现代数学物理方法

第一章，特殊函数

杨焕雄

中国科学技术大学近代物理系

hyang@ustc.edu.cn

September 11, 2017

References:

1. M. Stone and P. Goldbart, Mathematics for physics, CUP, 2009.
2. H. Georgi, Lie algebras in particle physics, 2ed, 1999.
3. G. B. Arfken and H. J. Weber, Mathematical methods for physicists, 6th Edition, Elsevier Inc., 2005.
4. E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytseuko and S. Zerbini, Zeta regularization techniques with applications, World Scientific, 1994.
5. M. Srednicki, Quantum field theory, CUP, 2007.
6. J. Polchinski, String theory, Vol.1 and Vol.2, CUP, 2005.

Outline

This chapter studies some special functions in mathematical physics. Special focus is on:

1. Gamma function $\Gamma(z)$
2. Theta functions

Γ-Function :

We begin by examining how Euler's *Gamma function*, i.e., $\Gamma(z)$, behaves when z is allowed to become complex.

$\Gamma(z)$ is usually defined as:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re z > 0. \tag{1}$$

The condition $\Re z > 0$ is necessary to make the integral converge.

1. It is obviously that,

$$\Gamma(1) = 1 \tag{2}$$

2. For $\Re z > 0$, we have:

$$\begin{aligned} \Gamma(z+1) &= \int_0^\infty t^z e^{-t} dt = -\int_0^\infty t^z de^{-t} \\ &= -(t^z e^{-t})\Big|_0^\infty + z \int_0^\infty t^{z-1} e^{-t} dt \\ &= z\Gamma(z) \end{aligned} \tag{3}$$

Eqs.(2) and (3) are the most important properties of Γ-function. From them we get,

$$\Gamma(n+1) = n! \tag{4}$$

for an arbitrary integer $n \geqslant 0$, i.e., $n = 0, 1, 2, 3, \cdots$.

Notice:

The recurrence relation $\Gamma(z+1) = z\Gamma(z)$ can be used to prolongate $\Gamma(z)$ to the left-half-plane, where $\Re z < 0$.

Let $\Re z < 0$ but $\Re z + n > 0$, where n is a positive integer. We have:

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)} = \frac{1}{z(z+1)\cdots(z+n-1)} \int_0^\infty t^{z+n-1} e^{-t} dt \tag{5}$$

Because $\Gamma(z+n)$ converges for $\Re(z+n) > 0$, the extended $\Gamma(z)$ has poles at zero and at the negative integers: $z = 0, -1, -2, \cdots, -n+1$.

The residue of $\Gamma(z)$ at the pole $z = -n + 1$ is:

$$a_{-1}\Big|_{z\to -n+1} = \frac{(-1)^{n-1}}{(n-1)!} \tag{6}$$

Remark:

Eq.(5) can also be used to establish the explicit expressions of $\Gamma(z)$ when $\Re z < 0$. The following is an example.

Let us suppose at the first place that $-1 < \Re z < 0$. It follows from Eq. (5) that,

$$\Gamma(z) = \frac{1}{z}\Gamma(z+1) = \frac{1}{z}\int_0^\infty t^z e^{-t} dt = \frac{1}{z}\int_\varepsilon^\infty t^z e^{-t} dt$$

In the last step we cut off the integral at the lower limit so as to avoid the divergence near $t = 0$. Therefore,

$$\Gamma(z) = -\frac{1}{z}(t^z e^{-t})\Big|_\varepsilon^\infty + \int_\varepsilon^\infty t^{z-1} e^{-t} dt = \frac{\varepsilon^z}{z} + \int_\varepsilon^\infty t^{z-1} e^{-t} dt$$

Since $-1 < \Re z < 0$, we have[1]:

$$\frac{\varepsilon^z}{z} = -\int_\varepsilon^\infty t^{z-1} dt \tag{7}$$

So,

$$\Gamma(z) = \int_\varepsilon^\infty t^{z-1}(e^{-t} - 1) dt$$

Though the integrand of the integral on the right-hand side of this last expression diverges as ε^z $(-1 < \Re z < 0)$ when $\varepsilon \to 0$, the integral itself behaves as

$$\varepsilon^{1+z}/(1+z)$$

when $\varepsilon \to 0$ and is convergent.

