this equation reads,

$$\Theta(\theta) = \sqrt{s_\theta}$$

**Checking:**
If $\Theta(\theta) = \sqrt{s_\theta}$ , we see that

$$(s_\theta \partial_\theta)\Theta = \frac{1}{2}\sqrt{s_\theta}\, c_\theta$$

$$\begin{aligned} s_\theta \partial_\theta (s_\theta \partial_\theta)\Theta &= \frac{1}{2} s_\theta \partial_\theta (\sqrt{s_\theta}\, c_\theta) \;= \frac{1}{4}\sqrt{s_\theta}\,(c_\theta^2 - 2s_\theta^2) \\ &= \frac{1}{4}\sqrt{s_\theta}\,(1 - 3s_\theta^2) \\ &= -\frac{1}{4}\left[3s_\theta^2 - 1\right]\Theta \end{aligned}$$

This is just what we have expected.

$Y(\theta, \phi) = \sqrt{s_\theta}\, e^{i\phi/2}$ appears to be an acceptable wave function in QM because $|Y|^2 = |s_\theta|$ is well defined in the unit spherical surface,

$$0 \leqslant \theta \leqslant \pi, \qquad 0 \leqslant \phi \leqslant 2\pi.$$

**Puzzle :**

What is wrong in the above argument ?

Go back to the primary definition of orbital angular momentum:[1]

$$\vec{L} = -i\vec{r} \times \vec{\nabla}$$

In Cartesian coordinates,

$$L_a = -i\epsilon_{abc} x_b \partial_{x_c}, \qquad (a = 1,\ 2,\ 3.)$$

Particularly, $L_3$ consists of four linear operators $\left\{x_1, x_2, \partial_{x_1}, \partial_{x_2}\right\}$ :

$$L_3 = -i\left[x_1\partial_{x_2} - x_2\partial_{x_1}\right]$$

[1] It holds only for the orbital angular momentum operator of a quantum particle.

To expose $L_3$'s interesting intrinsic structure, we now introduce four new linear operators:

$$q_1 = \tfrac{1}{\sqrt{2}}\left(x_1 - i\partial_{x_2}\right), \qquad q_2 = \frac{1}{\sqrt{2}}\left(x_1 + i\partial_{x_2}\right),$$

$$p_1 = -\tfrac{1}{\sqrt{2}}\left(x_2 + i\partial_{x_1}\right), \quad p_2 = \frac{1}{\sqrt{2}}\left(x_2 - i\partial_{x_1}\right).$$

Notice that $[\partial_{x_a},\, x_b] = \delta_{ab}$. The Lie brackets between these operators are

$$[q_a,\, q_b] = [p_a,\, p_b] = 0, \qquad [q_a,\, p_b] = i\delta_{ab}.$$

In terms of these *new* operators,

$$x_1 = \tfrac{1}{\sqrt{2}}\left(q_1 + q_2\right), \qquad x_2 = -\frac{1}{\sqrt{2}}\left(p_1 - p_2\right),$$

$$\partial_{x_1} = \tfrac{i}{\sqrt{2}}\left(p_1 + p_2\right), \qquad \partial_{x_2} = \frac{i}{\sqrt{2}}\left(q_1 - q_2\right).$$

and $L_3$ is recast as:

$$\begin{aligned}
L_3 &= -i\left(x_1\partial_{x_2} - x_2\partial_{x_1}\right) \\
&= \frac{1}{2}\Big[\big(q_1+q_2\big)\big(q_1-q_2\big) + \big(p_1-p_2\big)\big(p_1+p_2\big)\Big] \\
&= \frac{1}{2}\Big[(q_1^2+p_1^2) - (q_2^2+p_2^2)\Big] \\
&= H_1 - H_2
\end{aligned}$$

where

$$H_a = \frac{1}{2}(q_a^2+p_a^2), \qquad (a = 1,\ 2.)$$

are hamiltonian operators of two independent oscillators, each having mass $M = 1$ and angular frequency $\omega = 1$.

**Insight :**

*The eigenvalues of $L_3$ should be the difference of eigenvalues of two independent (but with identical parameters $M = \omega = 1$) harmonic oscillator Hamiltonians.*

The eigenvalues of a harmonic oscillator Hamiltonian $H_a = \frac{1}{2}(q_a^2 + p_a^2)$ are well-known,

$$E_{n_a} = n_a + \frac{1}{2}$$

with $n_a$ some nonnegative integers.

Consequently, the eigenvalues of orbital angular momentum $L_3$ are equal to,

$$m = \left(n_1 + \frac{1}{2}\right) - \left(n_2 + \frac{1}{2}\right) = n_1 - n_2 \quad \in Z$$

Namely, the orbital angular momentum eigenvalues must be some integers. *The possibility for $m$ being a half-integer is forbidden.* [2]

[2] This demonstration can be regarded as an indirect justification for the conventional boundary condition $Y(\theta, \phi + 2\pi) = Y(\theta, \phi)$ that leads to the same result.

