现代数学物理方法

第三章，李群

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December 13, 2017

Outline

Simple Roots :

Definition :

Simple roots are those positive root vectors that cannot be written as a sum of other positive root vectors.

Properties of Simple Roots :

- If $\vec{\alpha}$ and $\vec{\beta}$ are different simple roots, then $(\vec{\alpha}-\vec{\beta})$ is not a root vector.

  Proof : Let $\vec{\beta}$ be the larger so that $(\vec{\beta}-\vec{\alpha})>0$. The assumption that $\vec{\alpha}$ and $\vec{\beta}$ are simple roots and the fact

  $$\vec{\beta}=\vec{\alpha}+(\vec{\beta}-\vec{\alpha})$$

  indicate that $(\vec{\beta}-\vec{\alpha})$ is not a positive root vector.

- The angle $\theta_{\alpha\beta}$ between any pair of simple roots $\vec{\alpha}$ and $\vec{\beta}$ satisfies the constraint,

  $$\frac{\pi}{2}\leqslant\theta_{\alpha\beta}<\pi\ .$$

Proof : Consider two distinct simple roots $\vec{\alpha}$ and $\vec{\beta}$. Because $(\vec{\alpha}-\vec{\beta})$ is not a root vector, in the adjoint representation, we have:

$$E_{-\alpha}\left|E_\beta\right\rangle = E_{-\beta}\left|E_\alpha\right\rangle = 0.$$

Then, in the root vector chains $\{\vec{\beta}+n\vec{\alpha} \mid -q \leqslant n \leqslant p\}$ and $\{\vec{\alpha}+n'\vec{\beta} \mid -q' \leqslant n' \leqslant p'\}$, $q=q'=0$. The master formula between these two simple roots gives,

$$\frac{2\vec{\alpha}\cdot\vec{\beta}}{\alpha^2} = -p \leqslant 0, \quad \frac{2\vec{\beta}\cdot\vec{\alpha}}{\beta^2} = -p' \leqslant 0,$$

where p, p' are two nonnegative integers. Hence, $\cos\theta_{\alpha\beta} \leqslant 0$. Accurately, by combining the above two equations we get:

$$\cos\theta_{\alpha\beta} = -\sqrt{\frac{\vec{\alpha}\cdot\vec{\beta}}{\alpha^2}\cdot\frac{\vec{\beta}\cdot\vec{\alpha}}{\beta^2}} = -\frac{1}{2}\sqrt{pp'} \leqslant 0$$

Besides, the largest angle between any two positive root vectors cannot take values beyond π. As a result,

$$\frac{\pi}{2} \leqslant \theta_{\alpha\beta} < \pi \ .$$

- The simple roots are linearly independent from one another.

Proof : Consider a linear combination of the simple roots,

$$\vec{\gamma} = \sum_{\alpha} x_\alpha \vec{\alpha}$$

If all of the non-vanishing coefficients x_i have the same sign, $\vec{\gamma} \neq 0$. If there are some coefficients of each sign, we can write,

$$\vec{\gamma} = \vec{\mu} + \vec{\nu}$$

where $\vec{\mu} = \sum_\alpha x_\alpha \vec{\alpha}$ with all $x_\alpha > 0$, and $\vec{\nu} = \sum_\beta x_\beta \vec{\beta}$ with all $x_\beta < 0$. Relying on the fact $\frac{\pi}{2} \leqslant \theta_{\alpha\beta} < \pi$, $\vec{\alpha} \cdot \vec{\beta} \leqslant 0$.So,

$$\vec{\mu} \cdot \vec{\nu} = \sum_{\alpha, x_\alpha > 0} \sum_{\beta, x_\beta < 0} x_\alpha x_\beta \vec{\alpha} \cdot \vec{\beta} \geqslant 0.$$

From this we see,

$$\vec{\gamma}^2 = (\vec{\mu} + \vec{\nu})^2 = \vec{\mu}^2 + \vec{\nu}^2 + 2\vec{\mu} \cdot \vec{\nu} > 0 \ .$$

$\vec{\gamma} = 0$ is possible iff all coefficients x_α vanish. In conclusion, the simple roots are linearly independent of one another.

- Any positive root vector $\vec{\phi}$ can be written as a linear combination of all simple roots with non-negative integer coefficients k_α,

$$\vec{\phi} = \sum_{\alpha, k_\alpha \geqslant 0} k_\alpha \vec{\alpha}$$

Corolleries :

1. The simple roots are not only linearly independent of each other only, they are also complete.
2. Because the root vector space has dimension m, *the rank of the Lie algebra* $\mathfrak{g}$, the number of simple roots is equal to m (the rank of the algebra), which is also the number of Cartan generators.

Question :

How to determine all the root vectors of an algebra $\mathfrak{g}$?

- It is only necessary to find out all positive root vectors,

$$\vec{\phi}_k = \sum_{\alpha, k_\alpha \geqslant 0} k_\alpha \vec{\alpha}$$

where $\vec{\alpha}$ stands for simple roots and $k = \sum_\alpha k_\alpha$.

