现代数学物理方法

第四章，$SU(3)$

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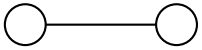
Fundamental weights of $su(3)$:

The algebra $su(3)$ is specified by Dynkin diagram

$su(3)$: 

It has two simple roots $\vec{\alpha_1}$ and $\vec{\alpha_2}$, with properties $\alpha_1^2 = \alpha_2^2 = 1$ and $\vec{\alpha_1} \cdot \vec{\alpha_2} = -1/2$. Therefore, $su(3)$ has 2 fundamental weight vectors :

$$\vec{M}_i = \left(a_i,\ b_i\right), \qquad \left\{i = 1,\ 2.\right\}$$

To find $\vec{M}_i$ ($i = 1,\ 2$), we first parameterize the simple roots as follows,

$$\vec{\alpha_1} = \left(1/2,\ \sqrt{3}/2\right), \qquad \vec{\alpha_2} = \left(1/2,\ -\sqrt{3}/2\right).$$

Because

$$\delta_{i1} = \frac{2\vec{M}_i \cdot \vec{\alpha_1}}{\alpha_1^2} = a_i + \sqrt{3}b_i, \quad \delta_{i2} = \frac{2\vec{M}_i \cdot \vec{\alpha_2}}{\alpha_2^2} = a_i - \sqrt{3}b_i,$$

we get

$$\begin{cases} a_1 + \sqrt{3}b_1 = 1 \\ a_1 - \sqrt{3}b_1 = 0 \end{cases} \qquad \begin{cases} a_2 + \sqrt{3}b_2 = 0 \\ a_2 - \sqrt{3}b_2 = 1 \end{cases}$$

The solution to this system of algebraic equations is unique,

$$\begin{cases} a_1 = 1/2 \\ b_1 = 1/2\sqrt{3} \end{cases}$$

$$\begin{cases} a_2 = 1/2 \\ b_2 = -1/2\sqrt{3} \end{cases}$$

We conclude that $su(3)$ has the following 2 fundamental weight vectors,

$$\begin{cases} \vec{M}_1 = \left(1/2, \quad 1/2\sqrt{3}\right) \\ \\ \vec{M}_2 = \left(1/2, \ -1/2\sqrt{3}\right) \end{cases}$$

Fundamental Rep. D_1 of $su(3)$:

D_1 is defined by the fundamental weight vector $\vec{M}_1$,

$$\vec{M}_1 = \left[\frac{1}{2}, \frac{1}{2\sqrt{3}}\right]$$

We now want to find all of the basis states of this representation. Our starting point is the highest weight state $|M_1\rangle$ satisfying

$$E_{\alpha_1}|M_1\rangle = E_{\alpha_2}|M_1\rangle = 0.$$

Procedure :
Build *two* $su(2)$ subalgebras associated to simple roots $\vec{\alpha_1}$ and $\vec{\alpha_2}$,

$$\left\{E_3 = \vec{\alpha_1}\cdot\vec{H},\ E_{\pm} = E_{\pm\alpha_1}\right\} \quad \rightsquigarrow \quad su(2)_1$$

$$\left\{E_3' = \vec{\alpha_2}\cdot\vec{H},\ E_{\pm}' = E_{\pm\alpha_2}\right\} \quad \rightsquigarrow \quad su(2)_2$$

The state $|M_1\rangle$ could be embedded into the spin-j representation of $su(2)_1$ with

$$j = \frac{1}{2}\Big[p + q\Big]$$

or the spin-j' representation of $su(2)_2$ with

$$j' = \frac{1}{2}\Big[p' + q'\Big]$$

so that

$$\begin{cases} (E_+)^{p+1} |M_1\rangle = (E_-)^{q+1} |M_1\rangle = 0 \\ (E'_+)^{p'+1} |M_1\rangle = (E'_-)^{q'+1} |M_1\rangle = 0 \end{cases}$$

Since $E_{\alpha_1} |M_1\rangle = 0$ and $2\vec{M}_1 \cdot \vec{\alpha_1} = 1$, we have $p = 0$, $q = 1$ and $j = 1/2$.

Hence,

$$|M_1\rangle = |1/2, 1/2\rangle_1$$

The second basis state in D_1 is found to be:

$$E_{-\alpha_1} |M_1\rangle = E_- |1/2, 1/2\rangle_1 = \frac{1}{\sqrt{2}} |1/2, -1/2\rangle_1$$

Similarly, the state $E_{-\alpha_1}\left|M_1\right\rangle$ can also be embedded into the spin-j'' representation of $su(2)_2$ with

$$j'' = \frac{1}{2}\Big[p'' + q''\Big]$$

where

$$(E'_+)^{p''+1} E_{-\alpha_1}\left|M_1\right\rangle = (E'_-)^{q''+1} E_{-\alpha_1}\left|M_1\right\rangle = 0.$$

