现代数学物理方法

第四章, $SU(N)$

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$SU(N)$:

Special unitary group $SU(N)$ has (N^2-1) hermitian generators T_a, $a = 1, 2, \cdots, (N^2-1)$.

In defining Rep., T_a are hermitian, traceless, $N \times N$ matrices with normalization

$$\mathrm{Tr}\left\{T_a T_b\right\} = \frac{1}{2}\delta_{ab}$$

They can be defined as a generalization of the Gell-Mann matrices:

$$[T_{ab}^{(1)}]_{ij} = \frac{1}{2}\left\{\delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi}\right\}$$

$$[T_{ab}^{(2)}]_{ij} = -\frac{i}{2}\left\{\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}\right\}$$

$$[T_c^{(3)}]_{ij} = \begin{cases} \delta_{ij}\frac{1}{\sqrt{2c(c-1)}}, & \text{if } \; i < c\,; \\ -\delta_{ij}\sqrt{\frac{(c-1)}{2c}}, & \text{if } \; i = c\,; \\ 0, & \text{if } \; i > c. \end{cases}$$

where $a, b = 1, 2, \cdots, N$ but $a < b$, and $c = 1, 2, \cdots, N-1$.

The $N-1$ generators $T_c^{(3)}$ form the Cartan subalgebra of $su(N)$. We relabel them as $H_m,\ m=1,2,\cdots,N-1$. In defining Rep.,

$$[H_m]_{ij} = \frac{1}{\sqrt{2m(m+1)}}\left[\sum_{k=1}^{m}\delta_{ik} - m\delta_{i,m+1}\right]\delta_{ij}$$

The generators of the raising and lowering operators are defined by,

$$E_{\pm\alpha_{ab}} = \frac{1}{\sqrt{2}}\left[T_{ab}^{(1)} \pm iT_{ab}^{(2)}\right]$$

so that

$$E^{\dagger}_{\pm\alpha_{ab}} = E_{\mp\alpha_{ab}}, \qquad \mathrm{Tr}\left\{E_{\alpha_{ab}}E_{-\alpha_{cd}}\right\} = \frac{1}{2}\delta_{ac}\delta_{bd}.$$

In defining Rep.,

$$[E_{\alpha_{ab}}]_{ij} = \frac{1}{\sqrt{2}}\delta_{ai}\delta_{bj}, \qquad [E_{-\alpha_{ab}}]_{ij} = \frac{1}{\sqrt{2}}\delta_{aj}\delta_{bi}.$$

Weights of defining Rep. of $SU(N)$:

The defining Rep. of $SU(N)$ has dimension N. It can be characterized by N independent weights

$$\nu^j, \qquad j = 1, 2, \cdots, N$$

Each weight ν^j is a $(N-1)$-dimensional vector in weight space, whose m-th component reads,

$$[\nu^j]_m = [H_m]_{jj} = \frac{1}{\sqrt{2m(m+1)}} \left[\sum_{k=1}^{m} \delta_{jk} - m\delta_{j,m+1} \right]$$

They satisfy,

$$\nu^i \cdot \nu^j = -\frac{1}{2N} + \frac{1}{2}\delta_{ij}$$

So the weights all have the same length, $|\nu^i|^2 = (N-1)/2N$, and the angles between any two distinct weights are equal:

$$\nu^i \cdot \nu^j = -\frac{1}{2N} \quad \text{for} \quad i \neq j.$$

Proof :

For $j = 1, 2, \cdots, N$, we have

$$
\begin{aligned}
(\nu^j)^2 &= \sum_{m=1}^{N-1} [\nu^j]_m [\nu^j]_m = \sum_{m=1}^{N-1} \frac{1}{2m(m+1)} \left[\sum_{k=1}^{m} \delta_{jk} - m\delta_{j,m+1} \right]^2 \\
&= \sum_{m=1}^{j-1} \frac{1}{2m(m+1)} [-m\delta_{j,m+1}]^2 \\
&\qquad + \sum_{m=j}^{N-1} \frac{1}{2m(m+1)} \left[\sum_{k=1}^{m} \delta_{jk} - m\delta_{j,m+1} \right]^2 \\
&= \frac{(j-1)^2}{2j(j-1)} + \sum_{m=j}^{N-1} \frac{1}{2m(m+1)} \\
&= \frac{(j-1)}{2j} + \frac{1}{2} \sum_{m=j}^{N-1} \left(\frac{1}{m} - \frac{1}{m+1} \right) = \frac{(j-1)}{2j} + \frac{1}{2} \left(\frac{1}{j} - \frac{1}{N} \right) \\
&= \frac{N-1}{2N}
\end{aligned}
$$

