CHAPTER 4

Applications of Ito's Formula

In this chapter, we discuss several basic theorems in stochastic analysis. Their proofs are good examples of applications of Itô's formula.

1. Lévy's martingale characterization of Brownian motion

Recall that B is a Brownian motion with respect to a filtration $\mathscr{F}_*$ if the increment $B_t - B_s$ has the normal distribution $N(0, t-s)$ independent of $\mathscr{F}_s$. A consequence of this definition is that B is continuous martingale with quadratic variation process $\langle B, B\rangle_t = t$. The following theorem shows that this property characterizes Brownian motion completely.

THEOREM 1.1. *(Lévy's characterization of Brownian motion) Suppose that B is a continuous local martingale with respect to a filtration $\mathscr{F}_*$ whose quadratic variation process is $\langle B, B\rangle_t = t$ (or equivalently, the process $\left\{B_t^2 - t,\, t \geq 0\right\}$ is also a local martingale), then it is a Brownian motion with respect to $\mathscr{F}_*$.*

PROOF. We will establish the following: for any $0 \leq s < t$ and any bounded $\mathscr{F}_s$-measurable random variable Z,

$$\mathbb{E}\left[Z \exp\left\{ia(B_t - B_s)\right\}\right] = \exp\left[-\frac{|a|^2}{2}(t-s)\right]\mathbb{E}Z. \tag{1.1}$$

Letting $Z = 1$, we see that $B_t - B_s$ is distributed as $N(0, t-s)$. Letting $Z = e^{ibY}$ for $Y \in \mathscr{F}_s$, we infer by the uniqueness of two-dimensional characteristic functions that $B_t - B_s$ is independent of any $Y \in \mathscr{F}_s$, which shows that $B_t - B_s$ is independent of $\mathscr{F}_s$. Therefore it is sufficient to show (1.1)

Denote the left side of (1.1) by $F(t)$. We first use Itô's formula to obtain

$$\begin{aligned}\exp\left\{ia\cdot(B_t - B_s)\right\} = & 1 + ia\cdot\int_s^t \exp\left\{ia\cdot(B_u - B_s)\right\} dB_u \\ & - \frac{|a|^2}{2}\int_s^t \exp\left\{ia\cdot(B_u - B_s)\right\} du.\end{aligned}$$

The stochastic integral is uniformly bounded because the other terms in this equality are uniformly bounded. Therefore the stochastic integral is not only a local martingale, but a martingale. Multiply both sides by Z and take the expectation. Noting that $Z \in \mathscr{F}_s$, we have

$$F(t) = \mathbb{E}Z - \frac{|a|^2}{2}\int_s^t F(u)\, du.$$

Solving for $F(t)$, we obtain

$$F(t) = \exp\left[-\frac{|a|^2}{2}(t-s)\right]\mathbb{E}Z.$$

This is what we wanted to prove. $\square$

A very useful corollary of Lévy's criterion is that every continuous local martingale M is a time change of Brownian motion. This can be seen as follows. We know that the quadratic variation process $\langle M, M\rangle$ is a continuous increasing process. Let us assume that as $t \to \infty$,

$$\langle M, M\rangle_t \to \infty$$

with probability 1. Let $\tau = \{\tau_t\}$ be the right inverse of $\langle M, M\rangle$ defined by

$$\tau_t = \inf\{s \geq 0 : \langle M, M\rangle_s > t\}.$$

It is easy to show that $t \mapsto \tau_t$ is a right-continuous, increasing process and $\langle M, M\rangle_{\tau_t} = t$ and $\tau_{\langle M,M\rangle_t} = t$. Furthermore, τ_t is a stopping time for each fixed t. Consider the time-changed process $B_t = M_{\tau_t}$. Let

$$\sigma_n = \inf\{t : |M_t| \geq n\}.$$

Since M is a continuous local martingale, we have $\sigma_n \uparrow \infty$ and each stopped process M^{σ_n} is a square integrable martingale. From $\langle M, M\rangle_t = t$, we have by Fatou's lemma

$$\mathbb{E}\left[B_t^2\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[M^2_{\sigma_n\wedge\tau_t}\right] = \liminf_{n\to\infty} \mathbb{E}\left[\langle M, M\rangle_{\sigma_n\wedge\tau_t}\right] \leq \mathbb{E}\left[\langle M, M\rangle_{\tau_t}\right] = t.$$

By the optional sampling theorem, B is a martingale.

We now show that B is continuous. There exists a sequence of partitions $\{\Delta_n\}$ of the time set $\mathbb{R}_+$ such that $|\Delta_n| \to 0$ and that with probability 1,

$$\forall t \geq 0 : \sum_j \left[M_{t_j^n\wedge t} - M_{t_{j-1}^n\wedge t}\right]^2 = \langle M, M\rangle_t$$

for all $t \geq 0$. Now for any fixed t, the jump

$$(B_t - B_{t-})^2 = (M_{\tau_t} - M_{\tau_{t-}})^2$$

is bounded by the quadratic variation of M in the time interval $[\tau_{t-\delta}, \tau_t]$ for any $\delta > 0$. Hence

$$|B_t - B_{t-}|^2 \leq \lim_{n\to\infty} \sum_{t_j^n \geq \tau_{t-\delta}} \left[M_{t_j^n\wedge\tau_t} - M_{t_{j-1}^n\wedge\tau_t}\right]^2 = \langle M, M\rangle_{\tau_t} - \langle M, M\rangle_{\tau_{t-\delta}} = \delta.$$

Since δ is arbitrary, we see that $B_t = B_{t-}$. This shows that B is continuous with probability 1.

So far we have proved that B is a continuous local martingale. We compute the quadratic variation of B. The process $M^2 - \langle M, M\rangle$ is a continuous local martingale. By the optional sampling theorem, we see that

$$M^2_{\tau_t} - \langle M, M\rangle_{\tau_t} = B_t^2 - t$$

is a continuous local martingale. It follows that the quadratic variation process of B is just $\langle B, B\rangle_t = t$. Now we can use Lévy's characterization to conclude that $B_t = M_{\tau_t}$ is a Brownian motion. Therefore we have shown that every continuous local martingale can be transformed into a Brownian motion by a time change.

From $B_t = M_{\tau_t}$ and $\tau_{\langle M,M\rangle_t} = t$ we have $M_t = B_{\langle M,M\rangle_t}$. In this sense we say that every continuous local martingale is the time change of a Brownian motion.

2. Exponential martingale

We want to find an analog of the exponential function e^x. The defining property of the exponential function is the differential equation

$$\frac{df(x)}{dx} = f(x), \quad f(0) = 1,$$

or equivalently

$$f(x) = 1 + \int_0^x f(t)\, dt.$$

So we define an*exponential martingale* E_t by the stochastic integral equation

$$E_t = 1 + \int_0^t E_s\, dM_s,$$

where M is a continuous local martingale. This equation can be solved explicitly. Instead of writing down the formula and verify it, let us discover the formula. Since $E_0 = 1$ we can take the logarithm of E_t at least for small time t. Let $C_t = \log E_t$. By Itô's formula we have

$$C_t = \int_0^t E_s^{-1}\, dE_s - \frac{1}{2}\int_0^t E_s^{-2}\, d\langle E, E\rangle_s.$$

Since $dE_s = E_s dM_s$, we have $d\langle E, E\rangle_s = E_s^2 d\langle M, M\rangle_s$. Hence

$$C_t = M_t - \frac{1}{2}\langle M, M\rangle_t.$$

Therefore the formula for exponential martingale is

$$E_t = \exp\left[M_t - \frac{1}{2}\langle M, M\rangle_t\right].$$

Now it is easy to verify directly that this process satisfies the defining equation for E_t.

Every strictly positive continuous local martingale can be written in the form of an exponential martingale:

$$E_t = \exp\left[M_t - \frac{1}{2}\langle M, M\rangle_t\right],$$

where the local martingale M can be expressed in terms of E by

$$M_t = \int_0^t E_s^{-1}\, dE_s.$$

We can say that an exponential martingale is nothing but a positive continuous local martingale.