Alternatively, $(q'' - p'')$ is given by

$$q'' - p'' = 2(\vec{M}_1 - \vec{\alpha_1}) \cdot \vec{\alpha_2} = -2\vec{\alpha_1} \cdot \vec{\alpha_2} = 1.$$

The difference of two simple roots is not a root vector,

$$\Big[E_{-\alpha_1},\, E_{\alpha_2}\Big] = 0$$

Therefore,

$$E_{\alpha_2}\Big\{E_{-\alpha_1}\left|M_1\right\rangle\Big\} = E_{-\alpha_1}\Big\{E_{\alpha_2}\left|M_1\right\rangle\Big\} = 0, \qquad \rightsquigarrow \quad p'' = 0$$

i.e., $j'' = 1/2$.

The state $E_{-\alpha_1} |M_1\rangle$ can be equivalently cast as,

$$E_{-\alpha_1} |M_1\rangle = \frac{1}{\sqrt{2}} |1/2, 1/2\rangle_2$$

The third state in D_1 reads,

$$E_{-\alpha_2} E_{-\alpha_1} |M_1\rangle = E'_- \left\{ \frac{1}{\sqrt{2}} |1/2, 1/2\rangle_2 \right\} = \frac{1}{2} |1/2, -1/2\rangle_2$$

There are no more basis states in D_1.

Discussions :

- The fundamental representation D_1 of $su(3)$ is 3-dimensional.
- D_1 is conveniently written as,

$$\text{Rep.}(1, 0)$$

or $\mathbf{3}$.

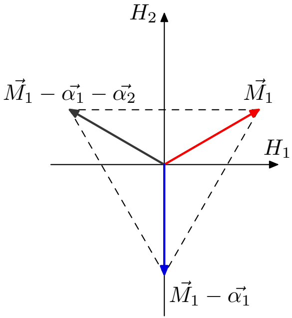
- The weight vectors in D_1 are,

$$\vec{M}_1 = \left[\frac{1}{2}, \frac{1}{2\sqrt{3}}\right] \qquad \left\{\text{Highest}\right\}$$

$$\vec{M}_1 - \vec{\alpha}_1 = \left[0, -\frac{1}{\sqrt{3}}\right]$$

$$\vec{M}_1 - \vec{\alpha}_1 - \vec{\alpha}_2 = \left[-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right]$$

In weight diagram,



- In D_1, three orthogonal basis states vectors are

$$|M_1\rangle, \quad E_{-\alpha_1}|M_1\rangle, \quad E_{-\alpha_2}E_{-\alpha_1}|M_1\rangle.$$

Let $\langle M_1|M_1\rangle = 1$. Then,

$$\begin{aligned}\langle M_1| E_{\alpha_1}E_{-\alpha_1}|M_1\rangle &= \langle M_1|[E_{\alpha_1},\ E_{-\alpha_1}]|M_1\rangle \\ &= \langle M_1|(\vec{\alpha_1}\cdot\vec{H})|M_1\rangle \\ &= (\vec{\alpha_1}\cdot\vec{M_1}) \\ &= 1/2\end{aligned}$$

and

$$\begin{aligned}&\langle M_1| E_{\alpha_1}E_{\alpha_2}E_{-\alpha_2}E_{-\alpha_1}|M_1\rangle \\ &\qquad = \langle M_1| E_{\alpha_1}[E_{\alpha_2},\ E_{-\alpha_2}]E_{-\alpha_1}|M_1\rangle \\ &\qquad = \alpha_{2i}\langle M_1| E_{\alpha_1}H_iE_{-\alpha_1}|M_1\rangle \\ &\qquad = \alpha_{2i}(\vec{M_1}-\vec{\alpha_1})_i\langle M_1| E_{\alpha_1}E_{-\alpha_1}|M_1\rangle \\ &\qquad = \frac{1}{2}\vec{\alpha_2}\cdot(\vec{M_1}-\vec{\alpha_1}) \\ &\qquad = 1/4\end{aligned}$$

Consequently,

$$|M_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad E_{-\alpha_1}|M_1\rangle = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$E_{-\alpha_2}E_{-\alpha_1}|M_1\rangle = \tfrac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

D_2 is defined by the fundamental weight vector $\vec{M}_2$,

$$\vec{M}_2 = \left[\frac{1}{2}, \ -\frac{1}{2\sqrt{3}}\right]$$

Highest weight state in D_2 :

The highest weight state $|M_2\rangle$ in D_2, i.e., Rep.$(0, 1)$ or $\bar{\mathbf{3}}$ satisfying

$$E_{\alpha_1}|M_2\rangle = E_{\alpha_2}|M_2\rangle = 0.$$

Besides,

$$\frac{2\vec{M}_2 \cdot \vec{\alpha_2}}{\alpha_2^2} = 1$$

Thus, $|M_2\rangle$ is also the highest weight state in the spin-$\frac{1}{2}$ representation of the accessory $su(2)_2$,

$$|M_2\rangle = |1/2, 1/2\rangle_2$$

Other states in D_2 :

