

## CHAPTER 2

### Brownian Motion

In stochastic analysis, we deal with two important classes of stochastic processes: Markov processes and martingales. Brownian motion is the most important example for both classes, and is also the most thoroughly studied stochastic process. In this chapter we discuss Brownian motion only to the extent of what is needed for these lectures.

#### 1. Stochastic processes

In this section we make some remarks about stochastic processes in general. We define a stochastic process  $X$  as a collection of random variables  $\{X_t, t \geq 0\}$  on a measurable space  $(\Omega, \mathcal{F})$ . If we have a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , then for each finite set of time points  $T = \{t_1, \dots, t_n\} \subset \mathbb{R}_+$ , we have a Borel measure  $\mu_T^X$  defined on  $\mathbb{R}^n$  by

$$\mu_T^X(C) = \mathbb{P} \{ (X_{t_1}, \dots, X_{t_n}) \in C \}.$$

Each  $\mu_T^X$  is called a finite dimensional marginal distribution of the process  $X$ . A theorem in measure theory (more precisely, Kolmogorov's extension theorem) says that the set of finite dimensional distributions

$$\left\{ \mu_T^X : T \text{ finite subsets of } \mathbb{R}_+ \right\}$$

determines a unique probability measure  $\mu^X$  on the product measurable space  $(\mathbb{R}^{\mathbb{R}_+}, \mathcal{B}^{\mathbb{R}_+})$  such that

$$\mu^X \left( \pi_T^{-1} C \right) = \mu_T^X(C),$$

where  $C \in \mathcal{B}(\mathbb{R}^n)$  and  $\pi_T : \mathbb{R}^{\mathbb{R}_+} \rightarrow \mathbb{R}^n$  is the projection

$$\pi_T(x) = \{x_{t_1}, \dots, x_{t_n}\}, \quad x = \{x_t, t \in \mathbb{R}_+\} \in \mathbb{R}^{\mathbb{R}_+}.$$

Equivalently,  $\mu^X$  is characterized by the relation

$$\mu^X \{x : (x_{t_1}, \dots, x_{t_n}) \in C\} = \mathbb{P} \{ \omega : (X_{t_1}(\omega), \dots, X_{t_n}(\omega)) \in C \}.$$

The probability measure  $\mu^X$  is called the law of the stochastic process.

We can regard a stochastic process  $X$  as a measurable map

$$X : (\Omega, \mathcal{F}) \rightarrow \left( \mathbb{R}^{\mathbb{R}_+}, \mathcal{B}(\mathbb{R})^{\mathbb{R}_+} \right)$$

defined by  $\omega \mapsto \{X_t(\omega)\}$ . Then the law of  $X$  is simply the induced measure  $\mu^X = \mathbb{P} \circ X^{-1}$  defined by

$$\mu^X(A) = \mathbb{P} \{ \omega : X(\omega) \in A \}, \quad A \in \mathcal{B}(\mathbb{R})^{\mathbb{R}_+}.$$

The sample path space  $(\mathbb{R}^{\mathbb{R}_+}, \mathcal{B}(\mathbb{R})^{\mathbb{R}_+})$  is too big to have any practical use. If the process  $X$  has certain special sample path properties, we can greatly reduce the sample path space. We say that  $X$  has continuous sample paths or  $X$  is a continuous process if

$$\mathbb{P} \{ \omega : t \mapsto X_t(\omega) \text{ is continuous} \} = 1.$$

If this is the case, then the process can be regarded as a map

$$X : \Omega \rightarrow C(\mathbb{R}_+, \mathbb{R})$$

from the underlying space  $\Omega$  to the space  $W(\mathbb{R}) = C(\mathbb{R}_+, \mathbb{R})$  of continuous functions from  $\mathbb{R}_+ = [0, \infty)$  to  $\mathbb{R}$ . The space  $W(\mathbb{R})$  is a metric space under the distance function

$$\|f - g\| = \sum_{N=1}^{\infty} \frac{1}{2^N} \frac{\|f - g\|_{\infty, N}}{1 + \|f - g\|_{\infty, N}},$$

where

$$\|f\|_{\infty, N} = \max_{0 \leq t \leq N} |f(t)|.$$

Under this metric  $f_n \rightarrow f$  if and only if  $f_n(t) \rightarrow f(t)$  uniformly on every bounded interval. We denote the Borel  $\sigma$ -algebra on  $W(\mathbb{R})$  by  $\mathcal{B}(W(\mathbb{R}))$ .

It is often convenient to consider the so-called coordinate process

$$\Pi = \{\Pi_t, t \geq 0\}$$

on  $(W(\mathbb{R}), \mathcal{B}(W(\mathbb{R})))$ . This is defined simply as  $\Pi_t(w) = w_t$  for all  $w \in W(\mathbb{R})$ . This process generates a standard filtration

$$\mathcal{B}_*(W(\mathbb{R})) = \{\mathcal{B}_t(W(\mathbb{R})), t \geq 0\}.$$

We have the following basic fact.