- All of the $\vec{\phi}_1$'s are roots because they are just the simple roots.
- Suppose we have determined the positive roots $\vec{\phi}_k$ for $k \leqslant n$. To find out $\{\vec{\phi}_{n+1}\}$, for all simple roots $\{\vec{\alpha}\}$, we consider the states

$$E_\alpha \left| E_{\phi_n} \right\rangle$$

in $\mathfrak{g}$'s adjoint representation. These states are related to the possible roots $\{\vec{\phi}_{n+1}\}$ of the form

$$\{\vec{\phi}_{n+1}\} = \{\vec{\phi}_n\} + \vec{\alpha}$$

Question :

Is $\{\vec{\phi}_{n+1}\}$ really a root ?

- $\{\vec{\phi}_{n+1}\}$ being a root means that $E_\alpha \left| E_{\phi_n} \right\rangle$ is a true state in the adjoint representation of the Lie algebra $\mathfrak{g}$.

From the perspective of accessory $su(2)$ (related to the simple root $\vec{\alpha}$),

$$E_3 = \alpha^{-2} \vec{\alpha} \cdot \vec{H}, \quad E_\pm = \alpha^{-1} E_{\pm\alpha},$$

this means that there must be a positive integer p such that,

$$(E_\alpha)^p \left| E_{\phi_n} \right\rangle \neq 0, \quad (E_\alpha)^{p+1} \left| E_{\phi_n} \right\rangle = 0.$$

Similarly, there must exist another non-negative integer q such that,

$$(E_{-\alpha})^q |E_{\phi_n}\rangle \neq 0, \quad (E_{-\alpha})^{q+1} |E_{\phi_n}\rangle = 0.$$

Claiming that these states form the spin-j representation of the above accessory $su(2)$, we have in $\mathfrak{g}$'s adjoint representation,

$$(E_{-\alpha})^q |E_{\phi_n}\rangle = |j, -j\rangle_{su(2)}, \quad (E_\alpha)^p |E_{\phi_n}\rangle = |jj\rangle_{su(2)}.$$

So,

$$\begin{aligned} -j(E_{-\alpha})^q |E_{\phi_n}\rangle &= E_3(E_{-\alpha})^q |E_{\phi_n}\rangle \\ &= \alpha^{-2}\alpha_i H_i (E_{-\alpha})^q |E_{\phi_n}\rangle \\ &= \alpha^{-2}(\vec{\alpha}\cdot\vec{\phi_n} - q\alpha^2)(E_{-\alpha})^q |E_{\phi_n}\rangle \end{aligned}$$

and

$$\begin{aligned} j(E_{\alpha})^p |E_{\phi_n}\rangle &= E_3(E_{\alpha})^p |E_{\phi_n}\rangle \\ &= \alpha^{-2}\alpha_i H_i (E_{\alpha})^p |E_{\phi_n}\rangle \\ &= \alpha^{-2}(\vec{\alpha}\cdot\vec{\phi_n} + p\alpha^2)(E_{\alpha})^p |E_{\phi_n}\rangle \end{aligned}$$

Hence,

$$\frac{\vec{\alpha}\cdot\vec{\phi_n}}{\alpha^2} + p = j, \quad \frac{\vec{\alpha}\cdot\vec{\phi_n}}{\alpha^2} - q = -j.$$

Summation of these two equations gives,

$$\frac{2\vec{\alpha}\cdot\vec{\phi_n}}{\alpha^2} = q - p$$

Warning !

The significance of equation $\frac{2\vec{\alpha}\cdot\vec{\phi_n}}{\alpha^2} = q - p$:

- The equation is used to determine the integer p.
  We always know q, because we know the history of how $\vec{\phi_n}$ got built up by the action of the raising operators from $\vec{\phi_k}$ with the smaller k.
- If $p > 0$, $\vec{\phi_n} + \vec{\alpha}$ is a (positive) root vector.

Example 1 : Suppose $\vec{\alpha}$ and $\vec{\beta}$ are two simple roots of a Lie algebra. Is $\vec{\alpha} + \vec{\beta}$ a root vector ?

Solution : Take $\vec{\phi_1} = \vec{\beta}$. Because $\vec{\alpha}$ and $\vec{\beta}$ are simple roots,

$$E_{-\alpha}\left|E_{\phi_1}\right\rangle = 0.$$

Comparing with $(E_{-\alpha})^{q+1}\left|E_{\phi_1}\right\rangle = 0$, we see that $q = 0$. So,

$$\frac{2\vec{\alpha}\cdot\vec{\phi_1}}{\alpha^2} = \frac{2\vec{\alpha}\cdot\vec{\beta}}{\alpha^2} = -p$$

If $\frac{2\vec{\alpha}\cdot\vec{\beta}}{\alpha^2} = 0$, $\theta_{\alpha\beta} = \pi/2$, $p = 0$, $\vec{\beta} + \vec{\alpha}$ is not a root vector. If $\frac{2\vec{\alpha}\cdot\vec{\beta}}{\alpha^2} < 0$, $\pi/2 < \theta_{\alpha\beta} < \pi$, $p > 0$, $\vec{\beta} + \vec{\alpha}$ is a positive root.

