现代数学物理方法

第三章，李群

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Lie Groups:

Lie groups G are groups where the group elements $g \in G$ depends smoothly on a set of continuous real parameters,

$$g = g(\alpha)$$

where

$$\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_N\} = \{\alpha_a \,|1 \leqslant a \leqslant N\}$$

In general, we choose parameters $\{\alpha_a\}$ so that the identity can be expressed as

$$e = g(\alpha)\,|_{\alpha=0} = g(0)$$

If we find a representation $D(G)$, we have similarly,

$$1 = D(\alpha)\,|_{\alpha=0} = D(0)$$

Generators:

In some neighborhood of the *identity*, the elements of a Lie group G or its representation $D(G)$ can be Taylor expanded as,

$$\begin{aligned} D(d\alpha) &= 1 + \sum_{a=1}^{N} d\alpha_a \left[\frac{\partial D(\alpha)}{\partial \alpha_a}\right]_{\alpha=0} + \cdots \\ &= 1 + i \sum_{a=1}^{N} d\alpha_a X_a + \cdots \\ &\approx 1 + i d\alpha_a X_a \end{aligned}$$

where

$$X_a = -i \frac{\partial D(\alpha)}{\partial \alpha_a} |_{\alpha=0}\,, \quad (a = 1,\ 2,\ \cdots,\ N)$$

are called the generators of group G in its representation $D(G)$.

Discussions:

1. X_a are independent of one another.
2. The factor i is included in the definition of generators X_a so that if the representation is unitary, X_a will be hermitian matrices.
3. The representation of the group elements for finite parameters $\alpha = \{\alpha_a\}$ can be defined as,

$$D(\alpha) = \lim_{k\to\infty} \left[1 + i\left(\frac{\alpha_a}{k}\right) X_a\right]^k = \exp(i\alpha_a X_a) = e^{i\alpha_a X_a}$$

   This procedure is called *exponential mapping*. It implies that, *at least in some neighborhood of identity*, the group elements can be written out in terms of the generators.
4. The exponential of a matrix is always defined as a power series,

$$e^{i\alpha_a X_a} = \sum_{n=0}^{\infty} \frac{i^n}{n!}(\alpha_a X_a)^n$$

We now consider the multiplication of two group elements of a Lie group G,

$$g_\alpha = e^{i\alpha_a X_a}, \qquad g_\beta = e^{i\beta_a X_a}.$$

That the generators X_a are matrices indicates,

$$g_\alpha g_\beta = e^{i\alpha_a X_a} e^{i\beta_a X_a} \neq e^{i(\alpha+\beta_a)X_a}$$

- Because the exponentials form a representation of the group G, it must be true that the product of two exponentials is also an exponential of the generators,

$$\begin{aligned} g_\alpha g_\beta &= e^{i\alpha_a X_a} e^{i\beta_a X_a} \\ &= e^{i\gamma_a X_a} \\ &= g_\gamma \end{aligned}$$

The parameters γ_a are determined by,

$$\begin{aligned} i\gamma_a X_a &= \ln(e^{i\alpha_a X_a} e^{i\beta_a X_a}) = \ln[1 + (e^{i\alpha_a X_a} e^{i\beta_a X_a} - 1)] \\ &= \ln(1+K) \\ &= K - \frac{K^2}{2} + \frac{K^3}{3} - \cdots \end{aligned}$$

where $K = e^{i\alpha_a X_a} e^{i\beta_a X_a} - 1$. Explicitly,

$$\begin{aligned} K &= \left[1 + i(\alpha_a X_a) - \frac{1}{2}(\alpha_a X_a)^2 + \cdots\right] \\ &\qquad \cdot\left[1 + i(\beta_b X_b) - \frac{1}{2}(\beta_b X_b)^2 + \cdots\right] - 1 \\ &= i(\alpha_a + \beta_a)X_a - \alpha_a\beta_b X_a X_b \\ &\qquad -\frac{1}{2}\left[(\alpha_a X_a)^2 + (\beta_a X_a)^2\right] + \cdots \end{aligned}$$

and

$$K^2 \approx \left[i(\alpha_a+\beta_a)X_a\right]^2 = -\alpha_a\beta_b(X_aX_b+X_bX_a)-\left[(\alpha_a X_a)^2+(\beta_a X_a)^2\right]$$

