CHAPTER 6

Financial Mathematics

1. Portfolio management

Let $\mathscr{F}_*$ be the filtration generated by the Brownian motion. It represents the information of the market up to time t. We consider the simplest case where the market consists of a risk-free investment called bond and a risky investment called stock. For a time being, a portfolio is a combination of the two investments. At time t it can be described by (ξ_t, η_t), where ξ_t is the number of shares of the bond, and η_t is the number of shares in the stock. The bond and the stock are described as follows:

The bond pays a variable interest at the rate r_t, which we assume is adapted to $\mathscr{F}_*$. If the bond price is $B_0 = 1$ at time 0, then its share price B_t at time t is given by the equation

$$dB_t = B_t r_t \, dt, \qquad B_0 = 1.$$

This can be solved as

$$B_t = \exp\left[\int_0^t r_s ds\right].$$

The bond is considered as a risk-free investment. It is completely characterized by the process B_t. We call B_t the growth factor and B_t^{-1} the discount factor.

Note that if we want to interpret the risk-free investment as a bank deposit, there is a slight confusion of terminology. For a cash deposit, it is customary to consider that the share price is always fixed at 1 per share and it is the number of shares that is changing, the extra shares being considered as dividend from the investment. In this case, B_t is interpreted as the number of shares instead of the price per share. In these notes, we considered B_t as the share price of the bond if the initial share price is $B_0 = 1$. Therefore the bond pays no dividend and has volatility zero (see below).

The stock is considered as a risky investment. It is characterized by the rate of return μ_t and the volatility σ_t. Its share price S_t is assumed to satisfy the equation

$$dS_t = S_t(\sigma_t \, dW_t + \mu_t \, dt).$$

Here W is a one-dimensional $\mathscr{F}_*$-Brownian motion. Whenever necessary we assume that $\mathscr{F}_* = \mathscr{F}_*^W$, i.e., the market filtration is generated by that of the Brownian motion. Note that we do not specify its initial price S_0 as we

did in the bond price. Explicitly, we have

$$S_t = S_0 \exp \left[\int_0^t \sigma_s \, dW_s s + \int_0^t \left(\mu_s - \frac{\sigma_s^2}{2} \right) \, ds \right].$$

A portfolio is described at time t by a pair of adapted processes $\pi = (\xi, \eta)$, where ξ_t and η_t are the numbers of shares of the bond and the stock, respectively, at time t. The wealth process X is simply the total value of the portfolio, i.e.,

$$X_t = \xi_t B_t + \eta_t W_t.$$

Suppose that at time $t = 0$, the investor has the initial asset $X_0 = x$. He invests his asset in the market to achieve a certain goal G at a terminal time T. He also needs to consume at a given rate c_t. The pair $C = (c, G)$ is called a contingent claim. To achieve these ends, he needs to constantly readjust his portfolio $\pi_t = (\xi_t, \eta_t)$ using the information up to time t of the market, i.e., ξ and η have to be adapted to the filtration $\mathscr{F}_*$. For him, the portfolio pair $\pi_t = (\xi_t, \eta_t)$ becomes a trading strategy. The question we want to solve is: What is the minimum initial investment $X_0 = x$ he must have in order to achieve his consumption and investment goal without outside infusion of capital? This minimum, if any, is defined to be the value of the contingent claim C. .

If we do not impose some conditions on the trading strategy π besides adaptability, pathological situations may arise. Roughly speaking, these pathological situations can be avoided if he is not allowed to be wildly in debt. There are various versions of such condition that work well mathematically. Here we take an easy version of it, namely, he is not allowed to be in debt at all.

DEFINITION 1.1. *A trading strategy π is said to be admissible if the wealth process is nonnegative: $X_t = \xi_t B_t + \eta_t S_t \geq 0$.*

We also need to give a mathematical definition of the condition which we phrased as "without outside infusion of capital." Such a portfolio is called self-financing. An arbitrary portfolio $\pi = (\xi_t, \eta_t)$ may need outside capital to maintain its value $X_t = \xi_t B_t + \eta_t S_t$. Suppose that this portfolio consists only of the bond. We have $X_t = \xi_t B_t$. How can we tell that it is self-financing? This is the case precisely when ξ_t is constant, i.e., its increase in value comes solely from the increase of the bond price B_t. Therefore we have

$$dX_t = \xi_t \, dB_t = X_t r_t \, dt.$$

Now if the portfolio has only the stock in it, then $X_t = \eta_t S_t$ and it is self-financing only when η_t is constant, i.e.,

$$dX_t = \eta_t \, dS_t.$$

If the portfolio contains both the stock and the bond, then exchanging between these two are allowed. We make the following definition.

