LECTURE 22: CHARACTERS

1. Schur Orthogonality for Characters

Let G be a Lie group. A continuous function $\phi : G \to \mathbb{C}$ is called a *class function* (or a *central function*) if it is conjugate invariant, i.e.

$$\phi(ghg^{-1}) = \phi(h)$$

holds for all $g, h \in G$. Obviously the set of all class functions on G form a linear subspace of $C(G)$.

Definition 1.1. Let (V, π) be a representation of Lie group G. The *character* of π is the function $\chi_\pi : G \to \mathbb{C}$ defined by

$$\chi_\pi(g) = \mathrm{Tr}(\pi(g)).$$

Obviously the character is a class function, and it depends only on the equivalence class of representations. Moreover, one can easily check

- $\chi_\pi(e) = \dim V$.
- $\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$.
- $\chi_{\pi_1 \otimes \pi_2} = \chi_{\pi_1}\chi_{\pi_2}$.
- $\chi_{\pi^*} = \overline{\chi_\pi} = \chi_\pi \circ \iota$.

Since the character $\chi_\pi(g) = \sum \pi_{ii}(g)$ is a linear combination of matrix coefficients, as a corollary of the Schur's orthogonality for matrix coefficients we get

Theorem 1.2 (Schur's orthogonality for characters)**.**

(1) *If (V, π) is an irreducible representation of a compact Lie group G, then*

$$\int_G |\chi_\pi(g)|^2 dg = 1.$$

(2) *If (V_1, π_1) and (V_2, π_2) are non-isomorphic irreducible representations of a compact Lie group G, then*

$$\int_G \chi_{\pi_1}(g)\overline{\chi_{\pi_2}(g)} dg = 0.$$

As a corollary, we can show that the character of a representation determines the representation itself:

Corollary 1.3. *Two representations (V_1, π_1) and (V_2, π_2) of a compact Lie group G are isomorphic if and only if $\chi_{\pi_1} = \chi_{\pi_2}$.*

Proof. Since G is compact, any representation of G is completely reducible. We decompose (V_1, π_1) and (V_2, π_2) into irreducible representations

$$(V_1, \pi_1) = \bigoplus_i m_i(W_i, \rho_i), \qquad (V_2, \pi_2) = \bigoplus_i n_i(W_i, \rho_i),$$

where (W_i, ρ_i)'s are non-isomorphic irreducible representations of G, and m_i, n_i are nonnegative integers. Then obviously $\chi_{\pi_1} = \sum m_i \chi_{\rho_i}$ and $\chi_{\pi_2} = \sum n_i \chi_{\rho_i}$. Suppose $\chi_{\pi_1} = \chi_{\pi_2}$, then we must have

$$m_i = \langle \chi_{\pi_1}, \chi_{\rho_i} \rangle = \langle \chi_{\pi_2}, \chi_{\rho_i} \rangle = n_i.$$

It follows that (V_1, π_1) is isomorphic to (V_2, π_2). □

Remark. In particular we see that for any completely reducible representation, there is a unique way to decompose it into irreducible ones

$$(V, \pi) = \bigoplus_{\rho \in \hat{G}} \langle \chi_\pi, \chi_\rho \rangle (W_\rho, \rho),$$

where $\hat{G}$ is the set of equivalence classes of all irreducible representations of G.

Corollary 1.4. *A representation* (V, π) *is irreducible if and only if*

$$\int_G |\chi_\pi(g)|^2 dg = 1.$$

Proof. Let $(V, \pi) = \bigoplus n_i(W_i, \rho_i)$ be the decomposition as above, then

$$\int_G |\chi_\pi(g)|^2 dg = \sum n_i^2.$$

□

Definition 1.5. We will call χ_π an *irreducible character* if π is irreducible.

