现代数学物理方法

第四章，$SU(N)$

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Tensor methods :

Lower & upper indices:

- We begin with relabeling the basis states of $su(3)$ fundamental representation $(1,0) = \mathbf{3}$,

$$\left.\begin{array}{rcl c l}
|M_1\rangle &=& |1/2,\ 1/2\sqrt{3}\rangle &=& |_1\rangle \\
E_{-\alpha_1}|M_1\rangle &=& |0,\ -1/\sqrt{3}\rangle &=& |_3\rangle \\
\sqrt{2}E_{-\alpha_2}E_{-\alpha_1}|M_1\rangle &=& |-1/2,\ 1/2\sqrt{3}\rangle &=& |_2\rangle
\end{array}\right\}$$

- The basis states of another $su(3)$ fundamental representation $(0,1) = \bar{\mathbf{3}}$ are re-labelled as:

$$\left.\begin{array}{rcl c l}
|M_2\rangle &=& |1/2,\ -1/2\sqrt{3}\rangle &=& |^2\rangle \\
E_{-\alpha_2}|M_2\rangle &=& |0,\ 1/\sqrt{3}\rangle &=& |^3\rangle \\
\sqrt{2}E_{-\alpha_1}E_{-\alpha_2}|M_2\rangle &=& |-1/2,\ -1/2\sqrt{3}\rangle &=& |^1\rangle
\end{array}\right\}$$

In Rep. $\mathbf{3}$, the matrices of $SU(3)$ generators X_a are expressed as

$$\left(X_a\right)^i{}_j$$

so that :

$$X_a \left|_j\right\rangle = \left|_i\right\rangle \left(X_a\right)^i{}_j$$

Because the Rep. $\bar{\mathbf{3}}$ is the complex conjugate of Rep. $\mathbf{3}$, with generators $-X_a^*$, i.e.,

$$-\left(X_a^*\right)_j{}^i = -\left(X_a^T\right)_j{}^i = -\left(X_a\right)^i{}_j$$

Then,

$$\begin{aligned} X_a \left|^i\right\rangle &= \left|^j\right\rangle \left(-X_a^*\right)_j{}^i \\ &= -\left|^j\right\rangle \left(X_a\right)^i{}_j \end{aligned}$$

Now, we can define the tensor product representation of $su(3)$.

A typical tensor product representation of $su(3)$ is:

$$\underbrace{\mathbf{3}\times\mathbf{3}\times\cdots\times\mathbf{3}}_{n}\times\underbrace{\bar{\mathbf{3}}\times\bar{\mathbf{3}}\times\cdots\times\bar{\mathbf{3}}}_{m}$$

The basis states of tensor product representation are:

$$\left|{}^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n}\right\rangle = \left|{}^{i_1}\right\rangle\left|{}^{i_2}\right\rangle\cdots\left|{}^{i_m}\right\rangle\ \left|{}_{j_1}\right\rangle\left|{}_{j_2}\right\rangle\cdots\left|{}_{j_n}\right\rangle$$

Recalling

$$X_a^{D_1\times D_2} = X_a^{D_1}\times 1 + 1\times X_a^{D_2}$$

under the generator action, these basis states transform as follows:

$$\begin{aligned} X_a\left|{}^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n}\right\rangle &= \sum_{l=1}^{n}\left|{}^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_{l-1} k j_{l+1} \cdots j_n}\right\rangle \left(X_a\right)^{k}{}_{j_l} \\ &\quad -\sum_{l=1}^{m}\left|{}^{i_1 i_2 \cdots i_{l-1} k i_{l+1} \cdots i_m}_{j_1 j_2 \cdots j_n}\right\rangle \left(X_a\right)^{i_l}{}_{k} \end{aligned}$$

An arbitrary state in this tensor product space is,

$$|v\rangle = \left|\begin{matrix} i_1 i_2 \cdots i_m \\ j_1 j_2 \cdots j_n \end{matrix}\right\rangle v^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m}$$

Discussions :

- $v = \left(v^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m}\right)$ is called a $SU(3)$ tensor.
- In analogy with the concept of *wave function* in QM, we can express the tensor's components as:

$$v^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m} = \left\langle \begin{matrix} i_1 i_2 \cdots i_m \\ j_1 j_2 \cdots j_n \end{matrix} \middle| v \right\rangle$$

- We can think of the action of the generator X_a on state $|v\rangle$ as an effective action of X_a on the tensor components:

$$X_a |v\rangle = |X_a v\rangle$$

Consequently,

$$
\begin{aligned}
(X_a v)^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m} &= \left\langle \begin{smallmatrix} i_1 i_2 \cdots i_m \\ j_1 j_2 \cdots j_n \end{smallmatrix} \middle| X_a v \right\rangle = \left\langle \begin{smallmatrix} i_1 i_2 \cdots i_m \\ j_1 j_2 \cdots j_n \end{smallmatrix} \middle| X_a \middle| v \right\rangle \\
&= \left\langle \begin{smallmatrix} i_1 i_2 \cdots i_m \\ j_1 j_2 \cdots j_n \end{smallmatrix} \middle| X_a \middle| \begin{smallmatrix} k_1 k_2 \cdots k_m \\ l_1 l_2 \cdots l_n \end{smallmatrix} \right\rangle v^{l_1 l_2 \cdots l_n}_{k_1 k_2 \cdots k_m} \\
&= \sum_{q=1}^{n} \left\langle \begin{smallmatrix} i_1 i_2 \cdots i_m \\ j_1 j_2 \cdots j_n \end{smallmatrix} \middle| \begin{smallmatrix} k_1 k_2 \cdots k_m \\ l_1 \cdots l_{q-1} p l_{q+1} \cdots l_n \end{smallmatrix} \right\rangle (X_a)^{p}{}_{l_q} v^{l_1 l_2 \cdots l_n}_{k_1 k_2 \cdots k_m} \\
&\quad - \sum_{q=1}^{m} \left\langle \begin{smallmatrix} i_1 i_2 \cdots i_m \\ j_1 j_2 \cdots j_n \end{smallmatrix} \middle| \begin{smallmatrix} k_1 \cdots k_{q-1} p k_{q+1} \cdots k_m \\ l_1 l_2 \cdots l_n \end{smallmatrix} \right\rangle (X_a)^{k_q}{}_{p} v^{l_1 l_2 \cdots l_n}_{k_1 k_2 \cdots k_m} \\
&= \sum_{q=1}^{n} (X_a)^{p}{}_{l_q} \, v^{j_1 \cdots j_{q-1} l_q j_{q+1} \cdots j_n}_{i_1 i_2 \cdots i_m} \delta^{j_q}_{p} \\
&\quad - \sum_{q=1}^{m} (X_a)^{k_q}{}_{p} \, v^{j_1 j_2 \cdots j_n}_{i_1 \cdots i_{q-1} k_q i_{q+1} \cdots i_m} \delta^{p}_{i_q}
\end{aligned}
$$