PROPOSITION 1.1. *We have*

$$\mathcal{B}(W(\mathbb{R})) = \bigvee_{t \geq 0} \mathcal{B}_t(W(\mathbb{R})),$$

*that is, the smallest  $\sigma$ -algebra generated by the coordinate process is precisely the Borel  $\sigma$ -algebra of the sample path space  $W(\mathbb{R})$ .*

PROOF. Exercise. □

Suppose that  $X$  is a continuous stochastic process on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Note that by definition  $X_t : \Omega \rightarrow \mathbb{R}$  is a random variable for each  $t$ . Then from the above PROPOSITION we see that

$$X : (\Omega, \mathcal{F}) \rightarrow (W(\mathbb{R}), \mathcal{B}(W(\mathbb{R})))$$

is a measurable map, or equivalently,  $X$  is a  $W(\mathbb{R})$ -valued random variable. The law  $\mu^X = \mathbb{P} \circ X^{-1}$  is a probability measure on  $C(\mathbb{R}_+, \mathbb{R})$ . It is clear that

the stochastic process  $X$  and the coordinate process  $\Pi$  have the same marginal distributions. In this sense  $\Pi$  on  $(W(\mathbb{R}), \mathcal{B}(W(\mathbb{R})), \mu^X)$  is a standard copy of  $X$ , and for all practical purpose, we can regard  $X$  and  $\Pi$  as the same process.

Regarding a continuous stochastic process  $X$  as a measurable map from the underlying probability space to the sample path space  $W(\mathbb{R})$  is a very convenient and useful point of view. A nice feature of  $W(\mathbb{R})$  is that it has a natural shift operator  $\theta_t : W(\mathbb{R}) \rightarrow W(\mathbb{R})$  defined by  $(\theta_t w)_s = w_{s+t}$ .

REMARK 1.2. It is a curious fact that the space of continuous functions  $C(\mathbb{R}_+, \mathbb{R})$  is *not* a measurable subset of the product measurable space  $(\mathbb{R}^{\mathbb{R}_+}, \mathcal{B}(\mathbb{R})^{\mathbb{R}_+})$ .

## 2. Brownian motion and its basic properties

DEFINITION 2.1. A stochastic process  $B = \{B_t, t \in \mathbb{R}_+\}$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is called a Brownian motion if it has the following two properties: (1)  $B$  has independent increments, i.e., for any finite set of increasing nonnegative numbers  $0 < t_1 < \dots < t_n$ , the random variables

$$(2.1) \quad B_0, B_{t_1} - B_0, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}$$

are independent;

(2) for any  $t > s \geq 0$ , the distribution of the increment  $B_t - B_s$  is  $N(0, t - s)$ , the Gaussian distribution with mean zero and variance  $t - s$ .

We can combine the two properties by saying that the joint distribution of the increments in (2.1) is the  $n$ -dimensional Gaussian distribution with zero mean vector and the diagonal covariance matrix

$$\text{diag}(t_1 - t_0, t_2 - t_1, \dots, t_n - t_{n-1}).$$

The distribution of  $B_0$  is called the *initial distribution* of the Brownian motion  $B$ . If  $\mathbb{P}\{B_0 = x\} = 1$  for a point  $x \in \mathbb{R}^1$ , we say that the Brownian motion  $B$  starts from  $x$ . It is clear that if  $B$  is a Brownian motion starting from 0, then  $B + x = \{B_t + x, t \geq 0\}$  is a Brownian motion starting from  $x$ .

The above definition is a description on the finite dimensional marginal distributions of the Brownian motion except that the initial distribution is not explicitly given. Given a distribution (a probability measure)  $\mu$  on  $\mathbb{R}^1$ , Brownian motion with initial distribution  $\mu$  is unique in the sense that any two such Brownian motions have the same finite-dimensional distributions. In fact, the finite-dimensional marginal probability

$$\mathbb{P}\{B_0 \in A_0, B_{t_1} \in A_1, B_{t_n} \in A_n\}$$

is given by

$$\int_{A_0} \mu(dx_0) \int_{A_1} p(t_1 - t_0, x_0, x_1) dx_1 \cdots \int_{A_n} p(t_n - t_{n-1}, x_{n-1}, x_n) dx_n.$$

where

$$p(t, x) = \frac{1}{\sqrt{2\pi t}} e^{-|x|^2/2t}.$$

By Kolmogorov's extension theorem, the existence of a Brownian motion with any given initial distribution is immediate.

Depending on one's taste, one can add more properties into the definition of a Brownian motion. One can require that  $B_0 = 0$ . This makes Brownian motion into a Gaussian process characterized uniquely by the covariance function

$$\mathbb{E}\{B_s B_t\} = \min\{s, t\}.$$

Let  $B$  be a Brownian motion on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  such that  $B_0 = 0$ . Let  $\mu$  be a probability distribution on  $\mathbb{R}$ . Then on the product probability space  $(\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mu \times \mathbb{P})$ , the random variable  $X(x, \omega) = x$  has the distribution  $\mu$  and is independent of  $B$ . It is easy to see that  $X + B = \{X + B_t, t \geq 0\}$  is a Brownian motion with initial distribution  $\mu$ .

Some authors also require that a Brownian motion has continuous sample paths. This requirement calls immediately into the question the existence of Brownian motion with continuous sample paths. We will show in the next section that such Brownian motion indeed exists (Wiener's theorem) A continuous Brownian motion with initial value zero is usually called a standard Brownian motion.

