现代数学物理方法

第三章，李群

杨焕雄

中国科学技术大学近代物理系

hyang@ustc.edu.cn

November 21, 2017

Outline

Goal :

In this lecture, we will define and discuss the tensor operators of the $su(2)$ [or equivalently $so(3)$] algebra.

A tensor operator transforming under the spin-s representation of $su(2)$ consists of a set of operators

$$\mathscr{O}^s_l, \quad (1 \leqslant l \leqslant 2s+1)$$

such that

$$[J_a, \ \mathscr{O}^s_l] = \mathscr{O}^s_m (J^s_a)_{ml}, \qquad (a = 1, \ 2, \ 3.)$$

Orbital angular momentum :

The $su(2)$ algebra can be realized by the orbital angular momentum operators of a quantum mechanics particle, $J_a = L_a = \epsilon_{abc} x_b p_c$.

Because $[x_a, \ p_b] = i\delta_{ab}$,

$$[J_a, \ x_b] = \epsilon_{acd} x_c [p_d, \ x_b] = \epsilon_{acd} x_c (-i\delta_{db}) = -i\epsilon_{acb} x_c$$

Recalling

$$(J_a^{\mathrm{adj}})_{cb} = -i\epsilon_{acb} \ ,$$

we get:

$$[J_a, \ x_b] = x_c(J_a^{\mathrm{adj}})_{cb} \quad \rightsquigarrow \quad x_c(J_a^1)_{cb}$$

Hence, the position vector $\vec{r} = \sum_{a=1}^{3} x_a \vec{e}_a$ is a tensor operator of $su(2)$ that transforms under the spin-1 representation.

Similarly,

$$\begin{aligned} [J_a, \ p_b] &= \epsilon_{acd}[x_c, \ p_b]p_d = \epsilon_{acd}(i\delta_{cb})p_d = i\epsilon_{abd}p_d = -i\epsilon_{acb}p_c = p_c(J_a^{\mathrm{adj}})_{cb} \\ [J_a, \ J_b] &= i\epsilon_{abc}J_c = -i\epsilon_{acb}J_c = J_c(J_a^{\mathrm{adj}})_{cb} \end{aligned}$$

The momentum $\vec{p} = \sum_{a=1}^{3} p_a \vec{e}_a$ and the angular momentum itself are also the tensor operators of $su(2)$ under the spin-1 representation.

we now consider the question about choosing an operator basis so that the standard spin-s representation generators J_a^s appears in the Lie brackets,

$$[J_a,\ \mathscr{O}_l^s] = \mathscr{O}_m^s (J_a^s)_{ml}, \qquad (a = 1, 2, 3.)$$

Suppose

1. we are given a tensor operator $\mathscr{O}$ that transforms under a representation D of $su(2)$ algebra,

$$[J_a,\ \mathscr{O}_\alpha] = \mathscr{O}_\beta (J_a^D)_{\beta\alpha}\ , \qquad (1 \leqslant \alpha, \beta \leqslant 2s+1)\ .$$

2. D is equivalent to the spin-s irreducible representation of $su(2)$. Namely, there is a nonsingular matrix S ($\det S \neq 0$) such that:

$$J_a^D = S^{-1} J_a^s S \qquad \rightsquigarrow \qquad (J_a^D)_{\beta\alpha} = (S^{-1})_{\beta j} (J_a^s)_{ji} S_{i\alpha}$$

we get,

$$[J_a,\ \mathscr{O}_\alpha] = \mathscr{O}_\beta (S^{-1})_{\beta j} (J_a^s)_{ji} S_{i\alpha}$$

i.e.,

$$[J_a,\ \mathscr{O}_\alpha](S^{-1})_{\alpha k} = \mathscr{O}_\beta (S^{-1})_{\beta j} (J_a^s)_{jk}$$

Definition :

$$\mathscr{O}_i^s \equiv \mathscr{O}_\beta (S^{-1})_{\beta i}$$

The above commutator is written as,

$$\rightsquigarrow \qquad [J_a,\ \mathscr{O}_i^s] = \mathscr{O}_j^s (J_a^s)_{ji}, \qquad -s \leqslant i, j \leqslant s.$$

In the standard basis, the $SU(2)$'s generator J_3 is a diagonal matrix: $(J_3^s)_{jk} = j\delta_{jk},\ (j, k = -s, -s+1, \cdots, s-1, s)$, i.e.,

$$J_3^s = \begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 & 0 \\ 0 & 0 & s-2 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & -s \end{bmatrix}$$

Therefore,

$$[J_3,\ \mathscr{O}_k^s] = \mathscr{O}_j^s (J_3^s)_{jk} = \mathscr{O}_j^s j \delta_{jk} = k \mathscr{O}_k^s$$

Remark :

What does the commutator $[J_3,\ \mathscr{O}_k^s] = k\mathscr{O}_k^s$ mean ?

- If we find a linear combination of the operators $\{\mathscr{O}_\alpha^s\}$ which has a definite value k of J_3 (with $|k| \leqslant s$),

$$[J_3,\ \mathscr{O}_\alpha^s] = k \sum_\beta c_{\alpha\beta}\mathscr{O}_\beta^s$$

  we can take that combination to be the tensor component $\mathscr{O}_k^s$,

$$\mathscr{O}_k^s = \sum_\alpha f_{k\alpha}\mathscr{O}_\alpha^s$$

- The other components $\{\mathscr{O}_i^s,\ i \neq k\}$ of the tensor operator $\mathscr{O}^s$ can be built up by applying raising and lowering operators.

Example :

Let $V^1 = \{V_1^1, V_0^1, V_{-1}^1\}$ be the position vector operator [the tensor operator in spin-1 representation of $su(2)$] in standard basis.

1. Since $[J_3,\ V_k^1] = kV_k^1$, we see $[J_3,\ V_0^1] = 0$. On the other hand, we have $[J_a,\ x_b] = -i\epsilon_{acb}x_c$ that implies $[J_3,\ x_3] = -i\epsilon_{3c3}x_c = 0$.Therefore, we can identify V_0^1 as x_3,
$$V_0^1 \equiv x_3$$
2. From the commutation relations $[J_a,\ \mathscr{O}_i^s] = \mathscr{O}_j^s(J_a^s)_{ji}$, we get
$$[J_\pm,\ V_0^1] = V_j^1(J_\pm^1)_{j0} = V_j^1\delta_{j,\pm 1} = V_{\pm 1}^1,$$
i.e.,
$$\begin{aligned} V_{\pm 1}^1 &= [J_\pm, V_0^1] = \tfrac{1}{\sqrt{2}}[J_1 \pm iJ_2,\ x_3] = \tfrac{1}{\sqrt{2}}(i\epsilon_{132}x_2 \pm i^2\epsilon_{231}x_1) \\ &= \tfrac{1}{\sqrt{2}}(-ix_2 \mp x_1) \\ &= \mp\tfrac{1}{\sqrt{2}}(x_1 \pm ix_2) \end{aligned}$$

In conclusion, we have
$$\begin{cases} V_1^1 &= -\frac{1}{\sqrt{2}}(x_1 + ix_2) \\ V_0^1 &= x_3 \\ V_{-1}^1 &= \frac{1}{\sqrt{2}}(x_1 - ix_2) \end{cases}$$

We now consider the $su(2)$ transformation of the state

$$\mathscr{O}_l^s\,|jm,\alpha\rangle$$

Straightforwardly,

$$\begin{aligned} J_a\mathscr{O}_l^s\,|jm,\alpha\rangle &= [J_a,\ \mathscr{O}_l^s]\,|jm,\alpha\rangle + \mathscr{O}_l^s J_a\,|jm,\alpha\rangle \\ &= \sum_{k=-s}^{s}\mathscr{O}_k^s(J_a^s)_{kl}\,|jm,\alpha\rangle + \mathscr{O}_l^s\sum_{k=-j}^{j}|jk,\alpha\rangle\,\langle jk,\alpha|J_a\,|jm,\alpha\rangle \\ &= \sum_{k=-s}^{s}\mathscr{O}_k^s(J_a^s)_{kl}\,|jm,\alpha\rangle + \mathscr{O}_l^s\sum_{k=-j}^{j}(J_a^j)_{km}\,|jk,\alpha\rangle \end{aligned}$$

In particular,

- J_3's value of the product of a tensor operator with a state is just the sum of the J_3's values of the operator and the state,

$$\begin{aligned} J_3\mathscr{O}_l^s\,|jm,\alpha\rangle &= \sum_{k=-s}^{s}\mathscr{O}_k^s(J_3^s)_{kl}\,|jm,\alpha\rangle + \sum_{k=-j}^{j}\mathscr{O}_l^s(J_3^j)_{km}\,|jk,\alpha\rangle \\ &= \sum_{k=-s}^{s}\mathscr{O}_k^s(k\delta_{kl})\,|jm,\alpha\rangle + \sum_{k=-j}^{j}\mathscr{O}_l^s(k\delta_{km})\,|jk,\alpha\rangle \\ &= (l+m)\mathscr{O}_l^s\,|jm,\alpha\rangle \end{aligned}$$

The product of a tensor operator and a state behaves under $su(2)$ just like the tensor products of two states. Therefore, it can be decomposed into the direct sum of irreducible representations of $su(2)$.