---

[1] If $-2 < \Re z < -1$, then we have

$$\frac{\varepsilon^z}{z} = -\int_\varepsilon^\infty t^{z-1} dt, \quad \frac{\varepsilon^{z+1}}{z+1} = -\int_\varepsilon^\infty t^z dt.$$

instead.

Therefore, we may safely take the limit $\varepsilon \to 0$ to obtain

$$\Gamma(z) = \int_0^\infty t^{z-1}(e^{-t} - 1)dt, \qquad -1 < \Re z < 0. \tag{8}$$

Secondly, we assume $-2 < \Re z < -1$. For such a z, it follows from Eq. (8) that

$$\Gamma(z+1) = \int_0^\infty t^z(e^{-t} - 1)dt$$

By introducing a cut-off ε at lower limit $t \to 0$, we have:

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{1}{z}\int_\varepsilon^\infty t^z(e^{-t} - 1)dt = \frac{1}{z}\int_\varepsilon^\infty t^z e^{-t}dt - \frac{1}{z}\int_\varepsilon^\infty t^z dt$$

The first term is evaluated below:

$$\begin{aligned}
\tfrac{1}{z}\textstyle\int_\varepsilon^\infty t^z e^{-t}dt &= -\frac{1}{z}(t^z e^{-t})\Big|_\varepsilon^\infty + \int_\varepsilon^\infty t^{z-1}e^{-t}dt \\
&= \frac{\varepsilon^z}{z}e^{-\varepsilon} + \int_\varepsilon^\infty t^{z-1}e^{-t}dt = \int_\varepsilon^\infty t^{z-1}(e^{-t} - e^{-\varepsilon})dt
\end{aligned}$$

The second term becomes:

$$\begin{aligned}
-\frac{1}{z}\int_\varepsilon^\infty t^z dt &= \frac{\varepsilon^{z+1}}{z(z+1)} = \left(\frac{1}{z} - \frac{1}{z+1}\right)\varepsilon^{z+1} \\
&= -\varepsilon \int_\varepsilon^\infty t^{z-1} dt + \int_\varepsilon^\infty t^z dt = \int_\varepsilon^\infty t^{z-1}(t-\varepsilon) dt
\end{aligned}$$

Adding these two terms gives:

$$\begin{aligned}
\Gamma(z) &= \int_\varepsilon^\infty t^{z-1}(e^{-t} - e^{-\varepsilon} + t - \varepsilon) dt \\
&= \int_\varepsilon^\infty t^{z-1}\left[e^{-t} - \left(1 - \varepsilon + \frac{\varepsilon^2}{2!} - \frac{\varepsilon^3}{3!} + \cdots\right) + t - \varepsilon\right] dt \\
&= \int_\varepsilon^\infty t^{z-1}(e^{-t} - 1 + t) dt - \sum_{n=2}^{+\infty} (-1)^n \frac{\varepsilon^n}{n!} \int_\varepsilon^\infty t^{z-1} dt
\end{aligned}$$

The integral above converges as $\varepsilon \to 0$. Therefore,

$$\Gamma(z) = \int_0^\infty t^{z-1}(e^{-t} - 1 + t) dt, \qquad -2 < \Re z < -1. \tag{9}$$

Remark:

The analytic continuation of the original integral is given by a new integral in which we have *subtracted* exactly as many as terms from the Taylor expansion of e^{-t} as are needed to just make the integral convergent at the lower limit $t \to 0$.

Because

$$e^{-t} \approx 1 - t + t^2/2 - t^3/6 + \cdots ,$$

we conjecture that in the region $-3 < \Re z < -2$,

$$\Gamma(z) = \int_0^\infty t^{z-1}\left(e^{-t} - 1 + t - \frac{t^2}{2}\right)dt, \qquad -3 < \Re z < -2. \tag{10}$$

Identities:

Gamma function satisfies some useful identities.

These identities, usually proved by elementary real-variable methods, include Euler's "Beta function" identity[2],

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{11}$$

where

$$B(a, b) = \int_0^1 (1-t)^{a-1} t^{b-1} dt \tag{12}$$

and

$$\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z) \tag{13}$$

[2] Eq.(11) is also known as the *Veneziano formula*, which was the original inspiration for string theory.