DEFINITION 1.2. *A portfolio (or a trading strategy) with consumption rate c is self-financing if its wealth process $X_t = \xi_t B_t + \eta_t S_t$ satisfies*

$$dX_t = \xi_t \, dB_t + \eta_t \, dS_t - c_t \, dt.$$

Now we can formulate the problem we want to solve as follows. For a given contingent claim $C = (c, G)$, what is the smallest initial capital $X_0 = x$ for that there is an admissible, self-financing trading strategy $\pi = (\xi, \eta)$ such that

$$\mathbb{P}\{X_T \geq C\} = 1.$$

It is intuitively clear that there will be no such admissible strategy if $X_0 = x$ is too small. Our task is to find the smallest x for the existence of an admissible strategy.

We will first assume that there is a trading strategy $\pi = (\xi, \eta)$ to satisfy the contingent claim and derive a lower bound for $X_0 = x$. It is clear that we need to eliminate π from the picture and find a lower bound in terms of the given parameters. We start with the two equations for the wealth process

$$X_t = \xi_t B_t + \eta_t S_t$$

and

$$dX_t = \xi_t \, dB_t + \eta_t \, dS_t - c_t dt.$$

From these two equations we can eliminate ξ_t. We have

$$dB_t = B_t r_t \, dt \quad \text{and} \quad dS_t = S_t(\sigma_t \, dW_t + \mu_t \, dt).$$

Hence

$$\xi_t \, dB_t = \xi_t B_t r_t \, dt = (X_t - \eta_t S_t) r_t \, dt$$

and we have an equation without ξ_t as follows:

$$dX_t - X_t r_t \, dt = \eta_t S_t \{\sigma_t \, dW_t + (\mu_t - r_t) \, dt\} - c_t \, dt.$$

Now we come to the key step in the argument. We need to eliminate η and solve for x. For this purpose we introduce the shifted Brownian motion

$$\tilde{W}_t = W_t - \int_0^t \theta_s \, ds,$$

where

$$\theta_t = \frac{r_t - \mu_t}{\sigma_t}.$$

The equation for X can be written

$$dX_t - r_t X_t \, dt = \eta_t S_t \sigma_t \, d\tilde{W}_t - c_t \, dt.$$

It is clear now that we can do away with η if integrate with respect to a measure under which $\tilde{W}$ is a Brownian motion. This measure is readily provided by the Girsanov theorem. We define the risk-neutral probability measure $\mathbb{P}^*$ by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left[\int_0^T \theta_t \, dW_t - \frac{1}{2}\int_0^T \theta_t^2 \, dt\right].$$

The process $\tilde{W}$ is a Brownian motion under $\mathbb{Q}$. Using B_t^{-1} as an integrating factor we can solve the equation for X in the form

$$B_t^{-1} X_t - x = M_t - \int_0^t B_s^{-1} c_s \, ds,$$

where

$$M_t = \int_0^t B_s^{-1} \eta_s S_s \sigma_s \, d\tilde{W}_s$$

is a local martingale under $\mathbb{Q}$. The wealth process is given by

$$X_t = x B_t + B_t M_t - B_t \int_0^t B_s^{-1} c_s \, ds.$$

In particular,

$$B_T^{-1} X_T - x = M_T - \int_0^T B_s^{-1} c_s \, ds$$

and

$$x = B_T^{-1} X_T + \int_0^T B_s^{-1} c_s \, ds - M_T.$$

It is clear that M is a local martingale with an integrable lower bound, hence using Fatou's lemma on

$$\mathbb{E}^* M_{\tau_n \wedge T} = 0,$$

we have $\mathbb{E}^* M_T \geq 0$. It follows that

$$x \geq \mathbb{E}^* \left[B_T^{-1} G + \int_0^T B_s^{-1} c_s \, ds \right].$$

Here $\mathbb{E}^*$ is the expectation with respect to the risk-neutral probability $\mathbb{P}^*$.

THEOREM 1.3. *The price of the contingent claim $C = (c, G)$ is given by*

$$V(c, G) = \mathbb{E}^* \left[B_T^{-1} G + \int_0^T B_s^{-1} c_s \, ds \right].$$

PROOF. We have shown that the price cannot be smaller than the quantity on the right side. Denote the right side by x. We need to construct a trading strategy π for our contingent claim. We do this by reversing the argument above. The existence of a trading strategy is achieved by using the martingale representation theorem.

