现代数学物理方法

第二章，群论基础

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Why Group Theory ?

Group Theory is the study of symmetries.

Symmetries in Physics :

- Gauss law in electrostatics,

$$\oint \vec{E} \cdot d\vec{s} = Q/\epsilon_0 \quad \rightsquigarrow \quad \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q\,\vec{r}}{r^3}$$

- The dynamical law for a charged particle in electromagnetic field,

$$\frac{d\vec{p}}{dt} = q\vec{E} + \vec{J} \times \vec{B}$$

- Lagrangian describing Strong, weak and electromagnetic interactions,

$$\mathscr{L}_{\text{int}} \sim ig\bar{\Psi}\gamma^{\mu}\Psi T^i A^i_{\mu}$$

Group:

A group G is a set of elements with a rule for assigning to every (ordered) pair of elements, satisfying

- If $f, g \in G$, then $fg \in G$.
- For $f, g, h \in G$, $f(gh) = (fg)h$.
- There is an identity element, e, such that for all $f \in G$, $ef = fe = f$.
- Every element $f \in G$ has an inverse, f^{-1}, such that $ff^{-1} = f^{-1}f = 1$.

Therefore, a group G is a multiplication table specifying $g_1 g_2$ for both g_1 and g_2 belonging to G.   e.g.,

	e	g_1	g_2
e	e	g_1	g_2
g_1	g_1	$g_1 g_1$	$g_1 g_2$
g_2	g_2	$g_2 g_1$	$g_2 g_2$

Focus:

Our focus in this course will be on the Group Representation Theory.

Group Representations:

A representation $D(G)$ of group G is a mapping between the elements $g \in G$ and a set of linear operators $D(g)$ with the properties,

1. $D(e) = 1$
2. $D(g_1)D(g_2) = D(g_1 g_2)$

The representation of a group G does also form a group.

Finite group: Z_3

A group is finite if it has a finite number of elements. The number of elements in a finite group G is called the order of G.
The group Z_3 is a finite group of order 3.

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Notice that every row and column of the multiplication table contains each group elements exactly once. This is because

$$a^2 = b, \;\; b^2 = a, \;\; ab = ba = e \quad \rightsquigarrow \; e^{-1} = e, \; a^{-1} = b$$

An Abelian group is one in which the multiplication of arbitrary two elements is commutative,

$$g_1 g_2 = g_2 g_1$$

Evidently, Z_3 is Abelian.

Finite group: Z_3

A representation of Z_3:

$$D(e) = 1, \;\; D(a) = e^{2\pi i/3}, \;\; D(b) = e^{-2\pi i/3}.$$

Multiplication table reads,

	$D(e)$	$D(a)$	$D(b)$
$D(e)$	$D(e)$	$D(a)$	$D(b)$
$D(a)$	$D(a)$	$D(b)$	$D(e)$
$D(b)$	$D(b)$	$D(e)$	$D(a)$

$=$

	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$
1	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$
$e^{2\pi i/3}$	$e^{2\pi i/3}$	$e^{-2\pi i/3}$	1
$e^{-2\pi i/3}$	$e^{-2\pi i/3}$	1	$e^{2\pi i/3}$

The dimension of a representation is the dimension of the linear space on which the operators in the representation act. Hence, *The above representation of Z_3 is 1-dimensional.*

Here is another representation of Z_3, which is 3-dimensional,

$$D(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D(a) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad D(b) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

This is called the regular representation of Z_3.

Definition :

The regular representation of a group is constructed by taking the group elements $\{g_1, \ g_2, \ \cdots\}$ themselves as the orthonormal base vectors $\{|g_1\rangle, \ |g_2\rangle, \ \cdots\}$ of the representation space,

$$D_{\mathsf{reg}}(g_1)\,|g_2\rangle = |g_1 g_2\rangle$$

Hence,

$$[D_{\mathsf{reg}}(g)]_{ij} = \langle g_i|\, D_{\mathsf{reg}}(g)\,|g_j\rangle = \langle g_i|g g_j\rangle$$

The dimension of $D_{\mathsf{reg}}(G)$ is the order of group G.