and for $i < j$,

$$
\begin{aligned}
\nu^i \cdot \nu^j &= \sum_{m=1}^{N-1} [\nu^i]_m [\nu^j]_m \\
&= \sum_{m=1}^{N-1} \frac{1}{2m(m+1)} \left[\sum_{k=1}^{m} \delta_{ik} - m\delta_{i,m+1} \right] \left[\sum_{l=1}^{m} \delta_{jl} - m\delta_{j,m+1} \right] \\
&= -\frac{1}{2j} \sum_{m=1}^{j-1} \left[\sum_{k=1}^{m} \delta_{ik} - m\delta_{i,m+1} \right] \delta_{m,j-1} \\
&\qquad + \sum_{m=j}^{N-1} \frac{1}{2m(m+1)} \left[\sum_{k=1}^{m} \delta_{ik} - m\delta_{i,m+1} \right] \\
&= -\frac{1}{2j} + \sum_{m=j}^{N-1} \frac{1}{2m(m+1)} \\
&= -\frac{1}{2j} + \frac{1}{2} \left(\frac{1}{j} - \frac{1}{N} \right) = -\frac{1}{2N}
\end{aligned}
$$

Explicitly, the m-th component[1] of $su(N)$ weights in its defining representation read

$$
\begin{aligned}
\left[\nu^1\right]_m &= \frac{1}{\sqrt{2m(m+1)}} \\
\left[\nu^2\right]_m &= \frac{1}{\sqrt{2m(m+1)}}\left(\sum_{k=1}^{m}\delta_{k2} - \delta_{m,1}\right) \\
\left[\nu^3\right]_m &= \frac{1}{\sqrt{2m(m+1)}}\left(\sum_{k=1}^{m}\delta_{k3} - 2\delta_{m,2}\right) \\
&\dots \\
\left[\nu^j\right]_m &= \frac{1}{\sqrt{2m(m+1)}}\left[\sum_{k=1}^{m}\delta_{kj} - (j-1)\delta_{m,j-1}\right] \\
&\dots \\
\left[\nu^N\right]_m &= -\sqrt{\frac{N-1}{2N}}\,\delta_{m,N-1}
\end{aligned}
$$

[1] Evidently, $1 \leqslant m \leqslant N-1$.

We see, for all possible m $(1 \leqslant m \leqslant N-1)$,

$$
\begin{aligned}
\textstyle\sum_{j=1}^{N}\left[\nu^{j}\right]_{m} &= \frac{1}{\sqrt{2m(m+1)}}\sum_{j=1}^{N}\left[\sum_{k=1}^{m}\delta_{kj} - m\delta_{j,m+1}\right] \\
&= \frac{1}{\sqrt{2m(m+1)}}\left[\sum_{j,k=1}^{m}\delta_{kj} - m\sum_{j=1}^{N}\delta_{j,m+1}\right] \\
&= \frac{1}{\sqrt{2m(m+1)}}\left[m-m\right] \\
&= 0
\end{aligned}
$$

It turns out to be the traceless condition of the Cartan generator H_m. Namely,

$$\sum_{j=1}^{N}\nu^{j} = 0$$

This result is an implication of the fact that *in $(N-1)$-dimensional weight space, the maximum number of independent vectors is $N-1$.*

The $su(N)$ weights in its defining representation are listed below:

$$\nu^1 = \left[\frac{1}{2}, \frac{1}{2\sqrt{3}}, \cdots, \frac{1}{\sqrt{2m(m+1)}}, \cdots, \frac{1}{\sqrt{2N(N-1)}}\right]$$

$$\nu^2 = \left[-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \cdots, \frac{1}{\sqrt{2m(m+1)}}, \cdots, \frac{1}{\sqrt{2N(N-1)}}\right]$$

$$\nu^3 = \left[0, -\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{6}} \cdots, \frac{1}{\sqrt{2m(m+1)}}, \cdots, \frac{1}{\sqrt{2N(N-1)}}\right]$$

$$\cdots$$

$$\nu^m = \left[0, 0, \cdots, \frac{1}{\sqrt{2m(m+1)}}, \cdots, \frac{1}{\sqrt{2N(N-1)}}\right]$$

$$\nu^{m+1} = \left[0, 0, \cdots, -\frac{m}{\sqrt{2m(m+1)}}, \cdots, \frac{1}{\sqrt{2N(N-1)}}\right]$$

$$\cdots$$

$$\nu^N = \left[0, 0, \cdots, 0, \cdots, -\frac{N-1}{\sqrt{2N(N-1)}}\right]$$

Discussions :