The action of the $SU(3)$ generators on an arbitrary tensor reads,

$$
(X_a v)^{j_1 \cdots j_n}_{i_1 \cdots i_m} = \sum_{l=1}^{n} (X_a)^{j_l}{}_{k} v^{j_1 \cdots j_{l-1} k j_{l+1} \cdots j_n}_{i_1 i_2 \cdots i_m} - \sum_{l=1}^{m} (X_a)^{k}{}_{i_l} v^{j_1 j_2 \cdots j_n}_{i_1 \cdots i_{l-1} k i_{l+1} \cdots i_m}
$$

Invariant tensors :

An invariant tensor of $SU(3)$ is referred to one that does not change under any $SU(3)$ transformations.

$SU(3)$ invariant tensors :

For $SU(3)$, three invariant tensors exist,

1. δ^i_j
2. ϵ_{ijk}
3. ϵ^{ijk}

Proof :

The invariance of δ^i_j is obvious,

$$\begin{aligned} (X_a\delta)^i{}_j &= (X_a)^i{}_k\delta^k_j - (X_a)^k{}_j\delta^i_k \\ &= (X_a)^i{}_j - (X_a)^i{}_j \\ &= 0 \end{aligned}$$

Next we consider the invariance of ϵ^{ijk} and ϵ_{ijk}. e.g.,

$$(X_a\epsilon)^{ijk} = (X_a)^i{}_l\epsilon^{ljk} + (X_a)^j{}_l\epsilon^{ilk} + (X_a)^k{}_l\epsilon^{ijl}$$

By definition,

$$\epsilon^{ijk} = \epsilon_{ijk} = \begin{cases} 1 & \text{if } (ijk) \text{ is an even permutation of } (123) \\ -1 & \text{if } (ijk) \text{ is an odd permutation of } (123) \\ 0 & \text{other cases} \end{cases}$$

Hence,

$$\begin{aligned}
(X_a\epsilon)^{123} &= (X_a)^1{}_i\epsilon^{i23} + (X_a)^2{}_j\epsilon^{1j3} + (X_a)^3{}_k\epsilon^{12k} \\
&= (X_a)^1{}_1 + (X_a)^2{}_2 + (X_a)^3{}_3 \\
&= \mathrm{Tr}(X_a) \ = 0 \\
(X_a\epsilon)^{112} &= (X_a)^1{}_3\epsilon^{312} + (X_a)^1{}_3\epsilon^{132} + (X_a)^2{}_k\epsilon^{11k} \\
&= (X_a)^1{}_3 - (X_a)^1{}_3 \ = 0 \\
(X_a\epsilon)^{111} &= (X_a)^1{}_i\epsilon^{i11} + (X_a)^1{}_j\epsilon^{1j1} + (X_a)^1{}_k\epsilon^{11k} \ = 0
\end{aligned}$$

Therefore, for arbitrary $i, j, k = 1, 2, 3$, we have

$$\left(X_a \epsilon\right)^{ijk} = 0$$

and similarly,

$$\left(X_a \epsilon\right)_{ijk} = 0$$

Namely, ϵ_{ijk} and ϵ^{ijk} are two *invariant tensors* of $SU(3)$.

Warning :

Though δ^i_j is a $SU(3)$ invariant, both δ^{ij} and δ_{ij} are not invariant under $SU(3)$ transformations.

Explanation :

Since,

$$\left(X_a \delta\right)^{ij} = \left(X_a\right)^i{}_k \delta^{kj} + \left(X_a\right)^j{}_k \delta^{ik}$$

we have:

$$\left(X_a \delta\right)^{11} = \left(X_a\right)^1{}_k \delta^{k1} + \left(X_a\right)^1{}_k \delta^{1k} = 2\left(X_a\right)^1{}_1 \neq 0$$

Irreducible representations and symmetry :

We now pick out the states in *tensor product representation* according to the irreducible Rep.(n, m).

The highest weight of Rep.(n, m) of $SU(3)$ reads:

$$\vec{M} = n\vec{M}_1 + m\vec{M}_2$$

where $\vec{M}_1 = \left(\frac{1}{2},\ \frac{1}{2\sqrt{3}}\right)$ and $\vec{M}_2 = \left(\frac{1}{2},\ -\frac{1}{2\sqrt{3}}\right)$. Therefore, the highest weight state of Rep. (n, m) is

$$\left|\begin{matrix}222\cdots\\111\cdots\end{matrix}\right\rangle, \qquad \left\{\#2 = m, \quad \#1 = n\right\}$$

which corresponds to the tensor v_H below,

$$(v_H)^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m} = \left\langle \begin{matrix} i_1 i_2 \cdots i_m \\ j_1 j_2 \cdots j_n \end{matrix} \middle| \begin{matrix}222\cdots\\111\cdots\end{matrix}\right\rangle$$

$$= \mathcal{N}\ \delta^{j_1 1}\delta^{j_2 1}\ldots\delta^{j_n 1}\delta_{i_1 2}\delta_{i_2 2}\cdots\delta_{i_m 2}$$

with $\mathcal{N}$ the normalization constant.