$D_{\mathrm{reg}}(Z_3)$:

We now construct the regular representation of Z_3. Let $|1\rangle = |e\rangle$, $|2\rangle = |a\rangle$ and $|3\rangle = |b\rangle$ and

$$\langle i|j\rangle = \delta_{ij}, \quad \sum_{i=1}^{3} |i\rangle\langle i| = 1,$$

we get

$$\begin{aligned}
&[D_{\mathrm{reg}}(a)]_{11} = \langle e|ae\rangle = \langle e|a\rangle = 0, \quad [D_{\mathrm{reg}}(a)]_{12} = \langle e|aa\rangle = \langle e|b\rangle = 0,\\
&[D_{\mathrm{reg}}(a)]_{13} = \langle e|ab\rangle = \langle e|e\rangle = 1, \quad [D_{\mathrm{reg}}(a)]_{21} = \langle a|ae\rangle = \langle a|a\rangle = 1,\\
&[D_{\mathrm{reg}}(a)]_{22} = \langle a|aa\rangle = \langle a|b\rangle = 0, \quad [D_{\mathrm{reg}}(a)]_{23} = \langle a|ab\rangle = \langle a|e\rangle = 0,\\
&[D_{\mathrm{reg}}(a)]_{31} = \langle b|ae\rangle = \langle b|a\rangle = 0, \quad [D_{\mathrm{reg}}(a)]_{32} = \langle b|aa\rangle = \langle b|b\rangle = 1,\\
&[D_{\mathrm{reg}}(a)]_{33} = \langle b|ab\rangle = \langle b|e\rangle = 0.
\end{aligned}$$

Namely,

$$D(a) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Similarly we can get another matrices $D_{\text{reg}}(e)$ and $D_{\text{reg}}(b)$ of the regular representation of group Z_3.

Trace of a matrix is defined as the sum of its diagonal elements. Therefore, for a regular representation of a group G, we have:

$$\mathrm{Tr}[D_{\text{reg}}(e)] = N, \quad \mathrm{Tr}[D_{\text{reg}}(g)] = 0 \ (g \neq e),$$

where N is the order of the group G.

- A general p-dimensional representation of G is spanned by p orthonormal base vectors $\{|1\rangle, \ |2\rangle, \cdots, |p\rangle\}$ satisfying the conditions $\langle i|j\rangle = \delta_{ij}$ and $\sum_i |i\rangle\langle i| = 1$. The representation matrices are defined as:

$$[D(g)]_{ij} = \langle i| D(g) |j\rangle, \quad g \in G$$

  These matrices do indeed form a representation of the G, relying on the fact $D(g_1 g_2) = D(g_1)D(g_2)$.

Equivalent Representations

What makes the idea of group representations so powerful is the fact that they live in linear spaces. The powerful thing about linear spaces is that we are free to choose the base vectors (states) by making a linear transformation, $|\psi\rangle \rightsquigarrow |\psi'\rangle = S^{-1}|\psi\rangle$.

Such a transformation on the base vectors of the linear space induces a similarity transformation on the linear operators:

$$D(g) \quad \rightsquigarrow D'(g) = S^{-1}D(g)S$$

Obviously, $D'(G)$ is a representation of G if $D(G)$ is,

1. $D'(e) = 1$
2. $D'(g_1 g_2) = D'(g_1)D'(g_2)$

$D'(G)$ and $D(G)$ are said to be equivalent because they differ just by a trivial choice of base vectors.

Unitary Representations:

1. A representation of group $G = \{g\}$ is unitary if and only if all the matrix elements $\{D(g)\}$ of $D(G)$ are unitary,

$$[D(g)]^\dagger = [D(g)]^{-1}, \ \ \forall g \in G$$

2. It will turn out that all representations of finite groups are equivalent to unitary representations.

Examples:

Both given two representations of Abelian group Z_3 are unitary:

- 1-dimensional representation:

$$D_1(e) = 1,\ \ D_1(a) = e^{2\pi i/3},\ \ D_1(b) = e^{-2\pi i/3}.$$

- 3-dimensional representation:

$$D_2(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ D_2(a) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$
$$D_2(b) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Reducible Representations:

A representation is called reducible if it has an invariant subspace: *the action of any $D(g)$ on any vector in the subspace is still in the subspace.*

Projection operator:

Let P_1 be the projection operator of the subspace S_1 of space S, then

1. $P_1 S = S_1$
2. $P_1^2 = P_1$

Consequently, P_1 is an identity operator on S_1: $P_1 |\varphi\rangle = |\varphi\rangle$, $\forall\, |\varphi\rangle \in S_1$.