Example 2 : The $su(3)$ algebra has rank 2. So among its 3 positive roots of $\vec{\alpha_1} = (1/2, \sqrt{3}/2)$, $\vec{\alpha_2} = (1/2, -\sqrt{3}/2)$ and $\vec{\alpha_3} = (1, 0)$, there are only 2 simple roots. Because

$$\vec{\alpha_3} = \vec{\alpha_1} + \vec{\alpha_2}$$

$\vec{\alpha_1}$ and $\vec{\alpha_2}$ are the expected simple roots of $su(3)$ algebra.

Question : Is $(\vec{\alpha_2} + 2\vec{\alpha_1})$ a root vector of $su(3)$?

Solution : Construct $SU(2)$ generators from the generators related to the simple root $\vec{\alpha_1}$,

$$E_\pm = \alpha_1^{-1} E_{\pm\alpha_1} = E_{\pm\alpha_1}, \qquad E_3 = \alpha_1^{-2} \vec{\alpha_1} \cdot \vec{H} = \vec{\alpha_1} \cdot \vec{H},$$

where we have noticed that

$$\alpha_1^2 = \alpha_2^2 = 1, \quad \vec{\alpha_1} \cdot \vec{\alpha_2} = -1/2.$$

Now focus on $(\vec{\alpha_2} + 2\vec{\alpha_1}) = \vec{\alpha_3} + \vec{\alpha_1}$:

$$\frac{2\vec{\alpha_3} \cdot \vec{\alpha_1}}{\alpha_1^2} = 2\vec{\alpha_3} \cdot \vec{\alpha_1} = 1 = q - p, \qquad \rightsquigarrow \quad q - p = 1.$$

On the other hand,
$\vec{\alpha_3} - \vec{\alpha_1} = \vec{\alpha_2}$ is a root but $\vec{\alpha_3} - 2\vec{\alpha_1} = \vec{\alpha_2} - \vec{\alpha_1}$ is not.
This implies $q = 1$.
So, $p = 0$. $\vec{\alpha_3} + \vec{\alpha_1} = 2\vec{\alpha_2} + \vec{\alpha_1}$ is not a $su(3)$ root vector.

Constructing Lie algebra :

Background :

1. The basis states of the adjoint representation space have a one-to-one correspondence with the generator,

$$T_a \Leftrightarrow |T_a\rangle , \qquad T_a |T_b\rangle = |[T_a,\ T_b]\rangle$$

Thus, knowing the states in adjoint representation enable us to obtain the Lie algebra

$$[T_a,\ T_b] = i f_{abc} T_c$$

itself.

2. There is also a one-to-one correspondence between root vectors and the non-Cartan generators. Therefore, in adjoint representation, each root vector $\vec{\beta}$ corresponds uniquely to a basis state $|E_\beta\rangle$.

3. Associated with a simple root $\vec{\alpha}$, we can define an accessory $su(2)_\alpha$ subalgebra,

$$E_\pm = \alpha^{-1} E_{\pm\alpha} , \quad E_3 = \alpha^{-2} \vec{\alpha} \cdot \vec{H} .$$

Some of the states $\{|E_\beta\rangle\}$ will form a spin-j representation of this $su(2)_\alpha$,

$$j = \frac{1}{2}(p + q)$$

where p, q are two integers, determined by

$$(E_-)^{q+1}|E_\beta\rangle = 0, \quad \frac{2\vec{\beta}\cdot\vec{\alpha}}{\alpha^2} = q - p.$$

Notice that,

$$E_3|E_\beta\rangle = \frac{\vec{\beta}\cdot\vec{\alpha}}{\alpha^2}|E_\beta\rangle$$

The state $|E_\beta\rangle$ can be recast as a standard $su(2)_\alpha$ form $|jm\rangle$,

$$|E_\beta\rangle = |j, \frac{\vec{\beta}\cdot\vec{\alpha}}{\alpha^2}\rangle$$

In this way, the knowledge of $su(2)$ enable us to know exactly how $E_\pm$ act (up to a phase).

Remark :

This procedure will enable us to determine $[E_\alpha,\ E_\beta] = \mathcal{N}_{\alpha\beta}E_{\alpha+\beta}$ and then the whole algebra.

Constructing $su(3)$:

Now we illustrate the above procedure by constructing the $su(3)$ algebra from the knowledge of its simple roots.

Starting point : The algebra $su(3)$ has 2 simple roots $\vec{\alpha_1}$ and $\vec{\alpha_2}$,

$$\vec{\alpha_1} = (1/2, \sqrt{3}/2), \quad \vec{\alpha_2} = (1/2, -\sqrt{3}/2).$$

Evidently, $\alpha_1^2 = \alpha_2^2 = 1$, $\vec{\alpha_1} \cdot \vec{\alpha_2} = -1/2$.

