LECTURE 1: INTRODUCTION

1. What is a Lie group ?

Mathematicians invented the concept of a *group* to describe symmetry. The collection of symmetries of any object is a group, and every group is the symmetries of some object. For example, the symmetry group of a square contains four rotations, two mirror images and two diagonal flip, which is a finite group of eight elements (usually denoted by H_4). Another simple example is a circle in the plane, whose symmetry group consists of all rotations and reflections, and can be identified with $O(2)$, the set of all two by two orthogonal matrices.

Among all groups, *Lie groups* are of particular importance. They were first studied by the Norwegian mathematician *Sophus Lie* at the end of nineteenth century. Roughly speaking, a Lie group is a group of symmetries where the symmetries varies smoothly. More precisely, a Lie group admits three structures

- an algebraic structure as a group,
- a geometric structure as a topological manifold,
- a smooth structure so that one can do analysis,

and moreover, these three structures are *compatible*, i.e. they are naturally related to each other. This make Lie groups into important objects as well as tools in mathematics.

We will postpone the official definition of Lie groups to later. Here are some basic examples of Lie groups:

- Any finite group (endowed with the discrete topology) is a Lie group (not very interesting)
- S^1 and $\mathbb{R}^1$ (endowed with the usual group structure and the usual topology structure) are Lie groups
- Matrix Lie groups: $GL(n)$, $SL(n)$, $O(n)$, $SO(n)$, $U(n)$ etc.

2. Why do we study Lie groups ?

Other than the fact that Lie groups have rich structures and that the mathematical theories of Lie groups are very beautiful, Lie groups also have applications to many different areas:

(1) (Lie's original work) Lie groups are important in studying differential equations

(2) Special functions such as Bessel functions, spherical harmonics etc can be unified by viewing them as coefficient of representation of Lie groups
(3) According to Klein's Erlanger program, the essence of geometry is to study the invariance of groups acting on homogeneous spaces
(4) Lie groups appears naturally in modern geometric theories, e.g. as the isometry group of a compact Riemannian manifold
(5) Lie groups describe symmetries in many physics theories, includes classical mechanics (conservative quantities), relativity theories (Lorenz group), quantum mechanics (Heisenberg group) etc.

To illustrate the power of symmetry in solving mathematical problems, let's look at the following baby example. (see A. Kirillov, *Introduction to Lie Groups and Lie Algebras.*)

Question: Suppose you have n ($n > 2$) numbers $a_1, \cdots, a_n$ arranged on a circle. You have a transformation A which replaces each number with the average of its neighborhoods, i.e. replaces a_1 with $\frac{a_n+a_2}{2}$, a_2 with $\frac{a_1+a_3}{2}$, and so on. If you do this sufficiently many times, will the numbers be roughly equal?

To answer this, we need to study the asymptotic behavior of A^n for n large. Alternately, it suffices to find the eigenvalues and the eigenvectors of A. While it is very hard to compute them from the characteristic polynomial, we can actually compute them via symmetry. Note that the problem has a rotational $\mathbb{Z}_n$-symmetry: if we denote by B the operator of rotating the circle by $\frac{2\pi}{n}$, i.e. sending $(a_1, a_2, \cdots, a_n)$ to $(a_2, a_3, \cdots, a_n, a_1)$, then $BAB^{-1} = A$. (This is $\mathbb{Z}_n$-symmetry since the operator B generates the group of cyclic permutation $\mathbb{Z}_n$.) We will use this symmetry to find the eigenvectors and eigenvalues of A.

We will make use of the following basic result from linear algebra: if two matrices A and B commutes, and V_λ is an eigenspace of B, then $AV_\lambda \subset V_\lambda$. (Check it if you never see it before!) In particular, A preserves the eigenspace decomposition of B. So to diagonalize A, we only need to diagonalize A on each eigenspace of B.

Back to our example. Since $B^n = I_n$ the identical matrix, all the eigenvalues of B must be n-th root of unity. Denote by $\omega = e^{2\pi i/n}$ the primitive n-th root of unity. It turns out that the eigenvalues of B are exactly $\lambda_k = \omega^k$ for $k = 0, 1, \cdots, n-1$, with corresponding eigenvectors $\vec{v}_k = (1, \omega^k, \omega^{2k}, \cdots, \omega^{(n-1)k})$. Since each eigenspace of B is one dimensional, each v_k must also be an eigenvector of A! It follows that the eigenvalues of A are $\lambda_k = \frac{\omega^k+\omega^{-k}}{2} = \cos\frac{2k\pi}{n}$.

Now we can answer the problem. If n is odd, then all eigenvalue except the first λ_0 (=1) satisfies $|\lambda| < 1$, so that $A^n\vec{a}$ will tends to an eigenvector of $\lambda_0 = 1$, i.e. to $(\frac{\sum a_i}{n}, \cdots, \frac{\sum a_i}{n})$. However, when n is even, we will have another eigenvector $\lambda_{n/2} = -1$, and so the numbers are not going to be roughly equal, but oscillating between two set of numbers, unless the projection of the initial data onto the -1 eigenspace is zero.

Let's summarize. There are two crucial ingredients in the previous solution. First we found that the problem has a $\mathbb{Z}_n$-symmetry. Second, instead of work on $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ itself, we realize the symmetry using the matrix B, or more precisely, the cyclic group $\{I, B, B^2, \cdots, B^{n-1}\}$, which is a subgroup of $GL(n)$. This is the idea of *representation* – trying to understand a group via its linear *action* on a linear space.

It turns out that the same idea works in many many other places. For example, consider the sphere $S^2 \subset \mathbb{R}^3$. Let Δ be the Laplace operator on S^2. It is very important in both mathematics and physics to compute the eigenvalues of Δ. Since Δ is a second order differential operator, it is very hard to compute the eigenvalues and eigenfunctions by solving the corresponding second order differential equation. But much as in the baby example above, it is easy to notice that this problem has a $SO(3)$-symmetry. The only difference is that now we have a very big group $SO(3)$ instead of the finite group $\mathbb{Z}_n$. The group $SO(3)$ has very nice structure, and is a Lie group. To find the eigenvalues and eigenfunctions of Δ, we need to study *irreducible representations* of $SO(3)$.

3. Course plan

In this course we would like to cover:

- Background: Basic theory of manifolds and vector fields – the geometry of the underlying spaces of Lie groups and Lie algebras.
- Basic Lie theory: Lie groups and their associated Lie algebras, with emphasis on the use of the exponential map.
- Lie groups as symmetric group: Lie group actions on smooth manifolds.
- Basic representation theory and its role in the harmonic analysis on a Lie group.
- Basic structural theory of compact Lie groups.

No textbook is required. I will post course notes after each lecture. The following two books could be used as references:

- J.Duistermaat and J.Kolk. *Lie Groups*.
- M. Sepanski, *Compact Lie Groups*.