We will also use a slight more general notion of a Brownian motion with respect to a filtration  $\mathcal{F}_* = \{\mathcal{F}_t, t \geq 0\}$ .

**DEFINITION 2.2.** A stochastic process  $B = \{B_t, t \in \mathbb{R}_+\}$  defined on a filtered probability space  $(\Omega, \mathcal{F}_*, \mathbb{P})$  is called a Brownian motion with respect to the filtration  $\mathcal{F}_*$  if it has the following two properties:

- (1)  $B$  is adapted to  $\mathcal{F}_*$  and  $B_t - B_s$  is independent of  $\mathcal{F}_s$  for all  $t \geq s \geq 0$ ;
- (2) for any  $t > s \geq 0$ , the distribution of the increment  $B_t - B_s$  is  $N(0, t - s)$ , the Gaussian distribution with mean zero and variance  $t - s$ .

It is clear that each Brownian motion  $B$  is a Brownian motion with respect to its own filtration  $\mathcal{F}_*^B$ . The following example shows why we need this slightly enlarged concept of Brownian motion.

**EXAMPLE 2.3.** An  $n$ -dimensional Brownian motion  $B$  is defined as  $B_t = (B_t^1, B_t^2, \dots, B_t^n)$ , where  $B^i$  are  $n$  independent Brownian motions. Let  $\mathcal{F}_*^B$  be the filtration generated by  $B$ . Then each component  $B^i$  is a Brownian motion with respect to  $\mathcal{F}_*^B$ .

In the rest of this section we discuss a few basic and useful properties of Brownian motion.

**PROPOSITION 2.4.** Let  $B = \{B_t, t \geq 0\}$  be a Brownian motion starting from zero.

- (1) **MARKOV PROPERTY.** The process  ${}^s B = \{B_{t+s} - B_s, t \geq 0\}$  is a Brownian motion independent of the  $\sigma$ -field  $\mathcal{F}_s^B = \sigma\{B_u, 0 \leq u \leq s\}$ ;

- (2) SCALING PROPERTY. For a nonzero  $c$ , the process  $\{cB_{c^{-2}t}, t \geq 0\}$  is also a Brownian motion; in particular  $-B = \{-B_t, t \geq 0\}$  is a Brownian motion;
- (3) TIME REVERSAL. If  $B$  is a Brownian motion 0, then

$$W = \{tB_{t^{-1}}, t \geq 0\}$$

(define  $W_0 = 0$ ) is also a Brownian motion from 0.

PROOF. We prove the time-reversal property. Brownian motion  $B$  starting from zero can be uniquely characterized as a Gaussian process with mean zero and the covariance function  $R(s, t) = \min\{s, t\}$ . It is clear that the process  $W_t = tB_{1/t}$  is a Gaussian process. We have

$$\mathbb{E}\{W_s W_t\} = st \mathbb{E}\{B_{1/s} B_{1/t}\} = st \min\{1/s, 1/t\} = \min\{s, t\}.$$

Hence  $W$  is a Gaussian process with the same covariance function as  $B$ . It therefore must also be a Brownian motion.  $\square$

PROPOSITION 2.5. Let  $B$  be a Brownian motion with respect to a filtration  $\mathcal{F}_*$ .

- (1)  $B$  is a martingale with respect to  $\mathcal{F}_*$  whose quadratic variation process is  $\langle B, B \rangle_t = t$ .
- (2) For any real  $\lambda$ , the process

$$\exp\left[\lambda B_t - \frac{\lambda^2}{2}t\right], \quad t \geq 0,$$

is a martingale.

PROOF. Since  $B_t - B_s$  independent of  $\mathcal{F}_s$  with the distribution  $N(0, t - s)$  we have

$$\mathbb{E}\{B_t - B_s | \mathcal{F}_s\} = \mathbb{E}(B_t - B_s) = 0.$$

Hence  $B$  is a martingale. Likewise, from

$$B_t^2 - B_s^2 = (B_t - B_s)^2 + 2(B_t - B_s)B_s,$$

we have

$$\mathbb{E}\{B_t^2 - B_s^2 | \mathcal{F}_s\} = \mathbb{E}(B_t - B_s)^2 + B_s \mathbb{E}\{B_t - B_s | \mathcal{F}_s\} = t - s.$$

This shows that  $B_t^2 - t$  is a martingale, which implies that the quadratic variation process is  $\langle B, B \rangle_t = t$ .

From  $e^{\lambda B_t} = e^{\lambda B_s} e^{\lambda(B_t - B_s)}$  we have

$$\mathbb{E}\{e^{\lambda B_t} | \mathcal{F}_s\} = e^{\lambda B_s} \mathbb{E}e^{\lambda(B_t - B_s)} = e^{\lambda B_s} e^{\lambda^2(t-s)/2}.$$

Therefore  $\exp[\lambda B_t - \lambda^2 t/2]$  is a martingale.  $\square$

### 3. Construction of Brownian motion

The goal of this section and the next section is to prove the existence of a Brownian motion with continuous sample paths. If  $\{B_t^n, 0 \leq t \leq 1\}$  are independent copies of Brownian motions with time interval  $[0, 1]$ , then the process defined by

$$B_t = B_n + B_{t-n}^{n+1}, \quad n \leq t < n+1,$$

is a Brownian motion with time interval  $\mathbb{R}_+ = [0, \infty)$ . Therefore it is enough to construct a Brownian motion on the time interval  $[0, 1]$ .

