LECTURE 21: SCHUR ORTHONONALITY

1. Schur's Lemma

Lemma 1.1 (Schur's Lemma)**.** *Let V, W be irreducible representations of G.*

(1) *If $f : V \to W$ is a G-morphism, then either $f \equiv 0$, or f is invertible.*
(2) *If $f_1, f_2 : V \to W$ are two G-morphisms and $f_2 \neq 0$, then there exists $\lambda \in \mathbb{C}$ such that $f_1 = \lambda f_2$.*

Proof. (1) Suppose f is not identically zero. Since $\ker(f)$ is a G-invariant subset in V, it must be $\{0\}$. So f is injective. In particular, $f(V)$ is a nonzero subspace of W. On the other hand, it is easy to check that $f(V)$ is a G-invariant subspace of W. It follows that $f(V) = W$, and thus f is invertible.

(2) Since $f_2 \neq 0$, it is invertible. So $f = f_2^{-1} \circ f_1$ is a G-morpism from V to V itself. Let λ be one of the eigenvalues of the linear map f. Then $f - \lambda \cdot \mathrm{Id}$ is a G-morphism from V to V which is not invertible. It follows that $f - \lambda \mathrm{Id} \equiv 0$, and thus $f_1 = \lambda f_2$. □

Note that in the proof we showed in particular

Corollary 1.2. *Let V be an irreducible representation of G, then* $\mathrm{Hom}_G(V, V) = \mathbb{C}\cdot\mathrm{Id}$.

Conversely, we have

Lemma 1.3. *If (π, V) is a unitary representation of G, and* $\mathrm{Hom}_G(V, V) = \mathbb{C}\cdot\mathrm{Id}$*, then (π, V) is an irreducible representation of G.*

Proof. Let $0 \neq W \subset V$ be a G-invariant subspace. We need to show that $W = V$. Let $P : V \to W$ be the orthogonal projection (with respect to the given G-invariant inner product). Since both W and $W^\perp$ are G-invariant, we have for any $g \in G$ and any $v = w + w^\perp \in V$,

$$P(g \cdot v) = P(g \cdot w + g \cdot w^\perp) = g \cdot w = g \cdot P(v),$$

i.e. $P : V \to W \subset V$ is a G-morphism. It follows that $P = \lambda \cdot \mathrm{Id}$ for some $\lambda \in \mathbb{C}$. Now $P^2 = P$ implies $\lambda = 1$, and thus $W = V$. □

Recall that the center $Z(G)$ of a Lie group G is

$$Z(G) = \{h \in G : gh = hg, \forall g \in G\}.$$

Corollary 1.4. *If (π, V) is an irreducible representation of G, then for any $h \in Z(G)$,* $\pi(h) = \lambda \cdot \mathrm{Id}$ *for some $\lambda \in \mathbb{C}$.*

Proof. Suppose $h \in Z(G)$, then for any $g \in G$,
$$\pi(h)\pi(g) = \pi(hg) = \pi(gh) = \pi(g)\pi(h).$$
In other words, $\pi(h) : V \to V$ is a G-morphism, and the conclusion follows. □

Corollary 1.5. *Any irreducible representation of an abelian Lie group is one dimensional.*

Proof. Since G is abelian, $Z(G) = G$. By the previous corollary, for any $g \in G$, $\pi(g)$ is a multiple of the identity map on V. It follows that any subspace of V is G-invariant. So V has no nontrivial subspace, which is equivalent to $\dim V = 1$. □

2. Schur Orthogonality for Matrix Coefficients

Let (V, π) be a representation of a Lie group G. If we choose a basis $e_1, \cdots, e_n$ of V, we can identify V with $\mathbb{C}^n$, and represent any $g \in G$ by a matrix:
$$\pi(g)v = \begin{pmatrix} \pi_{11}(g) & \cdots & \pi_{1n}(g) \\ \vdots & \vdots & \vdots \\ \pi_{n1}(g) & \cdots & \pi_{nn}(g) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$
for $v = \sum v_i e_i$. So if we take $L_j : V \to \mathbb{C}$ be the function
$$L_j(\sum_i v_i e_i) = v_j,$$
then π_{ij} is the function on G given by
$$\pi_{ij}(g) = L_i(\pi(g)e_j).$$

Definition 2.1. For any $v \in V, L \in V^*$, the map
$$\phi : G \to \mathbb{C}, \quad \phi(g) = L(\pi(g)v)$$
is called a *matrix coefficient* of G.

Obviously any matrix coefficients of G is a continuous function on G. In fact, they form a subring of $C(G)$:

Proposition 2.2. *If ϕ_1, ϕ_2 are matrix coefficients for G, so are $\phi_1 + \phi_2$ and $\phi_1 \cdot \phi_2$.*

Proof. Let (π_i, V_i) be representations of G, $v_i \in V_i$, $L_i \in V_i^*$ such that $\phi_i(g) = L_i(\pi_i(g)v_i)$. Then $(\pi_1 \oplus \pi_2, V_1 \oplus V_2)$ is a representation of G, $L_1 \oplus L_2 \in V_1^* \oplus V_2^* = (V_1 \oplus V_2)^*$ and
$$(L_1 \oplus L_2)((\pi_1 \oplus \pi_2)(g)(v_1, v_2)) = \phi_1(g) + \phi_2(g).$$
Similarly we have a linear functional $L_1 \otimes L_2$ on $V_1 \otimes V_2$ satisfying $(L_1 \otimes L_2)(v_1 \otimes v_2) = L_1(v_1)L_2(v_2)$, and thus
$$(L_1 \otimes L_2)((\pi_1 \otimes \pi_2)(g)(v_1 \otimes v_2)) = \phi_1(g)\phi_2(g).$$
□