In light of the argument given above, the stock component η of our trading strategy should satisfy the equality

$$\int_0^T B_s^{-1} \eta_s S_s \sigma_s \, d\tilde{W}_s = B_T^{-1} G + \int_0^T B_s^{-1} c_s \, ds - x.$$

By assumption the expected value of the right side is zero. If we let $H_s = \eta_s S_s \sigma_s$, then the above equality can be written as

$$B_T^{-1} G + \int_0^T B_s^{-1} c_s \, ds = x + \int_0^T H_s \, d\tilde{W}_s.$$

This equality looks very much like a martingale representation of the random variable on the left side. However, this random variable is measurable with respect to the filtration of the original Brownian motion W, whereas we want to express it as a stochastic integral with respect to the Brownian motion $\tilde{W}$ under $\mathbb{P}^*$. We know that martingales under $\mathbb{P}$ and $\mathbb{P}^*$ are related in a simple way as follows. Let

$$e_t = \exp\left[\int_0^t \theta_s \, dW_s - \frac{1}{2}\int_0^t \theta_s^2 \, ds\right].$$

Then $d\mathbb{P}^*/d\mathbb{P} = e_T$ and M_t is a martingale under $\mathbb{P}^*$ if and only if $M_t e_t$ is a martingale under $\mathbb{P}$. By the martingale representation theorem, there is a unique progressively measurable process H such that

$$\left[B_T^{-1}G + \int_0^T B_s^{-1}c_s \, ds\right]e_T = x + \int_0^T \tilde{H}_s \, dW_s.$$

By Ito's formula it is easy to verify that there exists a progressively measurable process H such that

$$e_T^{-1}\left[x + \int_0^T H_s \, dW_s\right] = x + \int_0^T H_s \, d\tilde{W}_s.$$

In fact we have

$$H_t = \left[\tilde{H}_t - \left(x + \int_0^t \tilde{H}_s \, dW_s\right)\theta_t\right]e_t^{-1}.$$

It follows that

$$B_T^{-1}G + \int_0^T B_s^{-1}c_s \, ds = x + \int_0^T H_s \, d\tilde{W}_s.$$

Having found H by applying the martingale representation theorem, we can proceed by reversing the argument we have carried out before. Define the stock component of the trading strategy by

$$H_t = B_t^{-1}\eta_t S_t \sigma_t, \qquad \eta_t = \frac{B_t H_t}{S_t \sigma_t}.$$

The wealth process is defined by

$$X_t = xB_t + B_t M_t - B_t \int_0^t B_s^{-1}c_s \, ds,$$

where

$$M_t = \int_0^t B_s^{-1}\eta_s S_s \sigma_s \, d\tilde{W}_s.$$

It is clear then the bond component of the trading strategy should be

$$\xi_t = \frac{X_t - \eta_t S_t}{B_t}.$$

Now we have automatically

$$X_t = \xi_t B_t + \eta_t S_t$$

by the definition of ξ. It remains to show that the trading strategy $\pi = (\xi, \eta)$ is
    (1) self-financing: $dX_t = \xi_t\, dB_t + \eta_t\, dS_t - c_t\, dt$;
    (2) admissible: $X_t \geq 0$;
    (3) having the correct payoff: $X_T = G$.

We first show that the trading strategy defined above is self-financing with consumption rate c. From the definition of X we have

$$B_t^{-1} X_t = x - \int_0^t B_s^{-1} c_s\, ds + \int_0^t B_s^{-1} \eta_s S_s \sigma_s\, d\tilde{W}_s.$$

Recall that previously we obtained this from its equivalent differential form

$$dX_t - X_t r_t\, dt = \eta_t S_t \sigma_t\, d\tilde{W}_t - c_t\, dt.$$

Using $X_t = \xi_t B_t + \eta_t S_t$ and

$$d\tilde{W}_t = dW_t - \left[\frac{r_t - \mu_t}{\sigma_t}\right] dt$$

we have

$$
\begin{aligned}
dX_t &= (\xi_t B_t + \eta_t S_t) r_t\, dt + \eta_t S_t \sigma_t\, dW_t - \eta_t S_t (r_t - \mu_t)\, dt - c_t\, dt \\
&= \xi_t\, dB_t + \eta_t S_t (\sigma_t\, dW_t + \mu_t\, dt) - c_t\, dt \\
&= \xi_t\, dB_t + \eta_t\, dS_t - c_t\, dt.
\end{aligned}
$$

This shows that the trading strategy $\pi = (\xi, \eta)$ is self-financing with consumption rate c.