Discussions :

- The tensor v_H is symmetric for the exchange of any two upper indices, and also symmetric for the exchange of any two lower indices.

$$\begin{aligned}(v_H)^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m} &= \mathcal{N}\, \delta^{j_1 1}\delta^{j_2 1}\cdots\delta^{j_n 1}\delta_{i_1 2}\delta_{i_2 2}\cdots\delta_{i_m 2} \\ &= (v_H)^{j_2 j_1 \cdots j_n}_{i_1 i_2 \cdots i_m} \quad = (v_H)^{j_1 j_2 \cdots j_n}_{i_2 i_1 \cdots i_m}\end{aligned}$$

- The tensor v_H is *traceless* for one upper and one lower indices,

$$\delta^{i_1}_{j_1}\,(v_H)^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m} = 0$$

Both properties of v_H are preserved by $SU(3)$ transformations, under which $v_H \rightsquigarrow X_a v_H$:

$$\left.\begin{aligned}&(X_a v_H)^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m} = (X_a v_H)^{j_2 j_1 \cdots j_n}_{i_1 i_2 \cdots i_m} = (X_a v_H)^{j_1 j_2 \cdots j_n}_{i_2 i_1 \cdots i_m}, \\ &\delta^{i_1}_{j_1}(X_a v_H)^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m} = 0.\end{aligned}\right\}$$

Dimension of $SU(3)$ Rep.(n, m) :

In Rep.(n, m) of $SU(3)$, the tensor related to the state $\left|\begin{smallmatrix} i_1 i_2 \cdots i_m \\ j_1 j_2 \cdots j_n \end{smallmatrix}\right\rangle$ is

$$v = v^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m}$$

- v has n upper and m lower indices.
- v is separately symmetric in each type of the indices. *If there were no further constraints*, the number of independent components of v would be:

$$B(n, m) = \frac{(n+2)!}{n!2!}\frac{(m+2)!}{m!2!} = \frac{1}{4}(n+1)(n+2)(m+1)(m+2)$$

- Unfortunately, v *has to be traceless*. As a result, v has to satisfy $B(n-1, m-1)$ additional constraints such as $v^{k j_2 j_3 \cdots j_n}_{i_1 k i_3 \cdots i_m} = 0$.

The correct number of independent components of $SU(3)$ tensor in its irreducible Rep. (n,m) is then,

$$\begin{aligned} D(n,m) &= B(n,m) - B(n-1,m-1) \\ &= \tfrac{1}{4}(n+1)(m+1)\big[(n+2)(m+2) - nm\big] \\ &= \tfrac{1}{2}(n+1)(m+1)(n+m+2) \end{aligned}$$

$D(n,m)$ could also be interpreted as the dimension of the irreducible Rep.(n,m).

Examples :

$$\begin{aligned} &D(1,0) = D(0,1) = 3, \\ &D(1,1) = 8, \\ &D(2,0) = D(0,2) = 6, \\ &D(2,1) = D(1,2) = 15, \\ &D(2,2) = 27, \\ &D(3,0) = D(0,3) = 10. \end{aligned}$$

Suppose u and v are two $SU(3)$ tensors in Rep. (n, m) and Rep. (p, q), respectively,

$$u = \left(u^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m}\right), \quad v = \left(v^{a_1 a_2 \cdots a_p}_{b_1 b_2 \cdots b_q}\right)$$

The tensor product of these two tensors

$$u \otimes v = (u \otimes v)^{j_1 \cdots j_n a_1 \cdots a_p}_{i_1 \cdots i_m b_1 \cdots b_q} = \left(u^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m} v^{a_1 a_2 \cdots a_p}_{b_1 b_2 \cdots b_q}\right)$$

yields a $SU(3)$ tensor in a *reducible* representation.

Strategy for picking out *irreducible representations* from the above reducible representation is,

- Making irreducible representations out of the product of tensors u and v;
- Expressing $u \otimes v$ as a sum of such terms that are proportional to some irreducible representations of $SU(3)$.

Consider the CG-decomposition of $\mathbf{3} \times \mathbf{3}$.

Because $\mathbf{3}$ is Rep.$(1, 0)$, the tensor of $\mathbf{3}$ has the form of $u = (u^i)$. Consequently, an arbitrary $SU(3)$ tensor of $\mathbf{3} \times \mathbf{3}$ can be written as

$$(u \otimes v)^{ij} = u^i v^j, \qquad i, j = 1,\ 2,\ 3$$

We do the Clebsch-Gordan decomposition as follows:

$$u^i v^j = \frac{1}{2}(u^i v^j + u^j v^i) + \frac{1}{2}(u^i v^j - u^j v^i)$$

- The number of the independent components of symmetric combination $\frac{1}{2}(u^i v^j + u^i v^i)$ is $\frac{1}{2} \cdot 3 \cdot 4 = 6$. This tensor belongs to the irreducible representation $\mathbf{6} = \text{Rep.}(2, 0)$.
- The second term (anti-symmetric combination) can be recast as

$$\frac{1}{2}(u^i v^j - u^j v^i) = \frac{1}{2}(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k) u^k v^l = \frac{1}{2} \epsilon^{ijm} \epsilon_{klm} u^k v^l$$

- In view of product $u^i v^j$, ϵ^{ijm} is an invariant tensor. The remaining factor $\epsilon_{klm} u^k v^l$ forms a tensor in $\bar{\mathbf{3}} = \text{Rep.}(0, 1)$ as it has only one *bare* lower index.

We conclude that

$$\mathbf{3} \times \mathbf{3} = \mathbf{6} + \bar{\mathbf{3}}$$

Alternatively but equivalently,

$$(1, 0) \otimes (1, 0) = (2, 0) \oplus (0, 1)$$

Consider the tensor product of $\mathbf{3} \times \bar{\mathbf{3}}$.