Therefore,

$$\begin{aligned} i\gamma_a X_a &= K - K^2/2 + \cdots \\ &= i(\alpha_a + \beta_a)X_a - \frac{1}{2}\alpha_a\beta_b\Big(X_aX_b - X_bX_a\Big) \\ &= i(\alpha_a + \beta_a)X_a - \frac{1}{2}\alpha_a\beta_b\left[X_a,\ X_b\right] \end{aligned}$$

where

$$[A,\ B] = AB - BA$$

is called the *Lie bracket* between two generators A and B.

- We conclude that,

$$(\alpha_a\beta_b)[X_a,\ X_b] = -2i(\gamma_c - \alpha_c - \beta_c)X_c$$

That is to say: *the generators of the Lie group G form an closed algebra under Lie brackets.* It is called the Lie algebra.

Lie algebras:

Lie algebras are generally written as,

$$[X_a,\ X_b] = i f_{abc} X_c$$

The coefficients f_{abc} are known as the structure constants of the Lie group G.

Properties of f_{abc}:

1. $f_{abc} = -f_{bac}$
2. *The generators of a unitary representation of Lie group G are hermitian matrices.* Consequently, all of the structure constants are real,

$$f^*_{abc} = f_{abc}$$

3. The structure constants satisfy the so-called Jacobi identity,

$$f_{abd} f_{dce} + f_{bcd} f_{dae} + f_{cad} f_{dbe} = 0.$$

Proof:
The *reality* of f_{abc} is proved as follows,

$$\begin{aligned}
-if^*_{abc}X_c &= (if_{abc}X_c)^\dagger = \{[X_a,\ X_b]\}^\dagger = (X_aX_b - X_bX_a)^\dagger \\
&= (X_b)^\dagger(X_a)^\dagger - (X_a)^\dagger(X_b)^\dagger \\
&= X_bX_a - X_aX_b = -[X_a,\ X_b] = -if_{abc}X_c
\end{aligned}$$

Hence, $f^*_{abc} = f_{abc}$.

Similar to the Poisson brackets in classical mechanics, the Lie brackets obey the so-called Jacobi identity,

$$[[X_a,\ X_b],\ X_c] + \text{Cyclic Permutations} = 0.$$

Explicitly,

$$[[X_a,\ X_b],\ X_c] + [[X_b,\ X_c],\ X_a] + [[X_c,\ X_a],\ X_b] = 0.$$

Here we check this formula. By definition of the Lie brackets

$$\begin{aligned}[[X_a,\ X_b],\ X_c] &= [X_aX_b - X_bX_a,\ X_c] \\ &= (X_aX_b - X_bX_a)X_c - X_c(X_aX_b - X_bX_a) \\ &= X_aX_bX_c - X_bX_aX_c - X_cX_aX_b + X_cX_bX_a\end{aligned}$$

Cyclic permutations of above equation lead to

$$\begin{aligned}[[X_b,\ X_c],\ X_a] &= X_bX_cX_a - X_cX_bX_a - X_aX_bX_c + X_aX_cX_b \\ [[X_c,\ X_a],\ X_b] &= X_cX_aX_b - X_aX_cX_b - X_bX_cX_a + X_bX_aX_c\end{aligned}$$

Obviously, the sum of these three terms vanishes:

$$[[X_a,\ X_b],\ X_c] + [[X_b,\ X_c],\ X_a] + [[X_c,\ X_a],\ X_b] = 0.$$

Because

$$[[X_a,\ X_b],\ X_c] = [if_{abd}X_d,\ X_c] = -f_{abd}f_{dce}X_e$$

The Jacobi identities put some stringent constraints on the structure constants:

$$f_{abd}f_{dce} + f_{bcd}f_{dae} + f_{cad}f_{dbe} = 0.$$

Adjoint Representation:

Define a set of hermitian matrices T_a from the structure constants,

$$(T_a)_{bc} = -i f_{abc} \,, \quad (T_a)_{bc} = (T_a)^*_{cb}.$$

We can rewrite the above Jacobi identities as,

$$\begin{aligned} 0 &= f_{abd} f_{dce} + f_{bcd} f_{dae} + f_{cad} f_{dbe} \\ &= -f_{abd} f_{cde} + f_{cbd} f_{ade} - f_{acd} f_{dbe} \\ &= (T_a)_{bd}(T_c)_{de} - (T_c)_{bd}(T_a)_{de} - i f_{acd}(T_d)_{be} \\ &= \left([T_a,\, T_c]\right)_{be} - i f_{acd}(T_d)_{be} \end{aligned}$$

Therefore, the structure constants themselves generate a representation of the Lie algebra:

$$[T_a,\, T_c] = i f_{acd}\, T_d$$

It is called the adjoint representation.

Discussions:

- For a unitary adjoint representation of a Lie group G, because

$$(T_a)_{bc} = -if_{abc}$$

its hermitian generators are pure imaginary and then antisymmetric matrices. Hence, f_{abc} becomes totally antisymmetric about its indices. In particular,

$$f_{abc} = -f_{acb}.$$

- The dimension of the adjoint representation is just the number of independent generators, which is also the number of real parameters required to describe a group element.

- The scalar product in the linear space of the generators is defined as the following trace,

$$\mathrm{Tr}(X_a X_b)$$

which is symmetric for interchanging indices a and b.
In the adjoint representation,

$$\begin{aligned}\mathrm{Tr}(T_a T_b) &= (T_a)_{cd}(T_b)_{dc} \\ &= (-if_{acd})(-if_{bdc}) \\ &= -f_{acd}f_{bdc} \\ &= f_{acd}f_{bcd}\end{aligned}$$

Since the basic symmetric quantity is δ_{ab}, this scalar product can be cast as a simple canonical form,[1]

$$\mathrm{Tr}(T_a T_b) = \lambda^a \delta_{ab}$$

Therefore,

$$f_{acd}f_{bcd} \propto \delta_{ab}$$

[1] There is no sum over index a.

Compact Lie algebras:

From now on we shall assume that all of the coefficients in $\{\lambda^a\}$ are positive and equal to each other. This defines a class of algebras called compact Lie algebras:

$$\mathrm{Tr}(T_a T_b) = \lambda\, \delta_{ab}$$

The structure constants of a compact Lie algebra are completely antisymmetric,

$$\begin{aligned} f_{abc} &= -i\lambda^{-1}(i f_{abd})\lambda\delta_{dc} \\ &= -i\lambda^{-1}(i f_{abd})\mathrm{Tr}(T_d T_c) \\ &= -i\lambda^{-1}\mathrm{Tr}\big[(i f_{abd} T_d) T_c\big] \\ &= -i\lambda^{-1}\mathrm{Tr}\big\{[T_a,\ T_b] T_c\big\} \\ &= -i\lambda^{-1}\mathrm{Tr}(T_a T_b T_c - T_b T_a T_c) \end{aligned}$$

Namely,

$$f_{abc} = -f_{bac} = f_{bca} = -f_{cba} = f_{cab} = -f_{acb}$$

Theorem:

The adjoint representation of a compact Lie algebra is *unitary*.

In fact, the reality of f_{abc} and its symmetry guarantee that the generators $(T_a)_{bc} = -if_{abc}$ are not only pure imaginary but anti-symmetric also.

Therefore,

$$\begin{aligned}
[(T_a)^\dagger]_{bc} &= [(T_a)^*]_{cb} \\
&= [(T_a)_{cb}]^* \\
&= (-if_{acb})^* \\
&= if_{acb} \\
&= -if_{abc} \\
&= (T_a)_{bc}
\end{aligned}$$

Namely,

$$(T_a)^\dagger = T_a$$

This is very the expected hermitility.

Invariant subalgebra:

An invariant subalgebra is some set of generators $\mathcal{H} = \{X_a\}$ which goes into itself under Lie brackets with any element Y_b of the whole algebra,

$$[X_a,\ Y_b] = i f_{abc} X_c$$

for an arbitrary generator Y_b of group G.

