LECTURE 27: THE WEYL GROUPS AND WEYL INTEGRATION FORMULA

1. The Weyl Groups

Let G be a compact connected Lie group, and $T \subset G$ a maximal torus. The normalizer of T is

$$N(T) = \{g \in G \mid gTg^{-1} = T\}.$$

Note that $N(T)$ is a closed subgroup of G, thus also a compact Lie group. By definition T is a normal subgroup of $N(T)$.

Definition 1.1. The quotient group $W = N(T)/T$ is called the *Weyl group* of G.

Obviously $N(gTg^{-1}) = gN(T)g^{-1}$. So the Weyl groups of G with respect to different maximal tori are isomorphic.

Proposition 1.2. $N(T)^0 = T$.

Proof. We first prove that the automorphism group $\mathrm{Aut}(T)$ of a torus $T = \mathbb{R}^k/\mathbb{Z}^k$ is isomorphic to $\mathrm{GL}(k,\mathbb{Z})$. In particular, it is discrete. To prove this, let $\varphi : T \to T$ be an automorphism. Then $d\varphi : \mathbb{R}^k \to \mathbb{R}^k$ is an invertible linear map, and we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{R}^k \arrow[r, "\exp"'] \arrow[d, "d\varphi"] & \mathbb{T}^k \arrow[d, "\varphi"] \\
\mathbb{R}^k \arrow[r, "\exp"'] & \mathbb{T}^k
\end{tikzcd}$$

It follows that $d\varphi(\ker(\exp)) \subset \ker(\exp)$. In other words, $d\varphi(\mathbb{Z}^k) \subset \mathbb{Z}^k$. So as a $k \times k$ matrix, $d\varphi$ is actually an *integer matrix*, i.e. $d\varphi \in \mathrm{GL}(k,\mathbb{Z})$. Conversely, any matrix in $\mathrm{GL}(k,\mathbb{Z})$ defines an invertible map on $\mathbb{R}^k$ that preserves $\mathbb{Z}^k$, and thus gives an automorphism of T.

It follows that any connected group of automorphisms must act trivially. Now $N(T)^0$ is a connected Lie group, and the conjugation action of $N(T)^0$ on T are automorphisms of T. So $N(T)^0$ acts trivially on T, i.e. any $h \in N(T)^0$ commutes with all elements in T. So $N(T)^0 \subset Z_G(T) = T$. On the other hand, by definition $N(T)^0 \supset T$. So $N(T)^0 = T$. $\square$

Corollary 1.3. *The Weyl group is a finite group.*

Proof. $W = N(T)/T$ is discrete by proposition 1.2. It is compact since $N(T)$ is. $\square$

Since T is abelian, the conjugation action of T on T itself is trivial. It follows that the Weyl group acts on T by conjugation.

Proposition 1.4. *The conjugation action of W on T is effective.*

Proof. This follows from the fact that $Z_G(T) = T = (N(T))^0$ since T is maximal. $\square$

Proposition 1.5. *Let G be a compact connected Lie group, and T a maximal torus. Then two elements $t_1, t_2 \in T$ are conjugate in G if and only if they sit on the same orbit of the Weyl group action.*

Proof. Obviously if $w(t_1) = wt_1w^{-1} = t_2$ for some $w \in W$, then t_1, t_2 are conjugate in G. Conversely if $gt_1g^{-1} = t_2$. Then $gTg^{-1} \subset gZ_G(t_1)g^{-1} = Z_G(t_2)$. It follows that both T and gTg^{-1} are maximal tori in $Z_G(t_2)^0$. So there exists $h \in Z_G(t_2)^0$ such that $hgTg^{-1}h^{-1} = T$. It follows that $hg \in N(T)$. Moreover,

$$hg(t_1) = hgt_1g^{-1}h^{-1} = ht_2h^{-1} = t_2.$$

This completes the proof. $\square$

Corollary 1.6. *All class (i.e. conjugate invariant) functions on G are in one-to-one correspondence to W-invariant functions on T.*

Example. As an example, we will calculate the Weyl group of $\mathrm{U}(n)$. We first notice that if $gt_1g^{-1} = t_2$ for $t_1, t_2 \in T$, then t_1 and t_2 have the same eigenvalues. In other words, as diagonal matrices the entries of t_2 are permutations of entries of t_1. It follows that the Weyl group acts on a generic element $t = \mathrm{diag}(e^{it_1}, \cdots, e^{it_n})$ by permuting t_i's. So W is a subgroup of the full symmetric group $S(n)$. On the other hand, since

$$\begin{pmatrix} 0 & e^{i\theta} \\ e^{i\mu} & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 0 & e^{-i\theta} \\ e^{-i\mu} & 0 \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}.$$

we see that any monomial matrix (matrices with a single nonzero entry in each row and column) in $U(n)$ is in the normalizer of T. It follows that $N(T)/T \supset S_n$. So the Weyl group of $\mathrm{U}(n)$ is $W(\mathrm{U}(n)) = S(n)$.