Because the tensors of $\mathbf{3}$ and $\bar{\mathbf{3}}$ are $u = (u^i)$ and $v = (v_j)$, respectively, the tensor in $\mathbf{3} \times \bar{\mathbf{3}}$ should be

$$(u \otimes v)^i_j = u^i v_j$$

The Clebsch-Gordan decomposition is,

$$u^i v_j = \left[u^i v_j - \frac{1}{3}\delta^i_j u^k v_k \right] + \frac{1}{3}\delta^i_j u^k v_k$$

As a result,

$$(1,0) \otimes (0,1) = (1,1) \oplus (0,0)$$

or

$$\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{8} + \mathbf{1}$$

Consider the tensor product of $\mathbf{3} \times \mathbf{8}$.

The tensors of $\mathbf{3}$ and $\mathbf{8}$ are $\boldsymbol{u} = (u^i)$ and $\boldsymbol{v} = (v^j{}_k)$, respectively[1]. Therefore, the tensor of $\mathbf{3} \times \mathbf{8}$ has the form

$$(u \otimes v)^{ij}_k = u^i v^j{}_k$$

---

[1] The tensor of $\mathbf{8}$ must be traceless, i.e., $v^j{}_j = 0$.

The Clebsch-Gordan decomposition is carried out in the way,

$$\begin{aligned} u^i v^j{}_k &= \frac{1}{2}(u^i v^j{}_k + u^j v^i{}_k) + \frac{1}{2}(u^i v^j{}_k - u^j v^i{}_k) \\ &= \frac{1}{2}(u^i v^j{}_k + u^j v^i{}_k) + \frac{1}{2}\epsilon^{ijm}\epsilon_{mnl}u^n v^l{}_k \end{aligned}$$

- The first term

$$\text{term 1} = \frac{1}{2}(u^i v^j{}_k + u^j v^i{}_k)$$

has been symmetrized about the upper indices i and j. To make it traceless further, we recast it as

$$\begin{aligned} \text{term 1} &= \frac{1}{2}\left[(u^i v^j{}_k + u^j v^i{}_k) - a\delta^i_k u^l v^j{}_l - b\delta^j_k u^l v^i{}_l\right] \\ &\quad + \frac{1}{2}\left(a\delta^i_k u^l v^j{}_l + b\delta^j_k u^l v^i{}_l\right) \end{aligned}$$

The first row is expected to be in Rep.$(2,1)$ but the second row in Rep.$(1,0)$.

The traceless condition in Rep.$(2,1)$ requires,

$$u^l v^j{}_l (1-3a-b) = 0, \quad u^l v^i{}_l (1-a-3b) = 0.$$

Hence $a = b = 1/4$. We finally recast the *first* term as:

$$\begin{aligned} \text{term 1} \quad &= \frac{1}{2}\left[(u^i v^j{}_k + u^j v^i{}_k) - \frac{1}{4}(\delta^i_k u^l v^j{}_l + \delta^j_k u^l v^i{}_l)\right] \\ &\quad + \frac{1}{8}\left(\delta^i_k u^l v^j{}_l + \delta^j_k u^l v^i{}_l\right) \end{aligned}$$

In the previous formula for decomposition of tensor product $u^i v^j{}_k$, the second term reads,

$$\text{term 2} = \frac{1}{2} \epsilon^{ijm} \epsilon_{mnl} u^n v^l{}_k$$

After discarding the invariant tensor ϵ^{ijm}, it has only two lower indices m and k, effectively.

- Irreducibility requires the symmetrization about these two indices. Therefore,

$$
\begin{aligned}
\text{term 2} &= \frac{1}{2}\epsilon^{ijm}\left[\frac{1}{2}(\epsilon_{mnl}u^n v^l{}_k + \epsilon_{knl}u^n v^l{}_m)\right. \\
&\qquad \left. + \frac{1}{2}(\epsilon_{mnl}u^n v^l{}_k - \epsilon_{knl}u^n v^l{}_m)\right] \\
&= \frac{1}{4}\epsilon^{ijm}\left(\epsilon_{mnl}u^n v^l{}_k + \epsilon_{knl}u^n v^l{}_m\right) \\
&\qquad + \frac{1}{4}\epsilon^{ijm}\epsilon_{pnl}u^n v^l{}_q(\delta^p_m\delta^q_k - \delta^q_m\delta^p_k) \\
&= \frac{1}{4}\epsilon^{ijm}\left(\epsilon_{mnl}u^n v^l{}_k + \epsilon_{knl}u^n v^l{}_m\right) \\
&\qquad + \frac{1}{4}\epsilon^{ijm}\epsilon_{pnl}u^n v^l{}_q\epsilon_{mkr}\epsilon^{pqr}
\end{aligned}
$$

On RHS, the first row stands for a symmetric tensor in Rep.$(0,2)$. Let us now focus on the second row.

$$
\begin{aligned}
\frac{1}{4}\epsilon^{ijm}\epsilon_{pnl}u^n v^l{}_q\epsilon_{mkr}\epsilon^{pqr} &= \frac{1}{4}u^n v^l{}_q\left(\delta^i_k\delta^j_r - \delta^j_k\delta^i_r\right)\left(\delta^q_n\delta^r_l - \delta^r_n\delta^q_l\right) \\
&= \frac{1}{4}u^n v^l{}_q\left[\delta^i_k(\delta^j_l\delta^q_n - \delta^j_n\delta^q_l) - \delta^j_k(\delta^i_l\delta^q_n - \delta^i_n\delta^q_l)\right] \\
&= \frac{1}{4}\left[\delta^i_k(u^l v^j{}_l - u^j v^l{}_l) - \delta^j_k(u^l v^i{}_l - u^i v^l{}_l)\right] \\
&= \frac{1}{4}\left(\delta^i_k u^l v^j{}_l - \delta^j_k u^l v^i{}_l\right)
\end{aligned}
$$

which stands for the tensor of Rep.$(1,0)$.

