2. Forces

In this section, we describe a number of different forces that arise in Newtonian mechanics. Throughout, we will restrict attention to the motion of a single particle. (We'll look at what happens when we have more than one particle in Section 5). We start by describing the key idea of energy conservation, followed by a description of some common and important forces.

2.1 Potentials in One Dimension

Let's start by considering a particle moving on a line, so its position is determined by a single function $x(t)$. For now, suppose that the force on the particle depends only on the position, not the velocity: $F = F(x)$. We define the *potential* $V(x)$ (also called the *potential energy*) by the equation

$$F(x) = -\frac{dV}{dx} \tag{2.1}$$

The potential is only defined up to an additive constant. We can always invert (2.1) by integrating both sides. The integration constant is now determined by the choice of lower limit of the integral,

$$V(x) = -\int_{x_0}^{x} dx' \ F(x')$$

Here x' is just a dummy variable. (Do not confuse the prime with differentiation! In this course we will only take derivatives of position x with respect to time and always denote them with a dot over the variable). With this definition, we can write the equation of motion as

$$m\ddot{x} = -\frac{dV}{dx} \tag{2.2}$$

For any force in one-dimension which depends only on the position, there exists a conserved quantity called the *energy*,

$$E = \frac{1}{2}m\dot{x}^2 + V(x)$$

The fact that this is *conserved* means that $\dot{E} = 0$ for *any* trajectory of the particle which obeys the equation of motion. While $V(x)$ is called the potential energy, $T = \frac{1}{2}m\dot{x}^2$ is called the *kinetic energy*. Motion satisfying (2.2) is called *conservative*.

It is not hard to prove that E is conserved. We need only differentiate to get

$$\dot{E} = m\dot{x}\ddot{x} + \frac{dV}{dx}\dot{x} = \dot{x}\left(m\ddot{x} + \frac{dV}{dx}\right) = 0$$

where the last equality holds courtesy of the equation of motion (2.2).

In any dynamical system, conserved quantities of this kind are very precious. We will spend some time in this course fishing them out of the equations and showing how they help us simplify various problems.

An Example: A Uniform Gravitational Field

In a uniform gravitational field, a particle is subjected to a constant force, $F = -mg$ where $g \approx 9.8 \ ms^{-2}$ is the acceleration due to gravity near the surface of the Earth. The minus sign arises because the force is downwards while we have chosen to measure position in an upwards direction which we call z. The potential energy is

$$V = mgz$$

Notice that we have chosen to have $V = 0$ at $z = 0$. There is nothing that forces us to do this; we could easily add an extra constant to the potential to shift the zero to some other height.

The equation of motion for uniform acceleration is

$$\ddot{z} = -g$$

Which can be trivially integrated to give the velocity at time t,

$$\dot{z} = u - gt \tag{2.3}$$

where u is the initial velocity at time $t = 0$. (Note that z is measured in the upwards direction, so the particle is moving up if $\dot{z} > 0$ and down if $\dot{z} < 0$). Integrating once more gives the position

$$z = z_0 + ut - \frac{1}{2}gt^2 \tag{2.4}$$

where z_0 is the initial height at time $t = 0$. Many high schools teach that (2.3) and (2.4) — the so-called "suvat" equations — are key equations of mechanics. They are not. They are merely the integration of Newton's second law for constant acceleration. Do not learn them; learn how to derive them.

Another Simple Example: The Harmonic Oscillator

The harmonic oscillator is, by far, the most important dynamical system in all of theoretical physics. The good news is that it's very easy. (In fact, the reason that it's so important is precisely because it's easy!). The potential energy of the harmonic oscillator is defined to be

$$V(x) = \frac{1}{2}kx^2$$

The harmonic oscillator is a good model for, among other things, a particle attached to the end of a spring. The force resulting from the energy V is given by $F = -kx$ which, in the context of the spring, is called *Hooke's law*. The equation of motion is

$$m\ddot{x} = -kx$$

which has the general solution

$$x(t) = A\cos(\omega t) + B\sin(\omega t) \qquad \text{with } \omega = \sqrt{\frac{k}{m}}$$

Here A and B are two integration constants and ω is called the *angular frequency*. We see that all trajectories are qualitatively the same: they just bounce backwards and forwards around the origin. The coefficients A and B determine the amplitude of the oscillations, together with the phase at which you start the cycle. The time taken to complete a full cycle is called the *period*

$$T = \frac{2\pi}{\omega} \tag{2.5}$$

The period is independent of the amplitude. (Note that, annoyingly, the kinetic energy is also often denoted by T as well. Do not confuse this with the period. It should hopefully be clear from the context).

If we want to determine the integration constants A and B for a given trajectory, we need some initial conditions. For example, if we're given the position and velocity at time $t = 0$, then it's simple to check that $A = x(0)$ and $B\omega = \dot{x}(0)$.

2.1.1 Moving in a Potential

Let's go back to the general case of a potential $V(x)$ in one dimension. Although the equation of motion is a second order differential equation, the existence of a conserved energy magically allows us to turn this into a first order differential equation,

$$E = \frac{1}{2}m\dot{x}^2 + V(x) \quad \Rightarrow \quad \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}(E - V(x))}$$

This gives us our first hint of the importance of conserved quantities in helping solve a problem. Of course, to go from a second order equation to a first order equation, we must have chosen an integration constant. In this case, that is the energy E itself. Given a first order equation, we can always write down a formal solution for the dynamics simply by integrating,

$$t - t_0 = \pm \int_{x_0}^{x} \frac{dx'}{\sqrt{\frac{2}{m}(E - V(x'))}} \tag{2.6}$$

As before, x' is a dummy variable. If we can do the integral, we've solved the problem. If we can't do the integral, you sometimes hear that the problem has been "reduced to quadrature". This rather old-fashioned phrase really means "I can't do the integral". But, it is often the case that having a solution in this form allows some of its properties to become manifest. And, if nothing else, one can always just evaluate the integral numerically (i.e. on your laptop) if need be.

Getting a Feel for the Solutions

Given the potential energy $V(x)$, it is often very simple to figure out the qualitative nature of any trajectory simply by looking at the form of $V(x)$. This allows us to answer some questions with very little work. For example, we may want to know whether the particle is trapped within some region of space or can escape to infinity.

Let's illustrate this with an example. Consider the cubic potential

$$V(x) = m(x^3 - 3x) \tag{2.7}$$

If we were to substitute this into the general form (2.6), we'd get a fearsome looking integral which hasn't been solved since Victorian times[1].

Even without solving the integral, we can make progress. The potential is plotted in Figure 2. Let's start with the particle sitting stationary at some position x_0. This means that the energy is

$$E = V(x_0)$$

and this must remain constant during the subsequent motion. What happens next depends only on x_0. We can identify the following possibilities

---

[1]Ok, I'm exaggerating. The resulting integral is known as an elliptic integral. Although it can't be expressed in terms of elementary functions, it has lots of nice properties and has been studied to death. 100 years ago, this kind of thing was standard fare in mathematics. These days, we usually have more interesting things to teach. Nonetheless, the study of these integrals later resulted in beautiful connections to geometry through the theory of elliptic functions and elliptic curves.
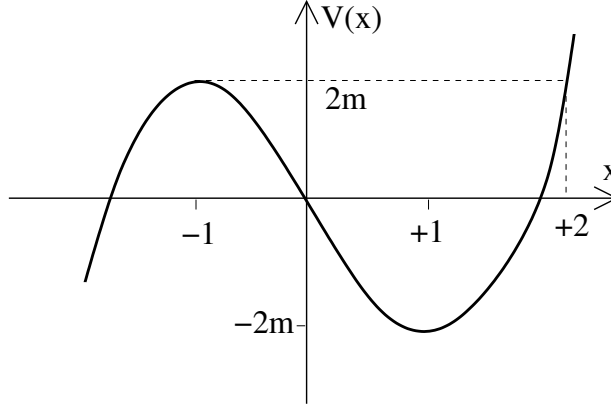
Figure 2: The cubic potential

- $x_0 = \pm 1$: These are the local maximum and minimum. If we drop the particle at these points, it stays there for all time.