Without sample path continuity, the existence of Brownian motion itself is guaranteed by the Kolmogorov extension theorem. We can also construct Brownian motion on  $[0, 1]$  more directly by using Fourier series. Suppose that  $B$  is a Brownian motion from 0. Let  $W_t = B_t - tB_1$ . The process  $W$  is called a Brownian bridge. Note that  $W_0 = W_1 = 0$ . Let us expand the sample paths of  $W$  into a Fourier sine series on  $[0, 1]$ :

$$W_t = \sum_{n=1}^{\infty} X_n \sin n\pi t.$$

Of course the Fourier coefficients  $X_n$  are random variables and we have a formula for them:

$$X_n = 2 \int_0^1 W_t \sin n\pi t \, dt.$$

From this formula it is clear that the random variables  $\{X_n\}$  form a Gaussian system. All  $X_n$  have zero mean. Let us compute the covariance matrix. We have

$$\mathbb{E}[W_s W_t] = \min\{s, t\} - st.$$

Using this we have

$$\begin{aligned} \mathbb{E}[X_m X_n] &= 4\mathbb{E}\left[\int_0^1 W_s \sin m\pi s \, ds \int_0^1 W_t \sin n\pi t \, dt\right] \\ &= 4 \int_0^1 \int_0^1 \mathbb{E}[W_s W_t] \sin m\pi s \sin n\pi t \, ds \, dt \\ &= 4 \int_0^1 \int_0^1 [\min\{s, t\} - st] \sin m\pi s \sin n\pi t \, ds \, dt. \end{aligned}$$

After calculating the last integral we obtain

$$\mathbb{E}[X_m X_n] = \frac{2}{\pi^2 n^2} \delta_{mn}.$$

It follows from the above relation that  $Z_n = n\pi X_n / \sqrt{2}$  is a sequence of i.i.d. random variables with the standard Gaussian distribution  $N(0, 1)$ . Let  $Z_0 = B_1$ . Now we can write

$$(3.1) \quad B_t = tZ_0 + \frac{\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{Z_n}{n} \sin n\pi t, \quad 0 \leq t \leq 1.$$

The covariance of  $Z_0$  and  $Z_n$  is

$$\mathbb{E}[Z_0 Z_n] = \frac{n\pi}{\sqrt{2}} \int_0^1 \mathbb{E}[B_1 W_s] \sin n\pi s \, ds = 0.$$

This is because

$$\mathbb{E}[B_1 W_s] = \mathbb{E}[B_1(B_s - sB_1)] = s - s = 0.$$

It follows that  $Z_0$  is independent of  $Z_n$ . There in the Fourier expansion (3.1) of Brownian motion  $\{Z_n, n \geq 0\}$  is an i.i.d. sequence with the normal distribution  $N(0, 1)$ .

We can construct a Brownian motion by reversing the above argument. More precisely, suppose that  $\{Z_n, n \geq 0\}$  is a sequence of i.i.d. random variables  $\{Z_n, n \geq 0\}$  with the distribution  $N(0, 1)$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  (e.g., a product probability space). Then we define a stochastic process  $B$  on this probability space by the formula (3.1). For each fixed  $t$ , the series converges in  $L^2(\Omega, \mathcal{F}, \mathbb{P})$  to a random variable  $B_t$ , and  $B = \{B_t, 0 \leq t \leq 1\}$  is a Gaussian process. It can be verified directly that its mean is zero and its covariance function is given by  $\mathbb{E}\{B_s B_t\} = \min\{s, t\}$ . This shows that  $B$  is indeed a Brownian motion.

#### 4. Continuity of Brownian motion

In the last section we have shown the existence of Brownian motion. In this section we show that there is a Brownian motion with continuous sample paths. There are many proofs of this result, but the classical proof presented here is still the best. It should be pointed out that this result only claims that there is a Brownian motion with continuous sample paths; it does not claim that every Brownian motion as defined before has continuous sample paths with probability one.

**DEFINITION 4.1.** *Let  $X$  and  $Y$  be two stochastic processes on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We say that  $X$  is a version of  $Y$  (and vice versa) if*

$$\mathbb{P}\{X_t = Y_t\} = 1 \quad \text{for all } t \geq 0.$$

Note that  $X_t = Y_t$  with probability one for each fixed  $t$ . Since the time set  $\mathbb{R}_+$  is uncountable, we cannot conclude that  $\mathbb{P}\{X_t = Y_t \text{ for all } t\} = 1$ . Thus  $X$  and  $Y$  may have very different sample paths properties. However,  $X$  and  $Y$  have the same finite dimensional marginal distributions. If one of them is a B Brownian motion, so will be the other. We will prove the following result.