Notice that,

1. The state $\mathscr{O}_s^s\,|jj,\alpha\rangle$ is the highest weight state in spin-$(j+s)$ Rep. of $su(2)$, with J_3 eigenvalue being $J_3 = j+s$. We can lower it to construct the rest states of the spin-$(j+s)$ representation.
2. We can find a linear combination of $J_3 = j+s-1$ states that is the highest weight state of the spin-$(j+s-1)$ representation. By lowering it we can get the entire states of the representation.
3. In this way, the explicit states of the irreducible representations of $su(2)$ algebra can be constructed in terms of linear combinations of the states $\{\mathscr{O}_l^s\,|jm,\alpha\rangle\}$,

$$|JM\rangle = \sum_{l=-s}^{s} d_{sjl,JM}\,\mathscr{O}_l^s\,|j,M-l,\alpha\rangle$$

where $|j-s| \leqslant J \leqslant j+s$ and $-J \leqslant M \leqslant J$.

Recalling,

$$|JM\rangle = \sum_{l=-s}^{s} c_{sjJ,l(M-l)M}\left[\,|sl\rangle \times |j,M-l\rangle\,\right]$$

with $c_{sjJ,l(M-l)M}$ C.G. coefficients.

Since the $su(2)$ transformation properties of states $\mathcal{O}_l^s |j, M-l, \alpha\rangle$ and $\left(|sl\rangle \times |j, M-l\rangle\right)$ are identical for a given J, the coefficients must be proportional:

$$d_{sjl,JM} = \frac{1}{k_J^\alpha} c_{sjJ,l(M-l)M}$$

Therefore,

$$k_J^\alpha |JM\rangle = \sum_{l=-s}^{s} c_{sjJ,l(M-l)M} \mathcal{O}_l^s |j, M-l, \alpha\rangle$$

Question :

What is the inverse relation ?

The C.G.coefficients are defined as,

$$c_{j_1 j_2 j, m_1(m-m_1)m} = \left(\langle j_1 m_1| \times \langle j_2, m-m_1|\right) |jm\rangle$$

we see:

$$c^*_{j_1 j_2 j, m_1(m-m_1)m} = \langle jm| \left(|j_1 m_1\rangle \times |j_2, m-m_1\rangle\right)$$

The completeness relation $\sum_{j,m} |jm\rangle \langle jm| = 1$ then implies that,

$$\sum_{j,m} c_{j_1 j_2 j, m_1(m-m_1)m} c^*_{j_1' j_2' j, m_1'(m-m_1')m} = \delta_{j_1 j_1'} \delta_{j_2 j_2'} \delta_{m_1 m_1'}$$

Consequently,

$$\mathscr{O}_l^s \, |jm, \alpha\rangle = \sum_{J=|j-s|}^{j+s} c^*_{sj\,J,lm(m+l)} k_J^\alpha \, |J, m+l\rangle$$

Wigner-Eckart Theorem :

The physics comes in when we express the state $k_J^\alpha \, |J, m+l\rangle$ in terms of the Hilbert space basis states $|J, m+l, \alpha\rangle$,

$$k_J^\alpha \, |J, m+l\rangle = \sum_\beta k_{\alpha\beta} \, |J, m+l, \beta\rangle$$

where $k_{\alpha\beta}$ are known as the reduced matrix elements which depend on α, j and $\mathscr{O}^s$.$k_{\alpha\beta}$ are generically denoted as,

$$k_{\alpha\beta} = \langle\langle J, \beta | \mathscr{O}^s | j, \alpha\rangle\rangle$$

As a result,

If we know any non-vanishing reduced matrix element of a tensor operator between states of some given $\{J, \beta\}$ and $\{j, \alpha\}$, we can compute all the other matrix elements using the algebra,

$$\begin{aligned}\langle J'm', \beta | \mathscr{O}^s_l | jm, \alpha\rangle &= \sum_\gamma k_{\alpha\gamma} \sum_{J=|j-s|}^{j+s} c^*_{sj\,J,lm(m+l)} \langle J'm', \beta | J, m+l, \gamma\rangle \\ &= \sum_\gamma k_{\alpha\gamma} \sum_{J=|j-s|}^{j+s} c^*_{sj\,J,lm(m+l)} \delta_{J'J} \delta_{m',m+l} \delta_{\beta\gamma} \\ &= k_{\alpha\beta} \delta_{m',m+l} c^*_{sj\,J',lm(m+l)}\end{aligned}$$

i.e.,

$$\langle J'm', \beta | \mathscr{O}^s_l | jm, \alpha\rangle = \delta_{m',m+l} c^*_{sj\,J',\ lm(m+l)} \cdot \left\langle\left\langle J', \beta | \mathscr{O}^s | j, \alpha \right\rangle\right\rangle$$

This conclusion is called Wigner-Eckart Theorem. Wigner-Eckart theorem has founded wide applications in quantum mechanics.

Problem :

Suppose $\langle 1/2, 1/2, \alpha | X_3 | 1/2, 1/2, \beta\rangle = \mathscr{A}$.
Find $\langle 1/2, 1/2, \alpha | X_1 | 1/2, -1/2, \beta\rangle = ?$

Solution :

The tensor operator related to the position vector is,

$$V^1_1 = -\frac{1}{\sqrt{2}}(X_1 + iX_2), \quad V^1_0 = X_3, \quad V^1_{-1} = \frac{1}{\sqrt{2}}(X_1 - iX_2).$$

Namely,

$$X_1 = \frac{1}{\sqrt{2}}(V^1_{-1} - V^1_1), \quad X_2 = \frac{i}{\sqrt{2}}(V^1_{-1} + V^1_1), \quad X_3 = V^1_0.$$

It follows from the Wigner-Eckart theorem that,

$$\begin{aligned}\mathscr{A} = \langle 1/2, 1/2, \alpha|\, V^1_0\, |1/2, 1/2, \beta\rangle &= c^*_{1\frac{1}{2}\frac{1}{2},0\frac{1}{2}\frac{1}{2}} \left\langle\left\langle 1/2, \beta|V^1|1/2, \alpha\right\rangle\right\rangle, \\ \langle 1/2, 1/2, \alpha|\, V^1_1\, |1/2, -1/2, \beta\rangle &= c^*_{1\frac{1}{2}\frac{1}{2},1-\frac{1}{2}\frac{1}{2}} \left\langle\left\langle 1/2, \beta|V^1|1/2, \alpha\right\rangle\right\rangle, \\ \langle 1/2, 1/2, \alpha|\, V^1_{-1}\, |1/2, -1/2, \beta\rangle &= 0.\end{aligned}$$

These equation imply,

$$\begin{aligned}\langle 1/2, 1/2, \alpha|\, X_1\, |1/2, -1/2, \beta\rangle &= \tfrac{1}{\sqrt{2}} \langle 1/2, 1/2, \alpha|\, (V^1_{-1} - V^1_1)\, |1/2, -1/2, \beta\rangle \\ &= -\tfrac{1}{\sqrt{2}} \langle 1/2, 1/2, \alpha|\, V^1_1\, |1/2, -1/2, \beta\rangle \\ &= -\tfrac{1}{\sqrt{2}} c^*_{1\frac{1}{2}\frac{1}{2},1-\frac{1}{2}\frac{1}{2}} \left\langle\left\langle 1/2, \beta|V^1|1/2, \alpha\right\rangle\right\rangle \\ &= -\tfrac{1}{\sqrt{2}} c^*_{1\frac{1}{2}\frac{1}{2},1-\frac{1}{2}\frac{1}{2}} \frac{\mathscr{A}}{c^*_{1\frac{1}{2}\frac{1}{2},0\frac{1}{2}\frac{1}{2}}}\end{aligned}$$