When exponentiated, an invariant subalgebra generates an subgroup $H = \{h\}$ of G,

$$h = e^{i\alpha_a X_a},\ \ \forall\ X_a \in \mathcal{H}.$$

For an arbitrary group element $g = e^{i\beta_b Y_b}$ in G, we see,

$$\begin{aligned}
g^{-1} h g \quad &= e^{-i\beta_b Y_b} e^{i\alpha_a X_a} e^{i\beta_c Y_c} = e^{-i\beta_b Y_b} \left[\sum_{n=0}^{\infty} \frac{i^n}{n!} (\alpha_a X_a)^n \right] e^{i\beta_c Y_c} \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} \left[e^{-i\beta_b Y_b} (\alpha_a X_a) e^{i\beta_c Y_c} \right]^n \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} (\alpha_a X'_a)^n = e^{i\alpha_a X'_a}
\end{aligned}$$

where

$$
\begin{aligned}
X'_a &= e^{-i\beta_b Y_b} X_a e^{i\beta_c Y_c} \\
&= X_a - i\beta_b [Y_b, X_a] - \frac{1}{2!}\beta_b\beta_c [Y_b,\ [Y_c,\ X_a]] + \cdots
\end{aligned}
$$

does still belong to the subalgebra $\mathcal{H}$. As a result, the considered exponentials form an invariant subgroup of G.

Remark:

The whole algebra and the null set ϕ are two trivial invariant subalgebras.

Simple Lie Algebras:

Definition:

A Lie algebra which has no nontrivial invariant subalgebras is called *simple Lie algebra*.
A simple Lie algebra generates a *simple Lie group*.

Theorem:

The adjoint representation of a simple Lie group G with generators $(T_a)_{bc} = -i f_{abc}$ satisfying

$$\mathrm{Tr}(T_a T_b) = \lambda\, \delta_{ab}$$

is irreducible.

Proof:

If the adjoint representation were reducible, there were an invariant subspace in the adjoint representation sapnned by some subset of generators,

$$T_j, \quad 1 \leqslant j \leqslant K$$

The rest of the generators are labeled as,

$$T_\alpha, \quad K+1 \leqslant \alpha \leqslant N$$

Because the indices j $(j = 1, 2, \cdots, K)$ label an invariant subspace, we must have

$$-i f_{aj\beta} = (T_a)_{j\beta} = 0, \qquad \begin{cases} 1 \leqslant a \leqslant N \\ 1 \leqslant j \leqslant K \\ K+1 \leqslant \beta \leqslant N \end{cases}$$

If $Tr(T_a T_b) = \lambda \delta_{ab}$, the structure constants are completely antisymmetric about their three indices. Consequently, $f_{aj\beta} = 0$ means:

$$f_{ij\beta} = f_{j\beta i} = f_{\beta ij} = 0, \quad (1 \leqslant i, j \leqslant K, \ K+1 \leqslant \beta \leqslant N)$$

and

$$f_{\alpha j\beta} = f_{j\beta\alpha} = f_{\beta\alpha j} = 0, \quad (1 \leqslant j \leqslant K, \ K+1 \leqslant \alpha, \beta \leqslant N)$$

The nonzero structure constants would be:

$$\begin{aligned} &f_{ijk}, \quad (1 \leqslant i, j, k \leqslant K) \\ &f_{\alpha\beta\gamma}, \quad (K+1 \leqslant \alpha, \beta, \gamma \leqslant N) \end{aligned}$$

The algebra contained two nontrivial invariant subalgebras, and not simple. *Contrary to the initial assumption !* Q.E.D.

Abelian invariant subalgebras:

An abelian invariant sub-algebra consists of a single generator which commutes with all of the generators of the Lie group G.

1. We call such a sub-algebra a $U(1)$ factor of the group.
2. If X_a is a $U(1)$ generator, $f_{abc} = 0$ for all possible b and c.

Semi-simple Lie algebras:

The Lie algebras without Abelian invariant sub-algebras are called semi-simple Lie algebras.