**THEOREM 4.2. (Wiener's theorem)** *Every Brownian motion has a continuous version with continuous sample paths.*

It is helpful to understand the above theorem from another point of view. Recall the path space

$$W(\mathbb{R}) = \text{the space of real-valued continuous functions on } \mathbb{R}_+ = [0, \infty).$$

For any fixed  $N$  let  $\|f - g\|_{\infty, N} = \sup_{0 \leq t \leq N} |f(t) - g(t)|$  for  $f$  and  $g$  in  $W(\mathbb{R})$ . If we equip  $W(\mathbb{R})$  with the distance function

$$d(f, g) = \sum_{N=1}^{\infty} \frac{1}{2^N} \frac{\|f - g\|_{\infty, N}}{1 + \|f - g\|_{\infty, N}},$$

then  $W(\mathbb{R})$  becomes a metric space and the convergence in this metric is the uniform convergence on every finite interval. As a metric space  $W(\mathbb{R})$  has its canonical Borel  $\sigma$ -field  $\mathcal{B}(W(\mathbb{R}))$ . Recall that the coordinate process  $X$  on  $W(\mathbb{R})$  is defined by  $X_t(w) = w_t$  for each  $w \in W(\mathbb{R})$ . The existence of a Brownian motion with continuous sample paths is equivalent to the following existence theorem.

**THEOREM 4.3.** *There is a probability measure  $\mu$  on the measurable space  $(W(\mathbb{R}), \mathcal{B}(W(\mathbb{R})))$  such that the coordinate process is a Brownian motion.*

The measure  $\mu$ , whose existence is claimed in the above theorem, is called the Wiener measure on the space  $(W(\mathbb{R}), \mathcal{B}(W(\mathbb{R})))$ . In the final analysis, we are claiming the existence of a probability measure in the space  $W(\mathbb{R}) = C(\mathbb{R}_+, \mathbb{R})$  of continuous functions such that

$$\mu \{w \in W(\mathbb{R}) : w_{t_1} \in A_1, w_{t_2} \in A_2, \dots, w_{t_n} \in A_n\} = \int_{A_1} p(t_1, x_1) dx_1 \int_{A_2} p(t_2 - t_1, x_2 - x_1) dx_2 \cdots \int_{A_n} p(t_n - t_{n-1}, x_n - x_{n-1}) dx_n.$$

From this point of view, we see that this is a highly nontrivial result.

The rest of this section is devoted to the proof of THEOREM 4.2. We start from a Brownian motion  $B = \{B_t, 0 \leq t \leq 1\}$  some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and proceed to construct a version  $\tilde{B}$  on the same probability space such that  $\mathbb{P} \{t \mapsto \tilde{B}_t \text{ is continuous for } 0 \leq t \leq 1\} = 1$ .

Let

$$D = \{k/2^n : n = 0, 1, \dots, k = 0, 1, \dots, 2^n\}$$

be the set of dyadic rational numbers in  $[0, 1]$ .  $D$  is a countable set. For the construction of a continuous Brownian motion, we only need  $B_t$  for  $t \in D$ . We will show that with probability one,  $B$  is uniformly continuous on  $D$ . In fact, we prove more: with probability one,  $B$  is Hölder continuous on  $D$  with any exponent  $\alpha < 1/2$ .

**PROPOSITION 4.4.** *Fix  $0 < \alpha < 1/2$ . There is a set  $\Omega_0 \subset \Omega$  of full measure  $\mathbb{P}(\Omega_0) = 1$  with the following property. For every  $\omega \in \Omega_0$ , there is a constant  $C(\omega)$  such that*

$$(4.1) \quad |B_t(\omega) - B_s(\omega)| \leq C(\omega) |t - s|^\alpha \quad \text{for all } s, t \in D.$$

**PROOF.** Consider the event:

$$A_n = \bigcup_{k=1}^{2^n} \left\{ |B_{k/2^n} - B_{(k-1)/2^n}| \geq 2^{-\alpha n} \right\}.$$



We estimate the probability  $\mathbb{P}[A_n]$ :

$$\begin{aligned}
\mathbb{P}[A_n] &\leq \sum_{k=1}^{2^n} \mathbb{P} \left[ |B_{k/2^n} - B_{(k-1)/2^n}| \geq 2^{-\alpha n} \right] \\
&= \sum_{k=1}^{2^n} \mathbb{P} \left[ |B_{1/2^n}| \geq 2^{-\alpha n} \right] \\
&= 2^n \mathbb{P} \left[ |B_1| \geq 2^{(1-2\alpha)n/2} \right] \\
&\leq 2^n \cdot \frac{2}{\sqrt{2\pi}} 2^{-(1-2\alpha)n/2} \exp \left[ -2^{(1-2\alpha)n} - 1 \right].
\end{aligned}$$

Here we have used the time-homogeneity in the second step, the scaling property in the third step, and the elementary inequality

$$\int_x^\infty e^{-t^2/2} dt \leq \frac{1}{x} e^{-x^2/2}$$

in the fourth step. It follows from the above bound for  $\mathbb{P}[A_n]$  and the hypothesis  $0 < \alpha < 1/2$  that  $\sum_{n=1}^\infty \mathbb{P}[A_n] < \infty$ . By the Borel-Cantelli lemma, we have  $\mathbb{P}\{A_n, \text{i.o.}\} = 0$ . Let

$$\Omega_0 = \{A_n \text{ i.o.}\}^c = \bigcup_{N=1}^\infty \bigcap_{n=N}^\infty \bigcap_{k=1}^{2^n} \left\{ |B_{k/2^n} - B_{(k-1)/2^n}| < 2^{-\alpha n} \right\}.$$

Then  $\mathbb{P}\{\Omega_0\} = 1$ .