We knew from the last lecture that

$$c_{1\frac{1}{2}\frac{1}{2},1-\frac{1}{2}\frac{1}{2}} = \sqrt{2/3}, \quad c_{1\frac{1}{2}\frac{1}{2},0\frac{1}{2}\frac{1}{2}} = -\sqrt{1/3}.$$

Hence,

$$\langle 1/2, 1/2, \alpha|\, X_1\, |1/2, -1/2, \beta\rangle = \mathscr{A}$$

Discussions :

- The similar applications of Wigner-Eckart theorem will yield,

$$\begin{aligned}&\langle 1/2, 1/2, \alpha|\, X_2\, |1/2, -1/2, \beta\rangle = -i\mathscr{A},\\&\langle 1/2, -1/2, \alpha|\, X_3\, |1/2, -1/2, \beta\rangle = -\mathscr{A},\\&\langle 1/2, 1/2, \alpha|\, X_3\, |1/2, -1/2, \beta\rangle = \langle 1/2, -1/2, \alpha|\, X_3\, |1/2, 1/2, \beta\rangle = 0,\\&\cdots\end{aligned}$$

- However, the Wigner-Eckart theorem is not enough for us to evaluate the matrix elements such as

$$\langle 3/2, 1/2, \alpha|\, X_3\, |1/2, 1/2, \beta\rangle$$

because we are not told the relevant reduced matrix element $\Big\langle\Big\langle 3/2, \beta|V^1|1/2, \alpha\Big\rangle\Big\rangle$.

Products of tensor operators :

One of the reason that tensor operators are important is that a product of two tensor operators, $\mathscr{O}^{s_1}_{m_1}$ and $\mathscr{O}^{s_2}_{m_2}$ in the spin-s_1 and spin-s_2 representations, transforms under the tensor product representation $s_1 \times s_2$:

$$\begin{aligned}
[J_a,\ \mathscr{O}^{s_1}_{m_1}\mathscr{O}^{s_2}_{m_2}] &= [J_a,\ \mathscr{O}^{s_1}_{m_1}]\mathscr{O}^{s_2}_{m_2} + \mathscr{O}^{s_1}_{m_1}[J_a,\ \mathscr{O}^{s_2}_{m_2}] \\
&= \mathscr{O}^{s_1}_{m_1'}\mathscr{O}^{s_2}_{m_2}(J^{s_1}_a)_{m_1' m_1} + \mathscr{O}^{s_1}_{m_1}\mathscr{O}^{s_2}_{m_2'}(J^{s_2}_a)_{m_2' m_2} \\
&= \mathscr{O}^{s_1}_{m_1'}\mathscr{O}^{s_2}_{m_2'}\left[(J^{s_1}_a)_{m_1' m_1}\delta_{m_2' m_2} + \delta_{m_1' m_1}(J^{s_2}_a)_{m_2' m_2}\right] \\
&= \mathscr{O}^{s_1}_{m_1'}\mathscr{O}^{s_2}_{m_2'}\left[J^{s_1}_a \times 1 + 1 \times J^{s_2}_a\right]_{m_1' m_2', m_1 m_2}
\end{aligned}$$

In particular,

$$[J_3,\ \mathscr{O}^{s_1}_{m_1}\mathscr{O}^{s_2}_{m_2}] = (m_1 + m_2)\mathscr{O}^{s_1}_{m_1}\mathscr{O}^{s_2}_{m_2}$$

Homework :

1. Consider an operator $\mathcal{O}_x$ for $x = 1$ to 2, transforming according to the spin-$1/2$ representation of $su(2)$ as follows, $[J_a,\ \mathcal{O}_x] = \mathcal{O}_y(\sigma_a/2)_{yx}$, where σ_a are Pauli matrices. Given $\langle 3/2, -1/2, \alpha|\,\mathcal{O}_1\,|1, -1, \beta\rangle = \mathcal{A}$, find $\langle 3/2, -3/2, \alpha|\,\mathcal{O}_2\,|1, -1, \beta\rangle$.

Goal :

We are going to generalize the analysis of the irreducible representations of $su(2)$ to those of an arbitrary simple Lie algebra.

Firstly, we are necessary to find the largest possible set of commuting hermitian generators and use their eigenvalues to label the states. These generators are the analog of J_3 in $su(2)$.

The rest of the generators will be analogous to the raising and lowering operators $J_{\pm}$.

Cartan subalgebra :

A subset of commuting Hermitian generators which is as large as possible is called a Cartan subalgebra.

These commuting generators are called the Cartan generators.

Rank : The total number m of the independent Cartan generators is called the rank of the Lie algebra.

In a particular irreducible representation D, the Cartan generators are formulated as m Hermitian matrices H_i $(i = 1, 2, \cdots, m)$,

$$H_i = H_i^\dagger, \quad [H_i,\ H_j] = 0.$$

For compact Lie algebra, we can choose a basis in which

$$\mathrm{Tr}(H_i H_j) = k_D \delta_{ij}$$

with k_D some constant that depends on the representation and on the normalization of the generators.

After simultaneously diagonalization of the Cartan generators, the basis vectors (states) of the representaton space (of Rep. D) can be cast as,

$$|\mu, \xi, D\rangle$$

such that

$$H_i\,|\mu, \xi, D\rangle = \mu_i\,|\mu, \xi, D\rangle\,, \qquad (i = 1, 2, \cdots, m.)$$

where ξ stands for any other parameters necessary for specifying the state.

Weights :

- The eigenvalues μ_i $(i = 1, 2, \cdots, m)$ of the Cartan generators $\{H_i\}$ are called weights.
- Weights are real.
- The whole set of weights $\{\mu_i\}$ forms a m-component vector $\vec{\mu}$,

$$\vec{\mu} = (\mu_1, \mu_2, \cdots, \mu_m)$$

in weight space, called weight vector.

Adjoint representation :

The adjoint representation of a Lie algebra $[X_a,\ X_b] = if_{abc}X_c$ is defined as,

$$(T_a)_{bc} = -if_{abc}$$

Due to the Jacobi identity, this definition leads to $[T_a,\ T_b] = if_{abc}T_c$.

Warning :

The rows and columns of the generators $\{T_a\}$ are labeled by the same indices as that labels the generators themselves.

Thus, the basis vectors (states) of the adjoint representation space have a one-to-one correspondence with the generators,

$$T_a \quad \Leftrightarrow \quad |T_a\rangle$$

which implies,

$$\alpha\,|T_a\rangle + \beta\,|T_b\rangle = |\alpha T_a + \beta T_b\rangle$$

The action of a generator on the basis states of the adjoint representation gives,

$$\begin{aligned} T_a\,|T_b\rangle &= \sum_c |T_c\rangle\,\langle T_c|\,T_a\,|T_b\rangle = \sum_c |T_c\rangle\,(T_a)_{cb} \\ &= \sum_c (if_{abc})\,|T_c\rangle = |\sum_c if_{abc}T_c\rangle \\ &= |[T_a,\ T_b]\rangle \end{aligned}$$

Its hermitian conjugate leads to:

$$\langle T_b|\,T_a^\dagger = \langle[T_a,\ T_b]|$$

In adjoint representation, the scalar product between any two basis states $|T_a\rangle$ and $|T_b\rangle$ is defined by[1],

$$\langle T_a|T_b\rangle = \lambda^{-1}\mathrm{Tr}(T_a^\dagger T_b)$$

[1]This formula is valid only for a compact Lie algebra.

Roots :

Weights of a Lie algebra in its adjoint representation are called roots.

Notice that,

- In the adjoint representation,

$$H_i \, |H_j\rangle = |[H_i, \; H_j]\rangle = |0\rangle = |0 \cdot H_j\rangle = 0 \, |H_j\rangle = 0$$

  the states $\{|H_j\rangle\}$ corresponding to the Cartan generators have zero weights.
- The Cartan states are orthonormal,

$$\langle H_i|H_j\rangle = \lambda^{-1}\mathrm{Tr}(H_i H_j) = \lambda^{-1} \cdot \lambda\delta_{ij} = \delta_{ij}.$$

- The other states $\{|E_\alpha\rangle\}$ in the adjoint representation, which do not correspond to Cartan generators, have non-zero weights:

$$H_i \, |E_\alpha\rangle = \alpha_i \, |E_\alpha\rangle \, , \qquad (i = 1, 2, \cdots, m.)$$

  i.e., $|[H_i, \; E_\alpha]\rangle = |\alpha_i E_\alpha\rangle$. This indicates,

$$[H_i, \; E_\alpha] = \alpha_i E_\alpha \quad , \qquad (i = 1, 2, \cdots, m.)$$

Definition :

- The weights $\{\alpha_i | \ i = 1, 2, \cdots, m\}$ of the adjoint representation are called roots.
- The special weight vector

$$\vec{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_m)$$

is called a root vector.