$su(2)_{\alpha_1}$: We construct a $su(2)_{\alpha_1}$ algebra $\{E_\pm = E_{\pm\alpha_1}, E_3 = \vec{\alpha_1} \cdot \vec{H}\}$ based on simple root $\vec{\alpha_1}$. Since $[E_{-\alpha_1}, \ E_{\alpha_2}] = 0$, in adjoint representation, we have:

$$0 = |[E_{-\alpha_1}, \ E_{\alpha_2}]\rangle = E_{-\alpha_1} |E_{\alpha_2}\rangle = E_- |E_{\alpha_2}\rangle$$

i.e., $q = 0$. Together with $(q - p) = 2\vec{\alpha_2} \cdot \vec{\alpha_1}/\alpha_1^2 = -1$ we see $p = 1$, $j = (p + q)/2 = 1/2$. So, in $su(2)_{\alpha_1}$ language, $|E_{\alpha_2}\rangle$ can be written as

$$|E_{\alpha_2}\rangle = \left| j, \frac{\vec{\alpha_2} \cdot \vec{\alpha_1}}{\alpha_1^2} \right\rangle_{\alpha_1} = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_{\alpha_1}$$

Consequently,

$$\left|[E_{\alpha_1},\ E_{\alpha_2}]\right\rangle = E_{\alpha_1}\left|E_{\alpha_2}\right\rangle = E_+\left|\frac{1}{2},-\frac{1}{2}\right\rangle_{\alpha_1} = \frac{1}{\sqrt{2}}\left|\frac{1}{2},\frac{1}{2}\right\rangle_{\alpha_1}$$

On the other hand, in adjoint representation, the state $\left|E_{\alpha_3}\right\rangle$ related to the positive root vector $\vec{\alpha_3} = \vec{\alpha_1} + \vec{\alpha_2}$ satisfies,

$$E_3\left|E_{\alpha_3}\right\rangle = \vec{\alpha_1}\cdot\vec{\alpha_3}\left|E_{\alpha_3}\right\rangle = \frac{1}{2}\left|E_{\alpha_3}\right\rangle$$

i.e.,

$$\left|E_{\alpha_3}\right\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle_{\alpha_1}$$

The consistency between the above results implies that,

$$\left|[E_{\alpha_1},\ E_{\alpha_2}]\right\rangle = \frac{1}{\sqrt{2}}\left|E_{\alpha_3}\right\rangle$$

i.e.,

$$[E_{\alpha_1},\ E_{\alpha_2}] = \frac{1}{\sqrt{2}}E_{\alpha_3}$$

For $su(3)$, the other Lie brackets can be calculated by using Jacobi identities. e.g,

$$\begin{aligned}[E_{-\alpha_1},\ E_{\alpha_3}] &= \sqrt{2}[E_{-\alpha_1},\ [E_{\alpha_1},\ E_{\alpha_2}]] \\ &= -\sqrt{2}[E_{\alpha_1},\ [E_{\alpha_2},\ E_{-\alpha_1}]] - \sqrt{2}[E_{\alpha_2},\ [E_{-\alpha_1},\ E_{\alpha_1}]] \\ &= \sqrt{2}\alpha_{1i}[E_{\alpha_2},\ H_i] \\ &= -\sqrt{2}(\vec{\alpha_1}\cdot\vec{\alpha_2})E_{\alpha_2} = \frac{1}{\sqrt{2}}E_{\alpha_2}\end{aligned}$$

i.e.,

$$[E_{-\alpha_1},\ E_{\alpha_3}] = \frac{1}{\sqrt{2}}E_{\alpha_2}$$

Similarly (Please check it yourself),

$$[E_{-\alpha_2},\ E_{\alpha_3}] = -\frac{1}{\sqrt{2}}E_{\alpha_1}$$

By taking the hermitian conjugation of above commutation relations, we further get

$$\begin{aligned}&[E_{\alpha_1},\ E_{-\alpha_2}] = 0, & &[E_{-\alpha_1},\ E_{-\alpha_2}] = -\frac{1}{\sqrt{2}}E_{-\alpha_3}, \\ &[E_{\alpha_1},\ E_{-\alpha_3}] = -\frac{1}{\sqrt{2}}E_{-\alpha_2}, & &[E_{\alpha_2},\ E_{-\alpha_3}] = \frac{1}{\sqrt{2}}E_{-\alpha_1}.\end{aligned}$$

Defintions :

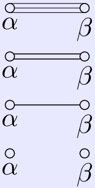
Cartan Matrix A : Let $\{\vec{\alpha_i}\}$ be simple roots of a Lie algebra $\mathfrak{g}$, its Cartan matrix is defined as,

$$A = (A_{ij}), \quad A_{ij} = \frac{2\vec{\alpha_i} \cdot \vec{\alpha_j}}{\alpha_j^2}$$

Dynkin Diagrm : A Dykin diagram is a short-hand notation for writing down the simple roots.