The second basis state in D_2 is

$$E_{-\alpha_2}|M_2\rangle = E'_-|M_2\rangle = \frac{1}{\sqrt{2}}|1/2, -1/2\rangle_2$$

Notice that $E_{\alpha_1}(E_{-\alpha_2}|M_2\rangle) = 0$. Moreover,

$$\frac{2(\vec{M}_2 - \vec{\alpha_2}) \cdot \vec{\alpha_1}}{\alpha_1^2} = -2\vec{\alpha_2} \cdot \vec{\alpha_1} = 1$$

Because of these two equalities, $E_{-\alpha_2}|M_2\rangle$ is not only the lowest weight state in spin-$1/2$ representation of $su(2)_2$, it is also the highest weight state in spin-$1/2$ representation of $su(2)_1$:

$$E_{-\alpha_2} |M_2\rangle = \frac{1}{\sqrt{2}} |1/2, 1/2\rangle_1$$

As a result, the third basis state in D_2 is probably to be,

$$E_{-\alpha_1} E_{-\alpha_2} |M_2\rangle = \frac{1}{2} |1/2, -1/2\rangle_1$$

There are no more basis states in D_1.

Conclusion :

- The fundamental representation D_2 of $su(3)$ is also 3-dimensional.
- D_2 is conveniently written as Rep.$(0, 1)$ or $\bar{\mathbf{3}}$.

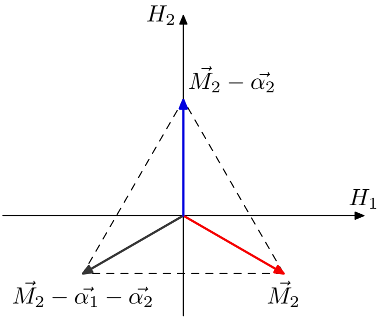
- The weight vectors in D_2 are,

$$\vec{M}_2 = \left[\frac{1}{2},\ -\frac{1}{2\sqrt{3}}\right] \qquad \left\{\text{Highest}\right\}$$

$$\vec{M}_2 - \vec{\alpha}_2 = \left[0,\ \frac{1}{\sqrt{3}}\right]$$

$$\vec{M}_2 - \vec{\alpha}_1 - \vec{\alpha}_2 = \left[-\frac{1}{2},\ -\frac{1}{2\sqrt{3}}\right]$$

In weight diagram,



Complex conjugation :

The weight vectors of $\bar{\mathbf{3}}$ are just the negatives of those of $\mathbf{3}$.

Weights in $\mathbf{3}$:

$$\vec{M}_1 = \left[\frac{1}{2},\ \frac{1}{2\sqrt{3}}\right],$$
$$\vec{M}_1 - \vec{\alpha}_1 = \left[0,\ -\frac{1}{\sqrt{3}}\right],$$
$$\vec{M}_1 - \vec{\alpha}_1 - \vec{\alpha}_2 = \left[-\frac{1}{2},\ \frac{1}{2\sqrt{3}}\right].$$

Weights in $\bar{\mathbf{3}}$:

$$\vec{M}_2 = \left[\frac{1}{2},\ -\frac{1}{2\sqrt{3}}\right],$$
$$\vec{M}_2 - \vec{\alpha}_2 = \left[0,\ \frac{1}{\sqrt{3}}\right],$$
$$\vec{M}_2 - \vec{\alpha}_1 - \vec{\alpha}_2 = \left[-\frac{1}{2},\ -\frac{1}{2\sqrt{3}}\right].$$

Question : What does this mean ?

This means that the two representations $\mathbf{3}$ and $\bar{\mathbf{3}}$ are related by complex conjugation.

Insight 1 :

Let $\left\{X_a\right\}$ be the generators of some representation D of some Lie group $\mathbb{G}$. The group elements can be expressed as

$$e^{i\alpha_a X_a}$$

As a result, we have the following expressions for the group elements of its complex conjugate $\bar{D}$:

$$\left(e^{i\alpha_a X_a}\right)^* = e^{-i\alpha_a X_a^*} = e^{i\alpha_a(-X_a^*)}$$

Besides, $\left\{-X_a^*\right\}$ obey the same Lie brackets as $\left\{X_a\right\}$,

$$[X_a,\ X_b] = if_{abc}X_c \qquad \rightsquigarrow\ [(-X_a^*),\ (-X_b^*)] = if_{abc}(-X_c^*)$$

Therefore, $\left\{-X_a^*\right\}$ are the generators of the complex conjugate Rep. $\bar{D}$ of the representation D.

Insight 2 :

The Cartan generators of the complex conjugate representation are $\left\{-H_i^*\right\}$. Because each H_i is a Hermitian matrix, H^* has the same eigenvalues as H_i.