Exponential martingale is related to iterated stochastic integrals defined by $I_0(t) = 1$ and

$$I_n(t) = \int_0^t I_{n-1}(s)\, dM_s.$$

By iterating the defining equation of the exponential martingale we have the expansion

$$E_t = \exp\left[M_t - \frac{1}{2}\langle M, M\rangle_t\right] = \sum_{n=0}^{\infty} I_n(t)$$

This formula can be verified rigorously, namely, it can be shown that the infinite series converges and the remainder from the iteration tends to zero. To continue our discussion let us introduce a parameter λ by replacing M with λM and obtain

$$\exp\left[\lambda M_t - \frac{\lambda^2}{2}\langle M, M\rangle_t\right] = \sum_{n=0}^{\infty} \lambda^n I_n(t). \tag{2.1}$$

If we set

$$x = \frac{M_t}{\sqrt{2\langle M, M\rangle_t}} \quad \text{and} \quad \theta = \lambda\sqrt{\frac{\langle M, M\rangle_t}{2}},$$

then the left side of (2.1) becomes $\exp\left[2x\theta - \theta^2\right]$. The coefficients of its Taylor expansion in θ are called Hermite polynomials

$$e^{2x\theta-\theta^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!}\, \theta^n.$$

It can be shown that

$$H_n(x) = e^{-x^2/2}\left(\frac{d}{dx}\right)^n e^{-x^2/2}.$$

and $\{H_n, n \geq 0\}$ is the orthogonal basis of $L^2(\mathbb{R}, e^{-x^2/2}dx/\sqrt{2\pi})$ obtained by the Gramm-Schmidt procedure from the complete system $\{x^n, n \geq 0\}$.

Now we can write

$$\exp\left[\lambda M_t - \frac{\lambda^2}{2}\langle M, M\rangle_t\right] = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\left(\frac{\langle M, M\rangle_t}{2}\right)^{n/2} H_n\left(\frac{M_t}{\sqrt{2\langle M, M\rangle_t}}\right).$$

It follows that the iterated stochastic integrals can be expressed in terms of M_t and $\langle M, M\rangle_t$ via Hermite polynomials as follows:

$$I_n(t) = \frac{1}{n!}\left(\frac{\langle M, M\rangle_t}{2}\right)^{n/2} H_n\left(\frac{M_t}{\sqrt{2\langle M, M\rangle_t}}\right).$$

Here are the first few iterated stochastic integrals:

$$\begin{aligned}
I_0(t) &= 1,\\
I_1(t) &= M_t,\\
I_2(t) &= \frac{1}{2}\left[M_t^2 - \langle M\rangle_t\right],\\
I_3(t) &= \frac{1}{6}\left[M_t^3 - \langle M\rangle_t M_t\right].
\end{aligned}$$

Further discussion on this topic can be found in Hida [**5**].

3. Uniformly integrable exponential martingales

In stochastic analysis exponential martingales (positive martingales) often appear in the following context. Suppose that $(\Omega, \mathscr{F}_*, \mathbb{P})$ is a filtered probability space. Let $\mathbb{Q}$ be another probability measure which is absolutely continuous with respect to $\mathbb{P}$ on the σ-algebra $\mathscr{F}_T$. Then $\mathbb{Q}$ is also absolutely continuous with respect to $\mathbb{P}$ on $\mathscr{F}_t$ for all $t \le T$. Let E_t be the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathscr{F}_t$:

$$E_t = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathscr{F}_t}.$$

Then it is easy to verify that

$$E_t = \mathbb{E}\left\{E_T \middle| \mathscr{F}_t\right\}.$$

This shows that $\{E_t,\, t \le T\}$ is a positive martingale. If it is continuous, then it is an exponential martingale. Not only that, it is a uniformly integrable martingale. Conversely, if $\{E_t,\, t \le T\}$ is a uniformly integrable exponential martingale, then it defines a change of measure on the filtered probability space $(\Omega, \mathscr{F}_*, \mathbb{P})$ up to time T.

It often happens that we know the local martingale M and wish that the exponential local martingale

$$E_t = \exp\left[M_t - \frac{1}{2}\langle M, M\rangle_t\right]$$

defines a change of probability measures by $d\mathbb{Q}/d\mathbb{P} = E_T$. For this it is necessary that $\mathbb{E}E_T = 1$. However, in general, we only know that $\{E_t\}$ is a positive local martingale. It is therefore a supermartingale and $\mathbb{E}E_T \le 1$. Therefore the requirement $\mathbb{E}E_T = 1$ is not automatic and has to be proved by imposing further conditions on the local martingale M.

PROPOSITION 3.1. *Let M be a continuous local martingale and*

$$E_t = \exp\left[M_t - \frac{1}{2}\langle M, M\rangle_t\right].$$

Then $\mathbb{E}E_T \le 1$ and $\{E_t,\, 0 \le t \le T\}$ is a uniformly integrable martingale if and only if $\mathbb{E}E_T = 1$.

PROOF. We know that E is a local martingale, hence there is a sequence of stopping times $\tau_n \uparrow \infty$ such that $E_{t\wedge\tau_n}$ is a martingale, hence

$$\mathbb{E}E_{T\wedge\tau_n} = \mathbb{E}E_0 = 1.$$

Letting $n \to \infty$ and using Fatou's lemma we have $\mathbb{E}E_T \le 1$. This inequality also follows from the fact that a nonnegative local martingale is always a supermartingale.

Suppose that $\mathbb{E}E_T = 1$. Since E is a supermartingale we have $E_t \ge \mathbb{E}\left\{E_T \mathscr{F}_t\right\}$. Taking expected value gives

$$1 \ge \mathbb{E}E_t \ge \mathbb{E}E_T = 1.$$

This shows that the equality must hold throughout and we have $E_t = \mathbb{E}\left\{E_T|\mathscr{F}_t\right\}$, which means that $\{E_t, 0 \le t \le T\}$ is a uniformly integrable martingale. □

We need to impose conditions on the local martingale M to ensure that the local exponential martingale E is uniformly integrable. We have the following result due to Kamazaki

THEOREM 3.2. *Suppose that M is a martingale and $\mathbb{E}e^{M_T/2}$ is finite. Then*

$$\left\{\exp\left[M_t - \frac{1}{2}\langle M\rangle_t\right],\, 0 \le t \le T\right\}$$

is uniformly integrable.

PROOF. This is a very interesting proof. Let

$$E(\lambda)_t = \exp\left[\lambda M_t - \frac{\lambda^2}{2}\langle M\rangle_t\right].$$

We need to show that $E(1)$ is uniformly integrable, but first we take a slight step back and prove that $E(\lambda)$ is uniformly integrable for all $0 < \lambda < 1$. We achieve this by proving that there is an $r > 1$ and C such that $\mathbb{E}E(\lambda)_\sigma^r \le C$ for all stopping time $\sigma \le T$. We have

$$E(\lambda)_\sigma^r = \exp\left[\left(\lambda r - \sqrt{\lambda^3 r}\right) M_\sigma\right] \exp\left[\sqrt{\lambda^3 r} M_\sigma - \frac{\lambda^2 r}{2}\langle M\rangle_\sigma\right].$$

Using Hölder's inequality with the exponents $1 - \lambda + \lambda = 1$ we see that $\mathbb{E}E(\lambda)_\sigma^r$ is bounded by

$$\left\{\mathbb{E}\exp\left[\left(\frac{\lambda r - \sqrt{\lambda^3 r}}{1-\lambda}\right) M_\sigma\right]\right\}^{1-\lambda} \left\{\mathbb{E}\exp\left[\sqrt{\lambda^2 r} M_\sigma - \frac{\lambda^2 r}{2}\langle M\rangle_\sigma\right]\right\}^{\lambda}.$$