2. Irreducible representations of $\mathbb{T}^n$

We had showed in lecture 11 that any connected abelian Lie group G has the form $\mathbb{T}^k \times \mathbb{R}^l$. So a compact connected abelian Lie group must be a torus $\mathbb{T}^n$. Last time we showed that any irreducible representation of $\mathbb{T}^n$ is one dimensional. In what follows we will find all its irreducible representations. Note that any one dimensional representation is automatically irreducible, so to find out all irreducible representations of $\mathbb{T}^n$ is equivalent to find out all one dimensional representations of $\mathbb{T}^n$. For any

$$\vec{m} = (m_1, \cdots, m_n) \in \mathbb{Z}^n,$$

there is a one-dimensional representation $\rho_{\vec{m}}$ of $\mathbb{T}^n$ on $\mathbb{C}$, given by

$$(e^{i\theta_1}, \cdots, e^{i\theta_n}) \cdot z = e^{i(m_1\theta_1 + \cdots + m_n\theta_n)} z.$$

Note that if $m_1 = \cdots = m_n = 0$, this gives the trivial representation.

Theorem 2.1. *The representations $\rho_{\vec{m}}$'s given above are pair-wise different, and give all the irreducible representations of $\mathbb{T}^n$.*

Proof. The character for the representation $\rho_{\vec{m}}$ is

$$\chi_{\rho_{\vec{m}}}(e^{i\theta_1}, \cdots, e^{i\theta_n}) = e^{i(m_1\theta_1+\cdots+m_n\theta_n)}.$$

So $\rho_{\vec{m}} \neq \rho_{\vec{m}'}$ for $\vec{m} \neq \vec{m}'$, i.e. these represenations are pairwise non-isomorphic.

On the other hand, by classical Fourier analysis, the characters

$$\{e^{i(m_1\theta_1+\cdots+m_n\theta_n)} \mid (m_1, \cdots, m_n) \in \mathbb{Z}^k\}$$

span $L^2(\mathbb{T}^n)$. So there is no irreducible character that is orthogonal to all the characters of $\rho_{\vec{m}}$'s. In other words, there is no other irreducible representations. □

Remark. We can formulate the representations of a general connected compact abelian Lie group G without identify it with $\mathbb{T}^n$, as follows: Let G be a connected compact abelian Lie group. The *group lattice* of G is the lattice

$$\mathbb{Z}_G = K = \ker(\exp) \subset \mathfrak{g},$$

and the *weight lattice* is the dual

$$\mathbb{Z}_G^* = \{\alpha \in \mathfrak{g}^* \mid \alpha(\xi) \in 2\pi\mathbb{Z} \text{ for } \xi \in \mathbb{Z}_G\}.$$

Then the irreducible representations of G are parametrized by $\mathbb{Z}_G^*$. In other words, any $\alpha \in \mathbb{Z}_G^*$ gives an irreducible representation ρ_α of G on $\mathbb{C}$ by

$$\exp(X) \cdot z = e^{i\alpha(X)} z.$$

Corollary 2.2. *Any representation of a connected compact abelian Lie group G is of the form*

$$\pi(e^X)(z_1, \cdots, z_n) = (e^{i\alpha_1(X)} z_1, \cdots, e^{i\alpha_n(X)} z_n)$$

for some weights $\alpha_1, \cdots, \alpha_n \in \mathbb{Z}_G^*$.

3. Irreducible Representations of SU(2)

Now we consider the group $\mathrm{SU}(2) = \mathrm{U}(2) \cap \mathrm{SL}(2, \mathbb{C})$. It is not hard to see

$$g \in \mathrm{SU}(2) \iff g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \quad \text{for } \alpha, \beta \in \mathbb{C} \text{ so that } |\alpha|^2 + |\beta|^2 = 1.$$

It follows that SU(2) is homeomorphic to the unit sphere $S^3 \subset \mathbb{C}^2$, and therefore is a compact, connect and simply connected Lie group. We can also write an element in SU(2) in real coordinates as

$$g = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}.$$

Then under the polar coordinates on S^3,

$$x_1 = \cos\theta, \quad x_2 = \sin\theta\cos\varphi, \quad x_3 = \sin\theta\sin\varphi\cos\psi, \quad x_4 = \sin\theta\sin\varphi\sin\psi,$$

where $0 \le \theta \le \pi$, $0 \le \varphi \le \pi$, $0 \le \psi \le 2\pi$, the normalized Haar measure on SU(2) is given by

$$dg = \frac{1}{2\pi^2} \sin^2\theta \sin\varphi d\theta d\varphi d\psi.$$

Moreover, the eigenvalues of g are $e^{i\theta}$ and $e^{-i\theta}$, so g is conjugate to the diagonal matrix

$$e(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Here are some known representations of SU(2):

