现代数学物理方法

第二章，群论基础

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Parity:

Parity is the operation of reflection in a mirror. *Reflecting twice gets you back to where you started,*

$$p^2 = e$$

The group including parity operation is Z_2:

	e	p
e	e	p
p	p	e

Representations of Z_2:

- Z_2 has only 2 irreducible representations. The first one is trivial,

$$D_1(e) = D_1(p) = 1.$$

- The second irreducible representation of Z_2 consists of

$$D_2(e) = 1, \;\; D_2(p) = -1.$$

- *Any representation of Z_2 is completely reducible.* The Hilbert space of any parity invariant system can be decomposed into states that behave like irreducible representations, on which $D(p)$ is either 1 or -1.
    1. The energy eigensates on which $D(p) = 1$ have an even parity.
    2. The energy eigensates on which $D(p) = -1$ have an odd parity.

Definition:

S_3 is the permutation group (or symmetric group) on 3 objects,

$$a_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} = (123) = (231) = (312)$$

$$a_2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = (132) = (213) = (321)$$

$$a_3 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} = (12) = (21)$$

$$a_4 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} = (23) = (32)$$

$$a_5 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} = (13) = (31)$$

$$e = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

Properties:

Basically,

1. $(ab) = (ba)$
2. $(ab)(ba) = e$
3. $(ab)(bc) = (abc)$

In general,

1. $(123 \cdots N) = (12)(23)(34) \cdots (N-1, N)$
2. $(123 \cdots N) = (1N)(1, N-1)(1, N-2) \cdots (13)(12)$

$$\rightsquigarrow \quad a_1 a_2 = (123)(321) = e, \quad a_1 a_3 = (123)(12) = (13) = a_5$$

Generators:

S_3 has *two* generators. They can be chosen as

$$\{a_1 = (123), \quad a_3 = (12)\}$$

From these generators, we have $a_2 = a_1 a_1$, $a_4 = a_3 a_1$, $a_5 = a_1 a_3$ and $e = a_1 a_1 a_1 = a_3 a_3$.

Non-Abelian:

S_3 is non-abelian because its multiplication law is not commutative. *e.g.*,

$$a_4 = a_3 a_1 \neq a_1 a_3 = a_5$$

It is the lack of commutativity that makes group theory very useful in *physics*.

Multiplication Table of S_3:

	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	a_2	a_3	a_4	a_5
a_1	a_1	a_2	e	a_5	a_3	a_4
a_2	a_2	e	a_1	a_4	a_5	a_3
a_3	a_3	a_4	a_5	e	a_1	a_2
a_4	a_4	a_5	a_3	a_2	e	a_1
a_5	a_5	a_3	a_4	a_1	a_2	e

Permutation group is an important transformation group in quantum mechanics, in particular in the system of identical particles.

An irreducible representation of S_3:

$$D(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D(a_1) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

$$D(a_2) = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix} \quad D(a_3) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D(a_4) = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \qquad D(a_5) = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$$

Discussions:

- The nontrivial representations of a non-Abelian group must be *matrices* rather than numbers. Only matrices can reproduce the non-commutative multiplication laws.
- In an irreducible representation, Not all of the matrices are diagonal.

Question:

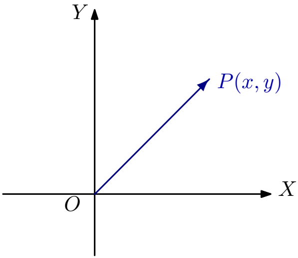
How to obtain this representation ?

My Explanation:

The two generators of S_3 obey,

$$(a_1)^3 = (a_3)^2 = 1$$

We can identify a_1 by a rotation in XY plane at an angle $2\pi/3$ with respect to X-axis, and a_3 a reflection about Y-axis. Therefore, on an arbitrary vector, $\vec{r} = x\vec{i} + y\vec{j} \sim \begin{bmatrix} x \\ y \end{bmatrix}$,



we have:

$$D(a_3)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

Hence,

$$D(a_3) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly,

$$D(a_1) = \begin{bmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

Based on these two generators, we get:

$$
\begin{aligned}
D(a_2) &= [D(a_1)]^2 \\
&= \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \\
&= \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
D(a_4) &= D(a_3)D(a_1) \\
&= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \\
&= \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
D(a_5) &= D(a_1)D(a_3) \\
&= \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}
\end{aligned}
$$

Of course,

$$
\begin{aligned}
D(e) &= [D(a_3)]^2 \\
&= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Addition of integers:

The integers form an infinite group $\mathbb{Z}$ under addition:

$$x \circ y := x + y$$

Checking:

1. If x and y are integers, $x + y$ is also an integer.
2. For three integers x, y and z, $(x + y) + z = x + (y + z)$.
3. Identity element exists, $e = 0$.
4. Inverse elements exist, $x^{-1} = -x$.