The second expectation on the right side does not exceed 1 (see PROPOSITION 3.1). For the first factor we claim that σ can be replaced by T and the coefficient can be replaced by $1/2$ if $r > 1$ is sufficiently close to 1. When $r = 1$ the coefficient is

$$\frac{\lambda - \sqrt{\lambda^3}}{1-\lambda} = \frac{\lambda}{1+\sqrt{\lambda}} < \frac{1}{2}$$

because $\lambda < 1$. Hence the coefficient is still less than $1/2$ if $r > 1$ but sufficiently close to 1. On the other hand, we have assumed that M is a martingale, hence $M_\sigma = \mathbb{E}\left\{M_T|\mathscr{F}_\sigma\right\}$. By Jensen's inequalty we have

$$e^{M_\sigma/2} \le \mathbb{E}\left\{e^{M_T/2}|\mathscr{F}_\sigma\right\}.$$

It follows that

$$\mathbb{E}\exp\left[\left(\frac{\lambda r - \sqrt{\lambda^3 r}}{1-\lambda}\right)M_\sigma\right] \le \mathbb{E}e^{M_\sigma/2} \le \mathbb{E}e^{M_T/2}.$$

We therefore have shown that for any $\lambda < 1$, there is an $r > 1$ such that

$$\mathbb{E}E(\lambda)_\sigma^r \le \mathbb{E}e^{M_T/2}$$

for all stopping times $\sigma \le T$. This shows that $E(\lambda)$ is a uniformly integrable martingale, which implies that $\mathbb{E}E(\lambda)_T = 1$ for all $\lambda < 1$.

We now use the same trick again to show $\mathbb{E}E(1)_T = 1$. We have

$$E(\lambda)_T = \exp\left[\lambda^2 M_T - \frac{\lambda^2}{2}\langle M\rangle_T\right]\exp\left[(\lambda-\lambda^2)M_T\right].$$

Using Hölder's inequality with the exponents $\lambda^2 + 1 - \lambda^2 = 1$ we have

$$1 = \mathbb{E}E(\lambda)_T \le \left\{\mathbb{E}\exp\left[M_T - \frac{1}{2}\langle M\rangle_T\right]\right\}^{\lambda^2}\left\{\mathbb{E}\exp\left(\frac{\lambda}{1+\lambda}M_T\right)\right\}^{1-\lambda^2}.$$

Because $\lambda/(1+\lambda) \le 1/2$, the second expectation on the right side can be replaced by $\mathbb{E}\exp[M_T/2]$. Letting $\lambda \downarrow 0$ we obtain

$$\mathbb{E}\exp\left[M_T - \frac{1}{2}\langle M\rangle_T\right] = 1.$$

□

The condition $\mathbb{E}\exp[M_T/2] < \infty$ is not easy to verify because we usually know the quadratic variation $\langle M\rangle_T$ much better than M_T itself. The following weaker criterion can often be used directly.

COROLLARY 3.3. *(Novikov's criterion) Suppose that M is a martingale. If $\mathbb{E}exp\left[\langle M\rangle_T/2\right]$ is finite, then*

$$\mathbb{E}\exp\left[M_T - \frac{1}{2}\langle M\rangle_T\right] = 1.$$

PROOF. We have

$$\exp\left[\frac{1}{2}M_T\right] = \exp\left[\frac{1}{2}M_T - \frac{1}{4}\langle M\rangle_T\right]\exp\left[\frac{1}{4}\langle M\rangle_T\right].$$

By the Cauchy-Schwarz inequality we have

$$\mathbb{E}\exp\left[\frac{1}{2}M_T\right] \le \sqrt{\mathbb{E}\exp\left[M_T - \frac{1}{2}\langle M\rangle_T\right]}\sqrt{\mathbb{E}\exp\left[\frac{1}{2}\langle M\rangle_T\right]}.$$

The factor factor on the right side does not exceed 1. Therefore

$$\mathbb{E}\exp\left[\frac{1}{2}M_T\right] \le \sqrt{\mathbb{E}\exp\left[\frac{1}{2}\langle M\rangle_T\right]}.$$

Therefore Novikov's condition implies Kamazaki's condition. $\square$

4. Girsanov and Cameron-Martin-Maruyama theorems

In stochastic analysis, we often need to change the base probability measure from a given $\mathbb{P}$ to another measure $\mathbb{Q}$. Suppose that B is a Brownian motion under $\mathbb{P}$. In general it will no longer be Brownian motion under $\mathbb{Q}$. The Girsanov theorem describes the decomposition of B as the same of a martingale (necessarily a Brownian motion) and a process of bounded variation under a class of change of measures.

We assume that B is an $\mathscr{F}_*$-Brownian motion and V a progressively measurable process such that

$$\exp\left[\int_0^t V_s\, dB_s - \frac{1}{2}\int_0^t |V_s|^2\, ds\right],\ 0 \le t \le T$$

is a uniformly integrable martingale. This exponential martingale therefore defines a change of measure on $\mathscr{F}_T$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left[\int_0^T V_s\, dB_s - \frac{1}{2}\int_0^T |V_s|^2\, ds\right],\ 0 \le t \le T.$$

This is the class of changes of measures we will consider.

THEOREM 4.1. *Suppose that $\mathbb{Q}$ is the new measure described above. Consider the Brownian motion with a drift*

$$X_t = B_t - \int_0^t V_\tau d\tau, \quad 0 \le t \le T. \tag{4.1}$$

Then X is Brownian motion under the probability measure $\mathbb{Q}$.

PROOF. Let $\{e_s\}$ be the exponential martingale

$$e_s = \exp\left\{\int_0^s \langle V_\tau, dB_\tau\rangle - \frac{1}{2}\int_0^s |V_\tau|^2 d\tau\right\}.$$

Then the density function of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathscr{F}_s$ is just e_s. From this fact it is easy to verify that if Y is an adapted process then

$$\mathbb{E}^{\mathbb{Q}}\left\{Y_t|\mathscr{F}_s\right\} = e_s^{-1}\mathbb{E}^{\mathbb{P}}\left\{Y_t e_t|\mathcal{F}_s\right\}.$$

This means that Y is a (local) martingale under $\mathbb{Q}$ if and only if $eY = \{e_s Y_s,\ s \ge 0\}$ is a (local) martingale under $\mathbb{Q}$.

Now e is a martingale and $de_s = e_s V_s\, dB_s$. On the other hand, using Itô's formula we have

$$d(e_s X_s) = e_s\, dB_s + e_s X_s V_s\, dB_s.$$

This shows that eX is local martingale under $\mathbb{P}$, hence X is a local martingale under $\mathbb{Q}$. Since $\mathbb{Q}$ and $\mathbb{P}$ are mutually absolutely continuous, it should be clear that the quadratic variation process X under $\mathbb{P}$ and under $\mathbb{Q}$ should be the same, i.e., $\langle X\rangle_t = t$. We can also verify this fact directly by applying Itô's formula to $Z_s = e_s(X_s^2 - s)$. We have

$$dZ_s = e_s\, d(X_s^2 - s) + (X_s^2 - s)\, de_s + 2X_s d\langle e, X\rangle_s.$$

The second term on the right side is a martingale. We have

$$d(X_s^2 - s) = 2X_s\, dX_s - d\langle X, X\rangle_s - ds = 2X_s\, dB_s - 2X_s V_s\, ds.$$

On the other hand, from $de_s = e_s V_s\, dB_s$ we have

$$d\langle e, X\rangle_s = e_s V_s\, ds.$$

It follows that

$$dZ_s = 2e_s X_s\, dB_s + (X_s^2 - s)\, de_s,$$

which shows that Z is a local martingale. Now we have shown that both X_s and $X_s^2 - s$ are local martingales under $\mathbb{Q}$. By Lévy's criterion we conclude that X is a Brownian motion under $\mathbb{Q}$. $\square$

The classical Cameron-Martin-Maruyama theorem is a special case of the Girsanov theorem but stated in a slightly different form. Let μ be the Wiener measure on the path space $W(\mathbb{R})$. Let $h \in W(\mathbb{R})$ and consider the shift in the path space $\xi_h w = w + h$. The shifted Wiener measure is $\mu^h = \mu \circ \xi_h^{-1}$. If X is the coordinate process on $W(\mathbb{R})$, then μ^h is just the law of $X + h$. We will prove the following dichotomy: either μ^h and μ are mutually absolutely continuous or they are mutually singular. In fact we have an explicit criterion for this dichotomy.