- $x_0 \in (-1, +2)$: Here the particle is trapped in the dip. It oscillates backwards and forwards between the two points with potential energy $V(x_0)$. The particle can't climb to the right because it doesn't have the energy. In principle, it could live off to the left where the potential energy is negative, but to get there it would have to first climb the small bump at $x = -1$ and it doesn't have the energy to do so. (There is an assumption here which is implicit throughout all of classical mechanics: the trajectory of the particle $x(t)$ is a continuous function).

- $x_0 > 2$: When released, the particle falls into the dip, climbs out the other side, before falling into the void $x \to -\infty$.

- $x_0 < -1$: The particle just falls off to the left.

- $x_0 = +2$: This is a special value, since $E = 2m$ which is the same as the potential energy at the local maximum $x = -1$. The particle falls into the dip and starts to climb up towards $x = -1$. It can never stop before it reaches $x = -1$ for at its stopping point it would have only potential energy $V < 2m$. But, similarly, it cannot arrive at $x = -1$ with any excess kinetic energy. The only option is that the particle moves towards $x = -1$ at an ever decreasing speed, only reaching the maximum at time $t \to \infty$. To see that this is indeed the case, we can consider the motion of the particle when it is close to the maximum. We write $x \approx -1 + \epsilon$ with $\epsilon \ll 1$. Then, dropping the ϵ^3 term, the potential is

$$V(x = -1 + \epsilon) \approx 2m - 3m\epsilon^2 + \dots$$

and, using (2.6), the time taken to reach $x = -1 + \epsilon$ is

$$t - t_0 = -\int_{\epsilon_0}^{\epsilon} \frac{d\epsilon'}{\sqrt{6\epsilon'}} = -\frac{1}{\sqrt{6}} \log\left(\frac{\epsilon}{\epsilon_0}\right)$$

The logarithm on the right-hand side gives a divergence as $\epsilon \to 0$. This tells us that it indeed takes infinite time to reach the top as promised.

One can easily play a similar game to that above if the starting speed is not zero. In general, one finds that the particle is trapped in the dip $x \in [-1, +1]$ if its energy lies in the interval $E \in [-2m, 2m]$.

2.1.2 Equilibrium: Why (Almost) Everything is a Harmonic Oscillator

A particle placed at an *equilibrium* point will stay there for all time. In our last example with a cubic potential (2.7), we saw two equilibrium points: $x = \pm 1$. In general, if we want $\dot{x} = 0$ for all time, then clearly we must have $\ddot{x} = 0$, which, from the form of Newton's equation (2.2), tells us that we can identify the equilibrium points with the critical points of the potential,

$$\frac{dV}{dx} = 0$$

What happens to a particle that is close to an equilibrium point, x_0? In this case, we can Taylor expand the potential energy about $x = x_0$. Because, by definition, the first derivative vanishes, we have

$$V(x) \approx V(x_0) + \frac{1}{2} V''(x_0)(x - x_0)^2 + \ldots \tag{2.8}$$

To continue, we need to know about the sign of $V''(x_0)$:

- $V''(x_0) > 0$: In this case, the equilibrium point is a minimum of the potential and the potential energy is that of a harmonic oscillator. From our discussion of Section 2.1.2, we know that the particle oscillates backwards and forwards around x_0 with frequency

$$\omega = \sqrt{\frac{V''(x_0)}{m}}$$

Such equilibrium points are called *stable*. This analysis shows that if the amplitude of the oscillations is small enough (so that we may ignore the $(x - x_0)^3$ terms in the Taylor expansion) then all systems oscillating around a stable fixed point look like a harmonic oscillator.

- $V''(x_0) < 0$: In this case, the equilibrium point is a maximum of the potential. The equation of motion again reads

$$m\ddot{x} = -V''(x_0)\,(x - x_0)$$

But with $V'' < 0$, we have $\ddot{x} > 0$ when $x - x_0 > 0$. This means that if we displace the system a little bit away from the equilibrium point, then the acceleration pushes it further away. The general solution is

$$x - x_0 = Ae^{\alpha t} + Be^{-\alpha t} \quad \text{with } \alpha = \sqrt{\frac{-V''(x_0)}{m}}$$

Any solution with the integration constant $A \neq 0$ will rapidly move away from the fixed point. Since our whole analysis started from a Taylor expansion (2.8), neglecting terms of order $(x - x_0)^3$ and higher, our approximation will quickly break down. We say that such equilibrium points are *unstable*.

Notice that there are solutions around unstable fixed points with $A = 0$ and $B \neq 0$ which move back towards the maximum at late times. These finely tuned solutions arise in the kind of situation that we described for the cubic potential where you drop the particle at a very special point (in the case of the cubic potential, this point was $x = 2$) so that it just reaches the top of a hill in infinite time. Clearly these solutions are not generic: they require very special initial conditions.

- Finally, we could have $V''(x_0) = 0$. In this case, there is nothing we can say about the dynamics of the system without Taylor expanding the potential further.

Yet Another Example: The Pendulum

Consider a particle of mass m attached to the end of a light rod of length l. This counts as a one-dimensional system because we need specify only a single coordinate to say what the system looks like at a given time. The best coordinate to choose is θ, the angle that the rod makes with the vertical. The equation of motion is

$$\ddot{\theta} = -\frac{g}{l}\sin\theta \tag{2.9}$$

The energy is

$$E = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta$$
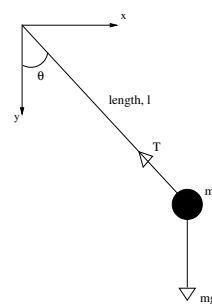
Figure 3:

(Note: Since θ is an angular variable rather than a linear variable, the kinetic energy is a little different. Hopefully this is familiar from earlier courses on mechanics. However, we will rederive this result in Section 4).

There are two qualitatively different motions of the pendulum. If $E > mgl$, then the kinetic energy can never be zero. This means that the pendulum is making complete circles. In contrast, if $E < mgl$, the pendulum completes only part of the circle before it comes to a stop and swings back the other way. If the highest point of the swing is θ_0, then the energy is

$$E = -mgl \cos \theta_0$$

We can determine the period T of the pendulum using (2.6). It's actually best to calculate the period by taking 4 times the time the pendulum takes to go from $\theta = 0$ to $\theta = \theta_0$. We have

$$T = 4 \int_0^{T/4} dt = 4 \int_0^{\theta_0} \frac{d\theta}{\sqrt{2E/ml^2 + (2g/l) \cos \theta}}$$
$$= 4 \sqrt{\frac{l}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{2 \cos \theta - 2 \cos \theta_0}} \qquad (2.10)$$

We see that the period is proportional to $\sqrt{l/g}$ multiplied by some dimensionless number given by (4 times) the integral. For what it's worth, this integral turns out to be, once again, an elliptic integral.