The Weyl groups for other classical groups:

- The Weyl group of $\mathrm{SU}(n)$ is still $S(n)$
- The Weyl group of $\mathrm{SO}(2l+1)$ is $G(l)$, the group of permutations φ of the set $\{-l, \cdots, -1, 1, \cdots, l\}$ with $\varphi(-k) = -\varphi(k)$ for all $1 \le k \le l$.
- The Weyl group of $\mathrm{SO}(2l)$ is the subgroup $SG(l)$ of $G(l)$ that consists of even permutations.
- The Weyl group of $\mathrm{Sp}(n)$ is still $G(n)$.

2. The Weyl Integration Formula

Suppose G is a compact Lie group, and $T \subset G$ a maximal torus. We have known from lecture 16 that the quotient G/T is a homogeneous G-manifold with tangent space

$$T_{eT}(G/T) = \mathfrak{g}/\mathfrak{t} := \mathfrak{p}.$$

We will fix an adjoint invariant inner product on $\mathfrak{g}$, and identify $\mathfrak{p}$ with the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}.$$

So in particular $\mathfrak{p} \subset \mathfrak{g}$.

We cite the following lemma without proof, and leave more details in the appendix:

Lemma 2.1. *There exists a normalized density $d(gT) = dg/dt$ on the quotient G/T which is invariant under the left G-action.*

Now we are ready to state the main theorem:

Theorem 2.2 (Weyl Integration Formula for Class Functions)**.** *Suppose G is compact, and f a class function on G. Denote by dg and dt the normalized Haar measures on G and T respectively. Then*

$$\int_G f(g)dg = \frac{1}{|W|}\int_T f(t)\left|\det([\mathrm{Ad}_{t^{-1}} - \mathrm{Id}]|_{\mathfrak{p}})\right| dt$$

Proof. Consider the map

$$\phi : G/T \times T \to G, \quad (gT, t) \mapsto gtg^{-1}.$$

We has to compute the Jacobian factor $|\det(d\phi)|$ at an arbitrary point (gT, t). To simplify the computations, we fix g, t and consider

$$\psi : G/T \times T \to G, \quad (hT, s) \mapsto htsh^{-1}t^{-1}.$$

We observe that

$$\psi = R_{t^{-1}} \circ c(g^{-1}) \circ \phi \circ (\tilde{L}_g \times L_t),$$

where $\tilde{L}_g$ is the "left multiplication by g" on G/T. So

$$(d\psi)_{(eT,e)} = (dR_{t^{-1}})_t \circ (dc(g^{-1})_{gtg^{-1}} \circ (d\phi)_{(gT,t)} \circ d(\tilde{L}_g \times L_t)_{(eT,e)}.$$

Since dg and $d(gT)$ are both G-invariant, the corresponding Jacobian factors must be 1, i.e.

$$|\det(dR_{t^{-1}})| = |\det(dc(g^{-1}))| = |\det(d(\tilde{L}_g \times L_t))| = 1.$$

It follows that

$$|\det(d\phi)_{(gT,t)}| = |\det(d\psi)_{(eT,e)}|.$$

It is easy to calculate $d\psi$ at (eT, e), which is given by

$$(d\psi)_{(eT,e)}(X, S) = (\mathrm{Id} - \mathrm{Ad}_t)(X) + \mathrm{Ad}_t S$$

for $X \in \mathfrak{b}$ and $S \in \mathrm{t}$. It follows

$$|\det(d\phi)_{(gT,t)}| = |\det([\mathrm{Ad}_{t^{-1}} - \mathrm{Id}]|_{\mathfrak{p}}) \det \mathrm{Ad}_t| = |\det([\mathrm{Ad}_{t^{-1}} - \mathrm{Id}]|_{\mathfrak{p}})|,$$

where in the last step we used the fact that $|\det \mathrm{Ad}_t| = 1$, since the map

$$g \mapsto |\det \mathrm{Ad}_g|$$

is a Lie group homomorphism from the compact group G to $\mathbb{R}^+$, whose image set has to be $\{1\}$. Another fact we will use without proof is