In summary,

$$
\begin{aligned}
u^i v^j{}_k \quad &= \frac{1}{2}\left[(u^i v^j{}_k + u^j v^i{}_k) - \frac{1}{4}(\delta^i_k u^l v^j{}_l + \delta^j_k u^l v^i{}_l)\right] \\
&\quad + \frac{1}{4}\epsilon^{ijm}\left(\epsilon_{mnl}u^n v^l{}_k + \epsilon_{knl}u^n v^l{}_m\right) \\
&\quad + \frac{1}{8}\left(3\delta^i_k u^l v^j{}_l - \delta^j_k u^l v^i{}_l\right)
\end{aligned}
$$

It implies:

$$(1,0) \otimes (1,1) = (2,1) \oplus (0,2) \oplus (1,0)$$

Equivalently,

$$\mathbf{3} \times \mathbf{8} = \mathbf{15} + \bar{\mathbf{6}} + \mathbf{3}$$

Consider the CG-decomposition of $\mathbf{6} \times \mathbf{3}$.

The tensors of $\mathbf{6}$ and $\mathbf{3}$ are $u = (u^{ij})$ and $v = (v^k)$, respectively. Consequently, the tensor of $\mathbf{6} \times \mathbf{3}$ has the form

$$\left(u \otimes v\right)^{ijk} = u^{ij} v^k$$

where u is a symmetric tensor of $SU(3)$ in Rep.$(2,0)$,

$$u^{ij} = u^{ji}$$

By symmetrizing all of the upper indices,

$$\begin{aligned} u^{ij}v^k &= \frac{1}{3}\left(u^{ij}v^k + u^{jk}v^i + u^{ki}v^j\right) \\ &\quad + \frac{1}{3}\left(2u^{ij}v^k - u^{jk}v^i - u^{ki}v^j\right) \end{aligned}$$

The first term on RHS

$$\frac{1}{3}\left(u^{ij}v^k + u^{jk}v^i + u^{ki}v^j\right)$$

is symmetric for exchanging any two indices. It describes a tensor in irreducible Rep.$(3,0)$ of $SU(3)$.

The second term is recast as:

$$\begin{aligned} &\frac{1}{3}(2u^{ij}v^k - u^{jk}v^i - u^{ki}v^j) \\ &\qquad = \frac{1}{3}\left(u^{ij}v^k - u^{jk}v^i\right) + \frac{1}{3}\left(u^{ij}v^k - u^{ki}v^j\right) \end{aligned}$$

$$= \frac{1}{3}\left(\delta^i_m\delta^k_n - \delta^i_n\delta^k_m\right)u^{mj}v^n + \frac{1}{3}\left(\delta^j_m\delta^k_n - \delta^j_n\delta^k_m\right)u^{im}v^n$$

$$= \frac{1}{3}\left[\epsilon^{ikl} \underbrace{\epsilon_{lmn}u^{mj}v^n}_{\text{traceless } \epsilon_{lmn}u^{ml}=0} + \epsilon^{jkl} \underbrace{\epsilon_{lmn}u^{im}v^n}_{\text{traceless } \epsilon_{lmn}u^{lm}=0}\right]$$

Apart from the invariant tensors ϵ^{ikl} and ϵ^{jkl}, the term is involved in some traceless tensors

$$\epsilon_{lmn}u^{mj}v^n, \quad \epsilon_{lmn}u^{im}v^n$$

Hence, it describes a tensor in the $SU(3)$ irreducible Rep.$(1,1)$.

In summary,

$$\begin{aligned} u^{ij}v^k &= \frac{1}{3}\left(u^{ij}v^k + u^{jk}v^i + u^{ki}v^j\right) \\ &\quad + \frac{1}{3}\left(\epsilon^{ikl}\epsilon_{lmn}u^{mj}v^n + \epsilon^{jkl}\epsilon_{lmn}u^{im}v^n\right) \end{aligned}$$

It implies that,

$$(2,0) \otimes (1,0) = (3,0) \oplus (1,1)$$

Equivalently,

$$\mathbf{6} \times \mathbf{3} = \mathbf{10} + \mathbf{8}$$

Corollary :

$$\mathbf{3} \times \mathbf{3} \times \mathbf{3} = (\mathbf{6} + \bar{\mathbf{3}}) \times \mathbf{3} = \mathbf{10} + \mathbf{8} + \mathbf{8} + \mathbf{1}$$

Equivalently,

$$(1,0) \otimes (1,0) \otimes (1,0) = (3,0) \oplus (1,1) \oplus (1,1) \oplus (0,0)$$

Homework :

Problems :

1. Decompose the product of tensor components $u^i v^{jk}$, where $v^{jk} = v^{kj}$ transforms like a tensor in Rep.**6** of $SU(3)$.

2. Find the matrix elements $\langle u|X_a|v\rangle$, where X_a stand for the $SU(3)$ generators and $|u\rangle$ and $|v\rangle$ are states in the adjoint representation of $SU(3)$ with tensor components u^i_j and v^i_j. Write the result in terms of the tensor components and the Gell-Mann Matrices.

3. In Rep. **6** of $SU(3)$, for each weight find the corresponding tensor component v^{ij}.

Young tableaux in $SU(3)$:

Young tableaux is very convenient in dealing with the Clebsch-Gordan decomposition of the Lie group representations. Here we consider its application in $SU(3)$.

A *crucial* observation:

The representation $\bar{\mathbf{3}}$ of $SU(3)$ is the antisymmetric product of two $\mathbf{3}$'s,

$$w_i = \epsilon_{ijk} u^j v^k$$

An irreducible $SU(3)$ tensor $\mathscr{A}$ in Rep.(n, m) has the component structure

$$\mathscr{A}^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_m}$$

1. $\mathscr{A}$ is symmetric in upper and lower indices, separately.
2. $\mathscr{A}$ is traceless for one upper and one lower indices.

