LECTURE 2: SMOOTH MANIFOLDS AND SMOOTH MAPS

1. Smooth Structures

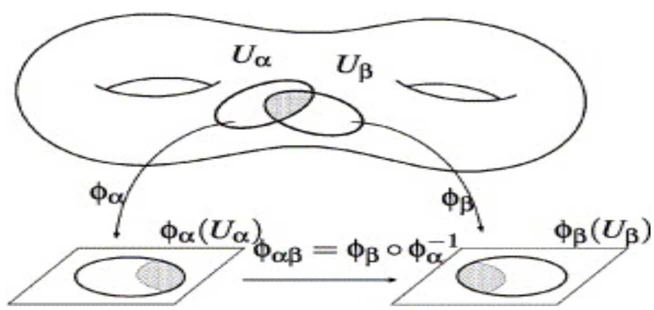
Roughly speaking, manifolds are topological spaces that locally looks like $\mathbb{R}^n$.

Definition 1.1. Let M be a (Hausdorff and second countable) topological space. It is said to be a n-dimensional *topological manifold* if for every $p \in M$, there exists a triple $\{\varphi, U, V\}$, where U is an open neighborhood of p in M, V an open subset of $\mathbb{R}^n$, and $\varphi : U \to V$ a homeomorphism. Such a triple is called a *chart* about p.

Two charts $\{\varphi_1, U_1, V_1\}$ and $\{\varphi_2, U_2, V_2\}$ are called *compatible* if the *transition map*

$$\varphi_{12} = \varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is a diffeomorphism. Note that both $\varphi_1(U_1 \cap U_2)$ and $\varphi_2(U_1 \cap U_2)$ are open in $\mathbb{R}^n$.



Definition 1.2. An *atlas* $\mathcal{A}$ on M is a collection of charts $\{\varphi_\alpha, U_\alpha, V_\alpha\}$ such that all charts in $\mathcal{A}$ are compatible to each other, and satisfies $\bigcup_\alpha U_\alpha = M$. Two atlas on M are said to be *equivalent* if there union is still an atlas on M.

Definition 1.3. An n-dimensional *smooth manifold* is an n-dimensional topological manifold M equipped with an equivalence class of atlas. This equivalence class is called its *smooth structure*.

So a smooth manifold is a pair $(M, \mathcal{A})$. Usually we will omit $\mathcal{A}$ and say M is a smooth manifold if there is no confusion of the smooth structure.

Example: Some smooth manifolds with the GOD-given smooth structures:

♣ $\mathbb{R}^n$ (or any finite dimensional vector space) is a smooth manifold.

♢ Open subsets of a smooth manifold are still smooth manifolds.

♡ If M and N are manifolds, so is their product $M \times N$.

♠ The graphs of smooth functions defined on open regions in Euclidean spaces are smooth mainfolds.

Example: The unit sphere

$$S^n = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

is a smooth manifold: Let $U_1 = S^n - \{(0, \cdots, 0, 1)\}$ and $U_2 = S^n - \{(0, \ldots, 0, -1)\}$. Consider the stereographic projection maps $\varphi_i : U_i \to \mathbb{R}^n$ defined by

$$\varphi_1(x) = \frac{1}{1 - x_{n+1}}(x_1, \cdots, x_n), \qquad \varphi_2(x) = \frac{1}{1 + x_{n+1}}(x_1, \cdots, x_n).$$

One can check that $\{\varphi_1, U_1, \mathbb{R}^n\}$ and $\{\varphi_2, U_2, \mathbb{R}^n\}$ are charts on S^n with transition map

$$\varphi_{12} : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}, \quad (y_1, \cdots, y_n) \mapsto \frac{1}{y_1^2 + \cdots + y_n^2}(y_1, \cdots, y_n)$$

which is a diffeomorphism. So they represent an atlas on S^n

Using charts, one can translate many mathematical conceptions from Euclidean spaces to smooth manifolds.

Definition 1.4. A map $\varphi : M \to N$ between two smooth manifolds is called *smooth* if for any chart $\{\varphi_\alpha, U_\alpha, V_\alpha\}$ of M and any chart $\{\psi_\beta, X_\beta, Y_\beta\}$ of N, the map

$$\psi_\beta \circ \varphi \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap \varphi^{-1}(X_\beta)) \to \psi_\beta(\varphi(U_\alpha) \cap X_\beta)$$

is smooth.

The set of all smooth maps from M to N is denoted by $C^\infty(M, N)$.

Definition 1.5. We say that $\varphi : M \to N$ is a *diffeomorphism* if it is bijective, and that both φ and φ^{-1} are smooth maps.

Obviously

- "Diffeomorphism" is an equivalence relation on the set of all smooth manifolds.
- For any smooth manifold M, the set

  $$\mathrm{Diff}(M) = \{\varphi : M \to M \mid \varphi \text{ is a diffeomorphism }\}$$

  is a (HUGE=infinitely dimensional) group. It is called the diffeomorphism group of M.

Remarks (on smooth structures).