- ν^1 is the highest weight of the defining representation of $su(N)$

$$\nu^1 = \left[\frac{1}{2}, \frac{1}{2\sqrt{3}}, \cdots, \frac{1}{\sqrt{2m(m+1)}}, \cdots, \frac{1}{\sqrt{2N(N-1)}}\right]$$

and

$$\nu^1 > \nu^2 > \nu^3 > \cdots > \nu^{N-1} > \nu^N$$

- The raising and lowering operators take us from one weight to another, so the $su(N)$ roots α_{ij} are differences of its weights, $\alpha_{ij} = \nu^i - \nu^j$ for $i \neq j$.
- The roots all have length 1.

$$\begin{aligned}
(\nu^i - \nu^j)^2 &= (\nu^i)^2 + (\nu^j)^2 - 2\nu^i \cdot \nu^j \\
&= 2\left(\frac{N-1}{2N}\right) - 2\left(\frac{1}{2}\delta_{ij} - \frac{1}{2N}\right) \\
&= 1
\end{aligned}$$

The last step has used the fact $i \neq j$.

For $su(N)$, the positive roots are $\alpha_{ij} = \nu^i - \nu^j$ for $i < j$. As expected, their number is $N(N-1)/2$.

The simple roots of $su(N)$ are

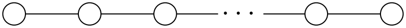
$$\alpha^i = \nu^i - \nu^{i+1}, \qquad i = 1,\ 2,\ \cdots,\ N-1.$$

Relying on the fact,

$$\begin{aligned}
\alpha^i \cdot \alpha^j &= (\nu^i - \nu^{i+1}) \cdot (\nu^j - \nu^{j+1}) \\
&= \nu^i \cdot \nu^j + \nu^{i+1} \cdot \nu^{j+1} - \nu^i \cdot \nu^{j+1} - \nu^{i+1} \cdot \nu^j \\
&= \delta_{ij} - \frac{1}{2}(\delta_{i,j+1} + \delta_{i,j-1})
\end{aligned}$$

$$\rightsquigarrow \qquad \theta_{i,i\pm 1} = 2\pi/3$$

the Dynkin diagram of $su(N)$ is:

$su(N)$: 

Explicit forms of positive roots of $su(N)$:

For completeness, we give the explicit expressions of $su(N)$ positive roots:

$$[\alpha_{ij}]_m = \frac{1}{\sqrt{2m(m+1)}}\left[\sum_{k=1}^{m}(\delta_{ki}-\delta_{kj}) - m(\delta_{m,i-1}-\delta_{m,j-1})\right]$$

where $m, i = 1,\ 2,\ \cdots,\ N-1;\ j = 2,\ 3,\ \cdots,\ N$ and $i < j$.

Equivalently,

$$[\alpha_{ij}]_m = \begin{cases} [-m\delta_{m,i-1}]/\sqrt{2m(m+1)} & \text{if} \quad m < i\ ; \\ [1+m\delta_{m,j-1}]/\sqrt{2m(m+1)} & \text{if} \quad i \leqslant m < j\ ; \\ 0 & \text{if} \quad m \geqslant j\ . \end{cases}$$

Exercise (optional) :

Please check

$$[H_m,\ E_{\pm\alpha_{ij}}] = \pm[\alpha_{ij}]_m E_{\pm\alpha_{ij}}$$

for $SU(N)$.

Fundamental weights of $su(N)$:

Group $SU(N)$ has $(N-1)$ inequivalent irreducible fundamental Reps. Each of them is characterized by a fundamental weight. e.g., D^j by μ^j, satisfying

$$\frac{2\alpha^i \cdot \mu^j}{(\alpha^i)^2} = \delta_{ij}$$

The $su(N)$ fundamental weights read explicitly,

$$\mu^j = \sum_{k=1}^{j} \nu^k, \qquad j = 1,\ 2,\ 3,\ \cdots,\ N-1.$$

$\mu^1 = \nu^1$ is the highest weight of D^1, the defining Rep. of $su(N)$.

1. The highest weight of any irreducible Rep. of $su(N)$ can be written as

$$\mu = \sum_{i=1}^{N-1} q_i \mu^i$$

q_is are non-negative integers, called the Dynkin coefficients.