The proofs of both formuae begin in the same way. Suppose that $a > 0,\ b > 0$, we have:

$$\Gamma(a) = \int_0^\infty dt t^{a-1} e^{-t} = 2\int_0^\infty dx x^{2a-1} e^{-x^2}$$

$$\Gamma(b) = \int_0^\infty dt t^{b-1} e^{-t} = 2\int_0^\infty dy y^{2a-1} e^{-y^2}$$

Hence[3],

$$\begin{aligned}
\Gamma(a)\Gamma(b) &= 4\int_0^\infty \int_0^\infty x^{2a-1} y^{2b-1} e^{-x^2-y^2}\, dxdy \\
&= 4\int_0^\infty r^{2a+2b-1} e^{-r^2} dr \int_0^{\pi/2} \sin^{2a-1}\theta \cos^{2b-1}\theta d\theta \\
&= 2\int_0^\infty \rho^{a+b-1} e^{-\rho} d\rho \int_0^{\pi/2} \sin^{2a-1}\theta \cos^{2b-1}\theta d\theta \\
&= 2\Gamma(a+b) \int_0^{\pi/2} \sin^{2a-1}\theta \cos^{2b-1}\theta d\theta
\end{aligned}$$

[3]where we have set $x = r\cos\theta$ and $y = r\sin\theta$ as well as $\rho = r^2$. The ranges for these new variables are obviously $0 \leqslant r, \rho < \infty$ and $0 \leqslant \theta \leqslant \pi/2$.

We can now change variable θ to $t = \sin^2\theta$ so that,

$$\int_0^{\pi/2} \sin^{2a-1}\theta \cos^{2b-1}\theta d\theta = \frac{1}{2}\int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{1}{2}B(a,b)$$

Therefore, the *Veneziano formula* is obtained:

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}dt = B(a,b) \tag{14}$$

On the other hand, if we set $\tan\theta = \zeta$, the range of such a ζ should be $0 \leqslant \zeta < \infty$, and

$$d\theta = \frac{d\zeta}{1+\zeta^2}$$

The Beta function is reexpressed as:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = 2\int_0^\infty \frac{\zeta^{2a-1}}{(1+\zeta^2)^{a+b}}d\zeta \tag{15}$$

Now we put $a = z$, $b = 1 - z$. Due to the fact that both a and b are positive real numbers, we see that $0 < z < 1$.

Eq.(15) tells us that,

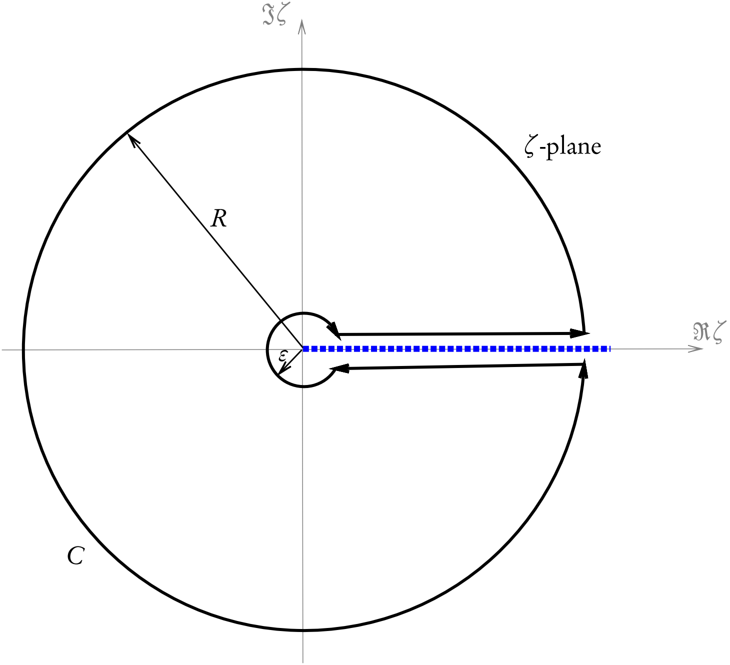
$$B(1 - z, z) = 2 \int_0^\infty \frac{\zeta^{2z-1}}{1 + \zeta^2} d\zeta \tag{16}$$

This integral can be evaluated by turning it into a complex integral with a slotted circle.