Rules :

1. Each simple root is expressed as an open or solid circle.
2. Pairs of circles are connected by lines, depending on the angle between the pair of roots to which the circles correspond $(\pi/2 \leqslant \theta_{\alpha\beta} < \pi)$:



$$\theta_{\alpha\beta} = 5\pi/6$$

$$\theta_{\alpha\beta} = 3\pi/4$$

$$\theta_{\alpha\beta} = 2\pi/3$$

$$\theta_{\alpha\beta} = \pi/2$$

Meaning of Cartan Matrix A_{ij} :

Let $\{\vec{\alpha_i}\}$ be simple roots of a Lie algebra $\mathfrak{g}$. The accessory $su(2)$ generators related to simple root $\vec{\alpha_j}$ are

$$E_3 = \alpha_j^{-2}\vec{\alpha_j} \cdot \vec{H}, \qquad E_\pm = \alpha_j^{-1} E_{\pm\alpha_j}.$$

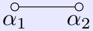
Therefore, in $\mathfrak{g}$'s adjoint representation, on the state $|E_{\alpha_i}\rangle$ related to some simple root $\vec{\alpha_i}$,

$$E_3 \, |E_{\alpha_i}\rangle = \frac{\vec{\alpha_i} \cdot \vec{\alpha_j}}{\alpha_j^2} \, |E_{\alpha_i}\rangle = \frac{A_{ij}}{2} \, |E_{\alpha_i}\rangle \,,$$

i.e., A_{ij} is twice of the eigenvalue of E_3 on state $|E_{\alpha_i}\rangle$.

Example : $su(3)$'s Dynkin diagram and Cartan matrix:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$



$$\theta_{\alpha_1\alpha_2} = 2\pi/3$$

G_2 :

Example: G_2

The algebra G_2 has 2 simple roots,

$$\vec{\alpha_1} = (0, 1), \qquad \vec{\alpha_2} = (\sqrt{3}/2, -3/2).$$

Obviously,

$$(\alpha_1)^2 = 1, \quad (\alpha_2)^2 = 3, \quad \vec{\alpha_1} \cdot \vec{\alpha_2} = -3/2.$$

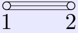
The Cartan matrix is,

$$A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

The angle θ_{12} between two simple roots is calculated through,

$$\cos\theta_{12} = \frac{\vec{\alpha_1} \cdot \vec{\alpha_2}}{\alpha_1 \alpha_2} = -\sqrt{3}/2 \qquad \rightsquigarrow \quad \theta_{12} = \frac{5\pi}{6}.$$

G_2's Dynkin diagram is:



$\theta_{12} = 5\pi/6$

The roots of G_2 :

Starting point :

We now search for all positive root vectors of G_2 algebra based on the its simple roots $\{\phi_1\}$,

$$\vec{\alpha_1} = (0,1), \qquad \vec{\alpha_2} = (\sqrt{3}/2, -3/2), \qquad\qquad (k=1).$$

Finding $\{\phi_2\}$:

Is $\vec{\alpha_1} + \vec{\alpha_2}$ a positive root vector of $k = 2$?

To answer this question, we examine the properties of states $E_{\pm\alpha_1}|E_{\alpha_2}\rangle$ in G_2's adjoint representation. Construct an accessory $su(2)$ algebra based on simple root $\vec{\alpha_1}$,

$$E_3 = \alpha_1^{-2}\vec{\alpha_1}\cdot\vec{H}, \qquad E_\pm = \alpha_1^{-1}E_{\pm\alpha_1}.$$

We claim that the states $E_{\pm\alpha_1}|E_{\alpha_2}\rangle$ are in the spin-j representation of this $su(2)_{\alpha_1}$. Because $(\vec{\alpha_1} - \vec{\alpha_2})$ is not a root, we have

$$E_{-\alpha_1}|E_{\alpha_2}\rangle = 0, \qquad \rightsquigarrow \qquad |E_{\alpha_2}\rangle = |j,-j\rangle_{\alpha_1}$$

So,

$$-j\,|E_{\alpha_2}\rangle = E_3\,|E_{\alpha_2}\rangle = \frac{1}{2}A_{21}\,|E_{\alpha_2}\rangle = -\frac{3}{2}\,|E_{\alpha_2}\rangle$$

i.e., $j = 3/2$ and

$$|E_{\alpha_2}\rangle = |3/2, -3/2\rangle_{\alpha_1}$$

Assuming

$$(E_{\alpha_1})^p |E_{\alpha_2}\rangle \neq 0, \quad (E_{\alpha_1})^{p+1} |E_{\alpha_2}\rangle = 0,$$

i.e.,

$$(E_+)^p |3/2, -3/2\rangle_{\alpha_1} = |3/2, 3/2\rangle_{\alpha_1}$$

This gives that $p = 3$ (> 0). Therefore, $\vec{\phi_2} = (\vec{\alpha_1} + \vec{\alpha_2})$ is a root vector of G_2 with $k = 2$.