Suppose that  $\omega \in \Omega_0$ . The sequence of the events  $\{A_n\}$  do not happen infinitely often. There is an  $N_0$  depending on  $\omega$  such that  $\omega \notin A_n$  for all  $n \geq N_0$ . This means that

$$|B_{k/2^n}(\omega) - B_{(k-1)/2^n}(\omega)| < 2^{-\alpha n} \quad \text{for all } n \geq N_0 \text{ and } 1 \leq k \leq 2^n.$$

It is enough to show the inequality (4.1) for dyadic points  $s < t$  such that  $t - s$  is less than a prefixed positive number. Suppose that  $0 < t - s < 2^{-N_0}$ . The difference  $t - s$  is also a dyadic number and we have

$$t - s = \sum_{n=N_1}^\infty \frac{k_n}{2^n}$$

with  $k_n = 0$  or  $1$  and  $k_{N_1} = 1$ . Since  $0 < t - s < 2^{-N_0}$  we must have  $N_1 \geq N_0$ . Let

$$s_n = s + \sum_{j=N_1}^n k_j / 2^j.$$

It is a finite sequence of dyadic points that leads from  $s$  to  $t$ . The difference of successive time points  $s_n - s_{n-1} = 0$  or  $2^{-n}$  with  $n \geq N_0$ , hence we have

$|B_{s_n}(\omega) - B_{s_{n-1}}(\omega)| \leq 2^{-\alpha n}$  for all  $n \geq N_1$ . Now we have

$$\begin{aligned} |B_t(\omega) - B_s(\omega)| &\leq \sum_{n=N_1}^{\infty} |B_{s_n} - B_{s_{n-1}}| \\ &\leq \sum_{n=N_1}^{\infty} 2^{-\alpha n} \\ &\leq \frac{2^{-\alpha N_1}}{1 - 2^{-\alpha}}. \end{aligned}$$

On the other hand  $|t - s| \geq 2^{-N_1}$ . It follows that

$$|B_t(\omega) - B_s(\omega)| \leq \frac{|t - s|^\alpha}{1 - 2^{-\alpha}}$$

for all dyadic points  $s$  and  $t$  such that  $|t - s| < 2^{-N_0}$ .  $\square$

We can now show the existence of a continuous Brownian motion. For each  $\omega \in \Omega_0$  and  $t \in [0, 1]$  we define

$$\tilde{B}_t(\omega) = \lim_{D \ni r \rightarrow t} B_r(\omega).$$

The above proposition shows that  $\tilde{B}_t(\omega)$  is well defined and  $t \mapsto B_t(\omega)$  is continuous. To complete the proof, it is enough to show that  $\tilde{B}$  is a version of  $B$ . From  $\mathbb{E}|B_s - B_t|^2 = |s - t|$ , we have

$$\mathbb{E}|\tilde{B}_t - B_t|^2 \leq \liminf_{D \ni r \rightarrow t} \mathbb{E}|B_r - B_t|^2 = 0.$$

Therefore  $\mathbb{P}\{\tilde{B}_t = B_t\} = 1$  for all  $t \geq 0$  and  $\tilde{B}$  is a version of  $B$ .

## 5. Strong Markov property

Recall the Markov property of Brownian motion. For any fixed time  $s \geq 0$ , the process  ${}^s B$  defined by  ${}^s B_t = B_{t+s} - B_s$ ,  $t \geq 0$  is a Brownian motion from zero independent of the  $\sigma$ -field  $\mathcal{F}_s^B = \sigma\{B_u; 0 \leq u \leq s\}$ . The strong Markov property is more general. It asserts that the above Markov property holds for a class of random times called stopping times. In many applications, the possibility of using stopping times is the source of power of stochastic methods

Let us first discuss briefly random times. A *random time* is simply a nonnegative measurable random variable  $\tau : \Omega \rightarrow [0, \infty]$ . Sometimes the value  $\infty$  is allowed. If  $\mathbb{P}\{\tau < \infty\} = 1$ , we say that  $\tau$  is a finite random time. A typical random time is the first passage time  $\tau_a$  of a point of  $a \in \mathbb{R}$  defined by

$$\tau_a = \inf\{t \geq 0 : B_t(\omega) = a\}.$$

We can show that  $\tau_a$  is a finite random time. Note that  $B_{\tau_a} = a$ . In general if  $\tau$  is a random time, then  $B_\tau$  stands for the random variable  $B_\tau(\omega) = B_{\tau(\omega)}(\omega)$ . Note that  $B_\tau$  is indeed a random variable.

