# 现代数学物理方法

## 第二章，群论基础

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# Projection Operator:

- Characters can be used to decompose an reducible representation into its irreducible ingredients. The key bridge to this end is the **Projection Operator** of an irreducible component representation.

Let $D(G)$ be an arbitrary representation of finite group $G = \{g\}$ (of order $N$) that contains an $n_a$-dimensional irreducible representation $D_a(G)$ with characters $\{\chi_a(g)\}$. Then

$$P_a = \frac{n_a}{N} \sum_{g \in G} \chi^*_{D_a}(g) D(g)$$

is the projection operator onto the subspace of $D_a(G)$.

The matrix elements of $P_a$ in a given representation space of $D(G)$ read

$$[P_a]_{ij} = \frac{n_a}{N} \sum_{g \in G} \chi^*_{D_a}(g) \, [D(g)]_{ij}$$

**Explanation:**

Recall that every representation of a finite group is equivalently unitary and completely reducible,

$$D(g) = \oplus_{a=1}^{s} c_a D_a(g), \quad \forall\, g \in G$$

we see,

$$[P_a]_{ij} = \frac{n_a}{N} \sum_{g \in G} \chi^*_{D_a}(g)\, [D(g)]_{ij} \; = \frac{n_a}{N} \sum_{g \in G} \chi^*_{D_a}(g)\, [\oplus_{b=1}^{s} c_b D_b(g)]_{ij}$$

Recall the orthogonality relations between irreducible representations:

$$\frac{n_a}{N} \sum_{g \in G} [D_a(g)]^*_{jk}\, [D_b(g)]_{lm} = \delta_{ab}\delta_{jl}\delta_{km}$$

We have

$$\frac{n_a}{N} \sum_{g \in G} \chi^*_{D_a}(g)\, [D_b(g)]_{lm} = \delta_{ab}\delta_{lm}$$

Hence, $P_a$ gives 1 on the subspaces that transform like $D_a(G)$ and 0 on all the other subspaces.

# An example:

**Question:**

Here is a 3-dimensional representation of $S_3$,

$$D(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad D(a_1) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad D(a_2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$D(a_3) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad D(a_4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad D(a_5) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

1. Is it irreducible ?
2. Is it the regular representation of $S_3$ ?
3. Evaluate the projection operators of the irreducible representations of $S_3$ in this 3-dimensional reducible representation.

**Solution:**

1. No. It is not an irreducible because its dimension is $n = 3$, violating the required relation $\sum_a n_a^2 = 6$.

2. No. The regular representation of $S_3$ should be 6-dimensional.

3. The projection operators of 3 irreducible representations of $S_3$ are evaluated from $P_a = \frac{n_a}{N} \sum_{g \in G} \chi^*_{D_a}(g) D(g)$. The results are as follows:

$$P_0 = \frac{1}{6} \sum_{g \in S_3} D(g) = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P_1 = \frac{1}{6} \left[ D(e) + \sum_{j=1}^{2} D(a_j) - \sum_{j=3}^{5} D(a_j) \right] = 0$$

$$P_2 = \frac{2}{6} \left[ 2D(e) - \sum_{j=1}^{2} D(a_j) \right] = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Simple calculations lead to $(P_j)^2 = P_j = (P_j)^\dagger$ for $j = 0, 1, 2$. Hence, $D = D_0 \oplus D_2$.

# QM Background:

In QM, we are interested in the eigenstates of an invariant hermitian operator, in particular the invariant hamiltonian under group $G$,

$$[D(g), H] = 0$$

where

$$H\left|n\right\rangle = \lambda_n \left|n\right\rangle, \quad n = 0, 1, 2, \cdots$$

**Theorem:**

1. If $H$ commutes with all the elements $\{D(g)\}$ of a representation of group $G$, then you can choose the eigenstates of $H$ to transform according to irreducible representations of $G$.
2. If an irreducible representation appears only once in the Hilbert space, every state in the irreducible representation is an eigenstate of $H$ with the same eigenvalue.

**Proof:**

- Due to the assumption that $[D(g), H] = 0$, the transformations in the representation $D(G)$ do not change the eigenvalues of operator $H$,

$$\begin{aligned} H\,|n\rangle &= \lambda_n\,|n\rangle\,, \\ H\,[D(g)\,|n\rangle] &= D(g)H\,|n\rangle \;=\; \lambda_n\,[D(g)\,|n\rangle] \end{aligned}$$

- If $G$ is finite, $D(G)$ can be decomposed into a direct sum of some irreducible representations $D_i(G)$:

$$D(G) = \oplus_i D_i(G)$$

Thus we can divide up the Hilbert space into some subspaces:

1. The $i$-th subspace is labelled by the eigenvalue $\lambda_i$ of $H$.
2. The $i$-th subspace furnishes an irreducible representation $D_i(G)$ of group $G$.