DEFINITION 4.2. *A path (function) $h \in W(\mathbb{R})$ is called Cameron-Martin path (function) if it is absolutely continuous and its derivative is square integrable. The Cameron-Martin norm is defined by*

$$|h|_{\mathscr{H}}^2 = \int_0^1 |\dot{h}_s|^2\, ds.$$

The space of Cameron-Martin paths is denoted by $\mathscr{H}$.

It is clear that $\mathscr{H}$ is a Hilbert space.

THEOREM 4.3. *(Cameron-Martin-Maruyama theorem) Suppose that $h \in \mathbb{H}$. Then the shifted Wiener measure μ^h and μ are mutually absolutely continuous and*

$$\frac{d\mu^h}{d\mu}(w) = \exp\left[\int_0^1 \dot{h}_s\, dw_s - \frac{1}{2}\int_0^1 |\dot{h}_s|^2\, ds\right].$$

PROOF. Denote the exponential martingale by $e_1(w)$. We need to show that for nonnegative measurable function F on $W(\mathbb{R})$,

$$\mathbb{E}^{\mu^h}(F) = \mathbb{E}^{\mu}(F e_1).$$

Let X be the coordinate process on $W(\mathbb{R})$. Then the left side is simply $\mathbb{E}^{\mu}F(X+h)$. Introduce the measure ν by

$$\frac{d\nu}{d\mu} = \exp\left[-\int_0^1 \dot{h}_s\,dw_s - \frac{1}{2}\int_0^1 |\dot{h}_s|^2\,ds\right].$$

We have

$$\begin{aligned}\frac{d\mu}{d\nu} &= \exp\left[\int_0^1 \dot{h}_s\,dw_s + \frac{1}{2}\int_0^1 |\dot{h}_s|^2\,ds\right] \\ &= \exp\left[-\int_0^1 \dot{h}_s\,d(w_s+h) - \frac{1}{2}\int_0^1 |\dot{h}_s|^2\,ds\right].\end{aligned}$$

Hence we can write $d\mu/d\nu = e_1(X+h)$. It follows that

$$\mathbb{E}^{\mu^h}F = \mathbb{E}^{\mu}\left[F(X+h)\right] = \mathbb{E}^{\nu}\left[F(X+h)\frac{d\mu}{d\nu}\right] = \mathbb{E}^{\nu}\left[F(X+h)e_1(X+h)\right].$$

By Girsanov's theorem $X+h$ is a Brownian motion under ν. On the other hand, X is a Brownian motion under μ. Therefore on the right side of the above equality we can replace ν by μ and at the same time replace $X+h$ by X, hence

$$\mathbb{E}^{\mu^h}F = \mathbb{E}^{\mu}\left[F(X)e_1(X)\right] = \mathbb{E}^{\mu}(Fe_1).$$

□

THEOREM 4.4. *Let $h \in W(\mathbb{R})$. The shifted Wiener measure μ^h mutually absolutely continuous or mutually singular with respect to μ according as $h \in \mathscr{H}$ or $h \notin \mathscr{H}$.*

PROOF. We need to show that if $h \notin \mathscr{H}$, then μ^h is singular with respect to μ.

First we need to convert the condition $h \notin \mathscr{H}$ into a more convenient condition. A function $\dot{h}$ to be square integrable if and only if

$$\int_0^1 f_s\dot{h}_s\,ds \le C|f|_2$$

for some constant C. Therefore it is conceivable that if $h \notin \mathscr{H}$, then for any C, there is a step function f such that

$$|f|_2^2 = \sum_{i=1}^n |f_i|^2(s_i - s_{i-1}) = 1$$

and

$$\int_0^1 f_s\,dh_s = \sum_{i=1}^n f_i(h_{s_i} - h_{s_{i-1}}) \ge C.$$

This characterization of $h \notin \mathscr{H}$ can indeed be verified rigorously.

Second, the convenient characterization that μ^h is singular with respect to μ is the following: for any positive ϵ, there is a set A such that $\mu^h(A) \ge 1-\epsilon$ and $\mu(A) \le \epsilon$.

Consider the random variable

$$Z(w) = \int_0^1 f_s \, dw_s = \sum_{i=1}^n f_i(w_{s_i} - w_{s_{i-1}}).$$

It is a Gaussian random variable with mean zero and variance $|f|_2^1 = 1$. Therefore it has the standard Gaussian distribution $N(0,1)$. Let

$$A = \{Z \geq C/2\}.$$

We have $\mu(A) \leq \epsilon$ for sufficiently large C. On the other hand,

$$\mu^h(A) = \mu\left\{w \in W(\mathbb{R}) : Z(w+h) \geq \frac{C}{2}\right\}.$$

By the definition of $Z(w)$ we have

$$Z(w+h) = Z(w) + \sum_{i=1}^n f_i(h_{s_i} - h_{s_{i-1}}) \geq Z(w) + C.$$

Therefore

$$\mu^h(A) = \mu\left\{Z + C \geq \frac{C}{2}\right\} = \mu\left\{Z \geq -\frac{C}{2}\right\}$$

and $\mu^h(A) \geq 1 - \epsilon$ for sufficiently large C. Thus we have shown that μ^h and μ are mutually singular. $\square$

5. Moment inequalities for martingales

Let M be a continuous martingale and

$$M_t^* = \max_{0 \leq s \leq t} |M_s|.$$

The moment $\mathbb{E}\left[M_t^{*p}\right]$ can be bounded both from above and from below by $\mathbb{E}\left[\langle M, M\rangle_t^{p/2}\right]$.

THEOREM 5.1. *Let M be a continuous local martingale. For any $p > 0$, there are positive constants c_p, C_p such that*

$$c_p \mathbb{E}\left[\langle M, M\rangle^{p/2}\right] \leq \mathbb{E}\left[(M_t^*)^p\right] \leq C_p \mathbb{E}\left[\langle M, M\rangle_t^{p/2}\right].$$

PROOF. The case $p = 2$ is obvious. We only prove the case $p > 2$. The case $0 < p < 2$ is slightly more complicated, see Ikeda and Watanabe **[6]**.

By the usual stopping time argument we may assume without loss of generality that M is uniformly bounded, so there is no problem of integrability. We prove the upper bound first. We start with the Doob's submartingale inequality

$$\mathbb{E}\left[M_t^{*p}\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|M_t|^p\right]. \tag{5.1}$$

We use Itô's formula to $|M_t|^p$. Note that $x \mapsto |x|^p$ is twice continuously differentiable because $p > 2$. This gives

$$|M_t|^p = p\int_0^t |M_s|^{p-1}\mathrm{sgn}(M_s)\, dM_s + \frac{p(p-1)}{2}\int_0^t |M_s|^{p-2}\mathrm{sgn}(M_s)\, d\langle M\rangle_s.$$

Take expectation and using the obvious bound $|M_s| \le M_s^*$ we have

$$\mathbb{E}\left[|M_t|^p\right] \le \frac{p(p-1)}{2}\mathbb{E}\left[M_t^{*(p-2)}\langle M, M\rangle_t\right].$$

We use Hölder's inequality on the right side and (5.1) on the left side to obtain

$$\mathbb{E}\left[M_t^{*p}\right] \le C\left\{\mathbb{E}\left[M_t^{*p}\right]\right\}^{(p-2)/p}\left\{\mathbb{E}\left[\langle M, M\rangle_t^{p/2}\right]\right\}^{2/p},$$

where C is a constant depending on p. The upper bound follows immediately.

