LECTURE 4: SMOOTH SUBMANIFOLDS

1. Smooth Submanifolds

Definition 1.1. Let M be a smooth manifold of dimension n. A subset $S \subset M$ is a k dimensional embedded *submanifold* of M if for every $p \in S$, there is a chart (φ, U, V) of M around p such that

$$\varphi(U \cap S) = V \cap \hat{\mathbb{R}}^k,$$

where $\hat{\mathbb{R}}^k = \{(x_1, \cdots, x_k, 0, \cdots, 0)\} \subset \mathbb{R}^n$.

Example. S^n is a smooth submanifold of $\mathbb{R}^{n+1}$.

Remark. If S is a k dimensional submanifold of M, then

- S is a k dimensional smooth manifold under the *relative topology.*
- The inclusion map $\iota : S \hookrightarrow M$ is smooth.
- The differential $d\iota_p : T_pS \to T_pM$ is injective for all $p \in S$.

In particular, we may identify any vector $X_p \in T_pS$ with the vector $\tilde{X}_p = d\iota_p(X_p) \in T_pM$ and thus think of T_pS as a vector subspace of T_pM. The following theorem characterize the vectors in T_pM that can be identified with vectors in T_pS:

Theorem 1.2. *Suppose $S \subset M$ is a submanifold, and $p \in S$. Then*

$$T_pS = \{X_p \in T_pM \mid X_pf = 0 \text{ for all } f \in C^\infty(M) \text{ with } f|_S = 0\}.$$

Now suppose X is a vector field on M. We say that X is *tangent* to a submanifold $S \subset M$ if $X_p \in T_pS \subset T_pM$ for every $p \in S$. According to theorem 1.2,

Corollary 1.3. *A vector field $X \in \Gamma^\infty(TM)$ is tangent to a submanifold $S \subset M$ if and only if $Xf = 0$ for every $f \in C^\infty(M)$ that vanishes on S.*

Corollary 1.4. *If vector fields $X_1, X_2 \in \Gamma^\infty(TM)$ are tangent to a submanifold $S \subset M$, so is $[X_1, X_2]$.*

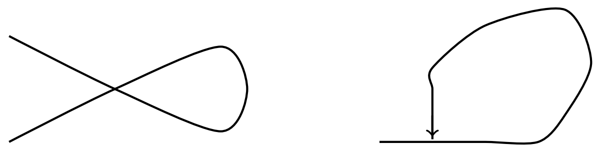
2. Smooth Submanifolds via Smooth Maps

The definition of submanifold above is not convenient to use in practice. A typical way to construct submanifolds is to present them as images of some smooth maps. With the inclusion map ι above at mind, we define

Definition 2.1. A smooth map $f : M \to N$ is *immersion* at p if the tangent map $df_p : T_pM \to T_{f(p)}N$ is injective. It is an *immersion* if it is immersion at all $p \in M$.

Let $f : M \to N$ be an immersion. One can show that locally the image of a neighborhood of each point $p \in M$ is a submanifold of N. Unfortunately globally the image of f could be more complicated than a submanifold:

Example. The following two graphs are the images of two immersions of $\mathbb{R}$ into $\mathbb{R}^2$. For the first one, the immersion is not injective. For the second one, the immersion is injective but not proper, and thus the image still has different topology than $\mathbb{R}$.



Definition 2.2. A smooth immersion $f : M \to N$ is called an *embedding* if it is a homeomorphism between M and $f(M) \subset N$, where the latte is endowed with the relative topology.

One can check that if an immersion f is injective and proper, then it is an embedding. Obviously if S is a submanifold of M, then $\iota : S \hookrightarrow M$ is an embedding. Conversely,

Proposition 2.3. *If $f : M \to N$ is an embedding, then $f(M)$ is a submanifold of N.*

Remark. A remarkable theorem in differential topology, the *Whitney embedding theorem*, claims that any smooth manifold of dimension n can be embedded into $\mathbb{R}^{2n+1}$ as a submanifold. In other words, abstract smooth manifolds are not abstract at all!

Remark. We will call the image $f(M)$ of an immersion $f : M \to N$ an *immersed* submanifold of N. Locally an immersed submanifold is as good as a regular submanifold. So in particular, an immersed submanifold is a smooth manifold by itself. However, as we have seen above, globally the topology of an immersed submanifold could be different from the relative topology.

A second way to construct smooth submanifold is to realize a submanifold as the level set of a smooth map.

Definition 2.4. A smooth map $f : M \to N$ is *submersion* at p if the tangent map $df_p : T_pM \to T_{f(p)}N$ is surjective.

Usually one can not hope that a smooth map $f : M \to N$ is a submersion at all $p \in M$. For example, if M is compact, then one can always find points $p \in M$ so that df_p is the zero map.

Definition 2.5. Suppose $f : M \to N$ is a smooth map between smooth manifolds. A point $q \in N$ is called *regular value* if f is a submersion at each $p \in f^{-1}(q)$.

The following theorem is very useful in justifying whether a subset of a smooth manifold is still a smooth manifold.