Checking :

$$
\begin{aligned}
\frac{2\alpha^i \cdot \mu^j}{(\alpha^i)^2} &= 2(\nu^i - \nu^{i+1}) \cdot \sum_{k=1}^{j} \nu^k \\
&= 2 \sum_{k=1}^{j} \left[(\nu^i \cdot \nu^k) - (\nu^{i+1} \cdot \nu^k) \right] \\
&= 2 \sum_{k=1}^{j} \left[\left(-\frac{1}{2N} + \frac{1}{2}\delta_{ki} \right) + \left(\frac{1}{2N} - \frac{1}{2}\delta_{k,i+1} \right) \right] \\
&= \sum_{k=1}^{j} \left[\delta_{ki} - \delta_{k,i+1} \right] \\
&= \delta_{ij}
\end{aligned}
$$

In the last step, we have analyzed three cases of $i < j$, $i = j$ and $i > j$.

$SU(N)$ tensors :

As in $SU(3)$, we can associate $SU(N)$ states with $SU(N)$ tensors.

The basis vectors of $SU(N)$ defining Rep. are $|\nu^i\rangle$, $i = 1, 2, \cdots, N$.

$$H_m |\nu^i\rangle = [\nu^i]_m |\nu^i\rangle$$

where $m = 1, 2, \cdots, N-1$ and

$$[\nu^i]_m = \frac{1}{\sqrt{2m(m+1)}} \left[\sum_{k=1}^{m} \delta_{ki} - m\delta_{i,m+1} \right]$$

Let us relabel the basis states $|\nu^i\rangle$ as $|_i\rangle$. An arbitrary state in $SU(N)$ defining Rep. could be

$$|u\rangle = u^i |_i\rangle$$

The *wave function* u^i is called a $SU(N)$ vector.

The arbitrary representations of $SU(N)$ could be built as the *tensor products* of the defining Reps.

Consider the antisymmetric tensor product of m defining Reps.. The basis vectors of such a tensor Rep. are

$$\left|i_1 i_2 \cdots i_m\right\rangle = \left|i_1\right\rangle \wedge \left|i_2\right\rangle \wedge \cdots \wedge \left|i_m\right\rangle$$

The general states in this Rep. are:

$$\left|A\right\rangle = A^{[i_1 i_2 \cdots i_m]} \left|i_1 i_2 \cdots i_m\right\rangle$$

where the wave function $A^{[i_1 i_2 \cdots i_m]}$ forms a completely antisymmetric $SU(N)$ tensor.

- Because of the antisymmetry, this set of states forms an irreducible representation of $SU(N)$.
- Because of antisymmetry, no two indices among $i_1, i_2, \cdots, i_m$ can take on the same value.

Consequently, the highest weight state in such Rep. is,

$$|A_H\rangle = A_H^{12\cdots m}\,|_{12\cdots m}\rangle \propto \left[\,|\nu^1\rangle \wedge |\nu^2\rangle \wedge \cdots \wedge |\nu^m\rangle\,\right]$$

The highest weight of this tensor Rep. reads,

$$\mu_{\text{highest}} = \sum_{k=1}^{m} \nu^k$$

It turns out to be the fundamental weight μ^m if $1 \leqslant m \leqslant N-1$.

Insight:

The antisymmetric tensor products of m defining Reps. of $SU(N)$ for $1 \leqslant m \leqslant N-1$ are the fundamental representations D^m.

Question :

What is the lowest weight of Rep. D^m ?

To answer this question, we have to notice the facts that

- Rep. D^m is the antisymmetric tensor product of m Rep. D^1s.
- In defining Rep. D^1, the weight sequence is:

$$\nu^1 > \nu^2 > \cdots > \nu^N$$

Thereby, the lowest weight state $|A_L\rangle$ in Rep. D^m should be:

$$|A_L\rangle \propto \left[|\nu^{N-m+1}\rangle \wedge |\nu^{N-m+2}\rangle \wedge \cdots \wedge |\nu^N\rangle \right]$$

The lowest weight of this tensor Rep. reads,

$$\mu_{\text{lowest}} = \sum_{k=N-m+1}^{N} \nu^k$$

The $SU(N)$ tensor $A^{[i_1 i_2 \cdots i_m]}$ associated with the fundamental Rep. D^m could be denoted as a Young tableau with one column of m boxes:

$$A^{[i_1 i_2 \cdots i_m]} \qquad \sim \quad \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline \\ \hline \end{array}$$