Corollaries : Relying on the facts,

$$(E_{\alpha_1})^3 |E_{\alpha_2}\rangle \neq 0, \quad (E_{\alpha_1})^4 |E_{\alpha_2}\rangle = 0,$$

the algebra G_2 has the following positive root vectors as well,

$$\begin{cases} \vec{\alpha_2} + 2\vec{\alpha_1}, & k = 3; \\ \vec{\alpha_2} + 3\vec{\alpha_1}, & k = 4. \end{cases}$$

Finding $\{\phi_3\}$:

We have found out a positive root vector of $k = 3$: $\vec{\alpha_2} + 2\vec{\alpha_1}$. The remaining candidate is then unique, which is $\vec{\alpha_1} + 2\vec{\alpha_2}$.
We define another accessory $su(2)$ related to the simple root $\vec{\alpha_2}$,

$$E'_3 = \alpha_2^{-2}\vec{\alpha_2} \cdot \vec{H}, \qquad E'_{\pm} = \alpha_2^{-1} E_{\pm\alpha_2}.$$

Notice that $\vec{\alpha_1} + 2\vec{\alpha_2} = (\vec{\alpha_1} + \vec{\alpha_2}) + \vec{\alpha_2}$. In adjoint representation of G_2, assume that

$$(E'_+)^{p'} |\alpha_1 + \alpha_2\rangle \neq 0, \quad (E'_+)^{p'+1} |\alpha_1 + \alpha_2\rangle = 0,$$

and

$$(E'_-)^{q'} |\alpha_1 + \alpha_2\rangle \neq 0, \quad (E'_-)^{q'+1} |\alpha_1 + \alpha_2\rangle = 0.$$

Because the difference between two simple roots is not a root vector,

$$(E_{-\alpha_2})^2 |\alpha_1 + \alpha_2\rangle = 0, \qquad \rightsquigarrow \quad q' = 1.$$

Besides,

$$(q' - p') = \frac{2\vec{\alpha_2} \cdot (\vec{\alpha_1} + \vec{\alpha_2})}{\alpha_2^2} = 2 + A_{12} = 1, \qquad \rightsquigarrow \quad p' = 0.$$

As a result, $\vec{\alpha_1} + 2\vec{\alpha_2}$ is not a root vector of G_2.

Finding $\{\phi_4\}$:

G_2 has a unique positive root vector of $k = 4$, which is the one founded previously,

$$\vec{\phi_4} = \vec{\alpha_2} + 3\vec{\alpha_1}.$$

Finding $\{\phi_5\}$:

There is a unique candidate for the positive root vector of $k = 5$,

$$\vec{\phi}_5 = 2\vec{\alpha}_2 + 3\vec{\alpha}_1 = (\vec{\alpha}_2 + 3\vec{\alpha}_1) + \vec{\alpha}_2.$$

Is it really a root vector of G_2 ?

As before, in G_2's adjoint representation, assume that

$$(E'_+)^{p''} |3\alpha_1 + \alpha_2\rangle \neq 0, \quad (E'_+)^{p''+1} |3\alpha_1 + \alpha_2\rangle = 0,$$

and

$$(E'_-)^{q''} |3\alpha_1 + \alpha_2\rangle \neq 0, \quad (E'_-)^{q''+1} |3\alpha_1 + \alpha_2\rangle = 0.$$

Because the integer multiple of a simple root is not a root vector,

$$E_{-\alpha_2} |3\alpha_1 + \alpha_2\rangle = 0, \qquad \rightsquigarrow \qquad q'' = 0.$$

Furthermore,

$$(q'' - p'') = \frac{2\vec{\alpha}_2 \cdot (3\vec{\alpha}_1 + \vec{\alpha}_2)}{\alpha_2^2} = 2 + 3A_{12} = -1, \qquad \rightsquigarrow \qquad p'' = 1.$$

Hence, $(2\vec{\alpha}_2 + 3\vec{\alpha}_1)$ is a true positive root vector of G_2 with $k = 5$.

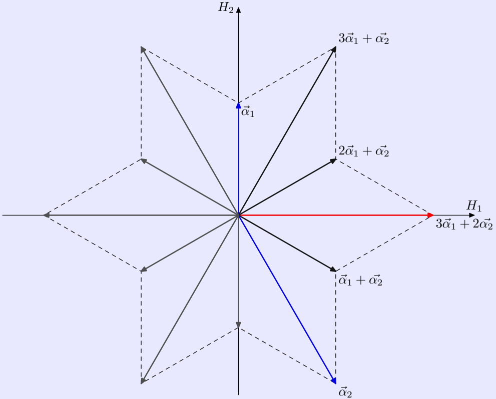
It is easy to know that G_2 has no more positive roots $\vec{\phi}_k$ with $k \geqslant 6$.

In conclusion, G_2 has 12 non-zero root vectors. They are listed as

$$\pm\vec{\alpha_1} = (0, \pm 1), \qquad \pm\vec{\alpha_2} = (\pm\sqrt{3}/2, \mp 3/2),$$

and $\pm(\vec{\alpha_1} + \vec{\alpha_2})$, $\pm(2\vec{\alpha_1} + \vec{\alpha_2})$, $\pm(3\vec{\alpha_1} + \vec{\alpha_2})$ and $\pm(3\vec{\alpha_1} + 2\vec{\alpha_2})$.