Conclusion :

If $\vec{\mu}$ is a weight vector of Rep.D, $-\vec{\mu}$ is a weight vector of the complex conjugate Rep.$\bar{D}$.

For $su(3)$, we have seen:

$$\text{Rep. } (1,0) = \mathbf{3}, \qquad \text{Rep. } (0,1) = \bar{\mathbf{3}}.$$

In general, for Lie algebra $su(3)$, the complex conjugate of Rep.(n,m) is Rep.(m,n).

Proof :

Because the lowest weight vector of Rep.$(1,0)$ is the minus of the highest weight vector of Rep.$(0,1)$, and vice versa. We have for Rep.(n,m),

$$\begin{array}{ll} \text{Highest weight :} & n\vec{M}_1 + m\vec{M}_2 \\ \text{Lowest weight :} & -n\vec{M}_2 - m\vec{M}_1 \end{array}$$

Consequently, the highest weight vector of its complex conjugate representation should be,

$$n\vec{M}_2 + m\vec{M}_1$$

Hence, Rep.(m,n) is the complex conjugate of Rep.(n,m).

Corollary:

- Rep.(n,n) are real representations of $su(3)$.

Rep.$(1,1)$ of $su(3)$:

We now look for the basis states of the real irreducible representation Rep.$(1,1)$ of $su(3)$.

Rep.$(1,1)$ is defined by the highest weight vector,

$$\vec{M} = \vec{M}_1 + \vec{M}_2 = (1,0)$$

so $2\vec{M} \cdot \vec{\alpha_1}/\alpha_1^2 = 1$, $2\vec{M} \cdot \vec{\alpha_2}/\alpha_2^2 = 1$.

Consider the highest weight state $|M\rangle$ in Rep.$(1,1)$, which satisfies,

$$E_{\alpha_1}|M\rangle = E_{\alpha_2}|M\rangle = 0.$$

$|M\rangle$ can also be regarded as the highest weight state of the spin-$1/2$ representations of either $su(2)_1$ or $su(2)_2$,

$$|M\rangle = |1/2, 1/2\rangle_1 = |1/2, 1/2\rangle_2 .$$

Consequently, the second and the third basis states in Rep.$(1,1)$ are found to be:

$$E_{-\alpha_1}|M\rangle = \frac{1}{\sqrt{2}}|1/2, -1/2\rangle_1$$
$$E_{-\alpha_2}|M\rangle = \frac{1}{\sqrt{2}}|1/2, -1/2\rangle_2$$

To find out the 4-th basis state in Rep.$(1,1)$, we examine $E_{-\alpha_1}|M\rangle$ in view of $su(2)_2$.

Notice that

$$E_{\alpha_2}\left\{E_{-\alpha_1}|M\rangle\right\} = 0$$

and

$$\frac{2(\vec{M}-\vec{\alpha_1})\cdot\vec{\alpha_2}}{\alpha_2^2} = 1 - \frac{2\vec{\alpha_1}\cdot\vec{\alpha_2}}{\alpha_2^2} = 1 - 2\left[-\frac{1}{2}\right] = 2$$

we alternatively have

$$E_{-\alpha_1}|M\rangle = \frac{1}{\sqrt{2}}|1,1\rangle_2$$

It leads to the following 4-th and 5-th basis states in Rep.$(1,1)$:

$$E_{-\alpha_2}E_{-\alpha_1}|M\rangle = \frac{1}{2}|1,0\rangle_2\,, \quad (E_{-\alpha_2})^2 E_{-\alpha_1}|M\rangle = \frac{1}{2\sqrt{2}}|1,-1\rangle_2\,.$$

Similarly,

$$E_{-\alpha_2}|M\rangle = \frac{1}{\sqrt{2}}|1,1\rangle_1$$

The 6-th and 7-th basis states of Rep.$(1,1)$ should be:

$$E_{-\alpha_1}E_{-\alpha_2}|M\rangle = \frac{1}{2}|1,0\rangle_1\,, \qquad (E_{-\alpha_1})^2 E_{-\alpha_2}|M\rangle = \frac{1}{2\sqrt{2}}|1,-1\rangle_1\,.$$

Recall that

$$E_{-\alpha_2}E_{-\alpha_1}|M\rangle = \frac{1}{2}|1,0\rangle_2$$

Remark :

The basis states $E_{-\alpha_1}E_{-\alpha_2}|M\rangle$ and $E_{-\alpha_2}E_{-\alpha_1}|M\rangle$ are linearly independent of each other, although they are not orthogonal.

Question :

Are there any other independent states in Rep.$(1,1)$?