The stochastic process  ${}^{\tau_a}B = \{B_{t+\tau_a} - a, t \geq 0\}$  is a continuous process starting from 0. We can ask the natural questions: Is  ${}^{\tau_a}B$  a Brownian motion independent of the Brownian motion up to time  $\tau_a$ ? To answer this question, we must first know how to define  $\mathcal{F}_\tau^B$  for a random time  $\tau$ . Intuitively this  $\sigma$ -algebra represents the information up to time  $\tau$ . Once this is properly done the answer to the above question is affirmative and follows from the usual Markov property of Brownian motion.

Before we state the strong Markov property, it is perhaps helpful to point out by a counterexample that this strong Markov property is not shared by all random times. Let  $\lambda_0$  be the last time the Brownian path is at the point  $z = 0$  before time 1:

$$\lambda_0 = \sup \{t \in [0, 1] : B_t = 0\}.$$

Since  $B_1 \neq 0$  with probability 1, we have  $\mathbb{P}\{\lambda_0 < 1\} = 1$ . The process  ${}^{\lambda_0}B = \{B_{t+\lambda_0}; t \geq 0\}$  is not a Brownian motion from 0. This is because  $t = 0$  is an isolated zero, whereas for a Brownian motion  $t = 0$  cannot be an isolated zero.

REMARK 5.1. The law of  $\lambda_0$  is known:

$$\mathbb{P}\{\lambda_0 \leq t\} = \frac{2}{\pi} \arcsin \sqrt{t}.$$

This is one of Lévy's arcsine laws.

The above counterexample shows that random times at which Brownian motion has the Markov property are special. The problem with the random time  $\lambda_0$  is that if we follow a Brownian traveler from time 0 to a fixed time  $t < 1$ , we cannot tell if the event  $\{\lambda_0 \leq t\}$  has occurred or not, since to find this out we have to look beyond time  $t$ ; in fact we have to look at the path all the way to time 1. On the other hand, the first passage time  $\tau_a$  is a time which, so to speak, does not depend on the future.

We now introduce a class of random times for which we will prove the strong Markov property.

DEFINITION 5.2. A nonnegative random variable  $\tau : \Omega \rightarrow [0, \infty]$  on a filtered measurable space  $(\Omega, \mathcal{F}_*)$  is called a *stopping time* (with respect to the filtration  $\mathcal{F}_*$ ) if  $\{\tau \leq t\} \in \mathcal{F}_t$  for all  $t \geq 0$ . If  $\mathcal{F}_* = \mathcal{F}_*^X$  is the filtration generated by a stochastic process, a stopping time with respect to  $\mathcal{F}_*^X$  is usually called a *stopping time of the process X*.

The intuitive meaning of the above definition is as follows. Suppose that  $\tau$  is the time a certain event associated to a process  $X$  takes place. Then  $\tau$  is a stopping time if the question of whether the event takes place or not before time  $t$  can be determined by looking at the path of the process  $\{X_s, 0 \leq s \leq t\}$  up to time  $t$ .

In the next proposition we list below a few properties of stopping times.

PROPOSITION 5.3. *The following statements hold.*

- (1) *If  $\sigma$  and  $\tau$  are stopping times, then  $\sigma + \tau$ ,  $\sigma \wedge \tau$ , and  $\sigma \vee \tau$  are stopping times;*
- (2) *If  $\sigma_n, n \geq 1$  is an increasing sequence of stopping times then the limit  $\tau_\infty = \lim_{n \rightarrow \infty} \sigma_n$  is a stopping time;*
- (3) *If  $\sigma_n, n \geq 1$  is a decreasing sequence of stopping times, and for each  $\omega \in \Omega$  there is an  $N(\omega)$  such that  $\sigma_n(\omega) = \sigma_{n+1}(\omega)$  for all  $n \geq N(\omega)$ , then the limit  $\sigma_\infty = \lim_{n \rightarrow \infty} \sigma_n$  is a stopping time.*

PROOF. These assertions follow more or less directly from the definition of a stopping time.  $\square$

EXAMPLE 5.4. When proving a statement involving a stopping time we often approximate it by a sequence of discrete stopping times and prove the statement on each set on which a discrete stopping time takes a fixed constant value. Let  $\tau$  be a stopping time and define

$$\tau_n = \frac{[2^n \tau] + 1}{2^n} = \frac{k}{2^n}, \quad \text{if } \frac{k-1}{2^n} \leq \tau < \frac{k}{2^n}.$$

Here  $[x]$  denotes the largest integer not exceeding  $x$ . We can verify that  $\{\tau_n\}$  is a decreasing sequence of stopping times and  $\tau_n \downarrow \tau$ .

We now define  $\mathcal{F}_\tau$ , the  $\sigma$ -algebra up to a stopping time  $\tau$ . If  $\mathcal{F}_* = \mathcal{F}_*^X$  is generated by a stochastic process  $X$ , then it is natural to define

$$\mathcal{F}_\tau^X = \sigma \{X_{t \wedge \tau}, t \geq 0\},$$

the  $\sigma$ -algebra generated by  $X^\tau = \{X_{t \wedge \tau}, t \geq 0\}$ , the process  $X$  stopped at time  $\tau$ . In general, when the filtration is not necessarily generated by a stochastic process, we use the following definition.