Fact: There exist dense open subsets $T^{reg} \subset T$ and $G^{reg} \subset G$ so that

- $\det([\mathrm{Ad}_{t^{-1}} - \mathrm{Id}]|_{\mathfrak{p}}) \neq 0$ on T^{reg}, so that ϕ is a locally a diffeomorphism from $G/T \times T^{reg}$ to G^{reg}
- Moreover, $\phi(g_1T, t_1) = \phi(g_2T, t_2)$ if and only if $t_1, t_2 \in T$ conjugate in G, or equivalently, lie in the same W-orbit. So ϕ is a $|W|$-to-one covering map from $G/T \times T^{reg}$ to G^{reg}

It follows that for any class function f,

$$\begin{aligned}\int_G f(g)dg &= \frac{1}{|W|}\int_{G/T\times T} f(\phi(gT,t))|\det(d\phi)_{(gT,t)}|d(gT)dt \\ &= \frac{1}{|W|}\int_T f(t)|\det([\mathrm{Ad}_{t^{-1}} - \mathrm{Id}]|_{\mathfrak{p}})dt.\end{aligned}$$

□

Note that for any continuous function $f \in C(G)$, the function

$$\tilde{f}(t) = \int_G f(gtg^{-1})dg$$

is a W-invariant function on T, which can be identified with a class function on G. Moreover,

$$\int_G f(g)dg = \int_G \tilde{f}(g)dg.$$

So we have

Corollary 2.3 (Weyl Integration Formula for Continuous Functions)**.** *For any continuous function f on G,*

$$\int_G f(g)dg = \frac{1}{|W|}\int_T \det([\mathrm{Ad}_{t^{-1}} - \mathrm{Id}]|_{\mathfrak{p}})(\int_G f(gtg^{-1})dg)dt.$$

As an example, let's write down an explicit formula for $G = \mathrm{U}(n)$. Let T be the maximal torus consists of all diagonal matrices in $\mathrm{U}(n)$. In other words, any $t \in T$ has the form

$$t = \begin{pmatrix} e^{it_1} & & \\ & \ddots & \\ & & e^{it_n}\end{pmatrix}.$$

Let dt be the normalized Haar measure on T.

Proposition 2.4. *In this setting,* $\det([\mathrm{Ad}_{t^{-1}} - \mathrm{Id}]|_{\mathfrak{p}}) = \prod_{j<k} |e^{it_j} - e^{it_k}|^2$.

Proof. We may think of $\mathrm{Ad}_{t^{-1}} - \mathrm{Id}$ as a linear transformation on the complexified vector space $\mathfrak{u}(n) \otimes \mathbb{C}$, which can be identified with $\mathfrak{gl}(n, \mathbb{C})$ since

- $\mathfrak{u}(n)$ consists of $n \times n$ skew-Hermitian matrices.
- A matrix A is skew-Hermitian if and only if iA is Hermitian.
- Any matrix in $\mathfrak{gl}(n, \mathbb{C})$ can be written uniquely as the sum of an Hermitian matrix and a skew-Hermitian matrix.

We will choose t with $t_1, \cdots, t_n$ distinct: These elements form a dense open subset in T. Then $\mathfrak{t} \otimes \mathbb{C}$ consists of all diagonal matrices in $\mathfrak{gl}(n, \mathbb{C})$. It follows that $\mathfrak{p} \otimes \mathbb{C}$ is the vector subspace spanned by the elementary matrices E_{jk}, $j \neq k$,(the matrices with the only nonzero entry a "1" at the (j, k)-position). Since the eigenvalues of $\mathrm{Ad}_{t^{-1}}$ on E_{jk} is $e^{-it_j} e^{it_k}$, we get

$$\begin{aligned}
\det([\mathrm{Ad}_{t^{-1}} - \mathrm{Id}]|_{\mathfrak{p}}) &= \prod_{j \neq k} (e^{-it_j} e^{it_k} - 1) \\
&= \prod_{j<k} (e^{it_j} e^{-it_k} - 1)(e^{it_k} e^{-it_j} - 1) \\
&= \prod_{j<k} (e^{it_j} - e^{it_k})(e^{-it_j} - e^{-it_k}) \\
&= \prod_{j<k} |e^{it_j} - e^{it_k}|^2.
\end{aligned}$$

□

It follows that for any class function f on $U(n)$,

$$\int_{U(n)} f(g) dg = \frac{1}{n!} \int_T f(t) \prod_{j<k} |e^{it_j} - e^{it_k}|^2 dt.$$