- Eigenvectors $\{|i, \alpha\rangle; \alpha = 1, 2, \cdots, n_i\}$ of $H$ belonging to $\lambda_i$

$$H\,|i, \alpha\rangle = \lambda_i\,|i, \alpha\rangle$$

can be chosen in terms of the irreducible representation $D_i(G)$:

$$g: \quad D_i(g)\,|i, \alpha\rangle = |i, \beta\rangle, \quad \forall g \in G$$

where $\alpha, \beta = 1, 2, \cdots, n_i$ and $i = 1, 2, 3, \cdots$.

- Consider an arbitrary vector in the whole Hilbert space,

$$|a,\ j,\ x\rangle, \quad 1 \leqslant j \leqslant n_a,$$

where $x$ stands for the times the $D_i(G)$ appearing in Hilbert space. Then,

$$H\,|a,\ j,\ x\rangle = \sum_y c_y\,|a,\ j,\ y\rangle$$

If $x$ and $y$ take only one value, $|a,\ j,\ x\rangle$ becomes an eigenvector of $H$.

# Tensor product representation:

**Question:**

How to put known representations together to form a new representation (with higher dimensions) ?

Suppose that $D_1$ is an $m$-dimensional representation acting on a space with basis vectors

$$|i\rangle, \quad (i = 1, 2, \cdots, m)$$

$D_2$ is an $n$-dimensional representation acting on a space with basis vectors

$$|\alpha\rangle, \quad (\alpha = 1, 2, \cdots, n)$$

We can make an $mn$-dimensional representation space, called the tensor product space, by defining its basis vectors as,

$$|i, \alpha\rangle = |i\rangle \otimes |\alpha\rangle, \qquad (i = 1, 2, \cdots, m; \;\; \alpha = 1, 2, \cdots, n)$$

In this space we define the so-called tensor product representation $D_{1\times 2} = D_1 \otimes D_2$,

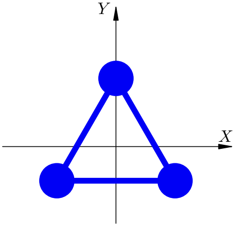
$$\langle i, \alpha| \, D_{1\times 2}(g) \, |j, \beta\rangle \equiv \langle i| \, D_1(g) \, |j\rangle \cdot \langle \alpha| \, D_2(g) \, |\beta\rangle$$

**Remarks:**

1. The tensor product representation is indeed a representation of group $G$ [Homework (optional)].
2. In general, the tensor product representation is not an irreducible representation.
3. One of our favorite pastimes is to decompose a reducible tensor representation into the direct sum of irreducible representations of the group $G$.

**Example:**

Three blocks are connected by springs in a triangle. The system is suposed to be free to slide on a frictionless surface.

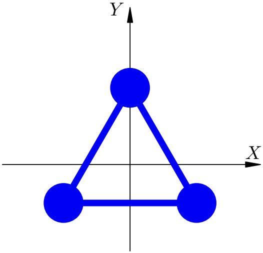


**Properties of the model:**

- The system has an $S_3$ symmetry.
- The system has 6 degrees of freedom, described by the $x$ and $y$ coordinates of the 3 blocks:

$$|\vec{r}\rangle = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} r_{11} \\ r_{12} \\ r_{21} \\ r_{22} \\ r_{31} \\ r_{32} \end{bmatrix} = |r_{i\alpha}\rangle$$

where $i$ labels coordinate $x$ or $y$, and $\alpha$ labels the blocks.



- This 6-dimensional configuration space can be viewed as a tensor product space of a 3-dimensional space of the blocks

$$|\xi\rangle = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

and a 2-dimensional space of coordinates $x$ and $y$,

$$|\zeta\rangle = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$$

That is:

$$|r_{i\alpha}\rangle = |\xi\rangle \otimes |\zeta\rangle$$

Namely,

$$r_{i\alpha} = \xi_i \zeta_\alpha, \quad (i = 1, 2, 3; \;\; \alpha = 1, 2.)$$

- Suppose that the representations of $S_3$ on 3-dimensional space $\{|\xi\rangle\}$ and 2-dimensional space $\{|\zeta\rangle\}$ could *respectively* be given by the previous $D_3$,

$$D_3(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad D_3(a_1) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$D_3(a_2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad D_3(a_3) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D_3(a_4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad D_3(a_5) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and $D_2$,

$$D_2(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D_2(a_1) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

$$D_2(a_2) = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix} \quad D_2(a_3) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D_2(a_4) = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \quad D_2(a_5) = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$$

we have a 6-dimensional representation $D_6(S_3)$ whose elements read,

$$[D_6(S_3)]_{i\alpha j\beta} = [D_3(S_3)]_{ij} \cdot [D_2(S_3)]_{\alpha\beta}$$

The characters of $D_6(S_3)$ are:

$$\begin{aligned} \chi_6(S_3) \quad &= \sum_{i\alpha} [D_6(S_3)]_{i\alpha i\alpha} = \left\{ \sum_i [D_3(S_3)]_{ii} \right\} \cdot \left\{ \sum_\alpha [D_2(S_3)]_{\alpha\alpha} \right\} \\ &= \chi_3(S_3)\chi_2(S_3) \end{aligned}$$