Remarks :

- Like the $su(2)$ raising and lowering operators, the generators $\{E_\alpha\}$ related to the non-zero root vectors are not hermitian.

The reason is as follows. Since $[H_i, \ E_\alpha] = \alpha_i E_\alpha$,

$$\alpha_i E_\alpha^\dagger = (\alpha_i E_\alpha)^\dagger = ([H_i, \ E_\alpha])^\dagger = -[H_i, \ E_\alpha^\dagger]$$

i.e.,

$$[H_i, \ E_\alpha^\dagger] = -\alpha_i E_\alpha^\dagger$$

By comparison we see that $E_\alpha \neq E_\alpha^\dagger$. Instead,

$$E_\alpha^\dagger = E_{-\alpha}$$

- States corresponding to different roots must be orthogonal, because they have different eigenvalues of at least one of the Cartan generators,

$$\langle E_\alpha | E_\beta \rangle = \delta_{\alpha\beta}$$

This gives moreover,

$$\mathrm{Tr}(E_\alpha^\dagger E_\beta) = \lambda \langle E_\alpha | E_\beta \rangle = \lambda \delta_{\alpha\beta}$$

- The generators $\{E_{\pm\alpha}\}$ are raising and lowering operators for the weights.

Proof : Consider a representation D of Lie algebra in which

$$H_i \,|\mu, D\rangle = \mu_i \,|\mu, D\rangle \,, \qquad (i = 1, 2, \cdots, m.)$$

Then,

$$\begin{aligned} H_i E_{\pm\alpha} \,|\mu, D\rangle &= [H_i,\ E_{\pm\alpha}] \,|\mu, D\rangle + E_{\pm\alpha} H_i \,|\mu, D\rangle \\ &= \pm\alpha_i E_{\pm\alpha} \,|\mu, D\rangle + E_{\pm\alpha} \mu_i \,|\mu, D\rangle \\ &= (\vec{\mu} \pm \vec{\alpha})_i E_{\pm\alpha} \,|\mu, D\rangle \end{aligned}$$

This result is valid for any representation, particularly true for the adjoint representation.

- Go back to the adjoint representation. We consider the state,

$$E_\alpha \left| E_{-\alpha} \right\rangle$$

This is an eigenstate of Cartan generators belonging to vanishing eigenvalue:

$$H_i E_\alpha \left| E_{-\alpha} \right\rangle = (\vec{\alpha} - \vec{\alpha})_i E_\alpha \left| E_{-\alpha} \right\rangle = 0.$$

Therefore,

$$E_\alpha \left| E_{-\alpha} \right\rangle = c_i \left| H_i \right\rangle \qquad \rightsquigarrow \qquad \left| [E_\alpha,\ E_{-\alpha}] \right\rangle = \left| c_i H_i \right\rangle$$

and from this we get the commutators,

$$[E_\alpha,\ E_{-\alpha}] = c_i H_i$$

We now determine the coefficients c_i:

$$\begin{aligned}
c_i = c_j \delta_{ij} &= c_j \left\langle H_i | H_j \right\rangle = \left\langle H_i | c_j H_j \right\rangle = \left\langle H_i | [E_\alpha,\ E_{-\alpha}] \right\rangle \\
&= \lambda^{-1} \mathrm{Tr}(H_i [E_\alpha,\ E_{-\alpha}]) \\
&= \lambda^{-1} \mathrm{Tr}(H_i E_\alpha E_{-\alpha} - H_i E_{-\alpha} E_\alpha) \\
&= \lambda^{-1} \mathrm{Tr}(E_{-\alpha} H_i E_\alpha - E_{-\alpha} E_\alpha H_i) \\
&= \lambda^{-1} \mathrm{Tr}(E_{-\alpha} [H_i,\ E_\alpha]) \\
&= \lambda^{-1} \mathrm{Tr}(E_\alpha^\dagger \alpha_i E_\alpha) \\
&= \lambda^{-1} \alpha_i \mathrm{Tr}(E_\alpha^\dagger E_\alpha) = \alpha_i
\end{aligned}$$

Thus,

$$[E_\alpha,\ E_{-\alpha}] = \alpha_i H_i = \vec{\alpha}\ \cdot\ \vec{H}$$

This is the analog of $[J_+,\ J_-] = J_3$ of $su(2)$ algebra.

- In adjoint representation, we now focus on the state,

$$E_\alpha \left|E_\beta\right\rangle$$

for $\vec{\alpha} + \vec{\beta} \neq 0$. This is an eigenstate of Cartan generators belonging to root vector $\vec{\alpha} + \vec{\beta}$,

$$H_i E_\alpha \left|E_\beta\right\rangle = (\vec{\alpha} + \vec{\beta})_i E_\alpha \left|E_\beta\right\rangle .$$

Therefore,

$$E_\alpha \left|E_\beta\right\rangle = \mathcal{N}_{\alpha\beta} \left|E_{\alpha+\beta}\right\rangle \qquad \rightsquigarrow \quad \left|[E_\alpha,\ E_\beta]\right\rangle = \left|\mathcal{N}_{\alpha\beta} E_{\alpha+\beta}\right\rangle$$

The relevant Lie brackets read,

$$[E_\alpha,\ E_\beta] = \mathcal{N}_{\alpha\beta} E_{\alpha+\beta}$$

Question:

$$\mathcal{N}_{\alpha\beta} = ?$$

We have formulated the Lie algebra into the so-called Cartan-Weyl basis,

$$\begin{aligned}
&[H_i,\ H_j] = 0\ ,\\
&[H_i,\ E_\alpha] = \alpha_i E_\alpha\ ,\\
&[E_\alpha,\ E_{-\alpha}] = \alpha_i H_i\ ,\\
&[E_\alpha,\ E_\beta] = \mathcal{N}_{\alpha,\beta} E_{\alpha+\beta}\ ,\quad (\text{ for }\ \vec{\alpha}+\vec{\beta}\neq 0.\ )
\end{aligned}$$

The structure constants $\mathcal{N}_{\alpha,\beta}$ will be determined systematically.

Lots of $su(2)$s :

For each pair of non-zero root vectors $\pm\vec{\alpha}$, there is an $su(2)$ subalgebra of the Lie algebra $\mathfrak{g}$, with generators,

$$E_\pm = \frac{E_{\pm\alpha}}{\alpha},\quad E_3 = \frac{\vec{\alpha}\cdot\vec{H}}{\alpha^2}$$

where $\alpha = |\vec{\alpha}|$.

Checking :

$$[E_3,\ E_\pm] = \alpha^{-3}\alpha_i[H_i,\ E_{\pm\alpha}] = \pm\alpha^{-3}\alpha_i\alpha_i E_{\pm\alpha} = \pm\alpha^{-1}E_{\pm\alpha} = \pm E_\pm\ ,$$

$$[E_+,\ E_-] = \alpha^{-2}[E_\alpha,\ E_{-\alpha}] = \alpha^{-2}\alpha_i H_i = E_3\ .$$

Corollaries :

- The 3 states $\{|E_3\rangle\,,\ |E_\pm\rangle\}$ in adjoint representation form a spin-1 representation of the associated $su(2)$ subalgebra $\{E_3,\ E_\pm\}$.

The nontrivial scalar products in subspace $\{|E_3\rangle\,,\ |E_\pm\rangle\}$ are,

$$\langle E_3|E_3\rangle = \alpha^{-4}\alpha_i\alpha_j\,\langle H_i|H_j\rangle = \alpha^{-2}\ ,$$
$$\langle E_\pm|E_\pm\rangle = \alpha^{-2}\,\langle E_{\pm\alpha}|E_{\pm\alpha}\rangle = \alpha^{-2}\ .$$

On these states, the action of generators $\{E_3,\ E_\pm\}$ is calculated below:

$$E_3\,|E_\pm\rangle = |[E_3,\ E_\pm]\rangle = |\pm E_\pm\rangle = \pm\,|E_\pm\rangle\ ,$$
$$E_3\,|E_3\rangle = |[E_3,\ E_3]\rangle = |0\rangle = 0\,|E_3\rangle = 0\ .$$

and

$$E_+\,|E_+\rangle = |[E_+,\ E_+]\rangle = |0\rangle = 0\ ,$$
$$E_+\,|E_3\rangle = |[E_+,\ E_3]\rangle = |-E_+\rangle = -\,|E_+\rangle\ ,$$
$$E_+\,|E_-\rangle = |[E_+,\ E_-]\rangle = |E_3\rangle\ .$$

By introducing the normalized basis states,

$$|1\rangle = \alpha\,|E_+\rangle = |E_\alpha\rangle\,,\quad |2\rangle = \alpha\,|E_3\rangle = \alpha^{-1}\alpha_i\,|H_i\rangle\,,\quad |3\rangle = \alpha\,|E_-\rangle = |E_{-\alpha}\rangle\,,$$

we get:

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad E_+ = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_- = (E_+)^\dagger = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

This is the very spin-1 representation of $su(2)$ algebra.

