# LIE GROUPS 2013 FALL
# PROBLEM SET 4

(1) Let $U$ be the Lie group of all $n \times n$ upper triangle matrices with nonzero diagonal entries. Consider the standard representation of $U$ on $\mathbb{C}^n$.
   (a) Find all its subrepresentations.
   (b) Show that this representation is reducible but not complete reducible for $n > 1$.

(2) (a) Prove: On $\mathrm{GL}(n, \mathbb{R})$, the measure $dx = (\det x)^{-n} \prod_{i,j=1}^n x_{ij}$ is both left invariant and right invariant. (So $\mathrm{GL}(n, \mathbb{R})$ is unimodular.)
   (b) Prove; On $U$ as in problem (1), the measure $dx_L = \prod_{i=1}^n |x_{ii}|^{i-n-1} dx_{ii} \prod_{i<j} dx_{ij}$ is left-invariant, while $dx_R = \prod_{i=1}^n |x_{ii}|^{-i} dx_{ii} \prod_{i<j} dx_{ij}$ is right invariant. What is the modular function?

(3) Suppose $G$ is a compact Lie group, and $dg$ the Haar measure on $G$.
   (a) Show that if $(\pi, V) \in \hat{G}$ is the trivial representation, $\chi_V(g) = 1$.
   (b) Show that if $(\pi, V) \in \hat{G}$ is not the trivial representation, then $\int_G \chi_V(g) dg = 0$.
   (c) Show that for any representation $(\pi, V)$ of $G$,

$$\int_G \chi_V(g) dg = \dim(V^G).$$

   (d) Prove: For any representations $(\pi, V)$ and $(\rho, W)$ of $G$, $\langle \chi_V, \chi_W \rangle_{L^2} = \dim \mathrm{Hom}_G(V, W)$.

(4) Let $G' = \langle g_1 g_2 g_1^{-1} g_2^{-1} \mid g_1, g_2 \in G \rangle$ be the *commutator subgroup* of a Lie group $G$.
   (a) Show that for any one dimensional representation $(\pi, V)$ of $G$, $G'$ acts on $V$ trivially.
   (b) Let $G$ be a compact Lie group. Prove: If all irreducible representations of $G$ are one dimensional, then $G' = \{e\}$, and thus $G$ is abelian.

(5) Let $(V_n, \pi_n)$ be the irreducible representations of $\mathrm{SU}(2)$ described in lecture 22. Prove:

$$\pi_n \otimes \pi_m \simeq \pi_{n+m} \oplus \pi_{n+m-2} \oplus \cdots \oplus \pi_{n-m}, \quad n \geq m.$$

(This is a special case of the so-called Clebsch-Gordan formula.)

(6) Let $G$ be a compact Lie group which is not a finite group. Prove: $\widehat{G}$ is countably infinite.

(7) (a) Describe all the irreducible *real* representations of $\mathbb{T}^n$.
   (b) Prove: If $G$ is a compact connected Lie group and $T$ a maximal torus, then $\dim G - \dim T$ is even.