We take contour C to be a circle of readius R centred at $\zeta = 0$, with a slot indentation designed to exclude the positive real axis, which we take as the branch cut of ζ^{2z-1}, and a small circle of radius ε about the origin. The branch of the fractional power is defined by setting

$$\zeta^{2z-1} = \exp\left[(2z - 1)(\ln|\zeta| + i\varphi)\right]$$

where we take φ to be zero immediately above the real axis, and 2π immediately below it.



$\Im\zeta$
ζ-plane
R
ε
$\Re\zeta$
C

There are two poles of the integrand of the contour integral

$$\oint_C \frac{\zeta^{2z-1}}{1+\zeta^2} d\zeta$$

at $\zeta = \pm i$, respectively.

The residue at the pole of $\zeta = i$ [i.e., $|\zeta| = 1, \varphi = \pi/2$] is

$$\frac{1}{2i} \exp\left[(2z-1)\left(0 + i\frac{\pi}{2}\right)\right] = \frac{1}{2i} e^{iz\pi} \exp\left(-i\frac{\pi}{2}\right) = -\frac{1}{2} e^{iz\pi}$$

The residue at the pole of $\zeta = -i$ [i.e., $|\zeta| = 1, \varphi = 3\pi/2$] is

$$-\frac{1}{2i} \exp\left[(2z-1)\left(0 + i\frac{3\pi}{2}\right)\right] = -\frac{1}{2i} e^{3iz\pi} \exp\left(-i\frac{3\pi}{2}\right) = -\frac{1}{2} e^{3iz\pi}$$

The residue theorem then tells us that,

$$\oint_C \frac{\zeta^{2z-1}}{1+\zeta^2} d\zeta = 2\pi i\left(-\frac{1}{2} e^{iz\pi} - \frac{1}{2} e^{3iz\pi}\right) = -2\pi i \cos(\pi z) e^{2iz\pi}$$

This contour integral can be decomposed into

$$\oint_C \frac{\zeta^{2z-1}}{1+\zeta^2} d\zeta = \oint_{|\zeta|=R} \frac{\zeta^{2z-1}}{1+\zeta^2} d\zeta + \left[1 - e^{i2\pi(2z-1)}\right] \int_\varepsilon^R \frac{\zeta^{2z-1}}{1+\zeta^2} d\zeta - \oint_{|\zeta|=\varepsilon} \frac{\zeta^{2z-1}}{1+\zeta^2} d\zeta$$

1. This contour integral is related our integral in Eq.(16) by setting $R \to \infty$ and $\varepsilon \to 0$.

2. As $R \to \infty$,

$$\oint_{|\zeta|=R} \frac{\zeta^{2z-1}}{1+\zeta^2} d\zeta \approx \oint_{|\zeta|=R} \zeta^{2z-3} d\zeta \leqslant 2\pi R \times R^{2z-3}$$

   This tends to zero provided that $z < 1$.

3. Similarly, provided that $z > 0$, the integral around the small circle about the rogin tends to zero with $\varepsilon \to 0$,

$$\oint\limits_{|\zeta|=\varepsilon} \frac{\zeta^{2z-1}}{1+\zeta^2} d\zeta \approx \oint\limits_{|\zeta|=\varepsilon} \zeta^{2z-1} d\zeta \leqslant 2\pi\varepsilon \times \varepsilon^{2z-1}$$

Thus,

$$\left[1 - e^{i2\pi(2z-1)}\right] \int_0^\infty \frac{\zeta^{2z-1}}{1+\zeta^2} d\zeta = -2\pi i \cos(\pi z) e^{2iz\pi}$$

Because

$$(1 - e^{i4\pi z}) = -2ie^{i2\pi z} \sin(2\pi z) = -4ie^{i2\pi z} \sin(\pi z) \cos(\pi z)$$

we have:

$$\int_0^\infty \frac{\zeta^{2z-1}}{1+\zeta^2} d\zeta = \frac{\pi}{2\sin(\pi z)} = \frac{\pi}{2} \csc(\pi z), \qquad (0 < z < 1) \qquad (17)$$

Therefore, we have shown the validness of the claimed formula:

$$\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z) \tag{18}$$

Discussion:

- Although the integral given in Eq.(17) has a restriction on the range of z, $0 < z < 1$, the formula (18) holds for all z by analytic continuation.
- If we put $z = 1/2$, we get from Eq.(18) that

  $$\left[\Gamma(1/2)\right]^2 = \pi \csc(\pi/2) = \pi$$

  Due to the fact $\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t} dt$, $\Gamma(1/2) > 0$. Hence,

  $$\Gamma(1/2) = \sqrt{\pi} \tag{19}$$

- Let n be an arbitrary positive integer. We have:

  $$\Gamma(n+1/2) = \frac{(2n-1)!}{2^{2n-1}(n-1)!}\sqrt{\pi} \tag{20}$$

Gamma function $\Gamma(z)$ has lots of interesting perperties. In view of its applications in theoretical physics, we focus on the famous *Sterling's formula* first.

