现代数学物理方法

第二章，群论基础

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Importance of Young tableaux:

1. Each different tableaux of n-boxes represents a different conjugacy class of S_n.
2. The Young tableaux are in *one-to-one* correspondence with the irreducible representations of S_n.

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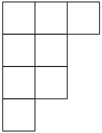

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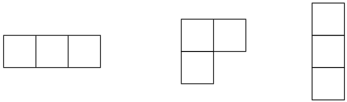

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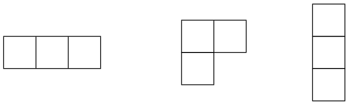

respectively.

The numbers of group elements in these conjugacy classes are:

$$\frac{3!}{3!} = 1, \qquad \frac{3!}{2} = 3, \qquad \frac{3!}{3} = 2.$$

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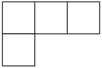

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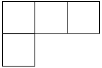

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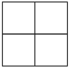

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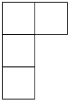

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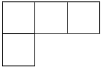

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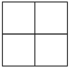

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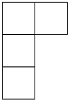

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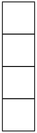

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Obviously, it is $(167425)(3)$.

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This state spans another 1-dimensional irreducible subspace which defines the so-called alternate representation D_1 of S_3:

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$$\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \rightsquigarrow |321\rangle + |231\rangle - |123\rangle - |213\rangle = -|\psi_{21}\rangle$$

Therefore, $\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ is associated with a 2-d irreducible representation of S_3.

Explanation:

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The state related to the Young tableau

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

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is determined as follows:

$$\begin{aligned}|\psi_{213}\rangle &= [e-(23)][e+(12)]\,|213\rangle \\ &= [e-(23)+(12)-(132)]\,|213\rangle\end{aligned}$$

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$$\begin{aligned} |\psi_{213}\rangle &= [e-(23)][e+(12)]\,|213\rangle \\ &= [e-(23)+(12)-(132)]\,|213\rangle \\ &= |213\rangle - |312\rangle + |123\rangle - |132\rangle \end{aligned}$$

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Recall that,

$$|\psi_{21}\rangle = |123\rangle + |213\rangle - |321\rangle - |231\rangle$$
$$|\psi_{22}\rangle = |132\rangle + |312\rangle - |231\rangle - |321\rangle$$

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is determined as follows:

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Recall that,

$$|\psi_{21}\rangle = |123\rangle + |213\rangle - |321\rangle - |231\rangle$$
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Hence,

$$|\psi_{213}\rangle = |\psi_{21}\rangle - |\psi_{22}\rangle$$

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1. In a standard Young tableau, the filled numbers increase within a row from left to right and within a column from top to down.

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Standard Young tableaux:

1. In a standard Young tableau, the filled numbers increase within a row from left to right and within a column from top to down.
2. For a given Young tableau, the number of the standard Young tableaux is the same as the dimensions of the corresponding irreducible representation.

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The standard Young tableaux of S_3 are as follows:

$$\boxed{1 \mid 2 \mid 3}\ ;$$

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- Go back to the construction of the 2-d irreducible representation of S_3.

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- Go back to the construction of the 2-d irreducible representation of S_3. On the states $|\psi_{21}\rangle$ and $|\psi_{22}\rangle$ that correspond to the standard Young tableaux,

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we have,

$$D_2[(12)]\,|\psi_{21}\rangle$$

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$D_2[(12)]\,|\Psi_{22}\rangle$

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D_2[(12)]\,|\Psi_{22}\rangle &= D_2[(12)]\Big\{\,|132\rangle + |312\rangle - |231\rangle - |321\rangle\,\Big\} \\
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By setting $|\psi_{21}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|\psi_{22}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$
\begin{aligned}
D_2[(12)]\,|\Psi_{22}\rangle &= D_2[(12)]\Big\{\,|132\rangle + |312\rangle - |231\rangle - |321\rangle\,\Big\} \\
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D_2[(12)]\,|\Psi_{22}\rangle \; &= \; D_2[(12)]\Big\{\,|132\rangle + |312\rangle - |231\rangle - |321\rangle\,\Big\} \\
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Hence,

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D_2[(23)] = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right]
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- In conclusion, the 2-d irreducible Rep. $D_2(S_3)$ is realized by,

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D_2[e] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D_2[(12)] = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}
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To get the equivalent unitary representation, we introduce an auxiliary hermitian matrix H,

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H = \sum_{g \in S_3} \left[D_2(g) \right]^{\dagger} D_2(g) = \begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix}
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The eigenvalue equation of matrix H reads,

$$\begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}$$

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where ϕ_1 and ϕ_2 are two arbitrary real parameters (phases). These two eigenvectors can be used to define a unitary matrix

$$u = \begin{bmatrix} \frac{e^{i\phi_1}}{\sqrt{2}} & \frac{e^{i\phi_2}}{\sqrt{2}} \\ \frac{e^{i\phi_1}}{\sqrt{2}} & -\frac{e^{i\phi_2}}{\sqrt{2}} \end{bmatrix}$$

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Explicitly,

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Warning:

The matrix forms of the 2-dimensional unitary irreducible representation of S_3 are still not unique, although they are equivalent to each other.

An alternative realization of this 2-d irreducible unitary representation for S_3 is,

$$D_2(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D_2[(123)] = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

$$D_2[(132)] = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$$

$$D_2[(12)] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D_2[(23)] = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

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Homework:

Question (optional):

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1. Please find a similarity transformation to relate these two equivalent unitary representations of S_3.

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Problem:

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Problem:

1. Find the group of all the discrete rotations that leave a regular tetrahedron invariant by labeling the four vertices and considering the rotations as permutations on the four vertices. This defines a four dimensional representation of a group. Find the conjugacy classes and the characters of the irreducible representations of this group.