If $D(G)$ has an invariant subspace (so that D is reducible), we have:

$$(1 - P_1)D(g)P_1 = 0, \quad \forall g \in G$$

$$\rightsquigarrow \qquad D(g)P_1 \sim P_1, \qquad \forall g \in G$$

Examples :

- The trivial $D = \{D(g) = 1,\ \forall g \in G\}$ of every group G is a reducible representation.

- The regular representation of Z_3 is reducible, due to the fact it has an invariant subspace projected on by

$$P = P^2 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Checking : Because

$$D_{\text{reg}}(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad D_{\text{reg}}(a) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$
$$D_{\text{reg}}(b) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

we have:

$$D_{\text{reg}}(g)P = P,\quad \forall g \in Z_3$$

Irreducible Representations:

A representation is *irreducible* if it has no nontrivial invariant space.

Completely Reducible Representations:

A representation is *completely reducible* if it is equivalent to a representation whose matrix elements have the following block diagonal form:

$$D(g) = \begin{bmatrix} D_1(g) & 0 & \cdots \\ 0 & D_2(g) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \qquad \forall \; g \in G$$

where $D_j(G) = \{D_j(g)\}$ are *irreducible* representations of G for all subscripts j.

1. A representation D in block diagonal form is said to be the direct sum of the sub-representations D_j,

$$D = D_1 \oplus D_2 \oplus \cdots \oplus D_M = \oplus_{j=1}^{M} D_j$$

Consequently, *A completely reducible representation can be decomposed into a direct sum of irreducible representations.*

Question:
Construct a similarity transformation so that the regular representation of Z_3 is written as the direct sum of some of its irreducible representations.

Solution:
Consider the unitary matrix S,

$$S = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \end{bmatrix}$$

we see:

1.

$$
\begin{aligned}
D'_{\text{reg}}(e) &= S^\dagger D_{\text{reg}}(e) S \\
&= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

2.

$$
\begin{aligned}
D'_{\text{reg}}(a) &= S^\dagger D_{\text{reg}}(a) S \\
&= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{bmatrix}\begin{bmatrix} 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{-2\pi i/3} \end{bmatrix}
\end{aligned}
$$

3.

$$
\begin{aligned}
D'_{\text{reg}}(b) &= S^\dagger D_{\text{reg}}(b) S \\
&= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{bmatrix}\begin{bmatrix} 1 & e^{-2\pi i/3} & e^{2\pi i/3} \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-2\pi i/3} & 0 \\ 0 & 0 & e^{2\pi i/3} \end{bmatrix}
\end{aligned}
$$

Hence, in $D_{\text{reg}}(Z_3)$, the involved irreducible representations of Abelian group $Z_3 = \{e, a, b\}$ are

1. $D_1(Z_3) = \{1, 1, 1\}$
2. $D_2(Z_3) = \{1, e^{2\pi i/3}, e^{-2\pi i/3}\}$
3. $D_3(Z_3) = \{1, e^{-2\pi i/3}, e^{2\pi i/3}\}$

All of these irreducible representations are 1-dimensional.

Transformation Groups:

There is a natural multiplication law for transformations of a physics system.

If the transformation group $G = \{g\}$ is the symmetry of a quantum mechanical system, then,

- For each group element g, there is a unitary operator $D(g)$ that maps the Hilbert space into itself,

$$D(g): \quad |\psi\rangle \quad \rightarrow \quad |\psi'\rangle = D(g)\,|\psi\rangle$$

- The full set of these unitary operators $\{D(g)\}$ form a unitary representation of G on the Hilbert space.

- The transformed states are subject to the same Schrödinger equation as the original states,

$$\left.\begin{array}{l} i\hbar\frac{d}{dt}\,|\psi\rangle = H\,|\psi\rangle \\ \\ i\hbar\frac{d}{dt}\,[D(g)\,|\psi\rangle] = H\,[D(g)\,|\psi\rangle] \end{array}\right\} \quad \rightsquigarrow \quad [D(g), H] = 0$$

$[D(g), H] = 0$ implies:

1. The transformed states have the same energy as the original states.
2. The full set of the energy eigenstates belonging to the same energy eigenvalue forms a complete set of basis vectors of an irreducible representation of the transformation group G.

Problems:

1. Find the multiplication table for a group with 3 elements and prove that it is unique.
2. Find all essentially different multiplication tables for groups with 4 elements (which can not be related by renaming elements).