It suggests that for sufficiently large values of z, i.e., $z \gg 1$,

$$\Gamma(z+1) \approx \sqrt{2\pi} z^{z+1} e^{-z} \tag{21}$$

Sterling's formula, i.e. Eq.(21), can approximately be "shown" as follows. Because z is assumed to be positive,

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t} dt = \int_0^\infty e^{-t+z\ln t} dt = \int_0^\infty e^{f(t)} dt$$

where $f(t) := -t + z\ln t$. Notice that,

$$0 = f'(t) = -1 + \frac{z}{t} \quad \rightsquigarrow \quad t = z \quad \rightsquigarrow \quad f''(t)\Big|_{t=z} = -\frac{1}{z} < 0$$

i.e., the function $f(t)$ has a relative maximum at $t = z$.

The substition of $t = z + x$ yields,

$$\Gamma(z+1) = \int_{-z}^{\infty} e^{-z-x+z\ln(z+x)}dx = z^z e^{-z}\int_{-z}^{\infty} \exp\left[-x+z\ln\left(1+\frac{x}{z}\right)\right]dx$$

Because $z \gg 1$, if $|x| \ll |z|$, we have:

$$\ln\left(1+\frac{x}{z}\right) \approx \frac{x}{z} - \frac{1}{2}\left(\frac{x}{z}\right)^2 + \cdots$$

Therefore,

$$\begin{aligned}
\Gamma(z+1) &\approx z^z e^{-z}\int_{-z}^{\infty} \exp\left[-x+z\left(\frac{x}{z}-\frac{x^2}{2z^2}+\cdots\right)\right]dx \\
&\approx z^z e^{-z}\int_{-z}^{\infty} e^{-x^2/2z}dx \\
&\approx z^z e^{-z}\int_{-\infty}^{\infty} e^{-x^2/2z}dx \approx \sqrt{2\pi}z^{z+1/2}e^{-z}
\end{aligned}$$

This result is roughly the same as Eq.(21) but not identical to it. Why the error happens is due to the use of Taylor expansion of $\ln(1+x/z)$ which does not behave good in the whole region $-z \leqslant x < \infty$.

Infinite product for $\Gamma(z)$:

Gamma function can also be expressed as an infinite product. Notice that for n being a positive integer,

$$\begin{aligned}
\Gamma(n+1) = n! &= \lim_{k\to\infty} \frac{n!(n+1)(n+2)\cdots(n+k)}{(n+1)(n+2)\cdots(n+k)} \\
&= \lim_{k\to\infty} \frac{(n+k)!}{(n+1)(n+2)\cdots(n+k)} \\
&= \lim_{k\to\infty} \frac{k!k^n}{(n+1)(n+2)\cdots(n+k)} \lim_{k\to\infty} \frac{(k+1)(k+2)\cdots(k+n)}{k^n} \\
&= \lim_{k\to\infty} \frac{k!k^n}{(n+1)(n+2)\cdots(n+k)}
\end{aligned}$$

This implies,

$$\Gamma(z+1) = \lim_{k\to\infty} \frac{k!k^z}{(z+1)(z+2)\cdots(z+k)} \tag{22}$$

where $z \neq -1, -2, \cdots, -k$. This expression must be read as an infinite product.

Introduce the so-called *Gauss function*:

$$
\begin{aligned}
\Pi(z,k) &= \frac{k!k^z}{(z+1)(z+2)\cdots(z+k)} \\
&= \frac{k^z}{(1+z)(1+z/2)\cdots(1+z/k)} = \frac{k^z}{\prod_{n=1}^{k}(1+z/n)}
\end{aligned}
$$

We have:

$$\Gamma(z+1) = \lim_{k\to\infty} \Pi(z,k) \tag{23}$$

To have an expression for $\Gamma(z)$, we need to introduce a parameter γ_k

$$\gamma_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} - \ln k \tag{24}$$

Such a parameter is shown to be a finite number and in paticular

$$\gamma = \lim_{k\to\infty} \gamma_k \approx 0.57721566 \tag{25}$$

γ is called the Euler's constant.