The lower bound is slightly trickier. Using Itô's formula we have

$$M_t\langle M, M\rangle_t^{(p-2)/4} = \int_0^t \langle M, M\rangle_s^{(p-2)/4} dM_s + \int_0^t M_s d\langle, M, M\rangle_s^{(p-2)/4}.$$

The first term on the right side is the term we are aiming at because its second moment is precisely the left side of the inequality we wanted to prove. This is the reason why choose the exponent $(p-2)/4$. We have

$$\left|\int_0^t \langle M, M\rangle_s^{(p-2)/4} dM_s\right| \le 2M_t^*\langle M, M\rangle_t^{(p-2)/4}.$$

Squaring the inequality and taking the expectation, we have after using Hölder's inequality,

$$\frac{2}{p}\mathbb{E}\left[\langle M, M\rangle_t^{p/2}\right] \le 4\left\{\mathbb{E}\left[M_t^{*p}\right]\right\}^{2/p}\left\{\mathbb{E}\left[\langle M, M\rangle_t^{p/2}\right]\right\}^{(p-2)/2}.$$

The lower bound follows immediately. □

6. Martingale representation theorem

Let B be a standard Brownian motion and $\mathscr{F}_t^B = \sigma\{B_s, s \le t\}$ be its associated filtration of σ–fields properly completed so that it satisfies the usual condition. Now suppose M is a square-integrable martingale with respect to this filtration. We will show it can always be represented as a stochastic integral with respect to the Brownian motion, namely, there exists a progressively measurable process H such that

$$M_t = \int_0^t H_s\, dB_s.$$

Thus every martingale with respect to the filtration generated by a Brownian motion is a stochastic integral with respect to this Brownian motion. This is a very important result in stochastic analysis. In this section we will give a proof of this result by an approach we believe is direct, short, and well motivated. It uses nothing more than Itô's formula in a very elementary way. In the next section we will discuss another approach to this useful theorem.

We make a few remarks before the proof. First of all, the martingale representation theorem is equivalent to the following representation theorem: if X is a square integrable random variable measurable with respect to $\mathscr{F}_T^B$, then it can be represented in the form

$$X = \mathbb{E}X + \int_0^T H_s \, dB_s,$$

for if we take $X = M_T$, then we have

$$M_T = \int_0^T H_s \, dB_s.$$

Since both sides are martingales, the equality must also holds if T is replaced by any $t \leq T$.

Second, the representation is unique because if

$$\int_0^T H_s \, dB_s = \int_0^T G_s \, dB_s,$$

then from

$$\mathbb{E} \left| \int_0^T H_s \, dB_s - \int_0^T G_s \, dB_s \right|^2 = \mathbb{E} \int_0^T |H_s - G_s|^2 \, ds$$

we have immediately $H = G$ on $[0, T] \times \Omega$ with respect to the measure $\mathbb{L} \times \mathbb{P}$, where $\mathbb{L}$ is the Lebesgue measure.

Third, if $X_n \to X$ in $L^2(\Omega, \mathscr{F}, \mathbb{P})$ and

$$X_n = \mathbb{E}X_n + \int_0^T H_s^n \, dB_s,$$

then $\mathbb{E}X_n \to \mathbb{E}X_n$ and

$$\int_0^T |H_s^m - H_s^n|^2 \, ds = \mathbb{E}|X_m - X_n|^2 \to 0.$$

This shows that $\{H^n\}$ is a Cauchy sequence in $L^2([0, T] \times \Omega, \mathbb{L} \times \mathbb{P})$. It therefore converges to a process H and

$$X = \mathbb{E}X + \int_0^T H_s \, dB_s.$$

The point here is that it is enough to show the representation theorem for a dense subset of random variables.

Finally we observe a square integrable martingale with respect to the filtration generated by a Brownian motion is necessarily continuous because every stochastic integral is continuous with respect to its upper time limit.

We may assume without loss generality that $T = 1$. Our method starts with a simple case $X = f(B_1)$ for a smooth bounded function f. It is easy to prove the theorem in this case and find the explicit formula for the process H.

PROPOSITION 6.1. *For any bounded smooth function f,*

$$f(W_1) = \mathbb{E}f(W_1) + \int_0^1 \mathbb{E}\left\{f'(W_1)|\mathscr{F}_s\right\} dW_s.$$

PROOF. We have $f(W_1) = f(W_t + W_1 - W_t)$. We know that $W_1 - W_t$ has the normal distribution $N(0, 1-t)$ and is independent of $\mathscr{F}_t$. Hence,

$$\mathbb{E}\left\{f(W_1)|\mathscr{F}_t\right\} = \frac{1}{\sqrt{2\pi(1-t)}} \int_{\mathbb{R}^1} f(W_t + x) e^{-|x|^2/2(1-t)} dx. \tag{6.1}$$

Now we regard the right side as a function of B_t and t and apply Itô's formula. Since we know that it is a martingale, we only need to find out its martingale part, which is very easy: just differentiate with respect to B_t and integrate the derivative with respect to B_t. We have

$$f(W_1) = \mathbb{E}f(W_1) + \int_0^1 \left[\frac{1}{\sqrt{2\pi(1-t)}} \int_{\mathbb{R}^1} f'(W_t + x) e^{-|x|^2/2(1-t)} dx\right] dW_s.$$

The difference between the integrand and the right side of (6.1) is simply that f is replaced by its derivative f'. This shows that

$$f(W_1) = \mathbb{E}f(W_1) + \int_0^1 \mathbb{E}\left\{f'(W_1)\mathscr{F}_t\right\}\, dt.$$

□

The general case can be handled by an induction argument.

THEOREM 6.2. *Let $X \in L^2(\Omega, \mathscr{F}_1, \mathbb{P})$. Then there is a progressively measurable process H such that*

$$X = \mathbb{E}\, X + \int_0^1 H_s\, dW_s.$$

PROOF. It is enough to show the representation theorem for random variables of the form

$$X = f(W_{s_1}, W_{s_2} - W_{s_1}, \cdots, W_{s_n} - W_{s_1}),$$

where f is a bounded smooth function with bounded first derivatives because the random variables of this type form a dense subset of $L^2(\Omega, \mathscr{F}_1, \mathbb{P})$ (see the remarks at the beginning of this section). By the induction hypothesis applied to the Brownian motion $\{W_s - W_{s_1},\, 0 \le s \le 1 - s_1\}$, we have

$$f(x, W_{s_2} - W_{s_1}, \cdots, W_{s_n} - W_{s_1}) = h(x) + \int_{s_1}^1 H_s^x dW_s,$$

where

$$h(x) = \mathbb{E}f(x, W_{s_2} - W_{s_1}, \cdots, W_{s_n} - W_{s_1}),$$

and the dependence of H^x on x is smooth. Hence, replacing x by W_{s_1} we have

$$X = h(W_{s_1}) + \int_{s_1}^{1} H_s \, dW_s,$$

where $H_s = H_s^{W_{s_1}}$. We also have

$$h(W_{s_1}) = \mathbb{E}\, h(W_{s_1}) + \int_0^{s_1} H_s \, dW_s.$$

It is clear that $\mathbb{E}\, h(W_{s_1}) = \mathbb{E}\, X$, hence

$$X = \mathbb{E} X + \int_0^1 H_s \, dW_s.$$

$\square$

We now give a general formula for the integrand in the martingale representation theorem for a wide class of random variables. Recall the definition of the Cameron-Martin space

$$\mathscr{H} = \left\{ h \in C[0,1] : h(0) = 0, \ \dot{h} \in L^2[0,1] \right\}$$

and the Cameron-Martin norm

$$|h|_{\mathscr{H}} = \sqrt{\int_0^1 |\dot{h}_s|^2 ds}.$$

A function $F : C[0,1] \to \mathbb{R}$ is called a cylinder function if it has the form

$$F(w) = f(w_{s_1}, \ldots, w_{s_l}),$$

where $0 < s_1 < \cdots < s_l \le 1$ and $f : \mathbb{R}^l \to \mathbb{R}$ is bounded smooth function. We also assume that f has bounded first derivatives and use the notation

$$F_{x_i}(w) = f_{x_i}(w_{s_1}, \ldots, w_{s_l}).$$

For a cylinder function F and $h \in C[0,1]$, the directional derivative along $h \in W(\mathbb{R})$ is defined by

$$D_h F(w) = \lim_{t \to 0} \frac{F(w + th) - F(w)}{t}.$$

If $h \in \mathscr{H}$, then it is easy to verify that

$$D_h F(w) = \langle DF(w), h \rangle_{\mathscr{H}},$$

where

$$DF(w)_s = \sum_{i=1}^{l} \min\{s, s_i\} \, F_{x_i}(w).$$

Note that the derivative

$$D_s F(w) = \frac{d\,\{DF(w)_s\}}{ds}$$

is given by

$$D_s F(w) = \sum_{i=1}^{l} F_{x_i}(w) I_{[0,s_i]}(s).$$

We have the following integration by parts formula.