Now suppose G is a compact Lie group, and dg the normalized Haar measure on G. Recall that $L^2(G)$, the space of square-integrable functions with respect to this Haar measure, is the completion of the space of continuous functions on G with respect to the inner product

$$\langle f_1, f_2 \rangle_{L^2} = \int_G f_1(g)\overline{f_2(g)}dg.$$

Theorem 2.3 (Schur's Orthogonality I)**.** *Let (π_1, V_1) and (π_2, V_2) are two non-isomorphic irreducible representations of a compact Lie group G. Then every matrix coefficient of π_1 is orthogonal in $L^2(G)$ to every matrix coefficient of π_2.*

Proof. Fix G-invariant inner products on V_1 and V_2 respectively. Suppose

$$\phi_i(g) = \langle \pi_i(g)v_i, w_i \rangle, i = 1, 2$$

are matrix coefficients for π_i, where $v_i, w_i \in V_i$. Fix a basis of V_1 such that $e_1 = v_1$. Define a linear map $f : V_1 \to V_2$ by $f(e_1) = v_2$ and $f(e_k) = 0$ for all $k \geq 2$. Consider the map

$$F : V_1 \to V_2, \quad v \mapsto F(v) = \int_G \pi_2(g)f(\pi_1(g^{-1})v)dg.$$

F is linear since f is. It is also G-equivariant, since

$$F(\pi_1(h^{-1})v) = \int_G \pi_2(g)f(\pi_1(hg)^{-1}v)dg = \pi_2(h^{-1})\int_G \pi_2(hg)f(\pi_1(hg)^{-1}v)dg = \pi_2(h^{-1})F(v).$$

By Schur's lemma, $F(v) = 0$ for any v, and in particular, $\langle F(v), w_2 \rangle = 0$. On the other hand, for any j,

$$\pi_2(g)f(\pi_1(g^{-1})e_j) = \pi_2(g)f(\sum_k \pi_1(g^{-1})_{kj}e_k) = \pi_1(g^{-1})_{1j}\pi_2(g)(v_2),$$

where $\pi_1(g^{-1})_{kj} = \langle \pi_1(g^{-1})e_j, e_k \rangle$ is the matrix coefficients of π_1 with respect to the basis $\{e_1, \cdots, e_n\}$. It follows that

$$\int_G \langle \pi_1(g^{-1})e_j, e_1 \rangle \langle \pi_2(g)v_2, w_2 \rangle dg = 0$$

for any j. So by linearity,

$$\int_G \langle \pi_1(g^{-1})w_1, v_1 \rangle \langle \pi_2(g)v_2, w_2 \rangle dg = 0.$$

Note that the inner product is G-invariant,

$$\langle \pi_1(g^{-1})w_1, v_1 \rangle = \langle w_1, \pi_1(g)v_1 \rangle = \overline{\langle \pi_1(g)v_1, w_1 \rangle} = \overline{\phi_1(g)}.$$

So the theorem follows. □

Theorem 2.4 (Schur's Orthogonality II)**.** *Let (π, V) be an irreducible representation of a compact Lie group G, with G-invariant inner product $\langle \cdot, \cdot \rangle$. Then*

$$\int_G \langle \pi(g)w_1, v_1 \rangle \overline{\langle \pi(g)w_2, v_2 \rangle} dg = \frac{1}{\dim \mathrm{V}} \langle w_1, w_2 \rangle \overline{\langle v_1, v_2 \rangle}.$$

Proof. Define the linear maps $f, F : V \to V$ as above. Then F is G-equivariant, and thus $F = \lambda \cdot \mathrm{Id}$ for some $\lambda = \lambda(v_1, v_2) \in \mathbb{C}$. On the other hand, when we take $\pi_1 = \pi_2 = \pi$, the computation in the previous proof shows

$$\lambda(v_1, v_2)\langle w_1, w_2\rangle = \langle F(w_1), w_2\rangle = \int_G \langle \pi(g^{-1})w_1, v_1\rangle\langle \pi(g)v_2, w_2\rangle dg.$$

Since the Haar measure is invariant under the inversion map $g \to g^{-1}$, the right hand side is invariant if we exchange w_1 with v_2, and exchange w_2 with v_1. It follows that

$$\lambda(v_1, v_2) = C\langle v_2, v_1\rangle = C\overline{\langle v_1, v_2\rangle}$$

for some constant C. Finally if we take $v_1 = v_2 = e_1$ a unit vector, then

$$\mathrm{Tr}(F) = \mathrm{Tr}\int_G \pi(g)\circ f\circ\pi(g^{-1})dg = \int_G \mathrm{Tr}(\pi(g)\circ f\circ\pi(g^{-1}))dg = \int_G \mathrm{Tr}(f)dg = \int_G 1dg = 1.$$

On the other hand, in this case $F = C\cdot\mathrm{Id}$. It follows from $\mathrm{Tr}(F) = 1$ that $C = \frac{1}{\dim \mathrm{V}}$. $\square$