In fact, for a continuous, monotonic descreasing function $f(x)$, there exist Maclaurin integral inequlities as follows,

$$\int_1^k f(x)dx \leqslant \sum_{n=1}^{k} f(n) \leqslant \int_1^k f(x)dx + f(1)$$

Let $f(x) = 1/x$. We have:

$$\int_1^k dx/x \leqslant \sum_{n=1}^{k} n^{-1} \leqslant \int_1^k dx/x + 1$$

i.e.[4],

$$\ln k \leqslant \sum_{n=1}^{k} n^{-1} \leqslant \ln k + 1, \quad \rightsquigarrow \quad 0 \leqslant \gamma_k \leqslant 1$$

Using Mathematica we can easily get that

$$\gamma_{100} \approx 0.58221, \quad \gamma_{1000} \approx 0.57772, \quad \gamma_{100000} \approx 0.57722, \quad \cdots$$

[4] Recall that γ_k is defined as: $\gamma_k = \sum_{n=1}^{k} n^{-1} - \ln k$.

They imply that:

$$\gamma = \lim_{k\to\infty} \gamma_k \approx 0.5772$$

Notice that $\ln k = \sum_{n=1}^{k} n^{-1} - \gamma_k$. We have:

$$k^z = e^{z \ln k} = \exp\left[-\gamma_k z + z \sum_{n=1}^{k} \frac{1}{n} \right] = e^{-\gamma_k z} \prod_{n=1}^{k} e^{z/n}$$

and

$$\Gamma(z+1) = \lim_{k\to\infty} \Pi(z,k) = e^{-\gamma z} \prod_{n=1}^{\infty} \frac{e^{z/n}}{(1+z/n)}$$

i.e.,

$$\frac{1}{\Gamma(z+1)} = e^{\gamma z} \prod_{n=1}^{\infty} (1+z/n) e^{-z/n} \tag{26}$$

Equivalently,

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} (1+z/n) e^{-z/n} \tag{27}$$

The derivative $\Gamma'(z)$:

Sometimes, the derivative $\Gamma'(z)$ of Gamma function with respect to its variable z is required. From $\Gamma(z+1) = z\Gamma(z)$ we see that

$$\ln \Gamma(z+1) = \ln z + \ln \Gamma(z)$$

Therefore,

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} = \frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{z} \tag{28}$$

Eq.(28) is a recurrence relation by which we can get $\Gamma'(z+1)$ from $\Gamma'(z)$.

The key task is to calculate $\Gamma'(1)$. By observing the expression of Gauss function $\Pi(z, k)$, we find:

$$\ln \Pi(z, k) = z \ln k - \sum_{n=1}^{k} \ln(1 + z/n)$$

so that,

$$\frac{\partial_z \Pi(z, k)}{\Pi(z, k)} = \ln k - \sum_{n=1}^{k} \frac{1}{(n+z)}$$

Taking the limit $k \to \infty$, we get

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} = \lim_{k\to\infty} \left[\ln k - \sum_{n=1}^{k} \frac{1}{(n+z)} \right] \tag{29}$$

Hence,

$$\frac{\Gamma'(1)}{\Gamma(1)} = \lim_{k\to\infty} \left(\ln k - \sum_{n=1}^{k} \frac{1}{n} \right) = -\lim_{k\to\infty} \gamma_k = -\gamma$$

i.e.,

$$\Gamma'(1) = -\gamma \approx -0.5772 \tag{30}$$

Eq.(30) has an important corollary:

$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!} \left[\frac{1}{\varepsilon} - \gamma + \sum_{k=1}^{n} \frac{1}{k} + \mathscr{O}(\varepsilon) \right], \quad (0 < \varepsilon \ll 1) \tag{31}$$

where n is required to be a non-negative integer, and ε a very small positive parameter.