# Tensor product representations:

Consider the tensor product representations of a Lie group $G$.

Suppose

$$D(g)\,|i\rangle = \sum_{j=1}^{N}\left[D_1(g)\right]_{ji}|j\rangle\,, \quad D(g)\,|\alpha\rangle = \sum_{\beta=1}^{M}\left[D_2(g)\right]_{\beta\alpha}|\beta\rangle$$

On states of tensor product $|i\rangle\,|\alpha\rangle$, we have:

$$\begin{aligned}
D_{1\times 2}(g)\,|i\rangle\,|\alpha\rangle &= \sum_{j=1}^{N}\sum_{\beta=1}^{M}\Big[D_1(g)\;D_2(g)\Big]_{j\beta,i\alpha}|j\rangle\,|\beta\rangle \\
&= \sum_{j=1}^{N}\sum_{\beta=1}^{M}\Big[D_1(g)\Big]_{ji}\Big[D_2(g)\Big]_{\beta\alpha}|j\rangle\,|\beta\rangle \\
&= \left\{\sum_{j=1}^{N}[D_1(g)]_{ji}\,|j\rangle\right\}\cdot\left\{\sum_{\beta=1}^{M}[D_2(g)]_{\beta\alpha}\,|\beta\rangle\right\}
\end{aligned}$$

i.e.,

$$\Big[D_{1\times 2}(g)\Big]_{j\beta,i\alpha} = \big[D_1(g)\big]_{ji}\,\big[D_2(g)\big]_{\beta\alpha}$$

Consider the infinitesimal group elements of the relevant representations,

$D_1(g) \approx 1 + i\xi_a J_a^1, \quad D_2(g) \approx 1 + i\xi_a J_a^2, \quad D_{1\times 2}(g) \approx 1 + i\xi_a J_a^{1\times 2}.$

The above relation can be recast as:

$$[1 + i\xi_a J_a^{1\times 2}]_{j\beta,i\alpha} = [1 + i\xi_b J_b^1]_{ji}[1 + i\xi_c J_c^2]_{\beta\alpha}$$

$$\rightsquigarrow \quad (J_a^{1\times 2})_{j\beta,i\alpha} = (J_a^1)_{ji}\delta_{\beta\alpha} + \delta_{ji}(J_a^2)_{\beta\alpha}$$

i.e.,

$$J_a^{1\times 2} = J_a^1 \times 1 + 1 \times J_a^2$$

The action of generators on the tensor product of states is as follows:

$$J_a^{1\times 2}\Big\{\,|i\rangle\,|\alpha\rangle\,\Big\} = \Big\{\,J_a^1\,|i\rangle\,\Big\}\cdot|\alpha\rangle + |i\rangle\cdot\Big\{\,J_a^2\,|\alpha\rangle\,\Big\}$$

## $J_3$'s value add :

Because we work in a basis $|jm\rangle$ in which $J_3$ ia diagonal, the $J_3$ values of tensor product states are just the sums of the $J_3$ values of the factors.

**Explanation :**

$$\begin{aligned} J_3\Big\{ |j_1m_1\rangle |j_2m_2\rangle \Big\} &= \Big\{ J_3 |j_1m_1\rangle \Big\} |j_2m_2\rangle + |j_1m_1\rangle \Big\{ J_3 |j_2m_2\rangle \Big\} \\ &= (m_1 + m_2)\Big\{ |j_1m_1\rangle |j_2m_2\rangle \Big\} \end{aligned}$$

The irreducible representation $\Big\{ |jm\rangle \Big\}$ of $SU(2)$ is related to its tensor product representation $\Big\{ |j_1m_1\rangle |j_2m_2\rangle \Big\}$ through,

$$|jm\rangle = \sum_{m_1=-j_1}^{j_1} c_{j_1j_2j,m_1(m-m_1)m}\Big\{ |j_1m_1\rangle |j_2, m - m_1\rangle \Big\}$$

**Remarks :**

1. The coefficients $c_{j_1 j_2 j, m_1 (m-m_1) m}$ are called Clebsch-Gordon coefficients of $SU(2)$.
2. In particular, we define:

$$c_{j_1 j_2 (j_1+j_2), j_1 j_2 (j_1+j_2)} = 1.$$

**Question :**

How to systematically determine the Clebsch-Gordon coefficients ?

**Answer :**

The highest weight procedure.