**Theorem:**

The characters of a tensor product representation are the products of the characters of the factor representations,

$$\chi_{D_1 \times D_2} = \chi_{D_1} \chi_{D_2}$$

The characters of $D_6(S_3)$ are then given by,

| | $\{e\}$ | $\{a_1, a_2\}$ | $\{a_3, a_4, a_5\}$ |
|---|---|---|---|
| $\chi_3$ | 3 | 0 | 1 |
| $\chi_2$ | 2 | $-1$ | 0 |
| $\chi_6$ | 6 | 0 | 0 |

$D_6(S_3)$ has the same characters as the regular representation $D_{\text{reg}}(S_3)$. Consequently,

1. $D_6(S_3)$ and $D_{\text{reg}}(S_3)$ are equivalent to each other (because the similarity transformations do not change the characters).
2. $D_6(S_3)$ contains $D_0$ and $D_1$ once but $D_2$ twice.

For completeness, we write down explicitly an element of $D_6(S_3)$:

$$
\begin{aligned}
D_6(a_1) &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 & -1/2 & -\sqrt{3}/2 \\ 0 & 0 & 0 & 0 & \sqrt{3}/2 & -1/2 \\ -1/2 & -\sqrt{3}/2 & 0 & 0 & 0 & 0 \\ \sqrt{3}/2 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & -\sqrt{3}/2 & 0 & 0 \\ 0 & 0 & \sqrt{3}/2 & -1/2 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

# Group $S_n$:

**Permutation group $S_n$:**

1. Any element of the permutation group $S_n$ on $n$-objects can be expressed in terms of cycles. e.g.,

$$\begin{cases} e = (1)(2)\cdots(n) \\ a_1 = (12)(3)(4)\cdots(n) \\ a_j = (1243)(5)(6)(79)(8)\cdots(n) \end{cases}$$

2. Each cycle is written as a set of numbers in parentheses, indicating the set of objects that are cyclically permuted.
3. Each element of $S_n$ involves each integer from 1 to $n$ in exactly one cycle.

**Illustration:**

- $(1)$ means $x_1 \to x_1$.
- $(1372)$ means $x_1 \to x_3 \to x_7 \to x_2 \to x_1$.

**Definition of $j$-cycle in $S_n$:**

In $S_n$, a $j$-cycle is defined as

$$(\xi_1\xi_2\xi_3\cdots\xi_j), \quad 1 \leqslant j \leqslant n.$$

If an element of $S_n$ has $k_j$ $j$-cycles, then

$$\sum_{j=1}^{n} jk_j = n$$

**An Example in $S_9$:**

$$(123)(456)(78)(9) \rightsquigarrow \begin{cases} k_1 = k_2 = 1 \\ k_3 = 2 \\ k_4 = k_5 = \cdots = k_9 = 0 \end{cases}$$

## Interchange:

An interchange is a 2-cycle, the permutation between two objects,

$$(\xi_i \xi_j), \quad 1 \leqslant i, j \leqslant n, \ \ (i \neq j)$$

**Remarks:**

- Except the trivial 1-cycle, each group element in $S_n$ can be written out in terms of the ordered product of interchanges. *e.g.* in $S_9$,

$$(123)(456)(78)(9) = (12)(23)(45)(56)(78)(9)$$

- The inner automorphism built from "interchanges" does not change the *cycle structure* $\{k_1 k_2 \cdots k_n\}$ of any element in $S_n$.

$$
\begin{aligned}
(\xi_j \xi_i)(\cdots \xi_1 \xi_i \xi_2 \cdots)(\cdots \xi_3 \xi_j \xi_4 \cdots)(\xi_i \xi_j) \\
= (\cdots \xi_1 \xi_j \xi_2 \cdots)(\cdots \xi_3 \xi_i \xi_4 \cdots) \\
(\xi_j \xi_i)(\cdots \xi_1 \xi_i \xi_2 \cdots \xi_3 \xi_j \xi_4 \cdots)(\xi_i \xi_j) \\
= (\cdots \xi_1 \xi_j \xi_2 \cdots \xi_3 \xi_i \xi_4 \cdots)
\end{aligned}
$$

Therefore, the inner automorphism $gg_1g^{-1}$ built from an arbitrary permutation $g \in S_n$ does not change the cycle structure of element $g_1 \in S_n$.