For small oscillations, we can write $\cos \theta \approx 1 - \frac{1}{2}\theta^2$ and the pendulum becomes a harmonic oscillator with angular frequency $\omega = \sqrt{g/l}$. If we replace the $\cos \theta$'s in (2.10) by their Taylor expansion, we have

$$T = 4 \sqrt{\frac{l}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\theta_0^2 - \theta^2}} = 4 \sqrt{\frac{l}{g}} \int_0^1 \frac{dx}{\sqrt{1 - x^2}} = 2\pi \sqrt{\frac{l}{g}}$$

This agrees with our result (2.5) for the harmonic oscillator.

2.2 Potentials in Three Dimensions

Let's now consider a particle moving in three dimensional $\mathbf{R}^3$. Here things are more interesting. Firstly, it is possible to have energy conservation even if the force depends on the velocity. We will see how this can happen in Section 2.4. Conversely, forces which only depend on the position do not necessarily conserve energy: we need an extra condition. For now, we restrict attention to forces of the form $\mathbf{F} = \mathbf{F}(\mathbf{x})$. We have the following result:

Claim: There exists a conserved energy if and only if the force can be written in the form

$$\mathbf{F} = -\nabla V \tag{2.11}$$

for some potential function $V(\mathbf{x})$. This means that the components of the force must be of the form $F_i = -\partial V/\partial x^i$. The conserved energy is then given by

$$E = \frac{1}{2}m\dot{\mathbf{x}} \cdot \dot{\mathbf{x}} + V(\mathbf{x}) \tag{2.12}$$

Proof: The proof that E is conserved if $\mathbf{F}$ takes the form (2.11) is exactly the same as in the one-dimensional case, together with liberal use of the chain rule. We have

$$\frac{dE}{dt} = m\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} + \frac{\partial V}{\partial x^i}\frac{\partial x^i}{\partial t} \quad \text{using summation convention}$$
$$= \dot{\mathbf{x}} \cdot (m\ddot{\mathbf{x}} + \nabla V) = 0$$

where the last equality follows from the equation of motion which is $m\ddot{\mathbf{x}} = -\nabla V$.

To go the other way, we must prove that if there exists a conserved energy E taking the form (2.12) then the force is necessarily given by (2.11). To do this, we need the concept of *work*. If a force $\mathbf{F}$ acts on a particle and succeeds in moving it from $\mathbf{x}(t_1)$ to $\mathbf{x}(t_2)$ along a trajectory $\mathcal{C}$, then the work done by the force is defined to be

$$W = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{x}$$

This is a line integral (of the kind you've met in the *Vector Calculus* course). The scalar product means that we take the component of the force along the direction of the trajectory at each point. We can make this clearer by writing

$$W = \int_{t_1}^{t_2} \mathbf{F} \cdot \frac{d\mathbf{x}}{dt}\, dt$$

The integrand, which is the rate of doing work, is called the *power*, $P = \mathbf{F} \cdot \dot{\mathbf{x}}$. Using Newton's second law, we can replace $\mathbf{F} = m\ddot{\mathbf{x}}$ to get

$$W = m\int_{t_1}^{t_2} \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}\, dt \;=\; \frac{1}{2}m\int_{t_1}^{t_2} \frac{d}{dt}(\dot{\mathbf{x}} \cdot \dot{\mathbf{x}})\, dt \;=\; T(t_2) - T(t_1)$$

where

$$T \equiv \frac{1}{2}m\,\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}$$

is the kinetic energy. (You might think that K is a better name for kinetic energy. I'm inclined to agree. Except in all advanced courses of theoretical physics, kinetic energy is always denoted T which is why I've adopted the same notation here).

So the total work done is proportional to the change in kinetic energy. If we want to have a conserved energy of the form (2.12), then the change in kinetic energy must be equal to the change in potential energy. This means we must be able to write

$$W = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{x} = V(\mathbf{x}(t_1)) - V(\mathbf{x}(t_2)) \tag{2.13}$$

In particular, this result tells us that the work done must be independent of the trajectory $\mathcal{C}$; it can depend only on the end points $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$. But a simple result (which you will prove in your *Vector Calculus* course) says that (2.13) holds only for forces of the form

$$\mathbf{F} = -\nabla V$$

as required $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$.

Forces in three dimensions which take the form $\mathbf{F} = -\nabla V$ are called *conservative*. You will also see in the *Vector Calculus* course that forces in $\mathbf{R}^3$ are conservative if and only if $\nabla \times \mathbf{F} = 0$.

2.2.1 Central Forces

A particularly important class of potentials are those which depend only on the distance to a fixed point, which we take to be the origin

$$V(\mathbf{x}) = V(r)$$

where $r = |\mathbf{x}|$. The resulting force also depends only on the distance to the origin and, moreover, always points in the direction of the origin,

$$\mathbf{F}(r) = -\nabla V = -\frac{dV}{dr}\hat{\mathbf{x}} \tag{2.14}$$

Such forces are called *central*. In these lectures, we'll also use the notation $\hat{\mathbf{r}} = \hat{\mathbf{x}}$ to denote the unit vector pointing radially from the origin to the position of the particle. (In other courses, you may see this same vector denoted as $\mathbf{e}_r$).

In the vector calculus course, you will spend some time computing quantities such as ∇V in spherical polar coordinates. But, even without such practice, it is a simple matter to show that the force (2.14) is indeed aligned with the direction to the origin. If $\mathbf{x} = (x_1, x_2, x_3)$ then the radial distance is $r^2 = x_1^2 + x_2^2 + x_3^2$, from which we can

compute $\partial r/\partial x_i = x_i/r$ for $i = 1, 2, 3$. Then, using the chain rule, we have

$$\begin{aligned}
\nabla V &= \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \frac{\partial V}{\partial x_3} \right) \\
&= \left(\frac{dV}{dr} \frac{\partial r}{\partial x_1}, \frac{dV}{dr} \frac{\partial r}{\partial x_2}, \frac{dV}{dr} \frac{\partial r}{\partial x_3} \right) \\
&= \frac{dV}{dr} \left(\frac{x_1}{r}, \frac{x_2}{r}, \frac{x_3}{r} \right) = \frac{dV}{dr} \hat{\mathbf{x}}
\end{aligned}$$

2.2.2 Angular Momentum

We will devote all of Section 4 to the study of motion in central forces. For now, we will just mention what is important about central forces: they have an extra conserved quantity. This is a vector $\mathbf{L}$ called *angular momentum*,

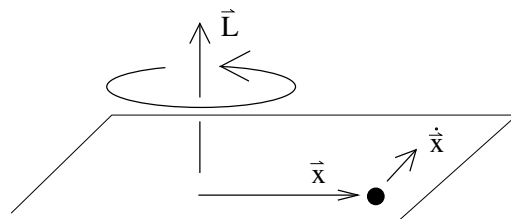
Figure 4:

$$\mathbf{L} = m\mathbf{x} \times \dot{\mathbf{x}}$$

Notice that, in contrast to the momentum $\mathbf{p} = m\dot{\mathbf{x}}$, the angular momentum $\mathbf{L}$ depends on the choice of origin. It is a perpendicular to both the position and the momentum.