In weight diagram,



Constructing G_2 :

Generators :

$$\begin{array}{llll} H_1, & H_2, & & \\ E_{\pm\alpha_1}, & E_{\pm\alpha_2}, & & \\ E_{\pm(\alpha_1+\alpha_2)}, & E_{\pm(2\alpha_1+\alpha_2)}, & E_{\pm(3\alpha_1+\alpha_2)}, & E_{\pm(3\alpha_1+2\alpha_2)}. \end{array}$$

Two $su(2)$ subalgebras based on simple roots :

1. $su(2)_{\alpha_1}$: $\quad E_3 = \vec{\alpha_1} \cdot \vec{H}, \quad E_\pm = E_{\pm\alpha_1}.$
2. $su(2)_{\alpha_2}$: $\quad E_3' = \frac{1}{3}\vec{\alpha_2} \cdot \vec{H}, \quad E_\pm' = \frac{1}{\sqrt{3}} E_{\pm\alpha_2}.$

Construction procedure :

Step 1 :

Obviously,

$$[E_{\alpha_1}, E_{-\alpha_2}] = [E_{-\alpha_1}, E_{\alpha_2}] = 0.$$

Step 2 :

Starting from the state $|E_{\alpha_2}\rangle$ in G_2's adjoint representation. For $su(2)_{\alpha_1}$, this state has:

$$q = 0, \quad p = 3, \quad j = (p+q)/2 = 3/2.$$

In the standard notation of $su(2)_{\alpha_1}$ representation, we rewrite this state as,

$$|E_{\alpha_2}\rangle = |3/2, -3/2\rangle_{\alpha_1}$$

Hence,

$$|[E_{\alpha_1},\ E_{\alpha_2}]\rangle = E_{\alpha_1}|E_{\alpha_2}\rangle = E_+|3/2, -3/2\rangle_{\alpha_1} = \sqrt{\frac{3}{2}}|3/2, -1/2\rangle_{\alpha_1}$$

Ignoring the possible phase factor, we define:

$$|E_{\alpha_1+\alpha_2}\rangle = |3/2, -1/2\rangle_{\alpha_1}$$

Consequently,

$$[E_{\alpha_1},\ E_{\alpha_2}] = \sqrt{\frac{3}{2}}E_{\alpha_1+\alpha_2}$$

- It is better to regard this commutator as the definition of generator $E_{\alpha_1+\alpha_2}$.

Applying E_+ once more gives,

$$\begin{aligned}|[E_{\alpha_1},\ [E_{\alpha_1},\ E_{\alpha_2}]]\rangle &= E_{\alpha_1}\,|[E_{\alpha_1},\ E_{\alpha_2}]\rangle = \sqrt{\tfrac{3}{2}}E_{\alpha_1}\,|E_{\alpha_1+\alpha_2}\rangle \\ &= \sqrt{\tfrac{3}{2}}E_+\,|3/2,-1/2\rangle_{\alpha_1} \\ &= \sqrt{3}\,|3/2,1/2\rangle_{\alpha_1}\end{aligned}$$

Defining:

$$|E_{\alpha_2+2\alpha_1}\rangle = |3/2,1/2\rangle_{\alpha_1}$$

Then,

$$E_{\alpha_2+2\alpha_1} = \frac{1}{\sqrt{3}}[E_{\alpha_1},\ [E_{\alpha_1},\ E_{\alpha_2}]]$$

Repeating this procedure, we get,

$$\begin{aligned}|[E_{\alpha_1},\ [E_{\alpha_1},\ [E_{\alpha_1},\ E_{\alpha_2}]]]\rangle &= E_{\alpha_1}\,|[E_{\alpha_1},\ [E_{\alpha_1},\ E_{\alpha_2}]\rangle = \sqrt{3}E_{\alpha_1}\,|E_{\alpha_2+2\alpha_1}\rangle \\ &= \sqrt{3}E_+\,|3/2,1/2\rangle_{\alpha_1} \\ &= \tfrac{3}{\sqrt{2}}\,|3/2,3/2\rangle_{\alpha_1}\end{aligned}$$

Defining:

$$|E_{\alpha_2+3\alpha_1}\rangle = |3/2,3/2\rangle_{\alpha_1}$$

Then,

$$E_{\alpha_2+3\alpha_1} = \frac{\sqrt{2}}{3}[E_{\alpha_1},\ [E_{\alpha_1},\ [E_{\alpha_1},\ E_{\alpha_2}]]]$$

Step 3 :

In view of $su(2)_{\alpha_2}$, the state $|E_{\alpha_2+3\alpha_1}\rangle$ in G_2's adjoint representation has the properties,

$$0 = E_{-\alpha_2}|E_{\alpha_2+3\alpha_1}\rangle \backsimeq E'_-|E_{\alpha_2+3\alpha_1}\rangle,$$
$$0 = (E_{\alpha_2})^2|E_{\alpha_2+3\alpha_1}\rangle \backsimeq (E'_+)^2|E_{\alpha_2+3\alpha_1}\rangle.$$

we see,

$$q' = 0, \qquad p' = 1, \qquad j' = (p' + q')/2 = 1/2$$

i.e.,

$$|E_{\alpha_2+3\alpha_1}\rangle = |1/2, -1/2\rangle_{\alpha_2}$$

Consequently,

$$\begin{aligned} |[E_{\alpha_2},\ E_{\alpha_2+3\alpha_1}]\rangle &= E_{\alpha_2}|E_{\alpha_2+3\alpha_1}\rangle = \sqrt{3}E'_+|E_{\alpha_2+3\alpha_1}\rangle \\ &= \sqrt{3}E'_+|1/2, -1/2\rangle_{\alpha_2} \\ &= \sqrt{\tfrac{3}{2}}|1/2, 1/2\rangle_{\alpha_2} \end{aligned}$$