**Examples in $S_4$:**

1. $(12) \cdot (1234) \cdot (12) = (1342)$
2. $(12) \cdot (23) \cdot (12) = (13)$
3. $(12) \cdot (13)(24) \cdot (12) = (14)(23)$

# Conjugacy classes in $S_n$:

1. In $S_n$, the conjugacy classes consist of all possible permutations with a particular cycle structure.

2. The conjugacy classes can be labeled by the set of integers $\{k_1, k_2, \cdots, k_n\}$, where $k_i$ is the number of $i$-cycle but $i$ the *length* of $i$-cycle[1].

3. The number of group elements in each conjugacy class $\{k_1, k_2, \cdots, k_n\}$ of $S_n$ is,

$$\# = \frac{n!}{\prod_{j=1}^{n} j^{k_j} (k_j)!}$$

[1] For example, the group elements $(1)(234)$, $(2)(341)$, $(3)(412)$ and $(4)(123)$ in $S_4$ are in the same conjugacy class.

**Proof:**

Each permutation of objects (from $1$ to $n$) gives a permutation in the class, the total number is $n!$. Hence, the number of group elements in class $\{k_1, k_2, \cdots, k_n\}$ should be proportional to $n!$,

$$\# \propto n!$$

But cyclic order doesn't matter within a cycle, e.g., $(1234)$ is the same as $(2341)$, $(3412)$ and $(4123)$,

$$\# \propto \frac{n!}{\prod_{j=1}^{n} j^{k_j}}$$

Furthermore, the order does not matter also at all between cycles of the same length, e.g., $(12)(34)$ is the same as $(34)(12)$,

$$\rightsquigarrow \quad \# = \frac{n!}{\prod_{j=1}^{n} j^{k_j}} \cdot \frac{1}{\prod_{j=1}^{n} (k_j)!} = \frac{n!}{\prod_{j=1}^{n} j^{k_j} (k_j)!}$$

In $S_3$, there are totally 3 conjugacy classes[2]:

$$\mathcal{C}_1 = \{e\}, \;\; \mathcal{C}_2 = \{(12), (23), (31)\}, \;\; \mathcal{C}_3 = \{(123), (321)\}$$

The number of group elements in each class is calculated as,

$$\begin{aligned}
\#\mathcal{C}_1 &= \frac{3!}{(1^3 \cdot 3!)(2^0 \cdot 0!)(3^0 \cdot 0!)} = 1 \\
\#\mathcal{C}_2 &= \frac{3!}{(1^1 \cdot 1!)(2^1 \cdot 1!)(3^0 \cdot 0!)} = 3 \\
\#\mathcal{C}_3 &= \frac{3!}{(1^0 \cdot 0!)(2^0 \cdot 0!)(3^1 \cdot 1!)} = 2
\end{aligned}$$

[2] In $S_3$, $e = (1)(2)(3)$ and the group element $(12)$ stands for $(12)(3)$, and so on.

# Example: $S_4$

There are totally 5 conjugacy classes in $S_4$,

$$
\begin{aligned}
\mathcal{C}_1 &= \{e\} \\
\mathcal{C}_2 &= \{(12),(13),(14),(23),(24),(34)\} \\
\mathcal{C}_3 &= \{(123),(124),(134),(234),(321),(421),(431),(432)\} \\
\mathcal{C}_4 &= \{(12)(34),(13)(24),(14)(23)\} \\
\mathcal{C}_5 &= \{(1234),(1243),(1324),(1342),(1423),(1432)\}
\end{aligned}
$$

The number of group elements in each class is calculated as follows:

$$
\begin{aligned}
\#\mathcal{C}_1 &= \frac{4!}{(1^4\cdot 4!)(2^0\cdot 0!)(3^0\cdot 0!)(4^0\cdot 0!)} = 1 \\
\#\mathcal{C}_2 &= \frac{4!}{(1^2\cdot 2!)(2^1\cdot 1!)(3^0\cdot 0!)(4^0\cdot 0!)} = 6 \\
\#\mathcal{C}_3 &= \frac{4!}{(1^1\cdot 1!)(2^0\cdot 0!)(3^1\cdot 1!)(4^0\cdot 0!)} = 8
\end{aligned}
$$

# Homework:

$$\#\mathcal{C}_4 = \frac{4!}{(1^0 \cdot 0!)(2^2 \cdot 2!)(3^0 \cdot 0!)(4^0 \cdot 0!)} = 3$$
$$\#\mathcal{C}_5 = \frac{4!}{(1^0 \cdot 0!)(2^0 \cdot 0!)(3^0 \cdot 0!)(4^1 \cdot 1!)} = 6$$

**Problems:**

1. How many conjugacy classes are there in symmetric group $S_6$ ?
   How many group elements are there in each of these classes ?