LECTURE 7: LINEAR LIE GROUPS

1. The general linear group

Recall that $M(n,\mathbb{R})$, the set of all $n \times n$ real matrices, is diffeomorphic to $\mathbb{R}^{n^2}$.

Definition 1.1. A *linear Lie group*, or *matrix Lie group*, is a submanifold of $M(n,\mathbb{R})$ which is also a Lie group, with group structure the matrix multiplication.

Let's begin with the "largest" linear Lie group, the *general linear group*

$$\mathrm{GL}(n,\mathbb{R}) = \{X \in M(n,\mathbb{R}) \mid \det X \neq 0\}.$$

Since the determinant map is continuous, $\mathrm{GL}(n,\mathbb{R})$ is open in $M(n,\mathbb{R})$ and thus a submanifold. Moreover, $\mathrm{GL}(n,\mathbb{R})$ is closed under the group multiplication and inversion operations, so it is a Lie group. Obviously $\mathrm{GL}(n,\mathbb{R})$ is an n^2-dimensional noncompact Lie group, and it is not connected. In fact, it consists of exactly two connected components,

$$\mathrm{GL}_+(n,\mathbb{R}) = \{X \in M(n,\mathbb{R}) \mid \det X > 0\}$$

and

$$\mathrm{GL}_-(n,\mathbb{R}) = \{X \in M(n,\mathbb{R}) \mid \det X < 0\}.$$

The fact that $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $M(n,\mathbb{R}) \simeq \mathbb{R}^{n^2}$ also implies that the Lie algebra of $\mathrm{GL}(n,\mathbb{R})$, as the tangent space at $e = I_n$, is the set $M(n,\mathbb{R})$ itself, i.e.

$$\mathfrak{gl}(n,\mathbb{R}) = \{A \mid A \text{ is an } n \times n \text{ real matrix}\}.$$

To figure out the Lie bracket operation, we take a matrix $A = (A_{ij})_{n\times n} \in \mathfrak{g}$, and take the global coordinate system by (X^{ij}). Then the corresponding tangent vector at $T_{I_n}\mathrm{GL}(n,\mathbb{R})$ is $\sum A_{ij}\frac{\partial}{\partial X^{ij}}$, and the corresponding left-invariant vector on G at the matrix $X = (X^{ij})$ is $\sum X^{ik}A_{kj}\frac{\partial}{\partial X^{ij}}$. It follows that the Lie bracket $[A,B]$ between matrices $A, B \in \mathfrak{g}$ is the matrix corresponding to

$$\begin{aligned}\left[\sum X^{ik}A_{kj}\frac{\partial}{\partial X^{ij}}, \sum X^{pq}B_{qr}\frac{\partial}{\partial X^{pr}}\right] &= \sum X^{ik}A_{kj}B_{jr}\frac{\partial}{\partial X^{ir}} - \sum X^{pq}B_{qr}A_{rj}\frac{\partial}{\partial X^{pj}} \\ &= \sum X^{ik}\left(A_{kr}B_{rj} - B_{kr}A_{rj}\right)\frac{\partial}{\partial X^{ij}}.\end{aligned}$$

In other words, the Lie bracket operation on $\mathfrak{g}$ is the matrix commutator

$$[A,B] = AB - BA.$$

Given any $A \in \mathfrak{gl}(n,\mathbb{R})$, we can define the matrix exponential

$$e^A = I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \cdots.$$

It is easy to check that the series converges, and

$$e^{sA}e^{tA} = e^{(s+t)A}.$$

Notice that $e^{0A} = I_n$, we have $(e^{tA})^{-1} = e^{-tA}$. In particular, $e^{tA} \in \mathrm{GL}(n, \mathbb{R})$. So e^{tA} is a one-parameter subgroup of $\mathrm{GL}(n, \mathbb{R})$. Since $\frac{d}{dt}|_{t=0} e^{tA} = A$, we conclude that the exponential map $\exp : \mathfrak{gl}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ is

$$\exp(A) = e^A = I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots .$$

Note that exp is not surjective, not even surjective to $\mathrm{GL}_+(n, \mathbb{R})$.

Remark. Alternatively, one can first show that e^{tA} is the one-parameter subgroup of $\mathrm{GL}(n, \mathbb{R})$ generated by $A \in \mathfrak{gl}(n, \mathbb{R})$, and thus the Lie bracket structure on $\mathfrak{gl}(n, \mathbb{R})$ is

$$[A, B] = \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} (e^{tA}e^{sB}e^{-tA}) = \left.\frac{d}{dt}\right|_{t=0} (e^{tA}Be^{-tA}) = AB - BA.$$

2. Lie subgroups of $\mathrm{GL}(n, \mathbb{R})$

Before we continue to study other linear Lie groups, we need some general results on Lie subgroups. More details and proofs of these results will be discussed later.