Let's look at what happens to angular momentum in the presence of a general force $\mathbf{F}$. When we take the time derivative, we get two terms. But one of these contains $\dot{\mathbf{x}} \times \dot{\mathbf{x}} = 0$. We're left with

$$\frac{d\mathbf{L}}{dt} = m\mathbf{x} \times \ddot{\mathbf{x}} = \mathbf{x} \times \mathbf{F}$$

The quantity $\boldsymbol{\tau} = \mathbf{x} \times \mathbf{F}$ is called the *torque*. This gives us an equation for the change of angular momentum that is very similar to Newton's second law for the change of momentum,

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}$$

Now we can see why central forces are special. When the force $\mathbf{F}$ lies in the same direction as the position $\mathbf{x}$ of the particle, we have $\mathbf{x} \times \mathbf{F} = 0$. This means that the torque vanishes and angular momentum is conserved

$$\frac{d\mathbf{L}}{dt} = 0$$

We'll make good use of this result in Section 4 where we'll see a number of important examples of central forces.

2.3 Gravity

To the best of our knowledge, there are four fundamental forces in Nature. They are

- Gravity

- Electromagnetism

- Strong Nuclear Force

- Weak Nuclear Force

The two nuclear forces operate only on small scales, comparable, as the name suggests, to the size of the nucleus ($r_0 \approx 10^{-15}\,m$). We can't really give an honest description of these forces without invoking quantum mechanics and, for this reason, we won't discuss them in this course. (A very rough, and slightly dishonest, classical description of the strong nuclear force can be given by the potential $V(r) \sim e^{-r/r_0}/r$). In this section we discuss the force of gravity; in the next, electromagnetism.

Gravity is a conservative force. Consider a particle of mass M fixed at the origin. A particle of mass m moving in its presence experiences a potential energy

$$V(r) = -\frac{GMm}{r} \tag{2.15}$$

Here G is Newton's constant. It determines the strength of the gravitational force and is given by

$$G \approx 6.67 \times 10^{-11}\ m^3 \text{Kg}^{-1} s^{-2}$$

The force on the particle is given by

$$\mathbf{F} = -\nabla V = -\frac{GMm}{r^2}\,\hat{\mathbf{r}} \tag{2.16}$$

where $\hat{\mathbf{r}}$ is the unit vector in the direction of the particle. This is Newton's famous inverse-square law for gravity. The force points towards the origin. We will devote much of Section 4 to studying the motion of a particle under the inverse-square force.

2.3.1 The Gravitational Field

The quantity V in (2.15) is the potential energy of a particle of mass m in the presence of mass M. It is common to define the *gravitational field* of the mass M to be

$$\Phi(r) = -\frac{GM}{r}$$

Φ is sometimes called the Newtonian gravitational field to distinguish it from a more sophisticated object later introduced by Einstein. It is also sometimes called the *gravitational potential*. It is a property of the mass M alone. The potential energy of the mass m is then given by $V = m\Phi$.

The gravitational field due to many particles is simply the sum of the field due to each individual particle. If we fix particles with masses M_i at positions $\mathbf{r}_i$, then the total gravitational field is

$$\Phi(\mathbf{r}) = -G \sum_i \frac{M_i}{|\mathbf{r} - \mathbf{r}_i|}$$

The gravitational force that a moving particle of mass m experiences in this field is

$$\mathbf{F} = -Gm \sum_i \frac{M_i}{|\mathbf{r} - \mathbf{r}_i|^3} (\mathbf{r} - \mathbf{r}_i)$$

The Gravitational Field of a Planet

The fact that contributions to the Newtonian gravitational potential add in a simple linear fashion has an important consequence: the external gravitational field of a spherically symmetric object of mass M – such as a star or planet – is the same as that of a point mass M positioned at the origin.

The proof of this statement is an example of the volume integral that you will learn in the *Vector Calculus* course. We include it here only for completeness. We let the planet have density $\rho(r)$ and radius R. Summing over the contribution from all points $\mathbf{x}$ inside the planet, the gravitational field is given by

$$\Phi(\mathbf{r}) = -\int_{|\mathbf{x}| \leq R} d^3x \, \frac{G\rho(\mathbf{x})}{|\mathbf{r} - \mathbf{x}|}$$
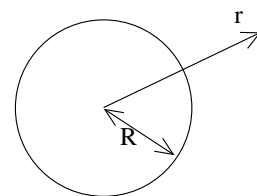
Figure 5:

It's best to work in spherical polar coordinates and to choose the polar direction, $\theta = 0$, to lie in the direction of $\mathbf{r}$. Then $\mathbf{r} \cdot \mathbf{x} = rx \cos\theta$. We can use this to write an expression for the denominator: $|\mathbf{r} - \mathbf{x}|^2 = r^2 + x^2 - 2rx \cos\theta$. The gravitational field then becomes

$$
\begin{aligned}
\Phi(\mathbf{r}) &= -G \int_0^R dx \int_0^\pi d\theta \int_0^{2\pi} d\phi \, \frac{\rho(x) x^2 \sin\theta}{\sqrt{r^2 + x^2 - 2rx\cos\theta}} \\
&= -2\pi G \int_0^R dx \int_0^\pi d\theta \frac{\rho(x) x^2 \sin\theta}{\sqrt{r^2 + x^2 - 2rx\cos\theta}} \\
&= -2\pi G \int_0^R dx \, \rho(x) x^2 \frac{1}{rx} \left[\sqrt{r^2 + x^2 - 2rx\cos\theta}\right]_{\theta=0}^{\theta=\pi} \\
&= -\frac{2\pi G}{r} \int_0^R dx \, \rho(x) x \left(|r + x| - |r - x|\right)
\end{aligned}
$$

So far this calculation has been done for any point $\mathbf{r}$, whether inside or outside the planet. At this point, we restrict attention to points external to the planet. This means that $|r + x| = r + x$ and $|r - x| = r - x$ and we have

$$\Phi(\mathbf{r}) = -\frac{4\pi G}{r} \int_0^R dx \; \rho(x) x^2 = -\frac{GM}{r}$$

This is the result that we wanted to prove: the gravitational field is the same as that of a point mass M at the origin.

2.3.2 Escape Velocity

Suppose that you're trapped on the the surface of a planet of radius R. (This should be easy). Let's firstly ask what gravitational potential energy you feel. Assuming you can only rise a distance $z \ll R$ from the planet's surface, we can Taylor expand the potential energy,

$$V(R + z) = -\frac{GMm}{R + z} = -\frac{GMm}{R} \left(1 - \frac{z}{R} + \frac{z^2}{R^2} + \dots \right)$$

If we're only interested in small changes in $z \ll R$, we need focus only on the second term, giving

$$V(z) \approx \text{constant} + \frac{GMm}{R^2} z + \dots$$

This is the familiar potential energy that gives rise to constant acceleration. We usually write $g = GM/R^2$. For the Earth, $g \approx 9.8 \, ms^{-2}$.

Now let's be more ambitious. Suppose we want to escape our parochial, planet-bound existence. So we decide to jump. How fast do we have to jump if we wish to truly be free? This, it turns out, is the same kind of question that we discussed in Section 2.1.1 in the context of particles moving in one dimension and can be determined very easily using gravitational energy $V = -GMm/r$. If you jump directly upwards (i.e. radially) with velocity v, your total energy as you leave the surface is
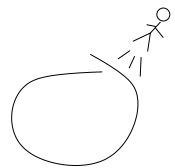
Figure 6:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{R}$$

For any energy $E < 0$, you will eventually come to a halt at position $r = -GMm/E$, before falling back. If you want to escape the gravitational attraction of the planet for

ever, you will need energy $E \geq 0$. At the minimum value of $E = 0$, the associated velocity

$$v_{\text{escape}} = \sqrt{\frac{2GM}{R}} \qquad (2.17)$$

This is the *escape velocity.*

Black Holes and the Schwarzchild Radius

Let's do something a little dodgy. We'll take the formula above and apply it to light. The reason that this is dodgy is because, as we will see in Section 7, the laws of Newtonian physics need modifying for particles close to the speed of light where the effects of special relativity are important. Nonetheless, let's forget this for now and plough ahead regardless.