- If $\vec{\alpha}$ is a root vector, no non-zero multiple of $\vec{\alpha}$ (except $-\vec{\alpha}$) is a root vector.

Proof :

Suppose $k\vec{\alpha}$ were a root vector for $k \neq \pm 1$. The corresponding generator and the state in adjoint representation read,

$$E_{k\alpha}, \qquad |E_{k\alpha}\rangle .$$

Then,

$$E_3 |E_{k\alpha}\rangle = |[E_3,\ E_{k\alpha}]\rangle = \alpha^{-2}\alpha_i |[H_i,\ E_{k\alpha}]\rangle = \alpha^{-2}\alpha_i |k\alpha_i E_{k\alpha}\rangle = k |E_{k\alpha}\rangle$$

i.e., $|E_{k\alpha}\rangle$ is the eigenstate of E_3 belonging to eigenvalue k. Recall that E_3 is a generator of $su(2)$ subalgebra, its eigenvalue k must be a half-integer.

There are two possibilities:

1. k is an integer.
   When k is an integer, $|E_{k\alpha}\rangle$ will be in such a $su(2)$ representation that contains another state $|E'_\alpha\rangle$ related to root vector $\vec{\alpha}$.

   We will show that a root vector corresponds uniquely to a generator. Hence,

   $$|E'_\alpha\rangle = |E_\alpha\rangle \quad \Leftrightarrow \quad E_\alpha$$

   Recall that $|E_\alpha\rangle$ is in the spin-1 representation of $su(2)$ subalgebra generated by $E_3 = \alpha^{-2}\vec{\alpha} \cdot \vec{H}$ and $E_\pm = \alpha^{-1}E_{\pm\alpha}$, $-1 \leqslant k \leqslant 1$. We conclude that,

   $|E_{k\alpha}\rangle$'s existence is impossible unless $k \neq \pm 1$.

2. k is half an odd integer.
   In this case, there were a state (and then a generator $E_{\alpha/2}$) with root vector $\vec{\alpha}/2$.

   We have seen that if $\vec{\alpha}$ is a root vector, $2\vec{\alpha}$ is not a root vector. Thus, if $\vec{\alpha}/2$ were a root vector, $\vec{\alpha} = 2(\vec{\alpha}/2)$ would not be a root vector $\rightsquigarrow$ absurd.

   We conclude that k cannot be half an odd integer.

- There is a one-to-one correspondence between root vectors and the generators.

Proof :

Suppose the contrary: there were 2 independent generators E_α and E'_α corresponding to the same root vector $\vec{\alpha}$.
Choosing appropriate linear combination of E_α and E'_α, we could have:

$$0 = \langle E_\alpha | E'_\alpha \rangle = \lambda^{-1} \mathrm{Tr}(E^\dagger_\alpha E'_\alpha) = \lambda^{-1} \mathrm{Tr}(E_{-\alpha} E'_\alpha)$$

Consider the action of $su(2)$ subalgebra (related to root $\vec{\alpha}$) on the state $|E'_\alpha\rangle$. Because,

$$[H_i,\ E_\alpha] = \alpha_i E_\alpha, \quad [H_i,\ E'_\alpha] = \alpha_i E'_\alpha, \qquad i = 1, 2, \cdots, m$$

In adjoint representation, we have:

$$\begin{aligned} H_i E_- \left| E'_\alpha \right\rangle &= \alpha^{-1} H_i E_{-\alpha} \left| E'_\alpha \right\rangle = \alpha^{-1} [H_i,\ E_{-\alpha}] \left| E'_\alpha \right\rangle + \alpha^{-1} E_{-\alpha} H_i \left| E'_\alpha \right\rangle \\ &= -\alpha^{-1} \alpha_i E_{-\alpha} \left| E'_\alpha \right\rangle + \alpha^{-1} E_{-\alpha} \left| [H_i,\ E'_\alpha] \right\rangle \\ &= -\alpha^{-1} \alpha_i E_{-\alpha} \left| E'_\alpha \right\rangle + \alpha^{-1} E_{-\alpha} \left| \alpha_i E'_\alpha \right\rangle = 0 \end{aligned}$$

It implies,

$$E_- \left| E'_\alpha \right\rangle = c_j \left| H_j \right\rangle$$

The coefficient c_j here is found to be vanishing,

$$
\begin{aligned}
c_j &= \langle H_j|\, E_- \,|E'_\alpha\rangle = \langle H_j|[E_-,\ E'_\alpha]\rangle = \lambda^{-1}\mathrm{Tr}(H_j[E_-,\ E'_\alpha]) \\
&= -\lambda^{-1}\mathrm{Tr}(E_-[H_j,\ E'_\alpha]) \\
&= -\lambda^{-1}\alpha^{-1}\alpha_j\mathrm{Tr}(E_{-\alpha}E'_\alpha) = 0
\end{aligned}
$$

Therefore

$$
E_- \,|E'_\alpha\rangle = 0
$$

It turns out to imply,

- $|E'_\alpha\rangle$ is the lowest E_3 state in the relevant $su(2)$ representation.

However,

$$
E_3\,|E'_\alpha\rangle = \alpha^{-2}\alpha_j H_j\,|E'_\alpha\rangle = \alpha^{-2}\alpha_j\,|[H_j,\ E'_\alpha]\rangle = \alpha^{-2}\alpha_j\,|\alpha_j E'_\alpha\rangle = |E'_\alpha\rangle
$$

This alternatively indicates that the state $|E'_\alpha\rangle$ is an eigenstate of E_3 belonging to eigenvalue $E_3 = 1$. As result, a contradiction emerges:

- $|E'_\alpha\rangle$ cannot be the lowest value of E_3.

The above contradiction shows that the generator E'_α cannot exist. E_α is the unique generator related to the root vector $\vec{\alpha}$.

Master formula :

More generaically, for any weight $\vec{\mu}$ of a representation D of Lie algebra $\mathfrak{g}$, the E_3 value is determined by,

$$E_3 |\mu, \xi, D\rangle = \alpha^{-2} \vec{\alpha} \cdot \vec{H} |\mu, \xi, D\rangle = \alpha^{-2} \vec{\alpha} \cdot \vec{\mu} |\mu, \xi, D\rangle$$

Because the E_3's value must be integers or half odd integers,

$$\frac{2\vec{\alpha} \cdot \vec{\mu}}{\alpha^2} = \text{integer}$$

From the perspective of E_3 related $su(2)$ subalgebra, this eigenvalue equation suggests that the state $|\mu, \xi, D\rangle$ is among the spin-j representation of this $su(2)$ for some non-negative half integer j.

Accurately, there is some non-negative integer p such that,

$$|jj\rangle_{su(2)} = (E_+)^p |\mu, \xi, D\rangle \neq 0$$

on which

$$E_3 |jj\rangle_{su(2)} = j |jj\rangle_{su(2)}, \quad E_+ |jj\rangle_{su(2)} = (E_+)^{p+1} |\mu, \xi, D\rangle = 0 \ .$$

Notice that

$$
\begin{aligned}
&[E_3,\ E_\pm] = \pm E_\pm \\
&[E_3,\ (E_\pm)^2] = E_\pm[E_3,\ E_\pm] + [E_3,\ E_\pm]E_\pm = \pm 2\ (E_\pm)^2 \\
&[E_3,\ (E_\pm)^3] = E_\pm[E_3,\ (E_\pm)^2] + [E_3,\ E_\pm]\ (E_\pm)^2 = \pm 3\ (E_\pm)^3 \\
&\qquad\qquad\cdots \\
&[E_3,\ (E_\pm)^p] = \pm p\ (E_\pm)^p
\end{aligned}
$$

we get,

$$
\begin{aligned}
j\,|jj\rangle_{su(2)} &= E_3(E_+)^p\,|\mu,\xi,D\rangle \\
&= [E_3,\ (E_+)^p]\,|\mu,\xi,D\rangle + (E_+)^p E_3\,|\mu,\xi,D\rangle \\
&= p(E_+)^p\,|\mu,\xi,D\rangle + (E_+)^p\left(\alpha^{-2}\vec{\mu}\cdot\vec{\alpha}\right)|\mu,\xi,D\rangle \\
&= (p+\alpha^{-2}\vec{\mu}\cdot\vec{\alpha})(E_+)^p\,|\mu,\xi,D\rangle = (p+\alpha^{-2}\vec{\mu}\cdot\vec{\alpha})\,|jj\rangle_{su(2)}
\end{aligned}
$$

i.e.,

$$
j = p + \frac{\vec{\mu}\cdot\vec{\alpha}}{\alpha^2}
$$

Likewise, there is some non-negative integer q such that,

$$
|j,-j\rangle_{su(2)} = (E_-)^q\,|\mu,\xi,D\rangle \neq 0
$$

$$
E_3\,|j,-j\rangle_{su(2)} = -j\,|j,-j\rangle_{su(2)}\,, \qquad E_-\,|j,-j\rangle_{su(2)} = (E_-)^{q+1}\,|\mu,\xi,D\rangle = 0\ .
$$