Defining:

$$|E_{3\alpha_1+2\alpha_2}\rangle = |1/2, 1/2\rangle_{\alpha_2}$$

we get,

$$\begin{aligned} E_{3\alpha_1+2\alpha_2} &= \sqrt{\tfrac{2}{3}}[E_{\alpha_2},\ E_{\alpha_2+3\alpha_1}] \\ &= \frac{2}{3\sqrt{3}}[E_{\alpha_2},\ [E_{\alpha_1},\ [E_{\alpha_1},\ [E_{\alpha_1},\ E_{\alpha_2}]]]] \end{aligned}$$

The above are enough for determining all the commutation relations of G_2. For example,

$$\begin{aligned}
[E_{-\alpha_1},\ E_{\alpha_1+\alpha_2}] &= \sqrt{\tfrac{2}{3}}[E_{-\alpha_1},\ [E_{\alpha_1},\ E_{\alpha_2}]] \\
&= -\sqrt{\tfrac{2}{3}}[E_{\alpha_2},\ [E_{-\alpha_1},\ E_{\alpha_1}]] \\
&= \sqrt{\tfrac{2}{3}}\alpha_{1i}[E_{\alpha_2},\ H_i] \\
&= -\sqrt{\tfrac{2}{3}}(\vec{\alpha_1}\cdot\vec{\alpha_2})E_{\alpha_2} \\
&= \sqrt{\tfrac{3}{2}}E_{\alpha_2}
\end{aligned}$$

Highest weights representation D:

Let $\{\vec{\alpha_i}\ |i=1,2,\cdots,m\}$ be the simple roots of a simple Lie algebra $\mathfrak{g}$. Consider an irreducible representation D of $\mathfrak{g}$, in which there is a state $|M\rangle$ satisfying,

$$E_{\alpha_i}|M\rangle = 0, \quad H_i|M\rangle = M_i|M\rangle$$

where $\vec{M}=(M_1,M_2,\cdots,M_m)$ is the weight vector related to $|M\rangle$.

Properties of $\vec{M}$:

- $\vec{M}$ is the highest weight vector in Representation D.
- There must exist some non-negative integers $\{l_i\}$ so that,

$$\frac{2\vec{M}\cdot\vec{\alpha_i}}{\alpha_i^2} = l_i$$

[$\{l_i\}$ are called Dynkin coefficients.]

Fundamental Weights :

Definition : The fundamental weights $\{\vec{M}_i\}$ of a simple Lie algebra $\mathfrak{g}$ is defined by,

$$\frac{2\vec{M}_i \cdot \vec{\alpha_j}}{\alpha_j^2} = \delta_{ij}, \qquad (i, j = 1, 2, \cdots, m.)$$

Properties of $\{\vec{M}_i\}$:

- Each $\vec{M}_i$ defines an irreducible representation of $\mathfrak{g}$, in which $\vec{M}_i$ is the highest weight vector.
- $\#\vec{M}_i = m$ (rank of $\mathfrak{g}$).
- The highest weight vectors $\{\vec{M}_i\}$ are called the fundamental weights of $\mathfrak{g}$. The cooresponding irreducible representations are called the fundamental representation.
- The highest weight vector $\vec{M}$ of an arbitrary irreducible representation D can be expressed as

$$\vec{M} = \sum_i l_i \vec{M}_i$$

or equivalently,

$$\vec{M} = (l_1, l_2, \cdots, l_m).$$

- The highest weight state $|M\rangle$ in an irreducible representation D is unique.

Proof : Obviously, if

$$H_i\,|M\rangle = M_i\,|M\rangle\,, \qquad H_i\,|M\rangle' = M_i\,|M\rangle'\,,$$

there will be some positive root vectors $\{\vec{\alpha}, \vec{\beta}, \cdots\}$ so that

$$|M\rangle' = E_\alpha \cdots E_\beta E_{-\alpha} \cdots E_{-\beta}\,|M\rangle\,.$$

It is enough to consider $\{\vec{\alpha}, \vec{\beta}, \cdots\}$ as the simple roots here, because

$$E_{\alpha+\beta} = [E_\alpha,\ E_\beta]/\mathcal{N}_{\alpha,\beta}$$

Hence, these two highest weight states are actually the same one:

$$|M\rangle' = (\vec{\alpha}\ \cdot\ \vec{M})\cdots(\vec{\beta}\ \cdot\ \vec{M})\,|M\rangle\,.$$

Homework :

1. Consider the algebra C_3 corresponding to the following Dynkin diagram. Let $\alpha_1^2 = \alpha_2^2 = 1$ and $\alpha_3^2 = 2$. Find the Cartan matrix A and all of the positive root vectors.