Multiplication table:

Since this group is infinite, the explicit *multiplication table* for it is impossible.

The additive group $\mathbb{Z}$ has a representation as follows:

$$D(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \quad \forall x \in \mathbb{Z}$$

Checking:

$$D(e) = \begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D(x)D(y) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+y \\ 0 & 1 \end{bmatrix} = D(x+y)$$

This representation is reducible but it is not completely reducible.

Reducibility:

Construct the projection operator P for subspace spanned by the base vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Because

$$D(x)P_1 = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = P_1$$

this representation is reducible.

However,

$$D(x)P_2 = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & 1 \end{bmatrix} \neq P_2$$

Therefore, it is not completely reducible.

Theorem 1:

Every representation of a finite group is equivalent to a unitary representation.

Proof:

Suppose $D(G)$ is a representation of a finite group $G = \{g\}$, from which we can construct a hermitian matrix S,

$$S = \sum_{g \in G} \left[D(g)\right]^\dagger D(g)$$

Consider the eigenvalue equation of this hermitian matrix,

$$S \left|\lambda_n\right\rangle = \lambda_n \left|\lambda_n\right\rangle, \quad n = 1,\ 2,\ 3,\ \cdots$$

Hence,

$$\lambda_n = \left\langle\lambda_n\right| S \left|\lambda_n\right\rangle = \left\langle\lambda_n\right| \sum_{g \in G} \left[D(g)\right]^\dagger D(g) \left|\lambda_n\right\rangle = \sum_{g \in G} \left\| D(g) \left|\lambda_n\right\rangle \right\|^2$$

Proof (continued):

i.e.,

$$\lambda_n = \|D(e)\,|\lambda_n\rangle\|^2 + \cdots \geqslant \|D(e)\,|\lambda_n\rangle\|^2 = \||\lambda_n\rangle\|^2 > 0$$

All of the eigenvalues of the hermitian matrix S are not only *real* but also *positive*.

As is well known, a hermitian matrix can be diagonalized via a unitary transformation,

$$S = U^\dagger \begin{bmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} U$$

Relying on the fact that $\lambda_n > 0$, the square root of S is also a hermitian matrix

$$X = \sqrt{S} = U^\dagger \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots \\ 0 & \sqrt{\lambda_2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} U$$

Proof (continued):

This hermitian matrix is invertible,

$$X^{-1} = \frac{1}{\sqrt{S}} = U^\dagger \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 & \cdots \\ 0 & \frac{1}{\sqrt{\lambda_2}} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} U$$

Construct a similarity transformation with this invertible X, we have:

$$D'(g) = X\, D(g)\, X^{-1}, \quad \forall g \in G$$

The new representation $D'(G)$ is equivalent to the old representation $D(G)$. Moreover, *it is unitary*.

Proof (continued):

$$
\begin{aligned}
\left[D'(g)\right]^\dagger D'(g) &= \left[XD(g)X^{-1}\right]^\dagger XD(g)X^{-1} \\
&= (X^{-1})^\dagger \left[D(g)\right]^\dagger X^\dagger XD(g)X^{-1} \\
&= X^{-1}\left[D(g)\right]^\dagger X^2 D(g)X^{-1} \\
&= X^{-1}\left[D(g)\right]^\dagger SD(g)X^{-1} \\
&= X^{-1}\left[D(g)\right]^\dagger \left\{ \textstyle\sum_{h\in G} \left[D(h)\right]^\dagger D(h) \right\} D(g)X^{-1} \\
&= X^{-1}\left\{ \textstyle\sum_{h\in G} \left[D(hg)\right]^\dagger D(hg) \right\} X^{-1} \\
&= X^{-1}SX^{-1} = 1
\end{aligned}
$$

Theorem 2:

Every representation of a finite group is completely reducible.

Proof:

- It is sufficient to consider unitary representations.
- If the representation is irreducible, the required proof is achieved because it is already in block diagonal form.
- If the representation $D(G) = \{D(g)\}$ is reducible, there exists a projection operator P_1 such that

$$(1 - P_1)D(g)P_1 = 0, \quad \forall\, g \in G$$

Taking its hermitian conjugation gives,

$$\begin{aligned} 0 &= P_1 \left[D(g)\right]^{\dagger} (1 - P_1) = P_1 \left[D(g)\right]^{-1} (1 - P_1) \\ &= P_1 D(g^{-1})(1 - P_1), \quad \forall g \in G \end{aligned}$$

Proof (continued):

- Equivalently,

$$P_1 D(g)(1 - P_1) = 0, \quad \forall g \in G$$

This equation demonstrates that the subspace of the complementary projection operator $P_2 = (1 - P_1)$ is also invariant under $D(G)$:

$$(1 - P_2)D(g)P_2 = 0, \quad \forall\, g \in G$$

- By induction, we eventually completely reduce the representation $D(G)$.