From these equations we see that there is another expression for the highest eigenvalue j of E_3,

$$\begin{aligned}
-j\,|j,-j\rangle_{su(2)} &= E_3(E_-)^q\,|\mu,\xi,D\rangle \\
&= [E_3,\ (E_-)^q]\,|\mu,\xi,D\rangle + (E_-)^q E_3\,|\mu,\xi,D\rangle \\
&= -q(E_-)^q\,|\mu,\xi,D\rangle + (E_-)^q\left(\alpha^{-2}\vec{\mu}\cdot\vec{\alpha}\right)|\mu,\xi,D\rangle \\
&= (-q+\alpha^{-2}\vec{\mu}\cdot\vec{\alpha})(E_-)^q\,|\mu,\xi,D\rangle \\
&= (-q+\alpha^{-2}\vec{\mu}\cdot\vec{\alpha})\,|j,-j\rangle_{su(2)}
\end{aligned}$$

i.e.,

$$j = q - \frac{\vec{\mu}\cdot\vec{\alpha}}{\alpha^2}$$

Comparison of the above two expressions of j yields $j=(p+q)/2$ and the so-called Master formula :

$$\frac{2\vec{\mu}\cdot\vec{\alpha}}{\alpha^2} = q - p$$

1. In master formula, p and q are two non-negative integers.
2. For a given weight $\vec{\mu}$ and root $\vec{\alpha}$, p and q are determined by

$$(E_\alpha)^{p+1}\,|\mu,\xi,D\rangle = 0, \qquad (E_{-\alpha})^{q+1}\,|\mu,\xi,D\rangle = 0$$

respectively.

For each weight vector $\vec{\mu}$ of the representation D of Lie algebra $\mathfrak{g}$, there is a spin-j representation $\left[j=(p+q)/2\right]$ of $su(2)$ subalgebra $\{E_3,\ E_\pm\}$ related to the root vector $\vec{\alpha}$,

- Its $(2j+1)$ basis states are as follows:

  $$(E_{-\alpha})^q\,|\mu,\xi,D\rangle\,,\ \ (E_{-\alpha})^{q-1}\,|\mu,\xi,D\rangle\,,\ \ \cdots,E_{-\alpha}\,|\mu,\xi,D\rangle\,,\ \ |\mu,\xi,D\rangle\,,$$
  $$E_\alpha\,|\mu,\xi,D\rangle\,,\ \ (E_\alpha)^2\,|\mu,\xi,D\rangle\,,\ \ \cdots,(E_\alpha)^{p-1}\,|\mu,\xi,D\rangle\,,\ \ (E_\alpha)^p\,|\mu,\xi,D\rangle\,.$$

  with

  $$\begin{cases} E_3(E_{-\alpha})^q\,|\mu,\xi,D\rangle & = & -\frac{(p+q)}{2}(E_{-\alpha})^q\,|\mu,\xi,D\rangle\,, \\ E_3(E_\alpha)^p\,|\mu,\xi,D\rangle & = & \frac{(p+q)}{2}(E_\alpha)^p\,|\mu,\xi,D\rangle\ \ . \end{cases}$$

- In view of the mother algebra $\mathfrak{g}$, the weights of these states are given by,

  $$\vec{\mu}+n\vec{\alpha},\qquad(-q\leqslant n\leqslant p).$$

- Because roots of Lie algebra $\mathfrak{g}$ are weights of its adjoint representation. For each root vector $\vec{\beta}$, there is a root vector chain as follows:

  $$\vec{\beta}+n\vec{\alpha},\qquad(-q\leqslant n\leqslant p).$$

  where the non-negative integers p and q are determined by conditions that both $\vec{\beta}+(p+1)\vec{\alpha}$ and $\vec{\beta}-(q+1)\vec{\alpha}$ are not roots.

The structure constants $\mathcal{N}_{\alpha,\beta}$ appear in the Lie brackets,

$$[E_\alpha,\ E_\beta] = \mathcal{N}_{\alpha,\beta} E_{\alpha+\beta}$$

Properties of $\mathcal{N}_{\alpha,\beta}$:

- Evidently, $\mathcal{N}_{\alpha,\beta} = -\mathcal{N}_{\beta,\alpha}$.
- There is a one-to-one correspondence between the generators and the root vectors. Therefore, only when all of $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\alpha}+\vec{\beta}$ are root vectors of Lie algebra $\mathfrak{g}$, $\mathcal{N}_{\alpha,\beta} \neq 0$. Otherwise, $\mathcal{N}_{\alpha,\beta} = 0$.
- For root vector chain $\{\ \vec{\beta} + n\vec{\alpha} \mid -q \leqslant n \leqslant p\ \}$,

$$\mathcal{N}_{\alpha,(\beta+p\alpha)} = \mathcal{N}_{-\alpha,(\beta-q\alpha)} = 0$$

- In adjoint representation, $\langle E_\alpha | E_\beta \rangle = \delta_{\alpha\beta}$. So, for three non-zero root vectors α, β and $\alpha+\beta$,

$$\begin{aligned} \langle E_\alpha |\, E_{-\beta}\, | E_{\alpha+\beta} \rangle &= \langle E_\alpha | [E_{-\beta},\ E_{\alpha+\beta}] \rangle = \langle E_\alpha | \mathcal{N}_{-\beta,\alpha+\beta} E_\alpha \rangle \\ &= \mathcal{N}_{-\beta,\alpha+\beta} \langle E_\alpha | E_\alpha \rangle = -\mathcal{N}_{\alpha+\beta,-\beta} \end{aligned}$$

Alternatively, $\langle E_\beta |\, E_{-\alpha} = \langle E_\beta |\, E_\alpha^\dagger = \langle [E_\alpha,\ E_\beta] |$ leads to,

$$
\begin{aligned}
\langle E_\alpha |\, E_{-\beta}\, | E_{\alpha+\beta} \rangle &= \langle [E_\beta,\ E_\alpha] | E_{\alpha+\beta}] \rangle \\
&= \langle \mathcal{N}_{\beta,\alpha} E_{\alpha+\beta} | E_{\alpha+\beta} \rangle \\
&= \mathcal{N}_{\beta,\alpha} \langle E_{\alpha+\beta} | E_{\alpha+\beta} \rangle \ = -\mathcal{N}_{\alpha,\beta}
\end{aligned}
$$

Therefore,

$$
\mathcal{N}_{\alpha+\beta,-\beta} = \mathcal{N}_{\alpha,\beta}\ .
$$

- Consider the generators related to the root vector chain $\{\ \vec{\beta} + n\vec{\alpha}\ \}$ with $-q \leqslant n \leqslant p$. Let

$$
F_n = -\mathcal{N}_{\beta+n\alpha,\alpha} \mathcal{N}_{\beta+(n+1)\alpha,-\alpha}
$$

we see $F_p = F_{-q-1} = 0$. Moreover,

$$
\begin{aligned}
0 &= [E_{\beta+n\alpha},\ [E_\alpha,\ E_{-\alpha}]] + [E_\alpha,\ [E_{-\alpha},\ E_{\beta+n\alpha}]] + [E_{-\alpha},\ [E_{\beta+n\alpha},\ E_\alpha]] \\
&= \alpha_j [E_{\beta+n\alpha},\ H_j] + \mathcal{N}_{-\alpha,\beta+n\alpha}[E_\alpha,\ E_{\beta+(n-1)\alpha}] \\
&\qquad + \mathcal{N}_{\beta+n\alpha,\alpha}[E_{-\alpha},\ E_{\beta+(n+1)\alpha}] \\
&= -\alpha_j(\beta_j + n\alpha_j) E_{\beta+n\alpha} + \mathcal{N}_{-\alpha,\beta+n\alpha}\mathcal{N}_{\alpha,\beta+(n-1)\alpha} E_{\beta+n\alpha} \\
&\qquad + \mathcal{N}_{\beta+n\alpha,\alpha}\mathcal{N}_{-\alpha,\beta+(n+1)\alpha} E_{\beta+n\alpha} \\
&= \Big[-\vec{\alpha}\ \cdot\ (\vec{\beta} + n\vec{\alpha}) - F_{n-1} + F_n \Big] E_{\beta+n\alpha}
\end{aligned}
$$