THEOREM 6.3. *Let $H : \Omega \to \mathscr{H}$ be $\mathscr{F}_*$-adapted and $\mathbb{E} \exp \left\{ |H|^2_{\mathscr{H}}/2 \right\}$ is finite. Then the following integration by parts formula holds for a cylinder function F:*

$$\mathbb{E} D_H F = \mathbb{E} \langle DF, H \rangle = \mathbb{E} \left[F \int_0^1 \dot{H}_s \, dW_s \right].$$

PROOF. By Girsanov's theorem, under the probability $\mathbb{Q}$ defined by

$$\frac{d\,\mathbb{Q}}{d\,\mathbb{P}} = \exp \left[t \int_0^1 \dot{H}_s dW_s - \frac{t^2}{2} \int_0^1 |\dot{H}_s|^2 ds \right],$$

the process

$$X_s = W_s - tH_s, \qquad 0 \le s \le 1,$$

is a Brownian motion, hence $\mathbb{E}^{\mathbb{Q}} F(X) = \mathbb{E}\, F(W)$. This means that

$$\mathbb{E}^{\mathbb{Q}} F(X) = \mathbb{E} \left\{ F(W - tH) \exp \left[t \int_0^1 \dot{H}_s \, dW_s - \frac{t^2}{2} \int_0^1 |\dot{H}_s|^2 ds \right] \right\}$$

is independent of t. Differentiating with respect to t and letting $t = 0$, we obtain the integration by parts formula. □

The explicit martingale representation theorem is given by the following Clark-Ocone formula.

THEOREM 6.4. *Let F be a cylinder function on the path space $W(\mathbb{R})$. Then*

$$F(W) = \mathbb{E}\, F(W) + \int_0^1 \mathbb{E} \left\{ D_s F(W) | \mathscr{F}_s \right\} dW_s.$$

PROOF. By the martingale representation theorem we have

$$F(W) = \mathbb{E}\, F(W) + \int_0^1 \dot{H}_s \, dW_s,$$

where $\dot{H}$ is $\mathscr{F}_*$-adapted $\dot{H}$. By definition,

$$\mathbb{E}\, D_G F = \mathbb{E} \langle DF, G \rangle_{\mathscr{H}}.$$

By the integration by parts formula, the left side is

$$\begin{aligned} \mathbb{E}\, D_G F &= \mathbb{E} \left[F \int_0^1 \dot{G}_s \, dW_s \right] \\ &= \mathbb{E} \left[\int_0^1 \dot{H}_s \, dW_s \int_0^1 \dot{G}_s \, dW_s \right] \\ &= \mathbb{E} \int_0^1 \dot{H}_s \dot{G}_s \, ds. \end{aligned}$$

The right side is

$$\mathbb{E}\langle DF, G\rangle_{\mathscr{H}} = \mathbb{E}\int \dot{G}_s D_s F\, ds = \mathbb{E}\int_0^1 \mathbb{E}\left\{D_s F|\mathscr{F}_s\right\} \dot{G}_s\, ds.$$

Hence for all $\mathscr{F}_*$-adapted process $\dot{G}$ we have

$$\mathbb{E}\int_0^1 \mathbb{E}\left\{D_s F|\mathscr{F}_s\right\} \dot{G}_s\, ds = \mathbb{E}\int_0^1 \dot{H}_s \dot{G}_s\, ds.$$

Since $\mathbb{E}\left\{D_s F|\mathscr{F}_s\right\} - \dot{H}_s$ is also adapted, we must have

$$\dot{H}_s = \mathbb{E}\left\{D_s F|\mathscr{F}_s\right\}.$$

□

7. Reflecting Brownian motion

Let B be a one-dimensional Brownian motion starting from zero. The process $X_t = |B_t|$ is called reflecting Brownian motion. We have $X_t = F(B_t)$, where $F(x) = |x|$. We want to apply Itô's formula to $F(B_t)$, but unfortunately F is not C^2. We approximate F by

$$F_\epsilon(x) = \frac{1}{\epsilon}\int_0^x du_1 \int_0^{u_1} I_{[-\epsilon,\epsilon]}(u_2)\, du_2.$$

The above function is still not C^2 because $F''_\epsilon = I_{[-\epsilon,\epsilon]}/\epsilon$, which is not continuous, but it is clear that $F_\epsilon(x) \to |x|$ and $F'_\epsilon(x) \to \operatorname{sgn}(x)$ as $\epsilon \to 0$. Now let ϕ be a continuous function and define

$$F_\phi(x) = \int_0^x du_1 \int_0^{u_1} \phi(u_2)\, du_2.$$

Itô's formula can be applied to $F_\phi(B_t)$ and we obtain

$$F_\epsilon(B_t) = \int_0^t F'_\phi(B_s)\, dB_s + \frac{1}{2}\int_0^t \phi(B_s) ds.$$

Now for a fixed ϵ we let ϕ in the above formula to be the continuous function

$$\phi_n(x) = \begin{cases} 0, & \text{if } |x| \ge \epsilon + n^{-1}, \\ \epsilon^{-1}, & \text{if } |x| \le \epsilon, \\ \text{linear}, & \text{in the two remaining intervals.} \end{cases}$$

Then $F_{\phi_n} \to F_\epsilon$, and $F'_{\phi_n} \to F'$. It follows that we have

$$F_\epsilon(B_t) = \int_0^t F'(B_s)\, dB_s + \frac{1}{2\epsilon}\int_0^t I_{[-\epsilon,\epsilon]}(B_s)\, ds.$$

In the last term,

$$\int_0^t I_{[-\epsilon,\epsilon]}(B_s)\, ds$$

is the amount of time Brownian paths spends in the interval $[-\epsilon, \epsilon]$ up to time t. Now let $\epsilon \to 0$ in the above identity for $F_\epsilon(B_t)$. The term on the left side converges to $|B_t|$ and the first term on the right side converges to the stochastic integral

$$\int_0^t \operatorname{sgn}(B_s)\, dB_s.$$

Hence the limit

$$L_t = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t I_{(-\epsilon,\epsilon)}(B_s)\, ds$$

must exist and we have

$$|B_t| = \int_0^t \operatorname{sgn}(B_s)\, dB_s + \frac{1}{2} L_t.$$

We see that L_t can be interpreted as the amount of time Brownian motion spends in the interval $(-\epsilon, \epsilon)$ properly normalized. It is called the local time of Brownian motion B at $x = 0$. Let

$$W_t = \int_0^t \operatorname{sgn}(B_s)\, dB_s.$$

Then W is a continuous martingale with quadratic variation process

$$\langle W, W \rangle_t = \int_0^t |\operatorname{sgn}(B_s)|^2 ds = t.$$

Note that Brownian motion spends zero amount of time at $x = 0$ because $\mathbb{E} I_{\{0\}}(B_s) = \mathbb{P}\{B_s = 0\} = 0$ and

$$\mathbb{E} \int_0^t I_{\{0\}}(B_s)\, ds = \int_0^t \mathbb{E} I_{\{0\}}(B_s)\, ds = 0.$$

We thus conclude that reflecting Brownian motion $|B_t|$ is submartingale with the decomposition

$$|B_t| = W_t + \frac{1}{2} L_t.$$

It is interesting to note that W can be expressed in terms of reflecting Brownian motion by

$$W_t = X_t - \frac{1}{2} L_t,$$

where

$$L_t = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t I_{[0,\epsilon]}(X_s)\, ds. \tag{7.1}$$

We now pose the question: Can X and L be expressed in terms of W? That the answer to this question is affirmative is the content of the so-called Skorokhod problem.

