LIE GROUPS 2013 FALL
PROBLEM SET 2

(1) (a) Prove: The exponential map $\exp : \mathfrak{so}(2) \to \mathrm{SO}(2)$ is surjective.
(b) Prove: The exponential map $\exp : \mathfrak{gl}(2,\mathbb{R}) \to \mathrm{GL}_+(2,\mathbb{R})$ is not surjective.
(c) Is $\mathrm{SL}(2,\mathbb{R})$ is connected? Is $\exp : \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ surjective?

(2) Suppose G is a Lie group, with Lie algebra $\mathfrak{g}$. Its center is by definition
$$Z(G) = \{g \in G \mid gg' = g'g \text{ for all } g' \in G\}$$
Prove $Z(G)$ is a normal Lie subgroup of G, and find its Lie algebra $Z(\mathfrak{g})$.

(3) (a) Let $\varphi : G \to G$ a Lie group automorphism. Let H be the set of fixed points of φ, i.e.
$$H = \{g \in G \mid \varphi(g) = g.\}$$
Prove H is a Lie subgroup of G, and find the Lie algebra of H.
(b) Realize $\mathrm{O}(n)$ and $\mathrm{SL}(n,\mathbb{R})$ as the fixed point set of some automorphism of $\mathrm{GL}(n,\mathbb{R})$.

(4) (a) Show that the universal cover group of $S^1 \times S^1$ is given by the map
$$\mathbb{R} \times \mathbb{R} \to S^1 \times S^1, \quad (s,t) \mapsto (e^{2\pi is}, e^{2\pi it}).$$
(b) Show that the group of Lie group automorphisms of $S^1 \times S^1$ is $\mathrm{SL}(2,\mathbb{Z})$.

(5) Let G be a connected Lie group, and H and K two commuting closed Lie subgroups of G such that $H \cap K = \{e\}$. Moreover, assume $\dim H + \dim K = \dim G$. Prove: G is isomorphic to $K \times H$.

(6) Let $H \subset \mathrm{GL}(3,\mathbb{R})$ be the Heisenberg group, i.e. the group of all 3×3 upper triangular real matrices whose diagonal entries are 1.
(a) Find the Lie algebra $\mathfrak{h}$ of H. What is $Z(H)$? What is $Z(\mathfrak{h})$?
(b) Prove: The exponential map $\exp : \mathfrak{h} \to H$ is a diffeomorphism.
(c) Find the formula for $\mu(X,Y)$ so that $\exp(\mu(X,Y)) = \exp X \exp Y$.

(7) (a) Let G be the Lie group $\mathbb{C} \times \mathbb{C}$ with group multiplication
$$(z,w) \cdot (z',w') = (z + z', w + e^z w').$$
Check that G is a simply connected Lie group, and find its center.
(b) For each $n \in \mathbb{N}$, let G_n be the Lie group $\mathbb{C}^* \times C$ with group multiplication
$$(z,w) \cdot (z',w') = (zz', w + z^n w').$$
Check that G_n is a Lie group, and G is the universal cover group of G.
(c) Show that for $n \neq m$, G_n is not isomorphic to G_m.

(8) (a) Let $\mathfrak{g}$ be a two dimensional Lie algebra. Prove: Either $\mathfrak{g}$ is abelian, or there exists a basis $\{X,Y\}$ of $\mathfrak{g}$ so that $[X,Y] = Y$.
(b) For each case in (a), find a simply connected Lie group G whose Lie algebra is $\mathfrak{g}$.