This yields a recursion relation :

$$
F_n = F_{n-1} + \vec{\alpha}\ \cdot\ (\vec{\beta} + n\vec{\alpha})
$$

Therefore,

$$
\begin{aligned}
F_n &= F_{n-1} + \vec{\alpha}\cdot(\vec{\beta}+n\vec{\alpha}) \\
&= F_{n-2} + \vec{\alpha}\cdot(\vec{\beta}+n\vec{\alpha}) + \vec{\alpha}\cdot\left[\vec{\beta}+(n-1)\vec{\alpha}\right] \\
&= F_{n-3} + \vec{\alpha}\cdot(\vec{\beta}+n\vec{\alpha}) + \vec{\alpha}\cdot\left[\vec{\beta}+(n-1)\vec{\alpha}\right] + \vec{\alpha}\cdot\left[\vec{\beta}+(n-2)\vec{\alpha}\right] \\
&= \cdots \\
&= F_{n-(n+q+1)} + \textstyle\sum_{i=0}^{n+q} \vec{\alpha}\cdot\left[\vec{\beta}+(n-i)\vec{\alpha}\right] \\
&= F_{-q-1} + (n+q+1)(\vec{\alpha}\cdot\vec{\beta}) \\
&\qquad + \left[n(n+q+1) - \tfrac{1}{2}(n+q+1)(n+q)\right](\vec{\alpha}\cdot\vec{\alpha}) \\
&= \tfrac{1}{2}(n+q+1)\left[2(\vec{\alpha}\cdot\vec{\beta}) + (n-q)\alpha^2\right]
\end{aligned}
$$

When $n=p$, this equation is reduced to the expected master formula,

$$\frac{2(\vec{\alpha}\cdot\vec{\beta})}{\alpha^2} = q-p$$

When $n=0$, it gives

$$F_0 = \frac{1}{2}(q+1)\left[2(\vec{\alpha}\cdot\vec{\beta}) - q\alpha^2\right] = -\frac{1}{2}p(q+1)\alpha^2$$

Notice that $F_0 = -\mathcal{N}_{\beta,\alpha}\mathcal{N}_{\beta+\alpha,-\alpha} = -\mathcal{N}_{\beta,\alpha}\mathcal{N}_{\beta,\alpha}$, we finally get:

$$(\mathcal{N}_{\alpha,\beta})^2 = \frac{1}{2}p(q+1)\alpha^2$$

Angle between two roots :

Consider the scalar product of root vectors $\vec{\alpha}$ and $\vec{\beta}$,

$$\frac{2(\vec{\alpha}\cdot\vec{\beta})}{\alpha^2} = q - p$$

or

$$\frac{2(\vec{\alpha}\cdot\vec{\beta})}{\beta^2} = q' - p'$$

The first master formula implies the existence of root vector chain $\{\ \vec{\beta} + n\vec{\alpha}\ \}$ with $-q \leqslant n \leqslant p$,while the second formula implies the existence of another root vector chain $\{\ \vec{\alpha} + n'\vec{\beta}\ \}$ with $-q' \leqslant n' \leqslant p'$. Hence,

$$\left(\cos\theta_{\alpha\beta}\right)^2 = \frac{(\vec{\alpha}\cdot\vec{\beta})^2}{\alpha^2\beta^2} = \frac{(q-p)(q'-p')}{4}$$

What is remarkable is that $(q-p)(q'-p')$ must be a non-negative integer.

Relying on the fact that

$$-1 \leqslant \cos\theta_{\alpha\beta} \leqslant 1$$

there are only 4 choices for the angle between two distinct root vectors:

Table : The possible angles between two distinct root vectors

$(q-p)(q'-p')$	$\theta_{\alpha\beta}$
0	$\pi/2$
1	$\pi/3$ or $2\pi/3$
2	$\pi/4$ or $3\pi/4$
3	$\pi/6$ or $5\pi/6$

The basic formula for such an angle is,

$$\cos\theta_{\alpha\beta} = \pm\frac{1}{2}\sqrt{(q-p)(q'-p')}$$

The possibility $(q-p)(q'-p') = 4$, which corresponds to $\theta_{\alpha\beta} = 0$ or $\theta_{\alpha\beta} = \pi$, is not interesting.

Problems :

1 Show that $[E_\alpha,\ E_\beta]$ must be proportional to $E_{\alpha+\beta}$. What happens if $\vec{\alpha} + \vec{\beta}$ is not a root vector ?

2 Suppose that the raising operators of some Lie algebra g satisfy $[E_\alpha,\ E_\beta] = \mathcal{N} E_{\alpha+\beta}$ for some nonzero $\mathcal{N}$. Calculate $[E_\alpha,\ E_{-\alpha-\beta}]$.

3 Consider the simple Lie algebra g formed by the 10 matrices

$$\{\sigma_a, \sigma_a\tau_1, \sigma_a\tau_3, \tau_2\}$$

for $a = 1$ to 3, where σ_a and τ_a are Pauli matrices in orthogonal spaces. Take $H_1 = \sigma_3$ and $H_2 = \sigma_3\tau_3$ as the Cartan generators. Find: (1) the weights of the 4-dimensional representation generated by these matrices; (2) the weights of the adjoint representation.

In its definition representation, $SU(3)$ is the group of 3×3 unitary matrices $\{u \mid uu^\dagger = u^\dagger u = 1\}$ with unity determinant ($\det u = 1$).

The group elements of $SU(3)$ have the form

$$u = e^{i \sum_{a=1}^{8} \alpha_a X_a}$$

with X_a a set of linearly independent 3×3 traceless hermitian generators:

$$\begin{array}{llll} X_1 = T_{12}^{(1)}, & X_2 = T_{12}^{(2)}, & X_3 = T_{2}^{(3)}, & X_4 = T_{13}^{(1)}, \\ X_5 = T_{13}^{(2)}, & X_6 = T_{23}^{(1)}, & X_7 = T_{23}^{(2)}, & X_8 = T_{3}^{(3)}. \end{array}$$

where $(T_{ab}^{(1)})_{ij} = \frac{1}{2}(\delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi})$, $(T_{ab}^{(2)})_{ij} = \frac{1}{2i}(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi})$ for $a \neq b$ and

$$(T_a^{(3)})_{ij} = \begin{cases} \delta_{ij}\dfrac{1}{\sqrt{2a(a-1)}}, & \text{if} \quad i < a\ ; \\ -\delta_{ij}\sqrt{\dfrac{a-1}{2a}}, & \text{if} \quad i = a\ ; \\ 0, & \text{if} \quad i > a. \end{cases}$$

We can recast the generators as $X_a = \lambda_a/2$. Such λ_a $(a = 1, 2, \cdots, 8)$ are called Gell-Mann matrices.

Gell-Mann matrices are explicitly written out as follows,

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \qquad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

The $SU(3)$ group is a compact Lie group, because its generators

$$X_a = \lambda_a/2 \qquad (a = 1, 2, \cdots, 8)$$

satisfy the uniform orthonormal conditions:

$$\mathrm{Tr}(X_a X_b) = \frac{1}{2}\delta_{ab}$$

Consequently, the structure constants $\{f_{abc}\}$ appearing in the Lie brackets $[X_a, \ X_b] = i f_{abc} X_c$ are completely antisymmetric.