To answer this question, we reexamine the 7-th basis state

$$(E_{-\alpha_1})^2 E_{-\alpha_2} |M\rangle = \frac{1}{2\sqrt{2}} |1, -1\rangle_1$$

in view of $\boldsymbol{su}(2)_2$.
Since $E_{-\alpha_1} |M\rangle \approx |1/2, -1/2\rangle_1$, we have $(E_{-\alpha_1})^2 |M\rangle = 0$.
Consequently,

$$\begin{aligned} E_{\alpha_2}(E_{-\alpha_1})^2 E_{-\alpha_2} |M\rangle &= (E_{-\alpha_1})^2 \left[E_{\alpha_2},\ E_{-\alpha_2}\right] |M\rangle \\ &= (\vec{\alpha_2} \cdot \vec{M})(E_{-\alpha_1})^2 |M\rangle \\ &= 0 \end{aligned}$$

and

$$2\vec{\alpha_2} \cdot (\vec{M} - 2\vec{\alpha_1} - \vec{\alpha_2})/\alpha_2^2 = 1 + 2 - 2 = 1.$$

This implies that

$$(E_{-\alpha_1})^2 E_{-\alpha_2} |M\rangle = \frac{1}{2\sqrt{2}} |1/2, 1/2\rangle_2$$

Followed which is the 8-th basis state in Rep.$(1,1)$,

$$E_{-\alpha_2}(E_{-\alpha_1})^2 E_{-\alpha_2}\,|M\rangle = \frac{1}{4}\,|1/2,-1/2\rangle_2$$

The procedure ends here[1].

Conclusion :

Rep.$(1,1)$ of $su(3)$ is 8-dimensional (i.e., adjoint), $\mathbf{8}$. It is spanned by the following independent basis states:

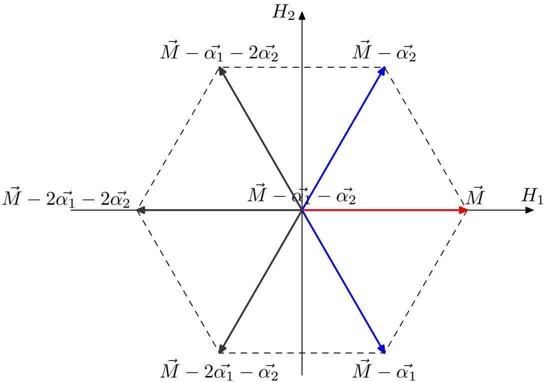
$$\begin{array}{ll} |M\rangle\,, & E_{-\alpha_1}E_{-\alpha_2}\,|M\rangle\,, \\ (E_{-\alpha_1})^2 E_{-\alpha_2}\,|M\rangle\,, & E_{-\alpha_1}\,|M\rangle\,, \\ E_{-\alpha_2}E_{-\alpha_1}\,|M\rangle\,, & (E_{-\alpha_2})^2 E_{-\alpha_1}\,|M\rangle \\ E_{-\alpha_2}\,|M\rangle & E_{-\alpha_2}(E_{-\alpha_1})^2 E_{-\alpha_2}\,|M\rangle \end{array}$$

[1] Because the 8-th state and $E_{-\alpha_1}(E_{-\alpha_2})^2 E_{-\alpha_1}\,|M\rangle$ are linearly dependent.

The corresponding weight vectors read,

$$\begin{array}{ll} \vec{M} = (1,0), & \vec{M} - \vec{\alpha_1} = (1/2, -\sqrt{3}/2), \\ \vec{M} - \vec{\alpha_2} = (1/2, \sqrt{3}/2), & \vec{M} - 2\vec{\alpha_1} - \vec{\alpha_2} = (-1/2, -\sqrt{3}/2), \\ \vec{M} - \vec{\alpha_1} - \vec{\alpha_2} = (0,0), & \text{(Degenerate)} \\ \vec{M} - \vec{\alpha_1} - 2\vec{\alpha_2} = (-1/2, \sqrt{3}/2), & \vec{M} - 2\vec{\alpha_1} - 2\vec{\alpha_2} = (-1, 0). \end{array}$$

Rep.$(1,1)$ of $su(3)$ is real. Its weight diagram is:



Appendix :

Now we examine the linear dependence between the basis states of Rep.$(1,1)$ of $su(3)$.

Theorem :

Two states $|A\rangle$ and $|B\rangle$ are linearly dependent iff

$$\langle A|B\rangle\langle B|A\rangle = \langle A|A\rangle\langle B|B\rangle\,.$$

Proof:

Consider the linear equation,

$$c_1\,|A\rangle + c_2\,|B\rangle = 0$$

The coefficients c_1 and c_2 can be viewed as the unknown quantities of

$$\begin{aligned} \langle A|A\rangle\, c_1 + \langle A|B\rangle\, c_2 &= 0, \\ \langle B|A\rangle\, c_1 + \langle B|B\rangle\, c_2 &= 0. \end{aligned}$$

Having non-zero c_1 and c_2 requires,

$$\begin{vmatrix} \langle A|A\rangle & \langle A|B\rangle \\ \langle B|A\rangle & \langle B|B\rangle \end{vmatrix} = 0. \qquad \text{(QED)}$$

Firstly, we examine the linear dependence of states $|A\rangle = E_{-\alpha_1} E_{-\alpha_2} |M\rangle$ and $|B\rangle = E_{-\alpha_2} E_{-\alpha_1} |M\rangle$.