    We now show that the trading strategy is admissible and $X_T = G$. Recall that

$$B_t^{-1} X_t = x - \int_0^t B_s^{-1} c_s\, ds + \int_0^t \eta_s S_s \sigma_s\, d\tilde{W}_s.$$

By the definition of η we also have

$$B_T^{-1} G + \int_0^T B_s^{-1} c_s\, ds = x + \int_0^T B_s^{-1} \eta_s S_s \sigma_s\, d\tilde{W}_s.$$

Hence,

$$\mathbb{E}^* \left[B_T^{-1} G + \int_0^T B_s^{-1} c_s\, ds \,\middle|\, \mathscr{F}_t \right] = x + \int_0^t B_t^{-1} \eta_s S_s \sigma_s\, d\tilde{W}_s.$$

It follows that

$$B_t^{-1} X_t = \mathbb{E}^* \left[B_T^{-1} G + \int_t^T B_s^{-1} c_s\, ds \,\middle|\, \mathscr{F}_t \right].$$

Therefore the wealth process is given by

(1.1) $$X_t = B_t\, \mathbb{E}^* \left[B_T^{-1} G + \int_t^T B_s^{-1} c_s\, ds \,\middle|\, \mathscr{F}_t \right].$$

It is clear from this expression that $X_t \geq 0$ and $X_T = G$. $\square$

REMARK 1.4. Let's take a closer look at the relation

$$B_t^{-1}X_t + \int_0^t B_s^{-1}c_s ds = x + \int_0^t B_s^{-1}\eta_s S_s \sigma_s \, d\tilde{W}_s.$$

Intuitively, this relation means that if the bond interest rate is equal to the mean stock return rate, which is true under the risk-neutral probability $\mathbb{P}^*$, then sum of the wealth and the total consumption properly discounted by the interest rate is a martingale (with respect to the risk-neutral probability $\mathbb{P}^*$) with mean value x. The local martingale on the right side has the quadratic variation

$$\epsilon_t = \int_0^t B_s^{-2}\eta_s^2 S_s^2 \sigma_s^2 \, ds.$$

Hence the local martingale has the form W_{ϕ_t} for some Brownian motion W and we have

$$W_{\epsilon_t} = B_t^{-1}X_t + \int_0^t B_s^{-1}c_s ds - x.$$

Since $X_t \geq 0$ and we have assumed that the interest rate and the consumption rate are uniformly bounded, there is a constant K such that

$$W_{\epsilon_t} \geq -K, \qquad 0 \leq t \leq T.$$

Now if $|\eta_t|S_t$ is bounded from below by a positive constant for $0 \leq t \leq T$, then $\epsilon_t \geq \epsilon t$ for some positive constant ϵ. This implies that the probability that the local martingale part is less than $-K$ somewhere on $[0, T]$ is strictly positive, which is impossible. Hence the trading strategy must vanish sometime, i.e., the investor has to invest all his asset in the bond.

REMARK 1.5. It is clear from the proof that the trading strategy $\pi = (\xi, \eta)$ is unique. The source of this strong conclusion is the assumption that the trading strategy is self-financing.

2. Black-Scholes equation and formula

We now consider a special case, where the market characteristics involved are functions of the stock price and time:

$$r_t = r(S_t, t), \quad \mu_t = \mu(S_t, t), \quad \sigma_t = \sigma(S_t, t).$$

We also assume that the contingent claim has the form

$$c_t = c(S_t, t), \quad G = G(S_T).$$

This case is generally considered as the Black-Scholes option price theory. We will see that it is intimately connected to the theory of partial differential equations. The price of a contingent claim is a function of time and stock price and it satisfies the so-called Black-Scholes partial differential equation . The case of constant interest rate and volatility and the call option $G(S_T) = (S_T - K)^+$ can be computed explicitly and gives the classical Black-Scholes formula.