Subgroups:

Subgroup :

A group H whose elements are all elements of a group G is called a subgroup of G.

Examples :

1. The identity e. (trivial)
2. The group G itself. (trivial)
3. $S_3 = \{e, a_1, a_2, a_3, a_4, a_5\}$ has the following nontrivial subgroups:

$$
\begin{aligned}
G_1 &= \{e, a_1, a_2\} \\
G_2 &= \{e, a_3\} \\
G_3 &= \{e, a_4\} \\
G_4 &= \{e, a_5\}
\end{aligned}
$$

Cosets:

Right Coset of subgroup H:

The right coset of subgroup H in G is the set of elements of the form Hg for some *fixed* element $g \in G$.

Examples:

The cosets of subgroup $Z_3 = \{e, a_1, a_2\}$ of the permutation group S_3 consist of the following elements,

$$Z_3 a_1 = \{e, a_1, a_2\} a_1 = \{a_1, a_2, e\} = Z_3$$
$$Z_3 a_4 = \{e, a_1, a_2\} a_4 = \{a_4, a_3, a_5\}$$

Properties:

- The number of elements in each coset is the order of subgroup H.
- Every element of G must belong to one and only one coset.
- For a finite group G, the order of its subgroup H must be a factor of the order of G.

Coset space G/H:

It is the linear space in which each coset of subgroup H is taken as a single element.

Normal Subgroup:

A subgroup H of G is called an invariant or normal subgroup if for every $g \in G$,

$$gH = Hg$$

- The trivial subgroups e and G are normal for any group G.
- If H is normal, $gH = Hg$, the coset space G/H forms a group under the same multiplication law in G:

$$(Hg_1)(Hg_2) = H(g_1 H)g_2 = H(Hg_1)g_2 = H(g_1 g_2) \in G/H$$

In this case, the coset space G/H is called Factor group of G by H.

Normal subgroup of S_3:

1. Among the nontrivial subgroups of S_3, only is Z_3 the normal subgroup:

$$\begin{aligned}
eZ_3 &= e\{e, a_1, a_2\} = \{e, a_1, a_2\} = \{e, a_1, a_2\}e = Z_3 e \\
a_1 Z_3 &= a_1\{e, a_1, a_2\} = \{a_1, a_2, e\} = \{e, a_1, a_2\}a_1 = Z_3 a_1 \\
a_2 Z_3 &= a_2\{e, a_1, a_2\} = \{a_2, e, a_1\} = \{e, a_1, a_2\}a_2 = Z_3 a_2 \\
a_3 Z_3 &= a_3\{e, a_1, a_2\} = \{a_3, a_4, a_5\} = \{e, a_2, a_1\}a_3 = Z_3 a_3 \\
a_4 Z_3 &= a_4\{e, a_1, a_2\} = \{a_4, a_5, a_3\} = \{e, a_2, a_1\}a_4 = Z_3 a_4 \\
a_5 Z_3 &= a_5\{e, a_1, a_2\} = \{a_5, a_3, a_4\} = \{e, a_2, a_1\}a_5 = Z_3 a_5
\end{aligned}$$

2. The other subgroups of S_3 are not normal subgroups. e.g.,

$$a_5\{e, a_4\} = \{a_5, a_2\} \neq \{a_5, a_1\} = \{e, a_4\}a_5$$

3. The factor group S_3/Z_3 is,

$$S_3/Z_3 = Z_2 \quad \rightsquigarrow \quad Z_2 \text{ is parity group.}$$

Center of a group:

The center of a group G is the set of all elements of G that commute with all elements of G.

Discussions:

1. The center is always an Abelian, normal subgroup of G.
2. It may be trivial, consisting only of the identity, or of the whole group G.

Homework:

1. There is a simple n-dimensional representation D of S_n called the defining representation, where the objects being permuted are just the basis vectors of an n-dimensional vector space:

$$|1\rangle\,,\ |2\rangle\,,\ \cdots\,,\ |n\rangle$$

The representation D is defined as $D[(\xi_j \xi_k)]\,|j\rangle = |k\rangle$. Show that this representation is reducible.