Cartan subalgebra:

In any Lie group, the maximum set of mutually commuting generators $H_a \;\; (a = 1, 2, \cdots, r)$ generates an abelian subalgebra $\mathbb{h}$,

$$[H_a,\; H_b] = 0$$

which is called the **Cartan subalgebra.**

1. The number of generators in $\mathbb{h}$ is the **rank** of the corresponding Lie algebra $\mathbb{g}$.
2. The Cartan generators H_a can be simultaneously diagonalized, and their eigenvalues or diagonal elements are the **weights**

$$H_a \left|\mu, x, D\right\rangle = \mu_a \left|\mu, x, D\right\rangle$$

   in which D labels the representation and x whatever other variables are needed to specify the state.

3. The vector $\vec{\mu} = (\mu_1, \mu_2, \cdots, \mu_r)$ is called the **weight vector**.
4. The weights of the *adjoint representation* is called the **roots**.

States and operators:

Consider a Lie group G and its representation spanned by the states or column vectors

$$|i\rangle, \quad i = 1,\, 2,\, 3,\, \cdots$$

Generators:

The generators $\{X_a\}$ of this representation can be thought of as either linear operators acting on the representation space,

$$X_a \,|i\rangle = \sum_j |j\rangle\langle j|\, X_a \,|i\rangle = \sum_j |j\rangle \,(X_a)_{ji}$$

Group elements:

The group elements $e^{i\alpha_a X_a}$ can be thought of as transformations of the states,

$$e^{i\alpha_a X_a}: \quad |i\rangle \rightsquigarrow |i'\rangle = e^{i\alpha_a X_a}\,|i\rangle, \quad \langle i| \rightsquigarrow \langle i'| = \langle i|\, e^{-i\alpha_a X_a}.$$

For a state generated from $|i\rangle$ by acting an operator $\mathscr{O}$: $\mathscr{O}\,|i\rangle$, we see,

$$e^{i\alpha_a X_a} : \quad \mathscr{O}\,|i\rangle \rightsquigarrow \mathscr{O}'\,|i'\rangle \quad \begin{aligned} &= e^{i\alpha_a X_a}\,\mathscr{O}\,|i\rangle \\ &= e^{i\alpha_a X_a}\,\mathscr{O} e^{-i\alpha_b X_b} e^{i\alpha_c X_c}\,|i\rangle \\ &= e^{i\alpha_a X_a}\,\mathscr{O} e^{-i\alpha_b X_b}\,|i'\rangle \end{aligned}$$

Hence,

$$e^{i\alpha_a X_a} : \quad \mathscr{O} \rightsquigarrow \mathscr{O}' = e^{i\alpha_a X_a}\,\mathscr{O} e^{-i\alpha_b X_b}$$

Invarant operators:

If $\mathscr{O}$ is an invariant operator under $G = \{e^{i\alpha_a X_a}\}$, then

$$[e^{i\alpha_a X_a},\ \mathscr{O}] = 0$$

Equivalently,

$$[X_a,\ \mathscr{O}] = 0, \quad \forall\, a$$

This conclusion can alternativley be obtained in the following manner. Under an infinitesimal transformation of Lie group G,

$$e^{i\alpha_a X_a} \approx 1 + i\alpha_a X_a$$

the variation of the operator $\mathscr{O}$ can be expressed as,

$$\begin{aligned}
\delta\mathscr{O} &= \mathscr{O}' - \mathscr{O} \\
&= e^{i\alpha_a X_a}\mathscr{O}e^{-i\alpha_b X_b} - \mathscr{O} \\
&= (1 + i\alpha_a X_a)\mathscr{O}(1 - i\alpha_b X_b) - \mathscr{O}
\end{aligned}$$

Namely,

$$\delta\mathscr{O} \approx i\alpha_a[X_a,\ \mathscr{O}]$$

- The invariance of $\mathscr{O}$ under this Lie group transformation is then recast as:

$$[X_a,\ \mathscr{O}] = 0, \quad \forall\ a.$$

As remarked previously, the exponential is alternatively defined as a power series expansion,

$$\exp(i\alpha_a X_a) = \sum_{n=0}^{\infty} \frac{i^n}{n!}(\alpha_a X_a)^n$$

In general, the generators do not commute mutually, $[X_a,\ X_b] \neq 0$. However,