DEFINITION 7.1. *Given a continuous path $f : \mathbb{R}_+ \to \mathbb{R}$ such that $f(0) \geq 0$. A pair of functions (g, h) is the solution of the Skorokhod problem if*

(1) *$g(t) \geq 0$ for all $t \geq 0$;*
(2) *h is increasing from $h(0) = 0$ and increases only when $g = 0$;*
(3) *$g = f + h$.*

The main result is that the Skorokhod problem can be solved uniquely and explicitly.

THEOREM 7.2. *There exists a unique solution to the Skorokhod equation.*

PROOF. It is interesting that the solution can be written down explicitly:

$$h(t) = -\min_{0\le s\le t} f(s)\wedge 0, \qquad g(t) = f(t) - \min_{0\le s\le t} f(s)\wedge 0.$$

Let's assume that $f(0) = 0$ for simplicity. If $f(0) > 0$, then $h(t) = 0$ and $g(t) = f(t)$ before the first time f reaches 0 and after this time it is as if the path starts from 0. The explicit solution in this case is

$$h(t) = \min_{0\le s\le t} f(s), \qquad g(t) = f(t) - \min_{0\le s\le t} f(s).$$

It is clear that $g(t) \ge 0$ for all t and h increases starting from $h(0) = 0$. The equation $f = g + h$ is also obvious. We only need to show that h increases only when $g(t) = 0$. This means that as a Borel measure h only charges the zero set $\{t : g(t) = 0\}$. This requirement is often written as

$$h(t) = \int_0^t I_{\{0\}}(g(s))\, dh(s).$$

Equivalently, it is enough to show that for any t such that $g(t) > 0$ there is a neighborhood $(t-\delta, t+\delta)$ of t such that h is constant on there. This should be clear, for if $g(t) > 0$, then $f(t) > \min_{0\le s\le t} f(s)$, which means that the minimum must be achieved at a point $\xi \in [0,t)$ and $f(t) > f(\xi)$. By continuity a small change of t will not alter this situation, which means that $h = f(\xi)$ in a neighborhood of t. More precisely, from $g(t) = f(t) - \min_{0\le s\le t} f(s) > 0$ and the continuity of f, there is a positive δ such that

$$\min_{t-\delta\le s\le t+\delta} f(s) > \min_{0\le s\le t-\delta} f(s).$$

Hence

$$\min_{0\le s\le t+\delta} f(s) = \min\left\{\min_{0\le s\le t-\delta} f(s), \min_{t-\delta\le s\le t+\delta} f(s)\right\} = \min_{0\le s\le t-\delta} f(s).$$

This means that $h(t+\delta) = h(t-\delta)$, which means that h must constant on $(t-\delta, t+\delta)$ because h is increasing.

We now show that the solution to the Skorokhod problem is unique. Suppose that (g_1, h_1) and let $\xi = h - h_1$. It is continuous and of bounded variation, hence

$$\xi(t)^2 = 2\int_0^t \xi(s)\, d\xi(s).$$

On the other hand, $\xi(s) = g(s) - g_1(s)$, hence

$$\xi(t)^2 = 2\int_0^t \{g(s) - g_1(s)\}\, d\{h(s) - h_1(s)\}.$$

There are four terms on the right side: $g(s)\, dh(s) = g_1(s)\, dh_1(s) = 0$ because h increases on when $g = 0$ and h_1 increases only when $g_1 = 0$;

$g(s)\,dh_1(s) \geq 0$ and $g_1(s)\,dh(s) \geq 0$ because $g(s) \geq 0$ and $g_1(s) \geq 0$. Putting these observations together we have $\xi(t)^2 \leq 0$, which means that $\xi(t) = 0$. This proves the uniqueness. □

If we apply Skorokhod equation to Brownian motion by replacing f with Brownian paths, we obtain some interesting results. We have shown that

$$|B_t| = W_t + \frac{1}{2}L_t,$$

where W is a Brownian motion. From the solution of the Skorokhod problem we conclude that $|B|$ and L are determined by W:

$$|B_t| = W_t - \min_{0\leq s\leq t} W_s, \quad \frac{1}{2}L_t = -\min_{0\leq s\leq t} W_s. \tag{7.2}$$

THEOREM 7.3. *Let W be a Brownian motion. (1) The processes*

$$\left\{\max_{0\leq s\leq t} W_s - W_t,\ t\geq 0\right\} \quad \text{and} \quad \{|W_t|,\ t\geq 0\}$$

have the same law, i.e, that of a reflecting Brownian motion. (2) We have

$$\max_{0\leq s\leq t} W_s = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_0^t I_{[0,\epsilon]}\big(\max_{0\leq u\leq s} W_u - W_s\big)\,ds.$$

PROOF. The assertions follow immediately from (7.2) by replacing W with $-W$, which is also a Brownian motion; in the second assertion, we use the fact that L is the normalized occupation time of reflecting Brownian motion (7.1). □

REMARK 7.4. We have calculated the joint distribution of $\max_{0\leq s\leq t} W_s$ and $|W_t$ for a fixed t and we know that $\max_{0\leq s\leq t} W_s - W_s$ and $|W_t|$ have the same distribution for each fixed t. The above theorem claims much more: they have the same distribution as two stochastic processes.

8. Brownian bridge

If we condition a Brownian motion to return to a fixed point x at time $t=1$ we obtain Brownian bridge from o to x with time horizon 1. Let

$$L_x(\mathbb{R}^n) = \{w \in W_o(\mathbb{R}^n) : w_1 = x\}.$$

The law of a Brownian bridge from o to x in time 1 is a probability measure μ_x on $L_x(\mathbb{R}^n)$, which we will call the Wiener measure on $L_x(\mathbb{R}^n)$. Note that $L_x(\mathbb{R}^n)$ is a subspace of $W_o(\mathbb{R}^n)$, thus μ_x is also a measure on $W_o(\mathbb{R}^n)$. By definition, we can write intuitively

$$\mu_x(C) = \mu\{C|w_1 = x\}.$$

Here μ is the Wiener measure on $W_o(\mathbb{R}^n)$. The meaning of this suggestive formula is as follows. If F is a nice function measurable with respect to $\mathscr{B}_s$ with $s<1$ and f a measurable function on $\mathbb{R}^n$, then

$$\mathbb{E}^\mu\{Ff(X_1)\} = \mathbb{E}^\mu\{\mathbb{E}^{\mu_{X_1}}(F)f(W_1)\},$$

where W denotes the coordinate process on $W_o(\mathbb{R}^n)$. Using the Markov property at time s we have

$$\mathbb{E}^{\mu}\left[F\int_{\mathbb{R}^n} p(1-s, W_s, y) f(y)\, dy\right] = \int_{\mathbb{R}^n} \mathbb{E}^{\mu_y}(F) p(1, o, y) f(y) dy,$$

where

$$p(t, y, x) = \left(\frac{1}{2\pi t}\right)^{n/2} e^{-|y-x|^2/2t}$$

is the transition density function of Brownian motion X. This being true for all measurable f, we have for all $F \in \mathscr{B}_s$,

$$\mathbb{E}^{\mu_x} F = \mathbb{E}^{\mu}\left[\frac{p(1-s, W_s, x)}{p(1, o, x)} F\right]. \tag{8.1}$$