- We will sometimes denote the representation corresponding to a Young tableau by giving the number of boxes in each column of the tableau, a series of non-increasing integers, $[l_1, l_2, \cdots]$. In this notation, D^m is $[m]$.
- The dimension of fundamental Rep.$[m]$ of $SU(N)$ is,

$$d_{[m]} = C_N^m = \frac{N!}{m!(N-m)!}$$

where $1 \leqslant m \leqslant N-1$. As expected,

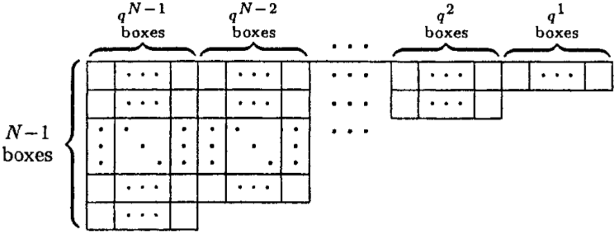
$$d_{[1]} = N$$

Now consider a general $SU(N)$ irreducible Rep. of highest weight

$$\mu = \sum_{k=1}^{N-1} q_k \mu^k$$

The Dynkin coefficients q_k are some non-negative integers.

- The tensor associated with this representation has, for each k from 1 to $N-1$, q_k sets of k indices that are antisymmetric within each set.
- The tensor can be identified to a Young tableau with q_k columns of k boxes:



Example :

Consider the $SU(N)$ irreducible Rep. with highest weight[2]

$$\mu = \mu^1 + \mu^2$$

The tensor associated with this Rep. is represented by Young tableau

$$\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

so the Rep. can be denoted as $[2, 1]$.

Let us study the dimension of Rep.$[2, 1]$ now. $[2, 1]$ tensor does only allow the following independent components:

$$\begin{array}{|c|c|}\hline i & j \\ \hline k & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline i & k \\ \hline j & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline i & i \\ \hline j & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline i & j \\ \hline j & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

where $i,\ j,\ k = 1,\ 2,\ \cdots,\ N$ but $i < j < k$.

[2]This highest weight can alternatively be cast as: $\mu = 2\nu^1 + \nu^2$.

The number of tensor components

$$\begin{array}{|c|c|}\hline i & j \\ \hline k & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \qquad \begin{array}{|c|c|}\hline i & k \\ \hline j & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

for $i < j < k$ are clearly,

$$d_1 = 2 \cdot C_N^3 = 2 \cdot \frac{N(N-1)(N-2)}{3!} = \frac{1}{3}N(N-1)(N-2)$$

The number of tensor components

$$\begin{array}{|c|c|}\hline i & i \\ \hline j & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \qquad \begin{array}{|c|c|}\hline i & j \\ \hline j & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

for $i < j$ are,

$$\begin{aligned} d_2 &= 2\Big[(N-1) + (N-2) + (N-3) + \cdots + 1\Big] \\ &= 2 \cdot \frac{1}{2}N(N-1) = N(N-1) \end{aligned}$$

Consequently, the dimension of $SU(N)$ Rep.$[2, 1]$ is,

$$d_{[2,1]} = d_1 + d_2 = \frac{1}{3}N(N-1)(N-2) + N(N-1) = \frac{1}{3}N(N+1)(N-1)$$

If $N = 3$, $d_{[2,1]} = 8$. As is well known, $[2, 1]$ is the adjoint Rep. of $SU(3)$.

Example :

Consider the $SU(N)$ irreducible Rep. with highest weight[3]

$$\mu = 3\mu^1$$

The tensor associated with this Rep. is represented by Young tableau

so the Rep. can be denoted as $[1, 1, 1]$.

The dimension of Rep.$[1, 1, 1]$ is calculated as follows. It is known that *the independent components of a tensor correspond to the standard Young tableaux.* Consequently,

[3] This highest weight can alternatively be cast as: $\mu = 3\nu^1$.

The tensor of Rep. $[1,1,1]$ has the following independent components:

$$\boxed{i \mid j \mid k}$$

where $i,\ j,\ k =\ 1,\ 2,\ \cdots,\ N$ and $i \leqslant j \leqslant k$. In other words,

$$i < j + 1 < k + 2$$

are 3 *different* integers from the set $1, 2, \cdots, (N+2)$.