Because

$$
\begin{aligned}
\langle A|A\rangle &= \langle M| E_{\alpha_2} E_{\alpha_1} E_{-\alpha_1} E_{-\alpha_2} |M\rangle \\
&= \left(\vec{\alpha_1} \cdot (\vec{M} - \vec{\alpha_2})\right)\left(\vec{\alpha_2} \cdot \vec{M}\right) = (1/2 + 1/2)1/2 = 1/2 \\
\langle B|B\rangle &= 1/2 \\
\langle A|B\rangle &= \langle M| E_{\alpha_2} E_{\alpha_1} E_{-\alpha_2} E_{-\alpha_1} |M\rangle \\
&= (\vec{\alpha_1} \cdot \vec{M})(\vec{\alpha_2} \cdot \vec{M}) = (1/2) \cdot (1/2) = 1/4 \\
\langle B|A\rangle &= 1/4
\end{aligned}
$$

we see,

$$
\begin{vmatrix} \langle A|A\rangle & \langle A|B\rangle \\ \langle B|A\rangle & \langle B|B\rangle \end{vmatrix} = (1/2)^2 - (1/4)^2 = \frac{3}{16} \neq 0.
$$

Hence, these two states are linearly independent.

Secondly, we examine the linearly dependence of states

$$
|\xi\rangle = E_{-\alpha_1} (E_{-\alpha_2})^2 E_{-\alpha_1} |M\rangle, \qquad |\eta\rangle = E_{-\alpha_2} (E_{-\alpha_1})^2 E_{-\alpha_2} |M\rangle .
$$

The norm of $|\xi\rangle$ is calculated below,

$$
\begin{aligned}
\langle \xi|\xi\rangle &= \langle M| E_{\alpha_1} (E_{\alpha_2})^2 E_{\alpha_1} E_{-\alpha_1} (E_{-\alpha_2})^2 E_{-\alpha_1} |M\rangle \\
&= \langle M| E_{\alpha_1} (E_{\alpha_2})^2 (\vec{\alpha_1} \cdot \vec{H} + E_{-\alpha_1} E_{\alpha_1}) (E_{-\alpha_2})^2 E_{-\alpha_1} |M\rangle \\
&= \left[\vec{\alpha_1} \cdot (\vec{M} - \vec{\alpha_1} - 2\vec{\alpha_2})\right] \langle M| E_{\alpha_1} (E_{\alpha_2})^2 (E_{-\alpha_2})^2 E_{-\alpha_1} |M\rangle \\
&\quad + \langle M| E_{\alpha_1} (E_{\alpha_2})^2 E_{-\alpha_1} E_{\alpha_1} (E_{-\alpha_2})^2 E_{-\alpha_1} |M\rangle
\end{aligned}
$$

where,

$$\begin{aligned}
\text{Term 2} &= (\vec{\alpha_1}\cdot\vec{M})^2 \langle M|\,(E_{\alpha_2})^2(E_{-\alpha_2})^2\,|M\rangle \\
&= (\vec{\alpha_1}\cdot\vec{M})^2 \langle M|\,E_{\alpha_2}(\vec{\alpha_2}\cdot\vec{H}+E_{-\alpha_2}E_{\alpha_2})E_{-\alpha_2}\,|M\rangle \\
&= (\vec{\alpha_1}\cdot\vec{M})^2(\vec{\alpha_2}\cdot\vec{M})\big[\vec{\alpha_2}\cdot(\vec{M}-\vec{\alpha_2})+\vec{\alpha_2}\cdot\vec{M}\big] \\
&= (1/2)^2(1/2)(1/2-1+1/2) \\
&= 0.
\end{aligned}$$

Rep.$(1,1)=\mathbf{8}$ is the adjoint representation of $su(3)$. Its highest weight vector is nothing but the positive root vector of the highest rank,