Light travels at speed $c \approx 3 \times 10^8$ ms^{-1}. Suppose that the escape velocity from the surface of a star is greater than or equal to the speed of light. From (2.17), this would happen if the radius of the star satisfies

$$R \leq R_S = \frac{2GM}{c^2}$$

What do we see if this is the case? Well, nothing! The star is so dense that light can't escape from it. It's what we call a *black hole.*

Although the derivation above is not trustworthy, by some fortunate coincidence it turns out that the answer is correct. The distance $R_s = 2GM/c^2$ is called the *Schwarzchild radius.* If a star is so dense that it lies within its own Schwarzchild radius, then it will form a black hole. (To demonstrate this properly, you really need to work with the theory of general relativity).

For what it's worth, the Schwarzchild radius of the Earth is around 1 *cm*. The Schwarzchild radius of the Sun is about 3 *km*. You'll be pleased to hear that, because both objects are much larger than their Schwarzchild radii, neither is in danger of forming a black hole any time soon.

2.3.3 Inertial vs Gravitational Mass

We have seen two formulae which involve mass, both due to Newton. These are the second law (1.2) and the inverse-square law for gravity (2.16). Yet the meaning of mass in these two equations is very different. The mass appearing in the second law represents the reluctance of a particle to accelerate under any force. In contrast, the

mass appearing in the inverse-square law tells us the strength of a particular force, namely gravity. Since these are very different concepts, we should really distinguish between the two different masses. The second law involves the *inertial mass*, m_I

$$m_I \ddot{\mathbf{x}} = \mathbf{F}$$

while Newton's law of gravity involves the *gravitational mass*, m_G

$$\mathbf{F} = -\frac{GM_G m_G}{r^2} \hat{\mathbf{r}}$$

It is then an experimental fact that

$$m_I = m_G \tag{2.18}$$

Much experimental effort has gone into determining the accuracy of (2.18), most notably by the Hungarian physicist Eötvösh at the turn of the (previous) century. We now know that the inertial and gravitational masses are equal to within about one part in 10^{13}. Currently, the best experiments to study this equivalence, as well as searches for deviations from Newton's laws at short distances, are being undertaken by a group at the University of Washington in Seattle who go by the name Eöt-Wash. A theoretical understanding of the result (2.18) came only with the development of the theory of General Relativity.

2.4 Electromagnetism

Throughout the Universe, at each point in space, there exist two vectors, $\mathbf{E}$ and $\mathbf{B}$. These are known as the *electric* and *magnetic* fields. Their role – at least for the purposes of this course – is to guide any particle that carries electric charge.

The force experienced by a particle with electric charge q is called the *Lorentz force*,

$$\mathbf{F} = q \Big(\mathbf{E}(\mathbf{x}) + \dot{\mathbf{x}} \times \mathbf{B}(\mathbf{x}) \Big) \tag{2.19}$$

Here we have used the notation $\mathbf{E}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ to stress that the electric and magnetic fields are functions of space. Both their magnitude and direction can vary from point to point.

The electric force is parallel to the electric field. By convention, particles with positive charge q are accelerated in the direction of the electric field; those with negative electric charge are accelerated in the opposite direction. Due to a quirk of history, the electron is taken to have a negative charge given by

$$q_{\text{electron}} \approx -1.6 \times 10^{-19} \text{ Coulombs}$$

As far as fundamental physics is concerned, a much better choice is to simply say that the electron has charge 1. All other charges can then be measured relative to this.

The magnetic force looks rather different. It is a velocity dependent force, with magnitude proportional to the speed of the particle, but with direction perpendicular to that of the particle. We shall see its effect in simple situations shortly.

In principle, both $\mathbf{E}$ and $\mathbf{B}$ can change in time. However, here we will consider only situations where they are static. In this case, the electric field is always of the form

$$\mathbf{E} = -\nabla \phi$$

For some function $\phi(\mathbf{x})$ called the *electric potential* (or *scalar potential* or even just the *potential* as if we didn't already have enough things with that name).

For time independent fields, something special happens: energy is conserved.

Claim: The conserved energy is

$$E = \frac{1}{2} m \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} + q\phi(\mathbf{x})$$

Proof:

$$\dot{E} = m\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} + q\nabla\phi \cdot \dot{\mathbf{x}} = \dot{\mathbf{x}} \cdot (\mathbf{F} + q\nabla\phi) = q\dot{\mathbf{x}} \cdot (\dot{\mathbf{x}} \times \mathbf{B}) = 0$$

where the last equality occurs because $\dot{\mathbf{x}} \times \mathbf{B}$ is necessarily perpendicular to $\dot{\mathbf{x}}$. Notice that this gives an example of something we promised earlier: a velocity dependent force which conserves energy. The key part of the derivation is that the velocity dependent force is perpendicular to the trajectory of the particle. This ensures that the force does no work. $\qquad \Box$.

2.4.1 The Electric Field of a Point Charge

Charged objects do not only respond to electric fields; they also produce electric fields. A particle of charge Q sitting at the origin will set up an electric field given by

$$\mathbf{E} = -\nabla \left(\frac{Q}{4\pi\epsilon_0 r} \right) = \frac{Q}{4\pi\epsilon_0} \frac{\hat{\mathbf{r}}}{r^2} \tag{2.20}$$

where $r^2 = \mathbf{x} \cdot \mathbf{x}$. The quantity ϵ_0 has the grand name *Permittivity of Free Space* and is a constant given by

$$\epsilon_0 \approx 8.85 \times 10^{-12} \ m^{-3}\mathrm{Kg}^{-1}s^2C^2$$

This quantity should be thought of as characterising the strength of the electric interaction.

The force between two particles with charges Q and q is given by $\mathbf{F} = q\mathbf{E}$ with $\mathbf{E}$ given by (2.20). In other words,

$$\mathbf{F} = \frac{qQ}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}$$

This is known as the *Coulomb force*. It is a remarkable fact that, mathematically, the force looks identical to the Newtonian gravitational force (2.16): both have the characteristic inverse-square form. We will study motion in this potential in detail in Section 4, with particular focus on the Coulomb force in 4.4.

Although the forces of Newton and Coulomb look the same, there is one important difference. Gravity is always attractive because mass $m > 0$. In contrast, the electrostatic Coulomb force can be attractive or repulsive because charges q come with both signs. Further differences between gravity and electromagnetism come when you ask what happens when sources (mass or charge) move; but that's a story that will be told in different courses.