The number of independent components of $SU(N)$ tensor $[1,1,1]$ is *therefore* equal to the number of ways of selecting 3 different integers from the set $1, 2, \cdots, (N+2)$:

$$d_{[1,1,1]} = C_{N+2}^{3} = \frac{(N+2)!}{3!(N-1)!} = \frac{1}{6}N(N+1)(N+2)$$

If $N = 3$,

$$d_{[1,1,1]} = 10.$$

Adjoint Rep. of $SU(N)$:

By definition, the adjoint Rep. of $SU(N)$ has dimension $(N^2 - 1)$. Because $SU(N)$ is compact, its adjoint Rep. is real.

In adjoint Rep., the $SU(N)$ tensor should have one upper index and one lower index, u^i_j, satisfying the traceless condition:

$$u^i_i = 0$$

Therefore,

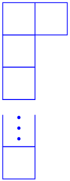
$$u^i_j \propto \epsilon_{j i_2 i_3 \cdots i_N} \left[v^i \otimes v^{i_2} \wedge v^{i_3} \wedge \cdots \wedge v^{i_N} \right]$$

where v^i is the $SU(N)$ vector in its defining Rep.[1], and

$$\epsilon_{i_1 i_2 \cdots i_N} = \begin{cases} 1 & \text{if } (i_1 i_2 \cdots i_N) \text{ is an even permutation of } (12 \cdots N); \\ -1 & \text{if } (i_1 i_2 \cdots i_N) \text{ is an odd permutation of } (12 \cdots N); \\ 0 & \text{other cases} \end{cases}$$

is an invariant tensor of $SU(N)$.

This implies that the $SU(N)$ tensor in its adjoint Rep. can be described by Young tableau[4]



The adjoint Rep. of $SU(N)$ is therefore denoted as Rep.$[N-1,1]$.

Question :

How to calculate the dimension $d_{[N-1,1]}$ of $SU(N)$ adjoint Rep. directly from the given Young tableau ?

[4] Hence, the $SU(N)$ adjoint Rep. is not among its fundamental irreducible representations.

Factors over hooks Rule :

The dimension of an irreducible Rep. of $SU(N)$ specified by a Young tableau can simply be calculated with the **factors over hooks** rule,

$$d = \frac{F}{H}$$

1. The factors are defined as follows. Put an N in the upper left hand corner of the Young tableau. Then put factors in all the other boxes, by adding 1 each time you move to the right, and subtracting 1 each time you move down. The product of all these factors is F.
2. There is one hook for each box. Call the number of boxes the hook passes through h. The product of all these hs for all hooks is H.

Sample : Please calculate the dimension $d_{[2,1]}$ of $SU(N)$ irreducible Rep.$[2, 1]$ by using factors over hooks rule.

Solution :

The $SU(N)$ tensor in Rep.$[2, 1]$ corresponds to Young tableau,

$$\begin{array}{|c|c|} \hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Hence[5],

$$F = \begin{array}{|c|c|} \hline x & y \\ \hline z & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = xyz = (N+1)N(N-1)$$

$$H = \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = 3$$

Therefore,

$$d_{[2,1]} = F/H = \frac{1}{3}N(N+1)(N-1)$$

[5] Here we set $x = N$, $y = N+1$ and $z = N-1$.

Sample : Please calculate the dimension $d_{[N-1,1]}$ of $SU(N)$ adjoint Rep.$[N-1,1]$ by using factors over hooks rule.

Solution :

The $SU(N)$ tensor in Rep.$[N-1,1]$ corresponds to Young tableau,

$$\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{c}{} \\ \cline{1-1} \multicolumn{2}{c}{} \\ \multicolumn{1}{|c|}{\vdots} & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Hence, the product of factors is[6],

$$F = \begin{array}{|c|c|}\hline a & b \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1} d & \multicolumn{1}{c}{} \\ \cline{1-1} \multicolumn{2}{c}{} \\ \multicolumn{1}{|c|}{\vdots} & \multicolumn{1}{c}{} \\ \cline{1-1} f & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = bacd\cdots f = (N+1)!$$

[6] Here we set $a = N$, $b = N+1$, $c = N-1$, $d = N-2$, $e = N-3$ and $f = 1$.

The product of hooks is[7],

$$H = \begin{array}{|c|c|}\hline a & 1 \\ \hline d & \multicolumn{1}{c}{} \\ \cline{1-1} e & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} f & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = ade \cdots f = N(N-2)!$$

As expected,

$$d_{[N-1,1]} = \frac{F}{H} = \frac{(N+1)!}{N(N-2)!} = (N+1)(N-1) = N^2 - 1$$

[7] Recall that $a = N$, $b = N+1$, $c = N-1$, $d = N-2$, $e = N-3$ and $f = 1$.

Complex Reps. of $SU(N)$:

Most of the representations of $SU(N)$ are complex.