We can raise all the lower tensor indices by using the invariant tensor ϵ^{ijk} of $SU(3)$,

$$\epsilon^{j_1 k_1 l_1} \epsilon^{j_2 k_2 l_2} \cdots \epsilon^{j_m k_m l_m} \mathscr{A}^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_m} = \mathscr{B}^{k_1 l_1 k_2 l_2 \cdots k_m l_m i_1 i_2 \cdots i_n}$$

- $\mathscr{B}^{k_1 l_1 k_2 l_2 \cdots k_m l_m i_1 i_2 \cdots i_n}$ is antisymmetric in each pair $\{k_a, l_a\}$ for interchange

$$k_a \leftrightsquigarrow l_a, \quad (a = 1,\ 2, \cdots,\ m)$$

and symmetric for exchange of pairs

$$\Big\{k_a, l_a\Big\} \leftrightsquigarrow \Big\{k_b, l_b\Big\}, \quad (a,\ b = 1,\ 2, \cdots,\ m)$$

- Traceless condition of $\mathscr{A}$ becomes:

$$\begin{aligned} &\epsilon_{i_1 k_1 l_1} \mathscr{B}^{k_1 l_1 k_2 l_2 \cdots k_m l_m i_1 i_2 \cdots i_n} \\ &\qquad = \epsilon_{i_2 k_2 l_2} \mathscr{B}^{k_1 l_1 k_2 l_2 \cdots k_m l_m i_1 i_2 \cdots i_n} \\ &\qquad = \cdots = 0 \end{aligned}$$

The traceless condition of tensor $\mathscr{B}$ could be shown as follows:

$$
\begin{aligned}
&\epsilon_{i_1 k_1 l_1} \mathscr{B}^{k_1 l_1 k_2 l_2 \cdots k_m l_m i_1 i_2 \cdots i_n} \\
&\quad = \epsilon_{i_1 k_1 l_1} \epsilon^{j_1 k_1 l_1} \epsilon^{j_2 k_2 l_2} \ldots \epsilon^{j_m k_m l_m} \mathscr{A}^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_m} \\
&\quad = 2\delta^{j_1}_{i_1} \epsilon^{j_2 k_2 l_2} \ldots \epsilon^{j_m k_m l_m} \mathscr{A}^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_m} \\
&\quad = 2\epsilon^{j_2 k_2 l_2} \ldots \epsilon^{j_m k_m l_m} \mathscr{A}^{i_1 i_2 \cdots i_n}_{i_1 j_2 \cdots j_m} \\
&\quad = 0
\end{aligned}
$$

With such a $SU(3)$ tensor $\mathscr{B}^{k_1 l_1 k_2 l_2 \cdots k_m l_m i_1 i_2 \cdots i_n}$ in Rep.(n, m), we associate a Young tableau

k_1	k_2	$\cdots$	k_m	i_1	i_2	$\cdots$	i_n
l_1	l_2	$\cdots$	l_m				

The Young tableau

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline k_1 & k_2 & \cdots & k_m & i_1 & i_2 & \cdots & i_n \\ \hline l_1 & l_2 & \cdots & l_m \\ \cline{1-4}\end{array}$$

describes a tensor

$$\mathscr{B} = \left(\mathscr{B}^{k_1 l_1 k_2 l_2 \cdots k_m l_m i_1 i_2 \cdots i_n} \right)$$

with the following properties:

- It has $(n + 2m)$ upper indices.
- It is antisymmetric for index interchange in every pair $\left\{k_a, l_a\right\}$, where $a = 1, 2, \cdots, m$.
- It is symmetric under arbitrary permutations of the indices i_b and k_a, and separately symmetric under arbitrary permutations of l_a, where $a = 1, 2, \cdots, m$ and $b = 1, 2, \cdots, n$.

Question : Why ?

Because $\mathscr{A} = E_- v_H$[2], and the $SU(3)$ transformation preserves the permutational symmetries in tensor indices, we are necessary to analyze the claimed symmetries for tensor $\mathscr{B}_H$,

$$\begin{aligned}
&\mathscr{B}_H^{k_1 l_1 k_2 l_2 \cdots k_m l_m i_1 i_2 \cdots i_n} \\
&\qquad = \epsilon^{j_1 k_1 l_1} \epsilon^{j_2 k_2 l_2} \ldots \epsilon^{j_m k_m l_m} \left(v_H\right)^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_m} \\
&\qquad = \mathcal{N}\ \epsilon^{j_1 k_1 l_1} \epsilon^{j_2 k_2 l_2} \ldots \epsilon^{j_m k_m l_m} \delta^{i_1 1} \delta^{i_2 1} \ldots \delta^{i_n 1} \delta_{j_1 2} \delta_{j_2 2} \cdots \delta_{j_m 2} \\
&\qquad = \mathcal{N}\ \epsilon^{2 k_1 l_1} \epsilon^{2 k_2 l_2} \ldots \epsilon^{2 k_m l_m} \delta^{i_1 1} \delta^{i_2 1} \ldots \delta^{i_n 1}
\end{aligned}$$

The **independent** components of $\mathscr{B}_H$ read,

$$\mathscr{B}_H^{1313\cdots 1311\cdots 1} = \mathcal{N}\ \epsilon^{213} \epsilon^{213} \ldots \epsilon^{213} = \pm \mathcal{N}$$

corresponding to

$$\begin{aligned}
&k_1 = k_2 = \cdots = k_m = i_1 = i_2 = \cdots = i_n = 1 \\
&l_1 = l_2 = \cdots = l_m = 3
\end{aligned}$$

[2] E_- stands for some $SU(3)$ generator.

Therefore,

1. $\mathscr{B}_H$ is symmetric for interchanging the indices in the same rows of the corresponding Young tableau.
2. $\mathscr{B}_H$ is antisymmetric for exchanging the indices in the same columns of the corresponding Young tableau.
3. Young tableaux can be directly used to represent the irreducible representations of $SU(3)$.

Example 1:

Young tableau

$$\boxed{i}$$

can be used to stand for *either* a $SU(3)$ tensor u^i of irreducible representation $\mathbf{3}$ *or* $\mathbf{3}$ itself[3].

[3] For $SU(3)$, $\mathbf{3}$ is Rep.$(1,0)$. Similarly, $\mathbf{6} = $ Rep.$(2,0)$.

Example 2:
Young tableau

$$\boxed{i \mid j}$$

describes *either* a symmetric $SU(3)$ tensor

$$u^{ij} = u^{ji}$$

in Rep.$(2,0) = \mathbf{6}$ *or* $\mathbf{6}$ itself.