Definition 2.1. A subgroup H of a Lie group G is called a *Lie subgroup* if it is a Lie group (with respect to the induced group operation), and the inclusion map $\iota_H : H \hookrightarrow G$ is an immersion (and therefore a Lie group homomorphism).

Suppose H is a Lie subgroup of G, and $\mathfrak{h}$ be the Lie algebra of H. Since $\iota : H \hookrightarrow G$ is an immersion and is a Lie group homomorphism, $d\iota_H : \mathfrak{h} \to \mathfrak{g}$ is injective and is a Lie algebra homomorphism. So we can think of $\mathfrak{h}$ as a *Lie subalgebra* (= a linear subspace that is closed under the Lie bracket) of $\mathfrak{g}$. Note that a one-parameter subgroup of H is automatically a one-parameter of G (with initial vector in T_eH), so the exponential map $\exp_H : \mathfrak{h} \to H$ is exactly the restriction of $\exp_G : \mathfrak{g} \to G$ onto $\mathfrak{h}$. The following theorem, which we will prove later, is very useful in determine the Lie algebra of a Lie subgroup.

Theorem 2.2. *Suppose H is a Lie subgroup of G. Then as a Lie subalgebra of $\mathfrak{g}$,*

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp_G(tX) \in H \text{ for all } t \in \mathbb{R}\}.$$

Now we are ready to study other linear Lie groups. By definition they are Lie subgroups of $\mathrm{GL}(n, \mathbb{R})$.

Example. **(The special linear group)** The special linear group is defined as

$$\mathrm{SL}(n, \mathbb{R}) = \{X \in \mathrm{GL}(n, \mathbb{R}) : \det X = 1\}.$$

It is easy see that $\mathrm{SL}(n, \mathbb{R})$ is a subgroup of $\mathrm{GL}(n, \mathbb{R})$. In PSET 1 we have seen that $\mathrm{SL}(n, \mathbb{R})$ is an $n^2 - 1$ dimensional submanifold of $\mathrm{GL}(n, \mathbb{R})$. It follows that $\mathrm{SL}(n, \mathbb{R})$ is a (connected non-compact) Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$.

To determine its Lie algebra $\mathfrak{sl}(n,\mathbb{R})$, we notice that

$$\det e^A = e^{\mathrm{Tr}(A)}.$$

So for an $n \times n$ matrix A, $e^A \in \mathrm{SL}(n,\mathbb{R})$ if and only if $\mathrm{Tr}(A) = 0$. We conclude

$$\mathfrak{sl}(n,\mathbb{R}) = \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid \mathrm{Tr}(A) = 0\}.$$

Example. **(The orthogonal group)** Next let's consider the *orthogonal group*

$$\mathrm{O}(n) = \{X \in \mathrm{GL}(n,\mathbb{R}) : X^T X = I_n\}.$$

This is another subgroup of $\mathrm{GL}(n,\mathbb{R})$. In PSET 1 we have seen that $\mathrm{O}(n)$ is an $\frac{n(n-1)}{2}$ dimensional submanifold of $GL(n,\mathbb{R})$. So $\mathrm{O}(n)$ is an $\frac{n(n-1)}{2}$ dimensional Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$. Note that $\mathrm{O}(n)$ is compact.

To figure out its Lie algebra $\mathfrak{o}(n)$, we note that $(e^A)^T = e^{A^T}$, so

$$(e^{tA})^T e^{tA} = I_n \Longleftrightarrow e^{tA^T} = e^{-tA}$$

For all t. Since the exponential map is locally bijective, we conclude that $A \in \mathfrak{o}(n)$ if and only if $A^T = -A$. So

$$\mathfrak{o}(n) = \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid A^T + A = 0\},$$

which is the space of $n \times n$ skew-symmetric matrices.

Notice that $\mathrm{O}(n)$ is not connected. It consists of two connected components, and the connected component of identity is the called the *special orthogonal groups*

$$\mathrm{SO}(n) = \{X \in \mathrm{GL}(n,\mathbb{R}) : X^T X = I_n, \det X = 1\} = \mathrm{O}(n) \cap \mathrm{SL}(n,\mathbb{R}).$$

Its Lie algebra $\mathfrak{so}(n)$ is the same as $\mathfrak{o}(n)$.