Example :
The lowest weight of the $SU(N)$ defining Rep. is ν^N. It follows from the traceless conditions of Cartan generators H_m that

$$\sum_{j=1}^{N} \nu^j = 0$$

Thus

$$\nu^N = -\sum_{j=1}^{N-1} \nu^j = -\mu^{N-1}$$

Therefore the Rep.[1] is complex. Its complex conjugate is Rep.$[N-1]$ or D^{N-1},

$$\overline{[1]} = [N-1]$$

Example :

The lowest weight of Rep.$[m]$ is the sum of the m smallest ν^is,

$$\mu_{\text{lowest}} = \sum_{j=N-m+1}^{N} \nu^j = -\sum_{j=1}^{N-m} \nu^j = -\mu^{N-m}$$

This result yields,

$$\overline{[m]} = [N-m]$$

General conclusion :

The complex conjugate of Rep. $[l_1, \cdots, l_n]$ of $SU(N)$ is,

$$\overline{[l_1, \cdots, l_n]} = [N - l_n, \cdots, N - l_1]$$

The Young tableau corresponding to a Rep. and its complex conjugate fit together into a rectangle N boxes high.

The adjoint Rep. $[N-1, 1]$ of $SU(N)$ is real,

$$\overline{[N-1, 1]} = [N-1, 1]$$

Symmetry breaking in $SU(N)$:

Symmetry breaking is a crucial concept in modern physics.

- The typical example in particle physics is the spontaneous breaking of electroweak gauge symmetries

$$SU(2) \times U(1) \;\rightarrow\; U(1)$$

- Another example is

$$SU(5) \;\rightarrow\; SU(3) \times SU(2) \times U(1)$$

in GUT, the so-called *Grand Unification Theory*. It is among the research frontiers beyond SM.

To understand the symmetry breaking mechanism better, we now study the subgroup structure of $SU(N)$.

$su(2) \times u(1) \in su(3)$:

We begin with the defining Rep.[1] of $SU(3)$.

Rep.[1] is generated by $T_a = \lambda_a/2$ ($a = 1, 2, \cdots, 8$), with λ_a the Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Generators T_a for $1 \leqslant a \leqslant 3$ could be recast as

$$T_a = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix}, \qquad (a = 1,\ 2,\ 3.)$$

Since

$$[\sigma_a,\ \sigma_b] = 2i\epsilon_{abc}\sigma_c$$

these generators generate a subgroup $SU(2)$ in $SU(3)$.

Besides, we can define a so-called hypercharge Y from the generator T_8, $Y = 2T_8/\sqrt{3}$, which could generate a subgroup $U(1) \in SU(3)$.
By introducing the 2×2 unit matrix, we can rewrite Y as

$$Y = \frac{1}{3}\begin{pmatrix} I & 0 \\ 0 & -2 \end{pmatrix}$$

Hence,

$$[Y,\ T_a] = 0, \quad 1 \leqslant a \leqslant 3.$$

Totally speaking, $SU(3)$ has a subgroup $SU(2) \times U(1)$.

Now we study the decomposition of a $SU(3)$ irreducible Rep. in terms of the irreducible Reps. of its subgroup $SU(2) \times U(1)$.

First consider the defining Rep.$\mathbf{3}$ of $SU(3)$. The $SU(3)$ vector in $\mathbf{3}$ is written as

$$v^\mu, \qquad (\mu = 1,\ 2,\ 3)$$

In terms of $SU(2) \times U(1)$,

$$v^\mu = \begin{cases} v^i, & \text{if} \quad \mu = i, \qquad Y = +1/3 \\ v^a, & \text{if} \quad \mu = a, \qquad Y = -2/3 \end{cases}$$

where $\mu = 1,\ 2,\ 3,\ i = 1,\ 2$ and $a = 3$.