The stock price satisfies the stochastic differential equation

$$dS_t = S_t(\sigma(S_t, t) \, dW_t + \mu(S_t, t) \, dt).$$

We assume that the time parameter t covers the range $\mathbb{R}_+ = [0, \infty)$. To create a more flexible theory, we sometimes consider the price history from a starting point s other than 0. In this case, if the initial stock price is $S_s = x$, then the stock price S_t is the solution of the following stochastic differential equation

$$dS_t = S_t(\sigma(S_t, t) \, dW_t + \mu(S_t, t) \, dt), \quad t \geq s; \quad S_s = x.$$

But what we really need is the equation under the risk-neutral probability, i.e.,

$$dS_t = S_t(\sigma(S_t, t) \, dW_t + r(S_t, t) \, dt), \quad t \geq s; \quad S_s = x.$$

We will assume that this equation has a unique solution. This assumption holds if we assume, for example, that the coefficients are globally Lipschitz. For the theory of stochastic differential equations, we know that the process $\{S_t, \, t \geq s\}$ is a strong Markov process. We will denote its law by $\mathbb{P}_{t,x}$.

We have shown that the price of the contingent claim $C = (c, G)$ is given by

$$V^C = \mathbb{E}^* \left[B_T^{-1} G(S_T) + \int_0^T B_s^{-1} c(S_s, s) \, ds \right].$$

Note that the stock price is the solution of the stochastic differential equation

$$dS_t = S_t(\sigma(S_t, t) \, dW_t + \mu(S_t, t) \, dt).$$

This can be written as

$$dS_t = S_t(\sigma(S_t, t) \, d\tilde{W}_t + r(S_t, t) \, dt).$$

Under the risk-neutral probability $\mathbb{P}^*$, the process $\tilde{W}$ is a Brownian motion. This is a diffusion process with the generator

$$\mathscr{L} = \frac{(\sigma S)^2}{2} \frac{\partial^2}{\partial x^2} + rS \frac{\partial}{\partial x}.$$

We also have

$$B_t^{-1} = \exp \left[- \int_0^t r(S_s, s) \, ds \right].$$

Recall that $\mathbb{P}_S = \mathbb{P}_{0,S}$ is the law of the stock price process $\{S_t, \, t \geq 0\}$ under the risk-neutral probability. We can now write

$$V(S; T) = \mathbb{E}_S \left[B_T^{-1} G(S_T) + \int_0^T B_s^{-1} c(S_s, s) \, ds \right].$$

Therefore the formula for V^C is precisely the probabilistic representation of an initial value problem. More precisely, the price $V^C = V(S; T)$ is a

function of the expiration time T and the initial stock price S and satisfies the Black-Scholes partial differential equation

$$V_T = \frac{(\sigma S)^2}{2} V_{SS} + rSV_S - rV + c$$

with the initial condition $V(0, S) = G(S)$.

We now consider the price of the contingent claim at a general time $t \le T$ instead of at time 0. We use $V(t, T; S)$ to denote the price of the contingent claim expiring at time T at the initial time $t \le T$ when the stock price is S at time t. Thus the old price function is $V(T; S) = V(0, T; S)$. The only difference between the old and the new price function is that now the initial time point is shifted from 0 to t. Therefore the formula for $V(T; S)$ applies to the new price function after an obvious modification, namely,

$$V(t, S; T) = \mathbb{E}_{t,S} \left[B_{t,T}^{-1} G(S_T) + \int_t^T B_{t,s}^{-1} c(S_s, s) \, ds \right],$$

where

$$B_{t,s} = \exp \left[\int_t^s r(S_u, u) \, du \right].$$

As before for a fixed t, the price function $V(t, T; S)$ is the solution of the initial boundary value problem

$$V_T = \frac{(\sigma S)^2}{2} V_{SS} + rSV_S - rV + c, \quad V(t, S; t) = G(S).$$

Similarly, $V(t, S; T)$, for a fixed expiration time T, is the solution of the following terminal boundary value problem:

$$V_t + \frac{(\sigma S)^2}{2} V_{SS} + rSV_s - rV + c = 0, \quad V(T, S; S) = 0.$$

This is similar to the equation of $V(t, S; T)$ if t is fixed, the difference being the sign of the time derivative and the change from a initial value problem to a terminal value problem. The above equation is also referred to as the Black-Scholes equation. In general diffusion theory, this is an example of backward diffusion equations.