2.4.2 Circles in a Constant Magnetic Field

Motion in a constant electric field is simple: the particle undergoes constant acceleration in the direction of $\mathbf{E}$. But what about motion in a constant magnetic field $\mathbf{B}$? The equation of motion is

$$m\ddot{\mathbf{x}} = q\,\dot{\mathbf{x}} \times \mathbf{B}$$

Let's pick the magnetic field to lie in the z-direction and write

$$\mathbf{B} = (0, 0, B)$$

We can now write the Lorentz force law (2.19) in components. It reads

$$m\ddot{x} = qB\dot{y} \tag{2.21}$$
$$m\ddot{y} = -qB\dot{x} \tag{2.22}$$
$$m\ddot{z} = 0$$

The last equation is easily solved and the particle just travels at constant velocity in the z direction. The first two equations are more interesting. There are a number of ways to solve them, but a particularly elegant way is to construct the complex variable $\xi = x + iy$. Then adding (2.21) to i times (2.22) gives

$$m\ddot{\xi} = -iqB\dot{\xi}$$

which can be integrated to give

$$\xi = \alpha e^{-i\omega t} + \beta$$

where α and β are integration constants and ω is given by

$$\omega = \frac{qB}{m}$$

If we choose our initial conditions to be that the particle starts life at $t = 0$ at the origin with velocity $-v$ in the y-direction, then α and β are fixed to be

$$\xi = \frac{v}{\omega} \left(e^{-i\omega t} - 1 \right)$$

Translating this back into x and y coordinates, we have

$$x = \frac{v}{\omega} \left(\cos \omega t - 1 \right) \quad \text{and} \quad y = -\frac{v}{\omega} \sin \omega t$$

The end result is that the particle undergoes circles in the plane with angular frequency ω, known as the *cyclotron frequency* The time to undergo a full circle is fixed: $T = 2\pi/\omega$. In contrast, the size of the circle is v/ω and arises as an integration constant. Circles of arbitrary sizes are allowed; the only price that you pay is that you have to go faster.
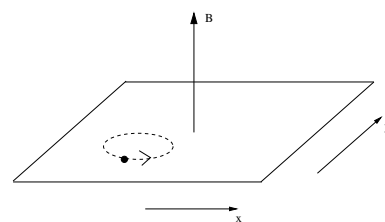
Figure 7:

A Comment on Solving Vector Differential Equations

The Lorentz force equation (2.19) gives a good example of a vector differential equation. The straightforward way to view these is always in components: they are three, coupled, second order differential equations for x, y and z. This is what we did above when understanding the motion of a particle in a magnetic field.

However, one can also attack these kinds of questions without reverting to components. Let's see how this would work in the case of Larmor circles. We start with the vector equation

$$m\ddot{\mathbf{x}} = q\dot{\mathbf{x}} \times \mathbf{B} \tag{2.23}$$

To begin, we take the dot product with $\mathbf{B}$. Since the right-hand side vanishes, we're left with

$$\ddot{\mathbf{x}} \cdot \mathbf{B} = 0$$

This tells us that the particle travels with constant velocity in the direction of $\mathbf{B}$. This is simply a rewriting of our previous result $\ddot{z} = 0$. For simplicity, let's just assume that the particle doesn't move in the $\mathbf{B}$ direction, remaining at the origin. This tells us that the particle moves in a plane with equation

$$\mathbf{x} \cdot \mathbf{B} = 0 \tag{2.24}$$

However, we're not yet done. We started with (2.23) which was three equations. Taking the dot product always reduces us to a single equation. So there must still be two further equations lurking in (2.23) that we haven't yet taken into account. To find them, the systematic thing to do would be to take the cross product with $\mathbf{B}$. However, in the present case, it turns out that the simplest way forwards is to simply integrate (2.23) once, to get

$$m\dot{\mathbf{x}} = q\mathbf{x} \times \mathbf{B} + \mathbf{c}$$

with $\mathbf{c}$ a constant of integration. We can now substitute this back into the right-hand side of (2.23) to find

$$\begin{aligned} m^2\ddot{\mathbf{x}} &= \mathbf{d} + q^2(\mathbf{x} \times \mathbf{B}) \times \mathbf{B} \\ &= \mathbf{d} + q^2\left((\mathbf{x} \cdot \mathbf{B})\mathbf{B} - (\mathbf{B} \cdot \mathbf{B})\mathbf{x}\right) \\ &= -q^2 B^2\left(\mathbf{x} - \mathbf{d}/q^2 B^2\right) \end{aligned}$$

where the integration constant now sits in $\mathbf{d} = q\mathbf{c} \times \mathbf{B}$ which, by construction, is perpendicular to $\mathbf{B}$. In the last line, we've used the equation (2.24). (Note that if we'd considered a situation in which the particle was moving with constant velocity in the $\mathbf{B}$ direction, we'd have to work a little harder at this point). The resulting vector equation looks like three harmonic oscillators, displaced by the vector $\mathbf{d}/q^2 B^2$, oscillating with frequency $\omega = qB/m$. However, because of the constraint (2.24), the motion is necessarily only in the two directions perpendicular to $\mathbf{B}$. The end result is

$$\mathbf{x} = \frac{\mathbf{d}}{q^2 B^2} + \boldsymbol{\alpha}_1 \cos\omega t + \boldsymbol{\alpha}_2 \sin\omega t$$

with α_i, $i = 1, 2$ integration constants satisfying $\boldsymbol{\alpha}_i \cdot \mathbf{B} = 0$. This is the same result we found previously.

Admittedly, in this particular example, working with components was somewhat easier than manipulating the vector equations directly. But this won't always be the case — for some problems you'll make more progress by playing the kind of games that we've described here.

2.4.3 An Aside: Maxwell's Equations

In the Lorentz force law, the only hint that the electric and magnetic fields are related is that they both affect a particle in a manner that is proportional to the electric charge. The connection between them becomes much clearer when things depend on time. A time dependent electric field gives rise to a magnetic field and vice versa. The dynamics of the electric and magnetic fields are governed by Maxwell's equations. In the absence of electric charges, these equations are given by

$$\nabla \cdot \mathbf{E} = 0 \quad , \quad \nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad , \quad \nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

with c the speed of light. You will learn more about the properties of these equations in the lectures on Electromagnetism.

For now, it's worth making one small comment. When we showed that energy is conserved, we needed both the electric and magnetic field to be time independent. What happens when they change with time? In this case, energy is still conserved, but we have to worry about the energy stored in the fields themselves.
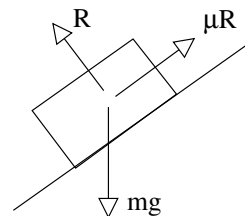
2.5 Friction

Friction is a messy, dirty business. While energy is always conserved on a fundamental level, it doesn't appear to be conserved in most things that you do every day. If you slide along the floor in your socks you don't keep going for ever. At a microscopic level, your kinetic energy is transferred to the atoms in the floor where it manifests itself as heat. But if we only want to know how far our socks will slide, the details of all these atomic processes are of little interest. Instead, we try to summarise everything in a single, macroscopic force that we call *friction*.

2.5.1 Dry Friction

Dry friction occurs when two solid objects are in contact. Think of a heavy box being pushed along the floor, or some idiot sliding in his socks. Experimentally, one finds that the complicated dynamics involved in friction is usually summarised by the force

$$F = \mu R$$

where R is the reaction force, normal to the floor, and μ is a constant called the *coefficient of friction*. Usually $\mu \approx 0.3$, although it depends on

Figure 8:

the kind of materials that are in contact. Moreover, the coefficient is usually, more or less, independent of the velocity. We won't have much to say about dry friction in this course. In fact, we've already said it all.