Example. **(The symplectic group)** The *symplectic group* is by definition

$$Sp(2n,\mathbb{R}) = \{X \in \mathrm{GL}(2n,\mathbb{R}) : X^T J X = J\},$$

where Let $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. It is a Lie group of dimension $n(2n+1)$ with Lie algebra

$$\begin{aligned} sp(2n,\mathbb{R}) &= \{A \in \mathfrak{gl}(2n,\mathbb{R}) \mid JA + A^T J = 0\} \\ &= \{A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \mid A_i \in M(n,\mathbb{R}), A_1 = -A_4^T, A_2 = A_2^T, A_3 = A_3^T\}. \end{aligned}$$

This, as well as the orthogonal group in the previous example, are special cases of the following more general example. Let $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear form on $\mathbb{R}^n$. Consider the set of all invertible $n \times n$ matrices that preserves β,

$$\mathrm{GL}_\beta(n,\mathbb{R}) = \{X \in \mathrm{GL}(n,\mathbb{R}) \mid \beta(Xu, Xv) = \beta(u,v) \text{for all } u, v \in \mathbb{R}^n\}.$$

In matrix form, there is a matrix B such that $\beta(u,v) = u^T B v$. Then

$$\mathrm{GL}_\beta(n,\mathbb{R}) = \{X \in \mathrm{GL}(n,\mathbb{R}) \mid X^T B X = B\}.$$

Lemma 2.3. $\mathrm{GL}_\beta(n,\mathbb{R})$ *is a linear Lie group with Lie algebra*

$$\mathfrak{gl}_\beta(n,\mathbb{R}) = \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid A^T B + BA = 0\}.$$

Proof. One can easily check that $\mathrm{GL}_\beta(n,\mathbb{R})$ is a subgroup of $\mathrm{GL}(n,\mathbb{R})$, and it is topologically a closed subset. According to the Cartan's closed subgroup theorem that we will prove later, it is a Lie subgroup.

To describe its Lie algebra, notice that $A \in \mathfrak{gl}_\beta(n,\mathbb{R})$ if and only if $e^{tA} \in \mathrm{GL}_\beta(n,\mathbb{R})$, i.e. $e^{tA^T} B e^{tA} = B$. By taking t derivative at $t=0$, we get $A^T B + BA = 0$. Conversely, if $A^T B + BA = 0$, i.e. $tA^T B = B(-tA)$, one can easily derive by definition that $e^{tA^T} B = Be^{-tA}$, i.e. $e^{tA^T} B e^{tA} = B$. This completes the proof. □

Notice that in the case $B = I_n$ ($\beta =$ the standard inner product on $\mathbb{R}^n$), we get the orthogonal group $O(n)$, and in the case $B = J$($\beta =$ the standard symplectic form on $\mathbb{R}^{2n}$), we get the symplectic group above. If we take β to be the standard inner product of signature $(p, n-p)$,

$$\beta(x,y) = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{n} x_i y_i,$$

then we will get the *indefinite orthogonal group* $\mathrm{O}(p, n-p)$,

$$\mathrm{O}(p,n-p) = \{X \in GL(n,\mathbb{R}) \mid X^T I(p,n-p) X = I(p,n-p)\},$$

where $I(p,n-p) = \mathrm{diag}(I_p, -I_{n-p})$. Its Lie algebra is

$$\mathfrak{o}(p,n-p) = \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid A^T I(p,n-p) + I(p,n-p)A = 0.\}.$$

Example. **(The unitary group)** All the previous examples generalize to complex matrices. In particular, the *unitary groups*

$$\mathrm{U}(n) = \{X \in \mathrm{GL}(n,\mathbb{C}) : \overline{X}^T X = I_n\}$$

is a Lie subgroup of $\mathrm{GL}(n,\mathbb{C})$ with Lie algebra

$$\mathfrak{u}(n) = \{A \in \mathfrak{gl}(n,\mathbb{C}) \mid \overline{A}^T + A = 0\},$$

the space of skew-Hermitian matrices.

Also the *special unitary groups*

$$\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n,\mathbb{C})$$

has Lie algebra

$$\mathfrak{su}(n) = \{A \in M(n,\mathbb{C}) \mid \overline{A}^T + A = 0, \mathrm{Tr}(A) = 0\}.$$

Also we have the *compact symplectic group*

$$\mathrm{Sp}(n) = \mathrm{U}(2n) \cap \mathrm{Sp}(2n,\mathbb{C}).$$

This is a real compact Lie group of dimension $n(2n+1)$.