Eq.(31) is widely used in the calculations of loop corrections in QFT. We now give a proof for this formula. Because $\varepsilon \approx 0$, we can expand $\Gamma(\varepsilon)$ as

$$\Gamma(\varepsilon) = \Gamma(1+\varepsilon)/\varepsilon = \frac{1}{\varepsilon}\left[\Gamma(1) + \Gamma'(1)\varepsilon + \mathscr{O}(\varepsilon^2)\right] = \frac{1}{\varepsilon} - \gamma + \mathscr{O}(\varepsilon)$$

Moreover,

$$\begin{aligned}
\Gamma(-n+\varepsilon) &= \frac{\Gamma(-n+\varepsilon+1)}{(-n+\varepsilon)} = \frac{\Gamma(-n+\varepsilon+2)}{(-n+\varepsilon)(-n+\varepsilon+1)} = \cdots \\
&= \frac{\Gamma(-n+\varepsilon+n)}{(-n+\varepsilon)(-n+\varepsilon+1)\cdots(-n+\varepsilon+n-1)} \\
&= \frac{(-1)^n\Gamma(\varepsilon)}{(n-\varepsilon)(n-1-\varepsilon)\cdots(1-\varepsilon)} \\
&= \frac{(-1)^n}{\prod_{k=1}^n (k-\varepsilon)}\left[\frac{1}{\varepsilon} - \gamma + \mathscr{O}(\varepsilon)\right] \\
&= \frac{(-1)^n}{n!\prod_{k=1}^n (1-\varepsilon/k)}\left[\frac{1}{\varepsilon} - \gamma + \mathscr{O}(\varepsilon)\right]
\end{aligned}$$

Let $f(\varepsilon) = 1/\prod_{k=1}^{n}(1-\varepsilon/k)$, we have

$$\ln f(\varepsilon) = -\sum_{k=1}^{n} \ln(1-\varepsilon/k) \approx \varepsilon \sum_{k=1}^{n} \frac{1}{k}$$

Hence,

$$f(\varepsilon) \approx \exp\left[\varepsilon \sum_{k=1}^{n} \frac{1}{k}\right] \approx 1 + \varepsilon \sum_{k=1}^{n} \frac{1}{k}$$

Substitution of this expression into the above formula for $\Gamma(1+\varepsilon)$, we obtain Eq.(31):

$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\varepsilon} - \gamma + \sum_{k=1}^{n} \frac{1}{k} + \mathcal{O}(\varepsilon)\right], \qquad (0 < \varepsilon \ll 1)$$

Some special but frequently used examples in QFT are as follows:

1.
$$\Gamma(1+\varepsilon) \approx 1 - \varepsilon\gamma + \mathcal{O}(\varepsilon^2) \tag{32}$$

2.
$$\Gamma(\varepsilon) \approx \frac{1}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon) \tag{33}$$

3.
$$\Gamma(-1+\varepsilon) \approx -\frac{1}{\varepsilon} - 1 + \gamma + \mathcal{O}(\varepsilon) \tag{34}$$

4.
$$\Gamma(-2+\varepsilon) \approx \frac{1}{2\varepsilon} + \frac{3}{4} - \frac{1}{2}\gamma + \mathcal{O}(\varepsilon) \tag{35}$$

Homework

1. Assume that $-3 < \Re z < -2$. Show that for such a z the Gamma function $\Gamma(z)$ is expressed as

$$\Gamma(z) = \int_0^\infty t^{z-1}\left[e^{-t} - 1 + t - \frac{t^2}{2}\right]dt$$

2. Show that $\Gamma(z)$ may be written

$$\Gamma(z) = \int_0^1 dt\,[\ln(1/t)]^{z-1}, \qquad \Re z > 0.$$

3. Show that

$$\int_0^\infty dx\, e^{-x^4} = \Gamma(5/4)$$

4. The wave function of a particle scattered by a Coulomb potential is $\psi(r,\theta)$. At the origin $\psi(0) = e^{-\pi\gamma/2}\Gamma(1+i\gamma)$, where γ is a real dimensionless constant. Show that:

$$|\psi(0)|^2 = \frac{2\pi\gamma}{e^{2\pi\gamma} - 1}$$

5. The so-called *digamma function* $\psi(z+1)$ is defined by

$$\psi(z+1) = \frac{d}{dz} \ln \Gamma(z+1)$$

Show that $\psi(z+1)$ has the series expansion

$$\psi(z+1) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) z^{n-1}$$

where $\gamma \approx 0.5772$ and $\zeta(n)$ is the Riemann zeta function $\zeta(n) = \sum_{i=1}^{\infty} i^{-n}$.