We now express the unique trading strategy $\pi_t = (\xi_t, \eta_t)$ in terms of the price function $V(t, T; S)$. Suppose that we are given a contingent claim as above. If we start with $X_0 = V(0, S_0)$, the value of the contingent claim at time 0 if the stock price is S_0. Then we expect that at time time t, the wealth process X_t should be equal to the value of the contingent claim when the stock price is S_t, i.e.,

$$X_t = V(t, S_t).$$

THEOREM 2.1. *For the trading strategy constructed in* THEOREM *1.3 we have* $X_t = V(t, S_t)$ *and*

$$\xi_t = B_t^{-1} \left\{ V(t, S_t) - S_t V_S(t, S_t) \right\},$$
$$\eta_t = V_S(t, S_t).$$

PROOF. We have shown that the wealth process is given by

$$X_t = B_t \, \mathbb{E}^* \left[B_T^{-1} G(S_T) + \int_t^T B_s^{-1} c(S_s, s) \, ds \, \middle| \, \mathscr{F}_t \right].$$

See (1.1). By the Markov property of the stock process and $B_T = B_t B_{t,T}$, we can write X_t can be written as

$$X_t = \mathbb{E}_{t,S_t} \left[B_{t,T}^{-1} G(S_T) + \int_t^T B_{t,s}^{-1} c(S_s, s) \, ds \right].$$

This shows that

$$X_t = V(t, S_t).$$

Calculating the stochastic differential of X_t we have

$$dX_t = V_t \, dt + V_S \, dS_t + \frac{1}{2} V_{SS} (\sigma S)^2 \, dt$$

$$= \left[V_t + \frac{(\sigma S)^2}{2} V_{SS} \right] dt + V_S \, dS_t.$$

Comparing this with

$$dX_t = \xi_t \, dB_t + \eta_t \, dS_t - c(S_t, t) \, dt$$

$$= \{ r(S_t, t) B_t \xi_t - c(S_t, t) \} \, dt + \eta_t \, dS_t$$

we have immediately $\eta_t = V_S(t, S_t)$. Using the Black-Scholes equation we have

$$V_t + \frac{(\sigma S)^2}{2} V_{SS} = rV - rSV_S - c.$$

Hence

$$r(S_t, t) B_t \xi_t - c(S_t, t) = r(S_t, t) V(t, S_t) - r(S_t, t) S_t V_S(t, S_t) - c(S_t, t).$$

The formula for ξ_t follows immediately.   □

We now prove the classical Black-Scholes formula for a call option.

PROPOSITION 2.2. *The call price is given by*

$$C(S; T) = SN(d_1) - Ke^{-rT} N(d_2),$$

where

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_\infty^z e^{-y^2/2} dy$$

is the distribution of the standard normal distribution and

$$d_1 = \frac{\log(S/K) + \left(r + \dfrac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}$$

and

$$d_2 = d_1 - \sigma \sqrt{T} = \frac{\log(S/K) + \left(r - \dfrac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}$$

PROOF. Under the risk-neutral probability $\mathbb{P}^*$, the stock price has the lognormal distribution. More precisely, $\ln S_T$ is a normal random variable with mean $\left(r - \dfrac{\sigma^2}{2}\right) T + \ln S$ and standard deviation $\sigma\sqrt{T}$. Therefore

$$\ln S_T = \sigma\sqrt{T}X + \left(r - \frac{\sigma^2}{2}\right) T + \ln S,$$

where X is a standard normal random variable. The call payoff is $(S_T - K)^+$, which is positive if

$$\sigma\sqrt{T}X + \left(r - \frac{\sigma^2}{2}\right) T + \ln S \geq \ln K, \quad \text{or} \quad X \geq -d_2.$$

It follows that

$$
\begin{aligned}
C(S;T) &= e^{-rT}\mathbb{E}\left\{\exp\left[\left(r - \frac{\sigma^2}{2}\right) T + \ln S + \sigma\sqrt{T}X\right] ; X \geq -d_2\right\} \\
&\quad - e^{-rT}K\mathbb{P}\left\{X \geq -d_2\right\} \\
&= S\,\mathbb{E}\left\{\exp\left[\sigma\sqrt{T}X - \frac{\sigma^2}{2} T\right] ; X - \sigma\sqrt{T} \geq -d_2 - \sigma\sqrt{T}\right\} \\
&\quad - Ke^{-rT}\left\{1 - N(-d_2)\right\} \\
&= S\,\mathbb{P}\left\{X \geq -d_1\right\} - Ke^{-rT}N(d_2) \\
&= SN(d_1) - Ke^{-rT}N(d_2).
\end{aligned}
$$

This completes the proof. $\qquad\square$

3. Sixth assignment