$$\begin{aligned}
[\alpha_a X_a,\ \alpha_b X_b] &= (\alpha_a \alpha_b)[X_a,\ X_b] = i(\alpha_a \alpha_b) f_{abc} X_c \\
&= \frac{i}{2}(\alpha_a \alpha_b) f_{abc} X_c + \frac{i}{2}(\alpha_a \alpha_b) f_{abc} X_c \\
&= \frac{i}{2}\big[(\alpha_a \alpha_b) f_{abc} X_c + (\alpha_b \alpha_a) f_{bac} X_c\big] \\
&= \frac{i}{2}\big[(\alpha_a \alpha_b) f_{abc} X_c - (\alpha_a \alpha_b) f_{abc} X_c\big] \\
&= 0
\end{aligned}$$

As a result, for an arbitrary real parameter ξ,

$$\begin{aligned}\frac{\partial}{\partial \xi}\exp(i\xi\alpha_a X_a) &= i(\alpha_b X_b)\exp(i\xi\alpha_a X_a)\\ &= i\exp(i\xi\alpha_a X_a)(\alpha_b X_b)\end{aligned}$$

Question:

$$\frac{\partial}{\partial \alpha_b} e^{i\alpha_a X_a} = ?$$

It follows from the above definition that,

$$\begin{aligned}\frac{\partial}{\partial \alpha_b} e^{i\alpha_a X_a} &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\partial_{\alpha_b}(\alpha_a X_a)^n\\ &= \sum_{n=1}^{\infty}\frac{1}{n!}\left[\sum_{m=0}^{n-1}(i\alpha_a X_a)^m iX_b(i\alpha_c X_c)^{n-1-m}\right]\end{aligned}$$

Using the famous mathematical identity,

$$\begin{aligned}\frac{(n-1-m)!m!}{n!} &= \frac{\Gamma(n-m)\Gamma(m+1)}{\Gamma(n+1)}\\ &= B(n-m, m+1)\\ &= \int_0^1 d\zeta \zeta^m (1-\zeta)^{(n-1-m)}\end{aligned}$$

i.e,

$$1 = \frac{n!}{m!(n-1-m)!} \int_0^1 d\zeta \zeta^m (1-\zeta)^{(n-1-m)}$$

we reexpress the above derivative as,

$$\begin{aligned}\frac{\partial}{\partial \alpha_b} e^{i\alpha_a X_a} = \sum_{n=1}^{\infty} \frac{1}{n!} \Bigg[\sum_{m=0}^{n-1} \frac{n!}{m!(n-1-m)!} \int_0^1 d\zeta \zeta^m (1-\zeta)^{(n-1-m)} \\ (i\alpha_a X_a)^m i X_b (i\alpha_c X_c)^{(n-1-m)} \Bigg]\end{aligned}$$

Notice that

$$(-n)! = \infty, \quad \forall\, n \in \mathbf{Z}^+$$

The upper limit $(n-1)$ of the summation inside the square bracket can be replaced with ∞. Consequently,

$$\begin{aligned}
\frac{\partial}{\partial \alpha_b} e^{i\alpha_a X_a} &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \int_0^1 d\zeta \left[\frac{(i\zeta\alpha_a X_a)^m}{m!} \right] (iX_b) \\
&\qquad \cdot \left\{ \frac{[i(1-\zeta)\alpha_c X_c]^{(n-1-m)}}{(n-1-m)!} \right\} \\
&= \int_0^1 d\zeta \left[\sum_{m=0}^{\infty} \frac{(i\zeta\alpha_a X_a)^m}{m!} \right] (iX_b) \left\{ \sum_{k=0}^{\infty} \frac{[i(1-\zeta)\alpha_c X_c]^k}{k!} \right\} \\
&= \int_0^1 d\zeta\; e^{i\zeta\alpha_a X_a}\; iX_b\; e^{i(1-\zeta)\alpha_c X_c}
\end{aligned}$$

Homework:

1. Find the explicit expression of the matrix $e^{i\alpha A}$ with

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

2. If $[A,\ B] = B$, calculate $e^{i\alpha A} B e^{-i\alpha A}$.

3. Carry out the expansion of γ_c in

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\gamma_c X_c}$$

to third order of α_a and β_b.