With Young tableaux, this decomposition reads:

$$\square = \left(\square \quad \bullet\right) \oplus \left(\bullet \quad \square\right)$$

where $\bullet$ stands for the trivial tableau with no boxes. Equivalently,

$$\mathbf{3} = \mathbf{2}_{1/3} \oplus \mathbf{1}_{-2/3}$$

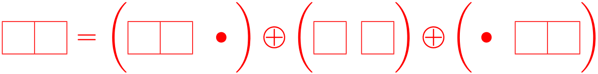
Second look at the **6**. The $SU(3)$ tensor in Rep.**6** is of rank-2

$$S^{\mu\nu}, \qquad (\mu,\ \nu = 1,\ 2,\ 3)$$

with symmetry $S^{\mu\nu} = S^{\nu\mu}$. In terms of subgroup $SU(2) \times U(1)$,

$$S^{\mu\nu} = \begin{cases} S^{ij}, & \text{if } \mu = i,\ \nu = j, \quad Y = +2/3 \\ S^{ib}, & \text{if } \mu = i,\ \nu = b, \quad Y = -1/3 \\ S^{ab}, & \text{if } \mu = a,\ \nu = b, \quad Y = -4/3 \end{cases}$$

where $i,\ j = 1,\ 2$ but $a,\ b = 3$.
With Young tableaux, this decomposition reads:



Equivalently,

$$\mathbf{6} = \mathbf{3}_{2/3} \oplus \mathbf{2}_{-1/3} \oplus \mathbf{1}_{-4/3}$$

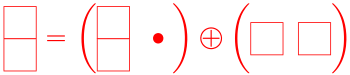
Thirdly we consider the $\bar{\mathbf{3}}$. The $SU(3)$ tensor in Rep.$\bar{\mathbf{3}}$ is of rank-2

$$A^{\mu\nu}, \qquad (\mu,\ \nu = 1,\ 2,\ 3)$$

with symmetry $A^{\mu\nu} = -A^{\nu\mu}$. In terms of subgroup $SU(2) \times U(1)$,

$$A^{\mu\nu} = \begin{cases} A^{ij}, & \text{if } \mu = i,\ \nu = j, \qquad Y = +2/3 \\ A^{ib}, & \text{if } \mu = i,\ \nu = b, \qquad Y = -1/3 \\ A^{ab}, & \text{if } \mu = a,\ \nu = b, \qquad Y = -4/3 \end{cases}$$

where $i,\ j = 1,\ 2$ but $a,\ b = 3$. Obviously, $A^{ab} = 0$.
With Young tableaux, this decomposition reads:



Equivalently,

$$\bar{\mathbf{3}} = \bar{\mathbf{1}}_{2/3} \oplus \mathbf{2}_{-1/3}$$

Next we consider the adjoint Rep.$\mathbf{8}$ of $SU(3)$. The $SU(3)$ tensor in $\mathbf{8}$ is represented by Young tableau

$$\ydiagram{2,1}$$

In terms of subgroup $SU(2) \times U(1)$,

$$\begin{aligned}
\ydiagram{2,1} \quad &= \left(\ydiagram{2,1} \quad \bullet \right) && \mathbf{2}_1 \\
&\oplus \left(\ydiagram{2} \quad \ydiagram{1} \right) && \mathbf{3}_0 \\
&\oplus \left(\ydiagram{1,1} \quad \ydiagram{1} \right) && \mathbf{1}_0 \\
&\oplus \left(\ydiagram{1} \quad \ydiagram{2} \right) && \mathbf{2}_{-1}
\end{aligned}$$

Namely,

$$\mathbf{8} = \mathbf{2}_1 \oplus \mathbf{3}_0 \oplus \mathbf{1}_0 \oplus \mathbf{2}_{-1}$$

Hypercharge :

Question :

How to determine the hypercharge of a tensor component in $SU(3)$ $\rightarrow$ $SU(2) \times U(1)$?

The $SU(3)$ tensor u in some irreducible Rep. forms the common eigenstates of $T_3 \in su(2)$ and hypercharge operator $Y \in u(1)$.

Hence,

$$Yu = yu$$

Consider a tensor u represented by a Young tableau of n boxes. We examine its components with j boxes belong to $su(2)$ and $(n-j)$ boxes belong to $u(1)$. The hypercharge of such components is:

$$y = \frac{j}{3} - \frac{2(n-j)}{3} = j - \frac{2}{3}n$$

Warning :

For $U(1) \in SU(3)$, the antisymmetric tensor such as

$$A^{ab} \sim \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$$

does not exist. Because $a = b = 3$, we see that $A^{ab} = -A^{ba} = 0$.

Problems :

1. Show that the $su(N)$ algebra has an $su(N-1)$ subalgebra. How do the fundamental Rep.[1] of $SU(N)$ decompose into $SU(N-1)$ representations ?
2. Find $[3] \otimes [1]$ in $SU(5)$. Check that the dimensions work out.
3. Find $[3,1] \otimes [2,1]$ in $SU(6)$.
4. Find $[2] \otimes [1,1]$ in $SU(N)$, using the factors over hooks rule to check that the dimensions work out for arbitrary N.