2.5.2 Fluid Drag

Drag occurs when an object moves through a fluid — either liquid or gas. The resistive force is opposite to the direction of the velocity and, typically, falls into one of two categories

- Linear Drag:

$$\mathbf{F} = -\gamma \mathbf{v}$$

  where the coefficient of friction, γ, is a constant. This form of drag holds for objects moving slowly through very viscous fluids. For a spherical object of radius L, there is a formula due to Stokes which gives $\gamma = 6\pi\eta L$ where η is the viscosity of the fluid.

- Quadratic Drag:

$$\mathbf{F} = -\gamma |\mathbf{v}| \mathbf{v}$$

  Again, γ is called the coefficient of friction. For quadratic friction, γ is usually proportional to the surface area of the object, i.e. $\gamma \sim L^2$. (This is in contrast to the coefficient for linear friction where Stokes' formula gives $\gamma \sim L$). Quadratic drag holds for fast moving objects in less viscous fluids. This includes objects falling in air such as, for example, the various farmyard animals dropped by Galileo from the leaning tower.

  Quadratic drag arises because the object is banging into molecules in the fluid, knocking them out the way. There is an intuitive way to see this. The force is proportional to the change of momentum that occurs in each collision. That gives one factor of v. But the force is also proportional to the number of collisions. That gives the second factor of v, resulting in a force that scales as v^2.

One can ask where the cross-over happens between linear and quadratic friction. Naively, the linear drag must always dominate at low velocities simply because $x \gg x^2$ when $x \ll 1$. More quantitatively, the type of drag is determined by a dimensionless number called the *Reynolds number*,

$$R \equiv \frac{\rho v L}{\eta} \tag{2.25}$$

where ρ is the density of the fluid while η is the viscosity. For $R \ll 1$, linear drag dominates; for $R \gg 1$, quadratic friction dominates.

What is Viscosity?

Above, we've mentioned the viscosity of the fluid, η, without really defining it. For completeness, I will mention here how to measure viscosity.

Place a fluid between two plates, a distance d apart. Keeping the lower plate still, move the top plate at a constant speed v. This sets up a velocity gradient in the fluid. But, the fluid pushes back. To keep the upper plate moving at constant speed, you will have to push with a force per unit area which is proportional to the velocity gradient,
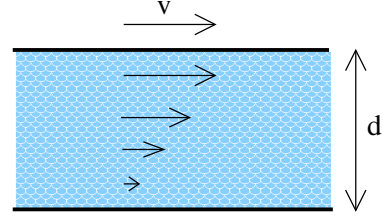

Figure 9:

$$\frac{F}{A} = \eta \frac{v}{d}$$

The coefficient of proportionality, η, is defined to be the *(dynamic) viscosity.*

2.5.3 An Example: The Damped Harmonic Oscillator

We start with our favourite system: the harmonic oscillator, now with a damping term. This was already discussed in your *Differential Equations* course and we include it here only for completeness. The equation of motion is

$$m\ddot{x} = -kx - \gamma\dot{x}$$

Divide through by m to get

$$\ddot{x} = -\omega_0^2 x - 2\alpha\dot{x}$$

where $\omega_0^2 = k/m$ is the frequency of the undamped harmonic oscillator and $\alpha = \gamma/2m$. We can look for solutions of the form

$$x = e^{i\beta t}$$

Remember that x is real, so we're using a trick here. We rely on the fact that the equation of motion is linear so that if we can find a solution of this form, we can take the real and imaginary parts and this will also be a solution. Substituting this ansatz into the equation of motion, we find a quadratic equation for β. Solving this, gives the general solution

$$x = Ae^{i\omega_+ t} + Be^{i\omega_- t}$$

with $\omega_\pm = i\alpha \pm \sqrt{\omega_0^2 - \alpha^2}$. We identify three different regimes,

- Underdamped: $\omega_0^2 > \alpha^2$. Here the solution takes the form,

$$x = e^{-\alpha t}\left(Ae^{i\Omega t} + Be^{-i\Omega t}\right)$$

  where $\Omega = \sqrt{\omega_0^2 - \alpha^2}$. Here the system oscillates with a frequency $\Omega < \omega_0$, while the amplitude of the oscillations decays exponentially.

- Overdamped: $\omega_0^2 < \alpha^2$. The roots $\omega_\pm$ are now purely imaginary and the general solution takes the form,

$$x = e^{-\alpha t}\left(Ae^{\Omega t} + Be^{-\Omega t}\right)$$

  Now there are no oscillations. Both terms decay exponentially. If you like, the amplitude decays away before the system is able to undergo even a single oscillation.

- Critical Damping: $\omega_0^2 = \alpha^2$. Now the two roots $\omega_\pm$ coincide. With a double root of this form, the most general solution takes the form,

$$x = (A + Bt)e^{-\alpha t}$$

  Again, there are no oscillations, but the system does achieve some mild linear growth for times $t < 1/\alpha$, after which it decays away.

2.5.4 Terminal Velocity with Quadratic Friction

You can drop a mouse down a thousand-yard mine shaft; and, on arriving at the bottom, it gets a slight shock and walks away, provided that the ground is fairly soft. A rat is killed, a man is broken, a horse splashes.

J.B.S. Haldane, *On Being the Right Size*

Let's look at a particle of mass m moving in a constant gravitational field, subject to quadratic friction. We'll measure the height z to be in the upwards direction, meaning that if $v = dz/dt > 0$, the particle is going up. We'll look at the cases where the particle goes up and goes down separately.

Coming Down

Suppose that we drop the particle from some height. The equation of motion is given by

$$m\frac{dv}{dt} = -mg + \gamma v^2$$

It's worth commenting on the minus signs on the right-hand side. Gravity acts downwards, so comes with a minus sign. Since the particle is falling down, friction is acting upwards so comes with a plus sign. Dividing through by m, we have

$$\frac{dv}{dt} = -g + \frac{\gamma v^2}{m} \tag{2.26}$$

Integrating this equation once gives

$$t = -\int_0^v \frac{dv'}{g - \gamma v'^2/m}$$

which can be easily solved by the substitution $v = \sqrt{mg/\gamma}\,\tanh x$ to get

$$t = -\sqrt{\frac{m}{\gamma g}}\,\tanh^{-1}\left(\sqrt{\frac{\gamma}{mg}}\,v\right)$$

Inverting this gives us the speed as a function of time

$$v = -\sqrt{\frac{mg}{\gamma}}\,\tanh\left(\sqrt{\frac{\gamma g}{m}}\,t\right)$$

We now see the effect of friction. As time increases, the velocity does not increase without bound. Instead, the particle reaches a maximum speed,

$$v \to -\sqrt{\frac{mg}{\gamma}} \qquad \text{as } t \to \infty \tag{2.27}$$

This is the *terminal velocity*. The sign is negative because the particle is falling downwards. Notice that if all we wanted was the terminal velocity, then we don't need to go through the whole calculation above. We can simply look for solutions of (2.26) with constant speed, so $dv/dt = 0$. This obviously gives us (2.27) as a solution. The advantage of going through the full calculation is that we learn how the velocity approaches its terminal value.

We can now see the origin of the quote we started with. The point is that if we compare objects of equal density, the masses scale as the volume, meaning $m \sim L^3$ where L is the linear size of the object. In contrast, the coefficient of friction usually scales as surface area, $\gamma \sim L^2$. This means that the terminal velocity depends on size. For objects of equal density, we expect the terminal velocity to scale as $v \sim \sqrt{L}$. I have no idea if this is genuinely a big enough effect to make horses splash. (Haldane was a biologist, so he should know what it takes to make an animal splash. But in his essay he assumed linear drag rather than quadratic, so maybe not).