$$\vec{M}=\vec{\alpha_1}+\vec{\alpha_2}.$$

Consequently,

$$\begin{aligned}
\langle\xi|\xi\rangle &= \big[\vec{\alpha_1}\cdot(\vec{M}-\vec{\alpha_1}-2\vec{\alpha_2})\big]\langle M|\,E_{\alpha_1}(E_{\alpha_2})^2(E_{-\alpha_2})^2E_{-\alpha_1}\,|M\rangle \\
&= -(\vec{\alpha_1}\cdot\vec{\alpha_2})\langle M|\,E_{\alpha_1}(E_{\alpha_2})^2(E_{-\alpha_2})^2E_{-\alpha_1}\,|M\rangle \\
&= -(\vec{\alpha_1}\cdot\vec{\alpha_2})\langle M|\,E_{\alpha_1}E_{\alpha_2}(\vec{\alpha_2}\cdot\vec{H}+E_{-\alpha_2}E_{\alpha_2})E_{-\alpha_2}E_{-\alpha_1}\,|M\rangle \\
&= -(\vec{\alpha_1}\cdot\vec{\alpha_2})\big[\vec{\alpha_2}\cdot(\vec{M}-\vec{\alpha_1}-\vec{\alpha_2})\big]\langle M|\,E_{\alpha_1}E_{\alpha_2}E_{-\alpha_2}E_{-\alpha_1}\,|M\rangle \\
&\quad -(\vec{\alpha_1}\cdot\vec{\alpha_2})\langle M|\,E_{\alpha_1}E_{\alpha_2}E_{-\alpha_2}E_{\alpha_2}E_{-\alpha_2}E_{-\alpha_1}\,|M\rangle \\
&= -(\vec{\alpha_1}\cdot\vec{\alpha_2})\langle M|\,E_{\alpha_1}E_{\alpha_2}E_{-\alpha_2}E_{\alpha_2}E_{-\alpha_2}E_{-\alpha_1}\,|M\rangle \\
&= -(\vec{\alpha_1}\cdot\vec{\alpha_2})\big[\vec{\alpha_2}\cdot(\vec{M}-\vec{\alpha_1})\big]\langle M|\,E_{\alpha_1}E_{\alpha_2}E_{-\alpha_2}E_{-\alpha_1}\,|M\rangle \\
&= -(\vec{\alpha_1}\cdot\vec{\alpha_2})\big[\vec{\alpha_2}\cdot(\vec{M}-\vec{\alpha_1})\big]^2(\vec{\alpha_1}\cdot\vec{M}) \\
&= (1/2)(1/2+1/2)^2(1/2) \\
&= 1/4
\end{aligned}$$

i.e. $\langle\xi|\xi\rangle = 1/4$. Similar calculations yield,

$$\langle\xi|\eta\rangle = \langle\eta|\xi\rangle = \langle\eta|\eta\rangle = 1/4$$

Therefore,

$$\begin{vmatrix} \langle\xi|\xi\rangle & \langle\xi|\eta\rangle \\ \langle\eta|\xi\rangle & \langle\eta|\eta\rangle \end{vmatrix} = (1/4)^2 - (1/4)^2 = 0$$

The involved two states $|\xi\rangle$ and $|\eta\rangle$ are linearly dependent.

Rep.$(2,0)$ of $su(3)$:

Rep.$(2,0)$ of $su(3)$ is defined by the highest weight vector

$$\vec{M} = 2\vec{M}_1 = \left[1,\ \frac{1}{\sqrt{3}}\right]$$

that obeys the master formulae $2\vec{M}\cdot\vec{\alpha_1}/\alpha_1^2 = 2$ and $2\vec{M}\cdot\vec{\alpha_2}/\alpha_2^2 = 0$.

- In Rep.$(2,0)$, the highest weight state $|M\rangle$ satisfies,

  $$E_{\alpha_1}|M\rangle = E_{\alpha_2}|M\rangle = 0.$$

  As a product of the Master formula $2\vec{M}\cdot\vec{\alpha_2}/\alpha_2^2 = 0$, it also satisfies,

  $$E_{-\alpha_2}|M\rangle = 0.$$

In view of the accessory $su(2)_1$ related to the simple root $\vec{\alpha_1}$, $|M\rangle$ can be formulated as,

$$|M\rangle = |1,1\rangle_1$$

Then two other basis states of Rep.$(2,0)$ follow,

$$E_{-\alpha_1}|M\rangle = |1,0\rangle_1\,, \qquad (E_{-\alpha_1})^2|M\rangle = |1,-1\rangle_1\,.$$

- Relying on the facts

$$E_{\alpha_2}E_{-\alpha_1}|M\rangle = 0, \qquad \frac{2(\vec{M}-\vec{\alpha_1})\cdot\vec{\alpha_2}}{\alpha_2^2} = 1,$$

  the second basis state $E_{-\alpha_1}|M\rangle$ can alternatively be regarded as the highest weight state

$$E_{-\alpha_1}|M\rangle = |1/2,1/2\rangle_2$$

  in the spin-$1/2$ representation of $su(2)_2$.

This observation leads to the 4-th basis state of Rep.$(2,0)$,

$$E_{-\alpha_2}E_{-\alpha_1}|M\rangle = \frac{1}{\sqrt{2}}|1/2,-1/2\rangle_2$$

- Notice that

$$E_{\alpha_2}(E_{-\alpha_1})^2 |M\rangle = 0, \qquad \frac{2(\vec{M} - 2\vec{\alpha_1}) \cdot \vec{\alpha_2}}{\alpha_2^2} = 2,$$

  the third basis state $(E_{-\alpha_1})^2 |M\rangle$ can alternatively be viewed as the highest weight state

$$(E_{-\alpha_1})^2 |M\rangle = |1, 1\rangle_2$$

  in the spin-1 representation of $su(2)_2$.