Theorem 2.6. *If q is a regular value of a smooth map $f : M \to N$, then $S = f^{-1}(q)$ is a submanifold of M of dimension* $\dim M - \dim N$*. Moreover, for every $p \in S$, T_pS is the kernel of the map $df_p : T_pM \to T_qN$.*

Example. The map

$$f : \mathbb{R}^{n+1} \to \mathbb{R}, (x_1, \cdots, x_{n+1}) \mapsto x_1^2 + \cdots + x_{n+1}^2$$

is smooth and 1 is a regular value. It follows that $S^n = f^{-1}(1)$ is a submanifold of $\mathbb{R}^n$.

One might ask the following question: Does regular value exist for a general smooth mapping? Of course one cannot hope to have regular value if $\dim M < \dim N$. On the other hand, we have the following remarkable result in differential topology

Theorem 2.7 (Sard's theorem)**.** *Let $f : M \to N$ be a smooth map. Then the set of* critical values

$$X = \{q \in M \mid \exists p \in f^{-1}(q) \text{ such that } f \text{ is not a submersion at } p\}$$

has measure zero.

So vaguely speaking, *most* points in N are either out of the image of f or are regular values of f. In particular, if $f(M)$ has positive measure in N, then most values are regular values. So using f one can construct plenty of submanifolds.

3. Smooth Submanifolds via Smooth Vector Fields

We have seen that any vector field can locally be integrate to integral curves. Note that if a vector field is zero at some point, then the corresponding integral curve is the constant path, i.e. whose image is a zero-dimensional point. Similarly any nonzero vector field will integrate to a one dimensional curve on the manifold. It is very nature to generalize these field-submanifold corresponding to higher dimensions.

Definition 3.1. A k-dimensional *distribution* $\mathcal{V}$ on M is a map which assigns to every $p \in M$ a k-dimensional vector subspace $\mathcal{V}_p$ of T_pM. $\mathcal{V}$ is called *smooth* if for every $p \in M$, there is a neighborhood U of p and smooth vector fields $X_1, \cdots, X_k$ on U such that for every $q \in U$, $X_1(q), \cdots, X_k(q)$ are a basis of $\mathcal{V}_q$.

For example, the vector fields $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_k}$ span a smooth k-dimensional distribution in $\mathbb{R}^n$. Note that these vector fields are tangent to any k-dimensional plane that is parallel to $\hat{\mathbb{R}}^k$. One can think of a distribution as a "multi-dimensional vector field", and we would like to integrate distributions to submanifolds as we did in the 1 dimensional case.

Definition 3.2. Suppose $\mathcal{V}$ is a k-dimensional distributions on M. An immersed submanifold $N \subset M$ is called an *integral manifold* for $\mathcal{V}$ if for every $p \in N$, the image of $d\iota_N : T_pN \to T_pM$ is $\mathcal{V}_p$. $\mathcal{V}$ is *integrable* if through each point of M there exists an integral manifold of $\mathcal{V}$.

We are interested in finding conditions to ensure a distribution to have an integral manifold through a given point. Suppose N is an integral manifold for $\mathcal{V}$ at p, and $X_p, Y_p \in \mathcal{V}_p = T_pN$. Then we necessarily have $[X_p, Y_p] \in T_pN$. In other words, an integrable distribution $\mathcal{V}$ must satisfies the following

$\boxed{\textbf{Frobenius Condition}}$ If $X, Y \in \Gamma^\infty(TM)$ belong to $\mathcal{V}$, so is $[X, Y]$.

Definition 3.3. A distribution $\mathcal{V}$ is *involutive* if it satisfies the Frobenius condition.

So the arguments above shows that any integrable distribution is involutive.

Example. Any nowhere vanishing smooth vector field gives a smooth 1-dimensional distribution (which is always involutive). The image of any integral curve is an integral manifold of this distribution.

Example. the smooth distribution on $\mathbb{R}^3$ spanned by vector fields $X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$ and $Z = \frac{\partial}{\partial y}$ is not involutive, since $[X, Y] = -\frac{\partial}{\partial z}$. So it is not integrable.

Remarks. An integral manifold doesn't have to be an embedded submanifold. For example, consider $M = S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$. Fix any irrational number a, the integral manifold of

$$X^a = (x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}) + a(y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2}) = \frac{\partial}{\partial \theta_1} + a\frac{\partial}{\partial \theta_2}$$

is dense in M. (However, it is an immersed submanifold.)

Conversely, we have

Theorem 3.4 (Frobenius Theorem)**.** *Any involutive distribution is integrable. If fact, if $\mathcal{V}$ is involutive, then through every point $p \in M$, there is a unique maximal connected integral manifold of $\mathcal{V}$.*

Example. Consider the distribution on $\mathbb{R}^3$ spanned by $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial z}$ on $\mathbb{R}^3 - \{x = y = 0\}$. Since $[X, Y] = 0$, it is involutive. What is its integral manifold? Well, let's first compute the integral curves of X and Y. Through any point (x, y, z), the integral curve of X is circle in the z-plane with origin the center, and the integral curve of Y is the line that is parallel to the z-axis. Note that the integral manifold passing (x, y, z) of the distribution should contains all points of the form $\varphi_t^X(\varphi_s^Y(x, y, z))$ for all t, s. In our case, this is the cylinders centering with the z-axis.