Example 3:
Young tableau

$$\begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array}$$

describes *either* the antisymmetric $SU(3)$ tensor

$$u^{ij} = -u^{ji} = \epsilon^{ijk} v_k$$

in Rep.$(0,1) = \bar{\mathbf{3}}$ *or* $\bar{\mathbf{3}}$ itself.

Example 4:

Young tableau

$$\begin{array}{|c|c|}\hline i & j \\ \hline k & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

describes *either* a $SU(3)$ tensor

$$u^{ijk} = u^{jik} = -u^{kji} = \epsilon^{ikl} v_l^j$$

in Rep.$(1,1) = \mathbf{8}$ *or* $\mathbf{8}$ itself.

Example 5:

Young tableau

$$\begin{array}{|c|}\hline i \\ \hline j \\ \hline k \\ \hline\end{array}$$

is related to the invariant $SU(3)$ tensor ϵ^{ijk}. It represents the trivial Rep.$(0,0) = \mathbf{1}$.

Example 6:

Young tableau

$$\begin{array}{|c|}\hline i \\ \hline j \\ \hline k \\ \hline l \\ \hline \end{array}$$

is not allowed in $SU(3)$. The antisymmetric $SU(3)$ tensor

$$u^{ijkl}, \qquad \left\{ i, j, k, l = 1,\ 2,\ 3 \right\}$$

does not exist in any of its representations.

Warning :

1. In Young tableaux of $SU(3)$, any columns with 3 boxes contribute a factor proportional to ϵ^{123} and should be ignored. e.g,

$$\begin{array}{|c|c|c|c|c|c|c|}\hline a & b & c & d & e & f & g \\ \hline h & i & j \\ \cline{1-3} k & l \\ \cline{1-2} \end{array}$$

should be reduced to

$$\begin{array}{|c|c|c|c|c|}\hline c & d & e & f & g \\ \hline j \\ \cline{1-1} \end{array}$$

2. The $SU(3)$ tensor which relates to a Young tableau with more than 3 boxes in any column vanishes!

Calculating $D(n,m)$ by using Young tableaux :

The irreducible Rep.(n,m) of $SU(3)$ has dimension

$$D(n,m) = \frac{1}{2}(n+1)(m+1)(n+m+2)$$

Question:

Can $D(n,m)$ be deduced from the corresponding Young tableau ?

The answer is absolutely *yes*. We draw the corresponding Young tableau

k_1	k_2	$\cdots$	k_m	i_1	i_2	$\cdots$	i_n
l_1	l_2	$\cdots$	l_m				

and represent $D(n,m)$ as a fraction:

$$D(n,m) = \frac{a(n,m)}{b(n,m)}$$

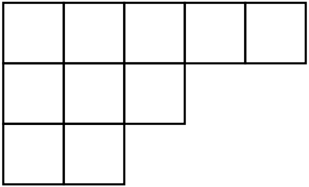
We now introduce the rules for calculating $a(n, m)$ and $b(n, m)$. To this end, we need define two concepts:

1. Content m_{ij}
2. Hook number h_{ij}

for related Young tableau. For later convenience, consider $SU(N)$ for a generic $N \geqslant 3$. The content m_{ij} for a box at the j-th column of the i-th row is,

$$m_{ij} = j - i$$

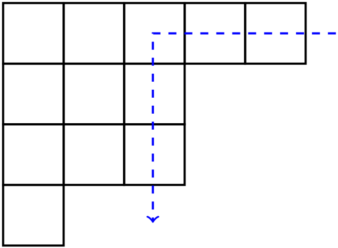
Example : For Young tableau



we have $m_{23} = 1, \; m_{14} = 3$ but $m_{32} = -1$.

To define hook number h_{ij}, we have to introduce the so-called *hook* for each box in Young tableau.

Here is the hook for box at the third column of the first row,



The hook number h_{ij} is the total number of boxes along the hook of the box at the j-th column of the i-th row in the Young tableau.

In given example, we have:

$$h_{13} = 5, \quad h_{22} = 3, \quad h_{21} = 5.$$

Dimensions of $SU(N)$ irreducible representations :

$d_{[\lambda]}(SU(N))$:

The dimension of the irreducible representation of $SU(N)$ described by Young tableau $[\lambda]$ is expressed by a quotient,

$$d_{[\lambda]}(SU(N)) = \prod_{ij} \frac{N + m_{ij}}{h_{ij}}$$

- For $SU(3)$, this formula reduces to:

$$D(n,m) = \frac{a(n,m)}{b(n,m)}$$

where

$$a(n,m) = \prod_{ij}(3 + m_{ij}), \qquad b(n,m) = \prod_{ij} h_{ij}.$$

By define the so-called Numerator Young tableau:

3	4	$\cdots$	$m+2$	$m+3$	$m+4$	$\cdots$	$m+n+2$
2	3	$\cdots$	$m+1$				

we can easily get:

$$a(n,m) = \prod_{i=3}^{n+m+2} \prod_{j=2}^{m+1} ij = \frac{1}{2}(n+m+2)!(m+1)!$$

We introduce the denominator Young tableau as follows:

h_{11}	h_{12}	$\cdots$	h_{1m}	n	$n-1$	$\cdots$	1
h_{21}	h_{22}	$\cdots$	h_{2m}				

where $h_{11} = n+m+1$, $h_{12} = n+m$, $h_{1m} = n+2$, $h_{21} = m$, $h_{22} = m-1$ and $h_{2m} = 1$. Therefore,

$$b(n,m) = \frac{(n+m+1)!m!}{(n+1)}$$

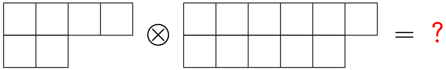
Consequently,

$$\begin{aligned} D(n,m) &= \frac{a(n,m)}{b(n,m)} \\ &= \frac{(n+m+2)!(m+1)!}{2} \cdot \frac{(n+1)}{(n+m+1)!m!} \\ &= \frac{1}{2}(n+1)(m+1)(n+m+2) \end{aligned}$$

This is what we have expected.