- The trivial representation of SU(2) on $\mathbb{C}$: $(\pi_0, V_0 = \mathbb{C})$
- The standard representation of SU(2) on $\mathbb{C}^2$: $(\pi_1, V_1 = \mathbb{C}^2)$
- The previous representation induces an representation of SU(2) on $C(G)$, given by
$$(g \cdot f)(z_1, z_2) = f(z \cdot g) = f(\alpha z_1 + \beta z_2, -\bar{\beta} z_1 + \bar{\alpha} z_2).$$
- The previous representation is too large. There is a much nicer subrepresentation: the representation space being the space of polynomials on $\mathbb{C}^2$. (If $f \in P(\mathbb{C}^2)$ is a polynomial, then $g \cdot f$ is again a polynomial.) This is still an infinite dimensional representation.
- The previous representation has an important *invariant*: the degree of the polynomial. It follows that for any $n \ge 0$, the space of homogeneous polynomials of degree n on $\mathbb{C}^2$,
$$V_n = \mathrm{Span}\{z_1^k z_2^{n-k} \mid 0 \le k \le n\}$$
is a representation of SU(2). Moreover, $\dim V_n = n + 1$. Note that the SU(2) action on V_n is given by
$$(g \cdot P)(z) = P(g^{-1} z) = P(\alpha z_1 + \beta z_2, -\bar{\beta} z_1 + \bar{\alpha} z_2)$$
for $g \in \mathrm{SU}(2), P \in V_n$.

Proposition 3.1. *The representations (V_n, π_n) described above are irreducible.*

Proof. For any $0 \le k \le n$, we denote by $p_k(z)$ the monomial $z_1^k z_2^{n-k}$. Then

$$\pi_n(e(\theta)) p_k = e^{i(2k-n)\theta} p_k.$$

It follows that the character

$$\chi_{\pi_n}(g) = \chi_{\pi_n}(e(\theta)) = \sum_{k=0}^{n} e^{i(2k-n)\theta} = \frac{\sin(n+1)\theta}{\sin\theta}.$$

Thus

$$\int_{\mathrm{SU}(2)} |\chi_{\pi_n}(g)|^2 dg = \frac{1}{2\pi^2} \int_0^\pi \int_0^\pi \int_0^{2\pi} \frac{\sin^2(n+1)\theta}{\sin^2\theta} \sin^2\theta \sin\varphi d\psi d\varphi d\theta = 1.$$

So these representations are irreducible. □

Of course the representations (V_n, π_n)'s are pairwise non-isomorphic since $\dim V_n \neq \dim V_m$ for $n \neq m$.

Theorem 3.2. *The representations* (V_n, π_n) *described above are the only irreducible representations of* $\mathrm{SU}(2)$.

Proof. According to Schur's orthogonality for characters, it is enough to show that the linear span of the irreducible characters

$$\mathrm{Span}\{\chi_{\pi_n}, n = 0, 1, \cdots\}$$

is dense in the space of continuous class functions. Since any matrix $g \in \mathrm{SU}(2)$ is conjugate to some $e(\theta)$, any class function on $\mathrm{SU}(2)$ is uniquely determined by its restriction on

$$\mathbb{T} = \{e(\theta) \mid \theta \in \mathbb{R}\},$$

which is isomorphic to S^1. Moreover, since $e(\theta)$ is conjugate to $e(-\theta)$, any class function restricted to $\mathbb{T}$ can be identified with an *even* function on S^1. On the other hand, since

$$\chi_{\pi_n}(e(\theta)) - \chi_{\pi_{n-2}}(e(\theta)) = e^{-in\theta} + e^{in\theta} = 2\cos(n\theta),$$

we have

$$\mathrm{Span}\{\chi_{\pi_n}|_{\mathbb{T}}, n = 0, 1, \cdots\} = \mathrm{Span}\{\cos(n\theta), n = 0, 1, 2, \cdots\}.$$

The basic theory on Fourier series tells us that $\mathrm{Span}\{\cos(n\theta), n = 0, 1, 2, \cdots\}$ is dense in the space of even functions on S^1. It follows that $\mathrm{Span}\{\chi_{\pi_n}, n = 0, 1, \cdots\}$ is dense in the space of continuous class functions. $\square$