With Gell-Mann matrices, the $su(3)$ algebra could be recast as:

$$[\lambda_a,\ \lambda_b] = 2if_{abc}\lambda_c$$

where f_{abc} are completely antisymmetric in the indices.
The nonzero f_{abc} are

$$\begin{aligned} &f_{123} = 1 \\ &f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = 1/2 \\ &f_{458} = f_{678} = \sqrt{3}/2 \end{aligned}$$

Besides, the Gell-Mann matrices have the following additional properties:

1. $\mathrm{Tr}(\lambda_a\lambda_b) = 2\delta_{ab}$
2. Completeness relation reads,

$$(\lambda_a)_{ij}(\lambda_a)_{kl} = -\frac{2}{3}\delta_{ij}\delta_{kl} + 2\delta_{il}\delta_{jk}$$

   where $i, j, k, l = 1,\ 2,\ 3.$
3. There exists a group of completely symmetric constants d_{abc} such that,

$$\{\lambda_a,\ \lambda_b\} = \frac{4}{3}\delta_{ab} + 2d_{abc}\lambda_c$$

For completeness, we list the nonzero components of d_{abc} below:

$$\begin{cases} d_{118} = d_{228} = d_{338} = 1/\sqrt{3} \\ d_{146} = d_{157} = d_{256} = d_{344} = d_{355} = 1/2 \\ d_{247} = d_{366} = d_{377} = -1/2 \\ d_{448} = d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}} \\ d_{888} = -1/\sqrt{3} \end{cases}$$

Casimir operators :

$SU(3)$ has two independent Casimir operators

$$C_2 = \sum_{a=1}^{8} X_a X_a, \qquad C_3 = \sum_{a,b,c=1}^{8} d_{abc} X_a X_b X_c$$

In definition representation, we have:

$$C_2 = 4/3, \qquad C_3 = 10/9.$$

Checking $\mathrm{Tr}(X_a X_b) = \frac{1}{2}\delta_{ab}$:

Notice that in $T^{(1)}_{ab}$ and $T^{(2)}_{ab}$, $a \neq b$. $T^{(3)}_{a}$ are diagonal matrices. Thus,

$$(T^{(1)}_{ab})_{ij}(T^{(1)}_{cd})_{ji} = \tfrac{1}{4}(\delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi})(\delta_{cj}\delta_{di} + \delta_{ci}\delta_{dj}) = \tfrac{1}{2}(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}),$$

$$(T^{(1)}_{ab})_{ij}(T^{(2)}_{cd})_{ji} = \tfrac{1}{4i}(\delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi})(\delta_{cj}\delta_{di} - \delta_{ci}\delta_{dj}) = 0,$$

$$(T^{(1)}_{ab})_{ij}(T^{(3)}_{c})_{ji} = \tfrac{1}{2}(\delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi})(T^{(3)}_{c})_{ji} = \tfrac{1}{2}\left[(T^{(3)}_{c})_{ab} + (T^{(3)}_{c})_{ba}\right] = 0,$$

$$(T^{(2)}_{ab})_{ij}(T^{(2)}_{cd})_{ji} = -\tfrac{1}{4}(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi})(\delta_{cj}\delta_{di} - \delta_{ci}\delta_{dj}) = \tfrac{1}{2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}),$$

$$(T^{(2)}_{ab})_{ij}(T^{(3)}_{c})_{ji} = \tfrac{1}{2i}(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi})(T^{(3)}_{c})_{ji} = \tfrac{1}{2i}\left[(T^{(3)}_{c})_{ba} - (T^{(3)}_{c})_{ab}\right] = 0,$$

Besides, when $a < b$,

$$(T^{(3)}_{a})_{ij}(T^{(3)}_{b})_{ji} = (a-1)\left[\frac{1}{\sqrt{2a(a-1)}} \cdot \frac{1}{\sqrt{2b(b-1)}}\right] - \sqrt{\frac{a-1}{2a}}\frac{1}{\sqrt{2b(b-1)}} = 0$$

while when $a = b$,

$$(T^{(3)}_{a})_{ij}(T^{(3)}_{a})_{ji} = (a-1)\left[\frac{1}{2a(a-1)}\right] + \frac{a-1}{2a} = \frac{1}{2}$$

Checking is finished.

Cartan generators :

Among these generators, there are two commute mutually and they form the Cartan generators of group $SU(3)$:

$$H_1 = X_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad H_2 = X_8 = \frac{1}{2\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Because H_1 and H_2 are already diagonal, the weights of $su(3)$ definition representation can be read off through

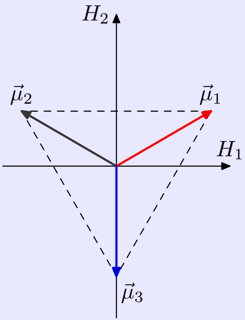
$$H_i \, |\vec{\mu}_a\rangle = (\vec{\mu}_a)_i \, |\vec{\mu}_a\rangle$$

with $i = 1, 2$ but $a = 1, 2, 3$. The result is as follows:

$\vec{\mu}_1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$	$\vec{\mu}_2 = \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$	$\vec{\mu}_3 = \left(0, -\frac{1}{\sqrt{3}}\right)$
$\|\vec{\mu}_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\|\vec{\mu}_2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\|\vec{\mu}_3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Weight diagram :

In weight diagram, these weight vectors form an equilateral triangle:



Here,

$$\vec{\mu}_1 = \Big(\frac{1}{2}, \frac{1}{2\sqrt{3}}\Big), \quad \vec{\mu}_2 = \Big(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\Big), \quad \vec{\mu}_3 = \Big(0, -\frac{1}{\sqrt{3}}\Big).$$

Among them, $\vec{\mu}_1$ is the highest weight vector.

Roots of $su(3)$:

Question :

How many root vectors does $su(3)$ algebra have ?

Because

- $su(3)$ has 6 non-Cartan generators.
- There is a one-to-one correspondence between the root vectors and the non-Cartan generators.

$su(3)$ has 6 distinct root vectors: half of which are positive, another half are negative.

The 3 distinct positive root vectors can be read off from the difference of weight vectors of the above definition representation:

$$\vec{\alpha}_1 = \vec{\mu}_1 - \vec{\mu}_2 = (1, 0)$$
$$\vec{\alpha}_2 = \vec{\mu}_1 - \vec{\mu}_3 = (1/2, \sqrt{3}/2)$$
$$\vec{\alpha}_3 = \vec{\mu}_3 - \vec{\mu}_2 = (1/2, -\sqrt{3}/2)$$

Their negative counterparts are,

$$-\vec{\alpha}_1 = (-1, 0), \quad -\vec{\alpha}_2 = (-1/2, -\sqrt{3}/2), \quad -\vec{\alpha}_3 = (-1/2, \sqrt{3}/2).$$

The corresponding generators are those that have only one offidiagonal entry,

$$E_{\pm\alpha_1} = \frac{1}{\sqrt{2}}(X_1 \pm iX_2), \qquad E_{\pm\alpha_2} = \frac{1}{\sqrt{2}}(X_4 \pm iX_5),$$
$$E_{\pm\alpha_3} = \frac{1}{\sqrt{2}}(X_6 \mp iX_7).$$

Explicitly,

$$E_{\alpha_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_{\alpha_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_{\alpha_3} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and

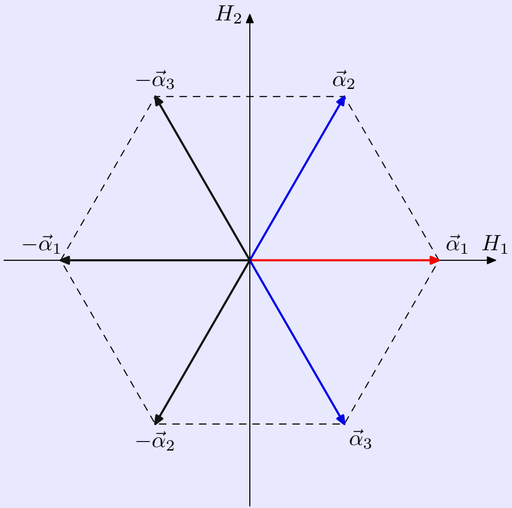
$$E_{-\alpha_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_{-\alpha_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$E_{-\alpha_3} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

In weight diagram, the 6 non-zero root vectors of $su(3)$

$$\pm\vec{\alpha}_1 = (\pm 1, 0), \quad \pm\vec{\alpha}_2 = (\pm 1/2, \pm\sqrt{3}/2), \quad \pm\vec{\alpha}_3 = (\pm 1/2, \mp\sqrt{3}/2),$$

form a regular hexagon:



Problems :

1. Calculate f_{147} and f_{458} in the $su(3)$ definition representation.
2. The $SU(3)$ structure constants have the property $f_{acd}f_{bcd} = 3\delta_{ab}$. Please show
$$f_{abc}\lambda_b\lambda_c = 3i\lambda_a$$
and
$$\lambda_b\lambda_a\lambda_b = -2\lambda_a/3$$
by making use of this relation.
3. Show that X_1, X_2 and X_3 generate an $su(2)$ subalgebra of $su(3)$. How does the representation generated by the Gell-Mann matrices transform under this subalgebra ?