Clebsch-Gordan decomposition :

Let us now to discuss the Young tableau rules for decomposing the tensor product of two $SU(3)$ irreducible representations. e.g.,



CG-decomposition rules :

- Mark each box of the second empty tableau with the corresponding number of its row. e.g.,

1	1	1	1	1	1
2	2	2	2	2	

- Continue by adding all the boxes of the second tableau to the first one. These boxes may only be added to the right or the bottom of the first tableau.

- Each resulting tableau has to be an allowed configuration, i.e., no row is longer than the row above.
- In the case of $SU(N)$, no column must contain more than N boxes.
- Within a row, the numbers in the boxes originating from the second tableau must not decrease from left to right.
- Within a column, the numbers in the boxes originating from the second tableau must increase from top to bottom.
- A box of the i-th row of the second Young tableau must not be attached to the first $(i-1)$ rows of the first Young tableau.
- If two tableaux of the same shape are produced, they are counted as different only if the labels are different.

Examples :

Focus on the tensor products of some irreducible representations of $SU(3)$.

The first example is,

$$\square \otimes \square = ?$$

By the studied rules,

$$\square \otimes \square \quad \rightsquigarrow \square \otimes \boxed{1} = \begin{array}{|c|c|}\hline & 1 \\ \hline \end{array} \oplus \begin{array}{|c|}\hline \\ \hline 1 \\ \hline \end{array}$$

Namely,

$$\mathbf{3} \times \mathbf{3} = \mathbf{6} + \bar{\mathbf{3}}$$

Our second example is about the CG-decomposition of

$$\ydiagram{1,1} \otimes \ydiagram{1}$$

By the studied rules,

$$\ydiagram{1,1} \otimes \ydiagram{1} \quad \rightsquigarrow \ydiagram{1,1} \otimes \begin{ytableau} 1 \end{ytableau} \; = \; \begin{ytableau} {} & 1 \\ {} \end{ytableau} \oplus \begin{ytableau} {} \\ {} \\ 1 \end{ytableau}$$

i.e.,

$$\bar{\mathbf{3}} \times \mathbf{3} = \mathbf{8} + \mathbf{1}$$

Another example is to ask

$$\ydiagram{1} \otimes \ydiagram{1,1} = \,?$$

By the studied rules, we have:

$$\ydiagram{1} \otimes \ydiagram{1,1} \rightsquigarrow \ydiagram{1} \otimes \ytableaushort{1,2} = \left\{ \ydiagram{1} \otimes \ytableaushort{1} \right\} \otimes \ytableaushort{2}$$

$$= \left\{ \ytableaushort{{}1} \oplus \ytableaushort{{},1} \right\} \otimes \ytableaushort{2}$$

$$= \ytableaushort{{}1,2} \oplus \ytableaushort{{},1,2} = \ydiagram{2,1} \oplus \ydiagram{1,1,1}$$

i.e.,

$$\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{8} + \mathbf{1}$$

As the 4-th example in $SU(3)$, we consider

$$\ydiagram{1,1} \otimes \ydiagram{1,1} = ?$$

By the studied rules, we see that

$$\ydiagram{1,1} \otimes \ydiagram{1,1} \quad \rightsquigarrow \quad \ydiagram{1,1} \otimes \begin{ytableau} 1 \\ 2 \end{ytableau} = \left\{ \ydiagram{1,1} \otimes \begin{ytableau} 1 \end{ytableau} \right\} \otimes \begin{ytableau} 2 \end{ytableau}$$

$$= \left\{ \begin{ytableau} {} & 1 \\ {} \end{ytableau} \oplus \begin{ytableau} {} \\ {} \\ 1 \end{ytableau} \right\} \otimes \begin{ytableau} 2 \end{ytableau}$$

$$= \begin{ytableau} {} & 1 \\ {} \\ 2 \end{ytableau} \oplus \begin{ytableau} {} & 1 \\ {} & 2 \end{ytableau} = \ydiagram{1} \oplus \ydiagram{2,2}$$

i.e.,

$$\bar{\mathbf{3}} \times \bar{\mathbf{3}} = \mathbf{3} + \bar{\mathbf{6}}$$

Finally, we consider the CG-decomposition of tensor product of

$$\ydiagram{2,1} \otimes \ydiagram{2,1}$$

By the studied rules, we have :

$$\begin{ytableau} \, & \, \\ \, \end{ytableau} \otimes \begin{ytableau} 1 & 1 \\ 2 \end{ytableau} = \begin{ytableau} \, & \, & 1 \\ \, & 1 \\ 2 \end{ytableau} \oplus \begin{ytableau} \, & \, & 1 \\ \, & 2 \\ 1 \end{ytableau} \oplus \begin{ytableau} \, & \, & 1 & 1 \\ \, & 2 \end{ytableau}$$

$$\oplus \begin{ytableau} \, & \, & 1 \\ \, & 1 & 2 \end{ytableau} \oplus \begin{ytableau} \, & \, \\ \, & 1 \\ 1 & 2 \end{ytableau}$$

$$\oplus \begin{ytableau} \, & \, & 1 & 1 \\ \, \\ 2 \end{ytableau}$$

i.e.,

$$\mathbf{8} \times \mathbf{8} = \mathbf{8} + \mathbf{8} + \mathbf{27} + \overline{\mathbf{10}} + \mathbf{1} + \mathbf{10}$$

Problems :

1. Find $(2,1) \otimes (2,1)$ for $SU(3)$. Can you determine which representations appear anti-symmetrically in the tensor product, and which appear symmetrically?
2. Find $\mathbf{10} \times \mathbf{8}$.
3. For any Lie group, the tensor product of the adjoint representation with any arbitrary nontrivial representation D must contain D (think about the action of the generators on the states of D and see if you can figure out why this is so.). In particular, you know that for any nontrivial $SU(3)$ representation D. How can you see this using Young tableaux?