(1) There exists topological manifolds that do not admits smooth structure. The first example was a compact 10-dimensional manifold found by M. Kervaire.
(2) It's possible that a topological manifold supports many different(=non-diffeomorphic) smooth structures. In fact, a remarkable result of J. Milnor and M. Kervaire asserts that the topological 7-sphere admits exactly 28 different smooth structures! However, on any Lie group there is only one smooth structure.

2. Smooth Functions

A smooth map from M to $\mathbb{R}$ is called a (real-valued) *smooth function*. It is not hard to prove that a real function $f : M \to \mathbb{R}$ is smooth if and only if for any chart $\{\varphi_\alpha, U_\alpha, V_\alpha\}$ on M, the function $f \circ \varphi_\alpha^{-1}$ is smooth on V_α. The set of all real-valued smooth functions on M is usually denoted by $C^\infty(M)$. This is a (HUGE=infinitely dimensional) vector space. Note that any smooth map $\varphi : M \to N$ induces a "pull-back" map

$$\varphi^* : C^\infty(N) \to C^\infty(M), \quad f \mapsto f \circ \varphi$$

which plays an important role in manifold theory.

An very important class of smooth functions on a smooth manifold M are so-called *bump functions*. Recall that the *support* of a smooth function f is by definition the set

$$\mathrm{supp}(f) = \overline{\{p \in M \mid f(p) \neq 0\}}.$$

We say that f is *compactly supported*, denoted by $f \in C_0^\infty(M)$, if the support of f is a compact subset in M. Obviously if M is compact, then any smooth function is compactly supported.

Theorem 2.1. *Let M be a smooth manifold, $K \subset M$ is a closed subset, and $U \subset M$ an open subset that contains K. Then there is a "bump" function $\rho \in C^\infty(M)$ so that $0 \leq \rho \leq 1$, $\varphi \equiv 1$ on K and $\mathrm{supp}(\rho) \subset U$. Moreover, if K is compact, one can choose ρ to be compactly-supported.*

The following theorem is well-known as "partition of unity" and is used in many settings to glue local data on manifolds into a global one.

Theorem 2.2 (Partition of unity)**.** *Let M be a smooth manifold, and $\{U_\alpha\}$ an open cover of M. Then there exists a collection of smooth functions $\{\rho_\alpha\}$, called a partition of unity subordinate to $\{U_\alpha\}$, so that*

(1) $0 \leq \rho_\alpha \leq 1$ *for all* α.
(2) $\mathrm{supp}(\rho_\alpha) \subset U_\alpha$ *for all* α.
(3) *Each point $p \in M$ has a neighborhood which intersects* $\mathrm{supp}(\rho_\alpha)$ *for only finitely many* α.
(4) $\sum_\alpha \rho_\alpha(p) = 1$ *for all* $p \in M$.

Remark. Note that the local finiteness condition (3) implies

- there are only countable ρ_α's whose support are non-empty.
- The summation in (4) is actually a finite sum near each point p.

One application of partion of unity is to define an integral on smooth manifolds. We would like to integrate functions on a smooth manifold. One can think of an *integral* on M as a linear map

$$I : C^0(M) \to \mathbb{R}, \quad f \mapsto I(f) = \text{“}\int_M f\text{”}$$

such that $f_1(x) \le f_2(x)$ for all $x \in M$ implies $I(f_1) \le I(f_2)$. There are too much integrals on manifolds and there is no canonical way to choose one. In manifold theory, to fix a choice of an integral is reduced to fix a *volume form*. A volume form ω on a n-dimensional manifold M is an object so that

- locally on any chart $\{\varphi_\alpha, U_\alpha, V_\alpha\}$ the volume form ω looks like
$$\omega = a(x)dx_1 \wedge \cdots \wedge dx_n,$$
where $x_1, \cdots, x_n$ are coordinate functions on V_α and $a(x)$ is a non-vanishing function.
- If on another chart $\{\varphi_\beta, U_\beta, V_\beta\}$ the volume form ω can be represented as
$$\omega = b(y)dy_1 \wedge \cdots \wedge dy_n,$$
then we must have
$$a(x) = b(\varphi_{\alpha\beta}(x))J\varphi_{\alpha\beta}(x)$$
where $J\varphi_{\alpha\beta}$ is the Jacobian determinant of $\varphi_{\alpha\beta}$.

Fixing such a volume form ω on M, one can get a positive measure $|\omega|$ on M and thus integrate compactly supported functions on M, as follows:

(1) If $f \in C_0^\infty(M)$ is supported in one coordinate chart $\{\varphi_\alpha, U_\alpha, V_\alpha\}$, then we define
$$\int_M f|\omega| = \int_{V_\alpha} f(x)|a(x)|dx_1 \cdots dx_n,$$
where $dx_1 \cdots dx_n$ is the Lebesgue measure on V_α.

(2) For general f, we cover $\mathrm{supp}(f)$ by coordinate charts, take a partition of unity $\{\rho_\alpha\}$ subordinate to that cover, and define
$$\int_M f|\omega| = \sum_\alpha \int_{U_\alpha} \rho_\alpha f|\omega|.$$

The fact that the above definition is independent of the choices of coordinate charts and the choices of partition of unity is a consequence of the change of variable formula in multivariable calculus.