LIE GROUPS 2013 FALL
PROBLEM SET 1

(1) For $i = 1, 2$, let M_i, N_i be smooth manifolds, $f_i : M_i \to N_i$ be smooth maps, and $p_i \in M_i$ be points.
 (a) Prove: $M_1 \times M_2$ is still a smooth manifold.
 (b) Prove: $T_{(p_1,p_2)}(M_1 \times M_2) \simeq T_{p_1}M_1 \oplus T_{p_2}M_2$.
 (c) Find the relation between $d(f_1 \times f_2)_{(p_1,p_2)}$ and $(df_i)_{p_i}$, $i = 1, 2$.

(2) (a) Let $f, g \in C^\infty(M)$, $X, Y \in \Gamma^\infty(TM)$. Show
$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$
 (b) Prove: Any one dimensional distribution is involutive.

(3) Let $X_1 = y^2\frac{\partial}{\partial x}$ and $X_2 = x^2\frac{\partial}{\partial y}$ be two vector fields on $\mathbb{R}^2$.
 (a) Prove: X_1 and X_2 are complete.
 (b) Prove: $X_1 + X_2$ is not complete. [Hint: Consider the integral curve starting at some point (c, c).]

(4) Recall that $\mathrm{GL}(n, \mathbb{R})$ is an n^2 dimensional smooth manifold.
 (a) Show that $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ is a smooth function.
 (b) Show that $d(\det)_A(B) = (\det A)\mathrm{tr}(A^{-1}B)$. (Hint: $d(\det)_A(B) = \frac{d}{dt}|_{t=0}\det(A + tB)$.)
 (c) Prove: For any $a \neq 0$, the set $S_a = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det A = a\}$ is a smooth submanifold of $\mathrm{GL}(n, \mathbb{R})$.

(5) Let $\mathrm{M}(n, \mathbb{R})$ be the set of all $n \times n$ real matrices, and $\mathrm{Sym}(n, \mathbb{R})$ the set of all $n \times n$ real symmetric matrices. Consider the map
$$f : \mathrm{M}(n, \mathbb{R}) \to \mathrm{Sym}(n, \mathbb{R}), \quad A \mapsto f(A) = A^T A.$$
 (a) Since both $\mathrm{M}(n, \mathbb{R})$ and $\mathrm{Sym}(n, \mathbb{R})$ are linear spaces, we can identify $T_A\mathrm{M}(n, \mathbb{R})$ with $\mathrm{M}(n, \mathbb{R})$, and $T_{f(A)}\mathrm{Sym}(n, \mathbb{R})$ with $\mathrm{Sym}(n, \mathbb{R})$. Prove: $df_A(B) = A^T B + B^T A$.
 (b) Prove: $O(n)$ is a $\frac{n(n-1)}{2}$ dimensional submanifold of $\mathrm{GL}(n, \mathbb{R})$.

(6) Prove: any left invariant vector field on a Lie group is complete.

(7) Let G be a Lie group and $i : G \to G$ be the inversion map $i(a) = a^{-1}$. For any $X_a \in T_aG$, express $di_a(X_a)$ in terms of the differential of the left and right multiplications. .

(8) Let G be a connected Lie group. Prove:
 (a) For any open neighborhood U of e in G, $G = \cup_{n=1}^\infty U^n$.
 (b) Any Lie group homomorphism $\varphi : G \to H$ is determined by the induced Lie algebra homomorphism $d\varphi : \mathfrak{g} \to \mathfrak{h}$.