**Example :**

Consider the spin-$1/2$ representation and spin-$1$ representation of $su(2)$,

$$j_1 = \frac{1}{2}, \quad j_2 = 1 \quad \rightsquigarrow \quad j_1 + j_2 = \frac{3}{2}.$$

The assumption $c_{j_1 j_2 (j_1+j_2), j_1 j_2 (j_1+j_2)} = 1$ means,

$$\left|\frac{3}{2},\frac{3}{2}\right\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle \cdot |1,1\rangle$$

Therefore,

$$\begin{aligned}
\sqrt{\tfrac{3}{2}}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle &= J_-\left|\tfrac{3}{2},\tfrac{3}{2}\right\rangle \\
&= J_-\left\{\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle \cdot |1,1\rangle\right\} \\
&= \left\{J_-^{1/2}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\right\} \cdot |1,1\rangle + \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle \cdot \left\{J_-^{1}\,|1,1\rangle\right\} \\
&= \tfrac{1}{\sqrt{2}}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle \cdot |1,1\rangle + \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle \cdot |1,0\rangle
\end{aligned}$$

Equivalently,

$$\left|\frac{3}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \cdot |1,1\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \cdot |1,0\rangle$$

Continuing this procedure yields:

$$\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \cdot |1,0\rangle + \sqrt{\frac{1}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \cdot |1,-1\rangle$$

$$\left|\frac{3}{2},-\frac{3}{2}\right\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle \cdot |1,-1\rangle$$

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \cdot |1,1\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \cdot |1,0\rangle$$

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \cdot |1,0\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \cdot |1,-1\rangle$$

# Clebsch-Gordon coefficients:

Hence, the decomposition of tensor product of spin-$1/2$ and spin-$1$ representations of $SU(2)$

$$D_{1/2} \times D_1 \sim \oplus_{j=1/2}^{3/2} D_j$$

is determined by the following non-vanishing Clebsch-Gordon coefficients $c_{j_1 j_2 j, m_1 (m-m_1) m}$:

| | |
|---|---|
| $c_{\frac{1}{2}1\frac{3}{2},\frac{1}{2}1\frac{3}{2}} = 1$ | $c_{\frac{1}{2}1\frac{3}{2},-\frac{1}{2}1\frac{1}{2}} = 1/\sqrt{3}$ |
| $c_{\frac{1}{2}1\frac{3}{2},\frac{1}{2}0\frac{1}{2}} = \sqrt{2/3}$ | $c_{\frac{1}{2}1\frac{3}{2},-\frac{1}{2}-1-\frac{3}{2}} = 1$ |
| $c_{\frac{1}{2}1\frac{3}{2},-\frac{1}{2}-1-\frac{1}{2}} = 1/\sqrt{3}$ | $c_{\frac{1}{2}1\frac{3}{2},-\frac{1}{2}0-\frac{1}{2}} = \sqrt{2/3}$ |
| $c_{\frac{1}{2}1\frac{1}{2},-\frac{1}{2}1\frac{1}{2}} = \sqrt{2/3}$ | $c_{\frac{1}{2}1\frac{1}{2},\frac{1}{2}0\frac{1}{2}} = -1/\sqrt{3}$ |
| $c_{\frac{1}{2}1\frac{1}{2},-\frac{1}{2}0-\frac{1}{2}} = \sqrt{1/3}$ | $c_{\frac{1}{2}1\frac{1}{2},\frac{1}{2}-1-\frac{1}{2}} = -\sqrt{2/3}$ |

## Homework :

1. Let $\left\{k\right\}$ be the spin-$k$ representation of $su(2)$. Show that

$$\left\{j\right\} \times \left\{s\right\} = \oplus_{l=|j-s|}^{j+s} \left\{l\right\}$$

2. Calculate

$$\exp\left[i\vec{\xi}\cdot\vec{\sigma}\right]$$

   where $\vec{\sigma} = \left\{\sigma_1, \sigma_2, \sigma_3\right\}$ are the pauli matrices and $\vec{\xi}$ a common 3-dimensional vector.

3. Show explicitly that the spin-1 representation of $su(2)$ obtained by the highest weight procedure with $j = 1$ is equivalent to the adjoint representation with $f_{abc} = \epsilon_{abc}$ by finding the similarity transformation that implements the equivalence.

4. Suppose that $\left(\sigma_a\right)_{ij}$ and $\left(\eta_a\right)_{xy}$ are pauli matrices in two different 2-dimensional spaces. In the 4-dimensional tensor product space, define the basis vectors as

$$\begin{aligned}|1\rangle &= |i=1\rangle\,|x=1\rangle\\ |2\rangle &= |i=1\rangle\,|x=2\rangle\\ |3\rangle &= |i=2\rangle\,|x=1\rangle\\ |4\rangle &= |i=2\rangle\,|x=2\rangle\end{aligned}$$

Write out the matrix elements of $\sigma_2 \times \eta_1$ in this basis.