As a result of $su(2)_2$, the 5-th and 6-th basis states of Rep.$(2, 0)$ emerge. They are

$$E_{-\alpha_2}(E_{-\alpha_1})^2 |M\rangle = |1, 0\rangle_2$$

and

$$(E_{-\alpha_2})^2(E_{-\alpha_1})^2 |M\rangle = |1, -1\rangle_2$$

respectively.

Question:

Does Rep.$(2, 0)$ *contain any more basis states ?*

Let us examine the 4-th basis state $E_{-\alpha_2} E_{-\alpha_1} |M\rangle$.

Obviously,

$$E_{\alpha_1} \left\{ E_{-\alpha_2} E_{-\alpha_1} |M\rangle \right\} = (\vec{\alpha_1} \cdot \vec{M}) E_{-\alpha_2} |M\rangle = 0,$$

$$\frac{2}{\alpha_1^2} \left[(\vec{M} - \vec{\alpha_1} - \vec{\alpha_2}) \cdot \vec{\alpha_1} \right] = 2 - 2 + 1 = 1.$$

This suggests that $E_{-\alpha_2} E_{-\alpha_1} |M\rangle$ forms the highest weight state

$$E_{-\alpha_2} E_{-\alpha_1} |M\rangle = \frac{1}{\sqrt{2}} |1/2, 1/2\rangle_1$$

of the spin-$1/2$ representation of $su(2)_1$.

Therefore, Rep.$(2, 0)$ does probably have the 7-th basis state as follows:

$$E_{-\alpha_1} E_{-\alpha_2} E_{-\alpha_1} |M\rangle = \frac{1}{2} |1/2, -1/2\rangle_1 .$$

However[2], $E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_1}|M\rangle$ and $E_{-\alpha_2}(E_{-\alpha_1})^2|M\rangle$, the 5-th basis state in Rep.$(2,0)$ are not only of the same weight, but linearly dependent also.

Conclusion :

Rep.$(2,0)$ of $su(3)$ is a 6-dimensional irreducible representation,

$$\text{Rep.}(2,0) = \mathbf{6}$$

Its 6 independent basis states read,

$$\begin{aligned}
&|M\rangle\,,\\
&E_{-\alpha_2}E_{-\alpha_1}|M\rangle\,,\\
&E_{-\alpha_1}|M\rangle\,,\\
&E_{-\alpha_2}(E_{-\alpha_1})^2|M\rangle\,,\\
&(E_{-\alpha_1})^2|M\rangle\,,\\
&(E_{-\alpha_2})^2(E_{-\alpha_1})^2|M\rangle\,.
\end{aligned}$$

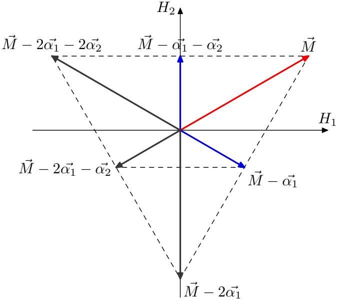
[2] Please check this claim yourself.

The weight vectors of Rep.$(2,0)$ are as follows:

$$\vec{M} = (1, 1/\sqrt{3}), \qquad \left\{\text{Highest}\right\}$$

$$\begin{aligned}
\vec{M} - \vec{\alpha_1} - \vec{\alpha_2} &= (0, 1/\sqrt{3}), \\
\vec{M} - \vec{\alpha_1} &= (1/2, -1/2\sqrt{3}), \\
\vec{M} - 2\vec{\alpha_1} - \vec{\alpha_2} &= (-1/2, -1/2\sqrt{3}), \\
\vec{M} - 2\vec{\alpha_1} &= (0, -2/\sqrt{3}), \\
\vec{M} - 2\vec{\alpha_1} - 2\vec{\alpha_2} &= (-1, 1/\sqrt{3}).
\end{aligned}$$

Its weight diagram is



1. Consider the following matrices defined in the 6-dimensional tensor product space of the Gell-Mann matrices λ_a and the Pauli matrices σ_i,

$$\tfrac{1}{2}\lambda_a\sigma_2, \qquad \text{for } a = 1, 3, 4, 6 \text{ and } 8;$$

$$\tfrac{1}{2}\lambda_a, \qquad \text{for } a = 2, 5, 7 \text{ and } 7.$$

   Show that these matrices generate a reducible representation of $su(3)$ and reduce it.

2. Decompose the tensor product of $\mathbf{3} \times \mathbf{3}$, using the highest weight techniques.