Therefore μ_x is absolutely continuous with respect to μ on $\mathscr{F}_s$ for any $s < 1$ and the Radon-Nikodym density is given by

$$\left.\frac{d\mu_x}{d\mu}\right|_{\mathscr{F}_s}(w) = \frac{p(1-s, w_s, x)}{p(1, o, x)} = e_s.$$

The process $\{e_s, 0 \le s < 1\}$ is a necessarily a positive (local) martingale under the probability μ. It therefore must have the form of an exponential martingale, which can be found explicitly by computing the differential of $\log e_s$. The density function $p(t, y, x)$ satisfies the heat equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\Delta_y p$$

in (t, y) for fixed x. This equation gives

$$\Delta_y \log p = \frac{\partial \log p}{\partial t} - |\nabla_y \log p|^2.$$

Using this fact and Iô's formula we find easily that

$$d\log e_s = \langle \nabla \log p(1-s, w_s, x), dw_s\rangle - \frac{1}{2}|\nabla p(1-s, w_s, x)|^2 ds.$$

Hence e_s is an exponential martingale of the form

$$\left.\frac{d\mu_o}{d\mu}\right|_{\mathscr{B}_s} = \exp\left[\int_0^s \langle V_u, dw_u\rangle - \frac{1}{2}\int_0^s |V_u|^2 du\right],$$

where

$$V_s = \nabla_y \log p(1-s, w_s, x).$$

By Girsanov's theorem, under probability μ_x, the process

$$B_s = W_s - \int_0^s \nabla \log p(1-\tau, W_\tau, x) d\tau, \qquad 0 \le s < 1$$

is a Brownian motion. The explicit formula for $p(t, y, x)$ gives

$$\nabla_y \log p(1-\tau, y, x) = -\frac{y-x}{1-\tau}.$$

Under the probability μ_x the coordinate process W is a reflecting Brownian motion from o to x in time 1. Therefore we have shown that reflecting Brownian motion is the solution to the following stochastic differential equation

$$dW_s = dB_s - \frac{W_s - x}{1-s}\, ds.$$

This simple equation can be solved explicitly. From the equation we have

$$d(W_s - x) + \frac{W_s - x}{1-s}\, ds = dB_s.$$

The left side after dividing by $1-s$ is the differential of $(W_s - x)/(1-s)$, hence

$$W_s = sx + (1-s)\int_0^s \frac{dB_u}{1-u}.$$

This formula shows that, like Brownian motion itself, Brownian bridge is a Gaussian procss.

The term "Brownian bridge" is often reserved specifically for Brownian bridge which returns to its starting point at the terminal time. In this case we have

$$dW_s = dB_s - \frac{W_s}{1-s}\, ds$$

and

$$W_s = (1-s)\int_0^s \frac{dB_u}{1-u}.$$

For dimension $n = 1$ it is easy verify that the covariance function is given by

$$\mathbb{E}\left\{W_s W_t\right\} = \min\left\{s, t\right\} - st.$$

Using this fact we obtain another representation of Brownian bridge.

THEOREM 8.1. *If $\{B_s\}$ is a Brownian motion, then the process*

$$W_s = B_s - sB_1$$

is a Brownian bridge.

PROOF. ($n = 1$) Verify directly that W defined above has the correct covariance function. □

The following heuristic discussion of $W_s = B_s - sB_1$ is perhaps more instructive. Let $\mathscr{F}_*$ be the filtration of the Brownian motion B. We enlarge the filtration to

$$\mathscr{G}_s = \sigma\left\{\mathscr{F}_s, B_1\right\}.$$

We compute the Doob-Meyer decomposition of W with respect to $\mathscr{G}_*$. Of course the martingale part is a Brownian motion because its quadratic variation process will be the same has that of W. Denote this Brownian motion by Ω. Doob's explicit decomposition formula for a semimartingale suggest that

$$W_s = \Omega_s + \int_0^s \mathbb{E}\left\{dW_s | \mathscr{G}_s\right\}.$$

We have $dW_s = dB_s - B_1 ds$. The differential $dW_s = W_{s+ds} - W_s$ is forward differential. We need to project dW_s to the L^2-space generated by

$$\mathscr{G}_s = \sigma\left\{B_u,\, u \le s; B_1\right\} = \left\{B_u,\, u \le s; B_1 - B_s\right\}.$$

Note that $\sigma\left\{B_u,\, u \le s\right\}$ is orthogonal to $B_1 - B_s$. The differential dB_s is orthogonal to the first part and its projection to the second part is

$$\frac{dB_s \cdot (B_1 - B_s)}{1-s} \cdot (B_1 - B_s) = \frac{B_1 - B_s}{1-s}\, ds.$$

The differential $B_1 ds$ is of course in the target space already, hence

$$\mathbb{E}\left\{dW_s | \mathscr{G}_s\right\} = \frac{B_1 - B_s}{1-s}\, ds - B_1\, ds = -\frac{B_s - sB_1}{1-s}\, ds = -\frac{W_s\, ds}{1-s}.$$

It follows that

$$W_s = \Omega_s - \int_0^s \frac{W_\tau}{1-\tau} d\tau,$$

which is exactly the stochastic differential equation for a reflecting Brownian motion.

9. Fourth assignment

EXERCISE 4.1. Let ϕ be a strictly convex function. If both N and $\phi(N)$ are continuous local martingales then N is trivial, i.e., there is a constant C such that $N_t = 1$ with probability 1 for all t.

EXERCISE 4.2. Let M be a continuous local martingale. Show that there is a sequence of partitions $\Delta_1 \subset \Delta_2 \subset \Delta_3 \subset \cdots$ such that $|\Delta_n| \to 0$ and with probability 1 the following holds: for all $t \ge 0$,

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} \left(M_{t_i^n \wedge t} - M_{t_{i-1}^n \wedge t} \right)^2 = \langle M, M \rangle_t.$$

EXERCISE 4.3. Let B be the standard Brownian motion. Then the reflecting Brownian motion $X_t = |B_t|$ is a Markov process. This means

$$\mathbb{P}\left\{X_{t+s} \in C | \mathscr{F}_s^X\right\} = \mathbb{P}\left\{X_{t+s} \in C | X_s\right\}.$$

What is its transition density function

$$q(t,x,y) = \frac{\mathbb{P}\left\{X_{t+s} \in dy | X_s = x\right\}}{dy}?$$

EXERCISE 4.4. Let L_t be the local time of Brownian motion at $x = 0$. Show that

$$\mathbb{E} L_t = \sqrt{\frac{8t}{\pi}}.$$

EXERCISE 4.5. Show that Brownian bridge from o to x in time 1 is a Markov process with transition density is

$$q(s_1, y; s_2, z) = \frac{p(s_2 - s_1, y, z)\, p(1 - s_2, z, x)}{p(1 - s_1, y, x)}.$$

EXERCISE 4.6. The finite-dimensional marginal density for Brownian bridge from o to x in time 1 is

$$p(1,o,x)^{-1}\prod_{i=0}^{l} p(s_{i+1}-s_i, x_i, x_{i+1}).$$

[Convention: $x_0 = o, s_0 = 0$.]

EXERCISE 4.7. Let $\{W_s,\ 0 \le s \le 1\}$ be a Brownian bridge at o. Then the reversed process

$$\{W_{1-s}, 0 \le s \le 1\}$$

is also a Brownian bridge.

EXERCISE 4.8. For an $a > 0$ define the stopping time

$$\sigma_a = \inf\{t:\ B_t - t = -a\}.$$

Show that for $\mu \ge 0$,

$$\mathbb{E}e^{-\mu\sigma_a} = e^{-(\sqrt{1+2\mu}-1)a}.$$

EXERCISE 4.9. Use the result of the previous exercise to show that $\mathbb{E}e^{\mu\sigma_a}$ is infinite if $\mu > 1/2$.

EXERCISE 4.10. By the martingale representation theorem there is a process H such that

$$W_1^3 = \int_0^1 H_s\, dW_s.$$

Fin an explicit expression for H.