Going Up

Now let's think about throwing a particle upwards. Since both gravity and friction are now acting downwards, we get a flip of a minus sign in the equation of motion. It is now

$$\frac{dv}{dt} = -g - \frac{\gamma v^2}{m} \tag{2.28}$$

Suppose that we throw the object up with initial speed u and we want to figure out the maximum height, h, that it reaches. We could follow our earlier calculation and integrate (2.28) to determine $v = v(t)$. But since we aren't asking about time, it's much better to instead consider velocity as a function of distance: $v = v(z)$. We write

$$\frac{dv}{dt} = \frac{dv}{dz}\frac{dz}{dt} = v\frac{dv}{dz} = -g - \frac{\gamma v^2}{m}$$

which can be rewritten as

$$\frac{1}{2}\frac{d(v^2)}{dz} = -g - \frac{\gamma v^2}{m}$$

Now we can integrate this equation to get velocity as a function of distance. Writing $y = v^2$, we have

$$\int_{u^2}^{0} \frac{dy}{g + \gamma y/m} = -2\int_0^h dz \quad \Rightarrow \quad \frac{m}{\gamma}\left[\log\left(g + \frac{\gamma y}{m}\right)\right]_{y=u^2}^{y=0} = -2h$$

which we can rearrange to get the final answer,

$$h = \frac{m}{2\gamma}\log\left(1 + \frac{\gamma u^2}{mg}\right)$$

It's worth looking at what happens when the effect of friction is small. Naively, it looks like we're in trouble here because as $\gamma \to 0$, the term in front gets very large. But surely the height shouldn't go to infinity just because the friction is small. The resolution to this is that the log is also getting small in this limit. Expanding the log, we have

$$h = \frac{u^2}{2g}\left(1 - \frac{\gamma u^2}{2mg} + \dots\right)$$

Here the leading term is indeed the answer we would get in the absence of friction; the subleading terms tell us how much the friction, γ, lowers the attained height.

Linear Drag and Ohm's Law

Consider an electron moving in a conductor. As we've seen, a constant electric field causes the electron to accelerate. A fairly good model for the physics of a conductor, known as the Drude model, treats the electron as a classical particle with linear damping. The resulting equation of motion is

$$m\ddot{x} = -eE - \gamma v$$

As in the previous example, we can figure out the terminal velocity by setting $\ddot{x} = 0$, to get

$$v = -\frac{eE}{\gamma}$$

In a conductor, the velocity of the electron v gives the current density, j,

$$j = -env$$

where n is the density of electrons. This then gives us a relationship between the current density and the electric field

$$j = \sigma E$$

The quantity $\sigma = e^2 n/\gamma$ is called the *conductivity*. This equation is Ohm's law. However, it's probably not yet in the form you know and love. If the wire has length L and cross-sectional area A, then the current I is defined as $I = jA$. Meanwhile, the voltage dropped across the wire is $V = EL$. With this in hand, we can rewrite Ohm's law as

$$V = IR$$

where the resistance is given by $R = L/\sigma A$.

A 3d Example: A Projectile with Linear Drag

All our examples so far have been effectively one-dimensional. Here we give a three dimensional example which provides another illustration of how to treat vector differential equations and, specifically, how to work with vector constants on integration. We will consider a projectile, moving under gravity, experiencing linear drag. (Think of a projectile moving very slowly in a viscous liquid). At time $t = 0$, we throw the object with velocity **u**. What is its subsequent motion?

The equation of motion is

$$m\frac{d\mathbf{v}}{dt} = m\mathbf{g} - \gamma\mathbf{v} \tag{2.29}$$

We can solve this by introducing the integrating factor $e^{\gamma t/m}$ to write the equation as

$$\frac{d}{dt}\left(e^{\gamma t/m}\mathbf{v}\right) = e^{\gamma t/m}\mathbf{g}$$

We now integrate, but have to introduce a vector integration constant – let's call it $\mathbf{c}$ – for our troubles. We have

$$\mathbf{v} = \frac{m}{\gamma}\mathbf{g} + \mathbf{c}e^{-\gamma t/m}$$

We specified above that at time $t = 0$, the velocity is $\mathbf{v} = \mathbf{u}$, so we can use this information to determine the integration constant $\mathbf{c}$. We get

$$\mathbf{v} = \frac{m}{\gamma}\mathbf{g} + \left(\mathbf{u} - \frac{m}{\gamma}\mathbf{g}\right)e^{-\gamma t/m}$$

Now we integrate $\mathbf{v} = d\mathbf{x}/dt$ a second time to determine x as a function of time. We get another integration constant, $\mathbf{b}$,

$$\mathbf{x} = \frac{m}{\gamma}\mathbf{g}t - \frac{m}{\gamma}\left(\mathbf{u} - \frac{m}{\gamma}\mathbf{g}\right)e^{-\gamma t/m} + \mathbf{b}$$

To determine this second integration constant, we need some further information about the initial conditions. Lets say that $\mathbf{x} = 0$ at $t = 0$. Then we have

$$\mathbf{x} = \frac{m}{\gamma}\mathbf{g}t + \frac{m}{\gamma}\left(\mathbf{u} - \frac{m}{\gamma}\mathbf{g}\right)\left(1 - e^{-\gamma t/m}\right)$$

We can now look at this in components to get a better idea of what's going on. We'll write $\mathbf{x} = (x, y, z)$ and we'll send the projectile off with initial velocity $\mathbf{u} = (u\cos\theta, 0, u\sin\theta)$. With gravity acting downwards, so $\mathbf{g} = (0, 0, -g)$, our vector equation becomes three equations. One is trivial: $y = 0$. The other two are

$$x = \frac{m}{\gamma}u\cos\theta\left(1 - e^{-\gamma t/m}\right)$$

$$z = -\frac{mgt}{\gamma} + \frac{m}{\gamma}\left(u\sin\theta + \frac{mg}{\gamma}\right)\left(1 - e^{-\gamma t/m}\right)$$

Notice that the time scale m/γ is important. For $t \gg m/\gamma$, the horizontal position is essentially constant. By this time, the particle is dropping more or less vertically.

Finally, we can revisit the question that we asked in the last example: what happens when friction is small? Again, there are a couple of terms that look as if they are going to become singular in this limit. But that sounds very unphysical. To resolve this, we should ask what γ is small relative to. In the present case, the answer lies in the exponential terms. To say that γ is small, really means $\gamma \ll m/t$ or, in other words, it means that we are looking at short times, $t \ll m/\gamma$. Then we can expand the exponential. Reverting to the vector form of the equation, we find

$$
\begin{aligned}
\mathbf{x} &= \frac{m}{\gamma}\mathbf{g}t + \frac{m}{\gamma}\left(\mathbf{u} - \frac{m}{\gamma}\mathbf{g}\right)\left(1 - 1 + \frac{\gamma t}{m} - \frac{1}{2}\left(\frac{\gamma t}{m}\right)^2 + \ldots\right) \\
&= \mathbf{u}t + \frac{1}{2}\mathbf{g}t^2 + \mathcal{O}\left(\frac{\gamma t}{m}\right)
\end{aligned}
$$

So we see that, on small time scales, we indeed recover the usual story of a projectile without friction. The friction only becomes relevant when $t \sim m/\gamma$.