DEFINITION 5.5. *Let  $\tau$  be a stopping time on a filtered measurable space  $(\Omega, \mathcal{F}_*)$ . The  $\sigma$ -algebra  $\mathcal{F}_\tau$  is the collection of events  $A \in \mathcal{F}_\infty$  such that*

$$A \cap \{\tau \leq t\} \in \mathcal{F}_t \quad \text{for all } t \geq 0.$$

We can verify that  $\mathcal{F}_\tau$  is indeed a  $\sigma$ -algebra and  $\mathcal{F}_\tau = \mathcal{F}_t$  if  $\tau$  is the constant stopping time  $\tau = t$  (EXERCISE ??). For  $\mathcal{F}_*^X$ , the filtration generated by a process  $X$ , the  $\sigma$ -algebra  $\mathcal{F}_\tau^X$  defined above coincides with the  $\sigma$ -algebra generated by the stopped process  $X^\tau$  (see EXERCISE ??).

Here are some properties of  $\mathcal{F}_\tau$  we will need in the future.

PROPOSITION 5.6. *Let  $\sigma, \tau$  be  $\mathcal{F}_t$ -stopping times.*

- (1) *If  $\sigma \leq \tau$ , then  $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$ ;*
- (2)  *$\tau \in \mathcal{F}_\tau$ ;*
- (3) *If  $A \in \mathcal{F}_\tau$ , then  $A \cap \{\sigma \leq \tau\} \in \mathcal{F}_\tau$  and  $A \cap \{\sigma = \tau\} \in \mathcal{F}_\tau$ ; in particular,  $\{\sigma \leq \tau\} \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau$ ;*

PROOF. These properties follow easily from the definition of stopping times.  $\square$

PROPOSITION 5.7. *Suppose that  $\mathcal{F}_*$  satisfies the usual conditions. If  $X$  is a continuous stochastic process adapted to  $\mathcal{F}_*$  and  $\tau$  a finite  $\mathcal{F}_*$ -stopping time, then  $X_\tau \in \mathcal{F}_\tau$ .*

PROOF. We consider the case where  $X$  is a real-valued process. Let

$$\tau_n = \frac{[2^n \tau] + 1}{2^n}$$

be the discrete approximation of  $\tau$  in EXAMPLE 5.4. By considering each set on which  $\tau_n$  is constant, we can show easily that  $X_{\tau_n} \in \mathcal{F}_{\tau_n}$ . In particular, for  $a \in \mathbb{R}$  and  $t > 0$  we have

$$\{X_{\tau_n} \leq a\} \cap \{\tau_n < t\} \in \mathcal{F}_t.$$

Using the sample path continuity of  $X$  and the fact that  $\tau_n \downarrow \tau$  we can verify that

$$\{X_\tau \leq a\} \cap \{\tau < t\} = \bigcap_{l=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left\{ X_{\tau_k} \leq a + \frac{1}{l} \right\} \cap \{\tau_k < t\}.$$

This implies that  $\{X_\tau \leq a\} \cap \{\tau < t\} \in \mathcal{F}_t$ , hence by the usual conditions

$$\{X_\tau \leq a\} \cap \{\tau \leq t\} \in \mathcal{F}_{t+} = \mathcal{F}_t.$$

From this relation and the definition of  $\mathcal{F}_\tau$  we see that  $\{X_\tau \leq a\} \in \mathcal{F}_\tau$  for all  $a$ , which implies immediately that  $X_\tau$  is measurable with respect to  $\mathcal{F}_\tau$ .  $\square$

Now we are in a position to state and prove the strong Markov property of Brownian motion. Let  $B$  be a Brownian motion with respect to a filtration  $\mathcal{F}_*$  and  $\tau$  a finite stopping time with respect to the same filtration. The strong Markov property is the assertion that the shifted process  ${}^\tau B = \{B_{t+\tau} - B_\tau, t \geq 0\}$  is a Brownian motion independent of the  $\sigma$ -algebra  $\mathcal{F}_\tau$ . There are two assertions in this theorem: (1) the process  ${}^\tau B$  is a Brownian motion (2) this Brownian motion is independent of  $\mathcal{F}_\tau^B$ . They are immediate consequences of the equality in the next theorem.

PROPOSITION 5.8. *Let  $0 \leq t_1 < t_2 < \dots < t_m$  be  $m$  time points. Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  be a bounded continuous function on  $\mathbb{R}^m$ . Let  $C \in \mathcal{F}_\tau$ . Then*

$$(5.1) \quad \mathbb{E}[f({}^\tau B_{t_1}, \dots, {}^\tau B_{t_m}); C] = \mathbb{P}[C] \mathbb{E}[f(B_{t_1}, \dots, B_{t_m})].$$

PROOF. We first prove the equality for discrete stopping times and then approximate a general finite stopping time by discrete stopping times as in EXAMPLE 5.4.

Assume that  $\tau$  is a discrete stopping time whose possible values are  $s_i$ . Since  $B$  is a Brownian motion with respect to  $\mathcal{F}_*$ , we have the usual Markov property for a constant time:  ${}^s B = \{B_{t+s} - B_s; t \geq 0\}$  is a Brownian motion

independent of  $\mathcal{F}_s$ . Assume that  $C \in \mathcal{F}_\tau$  and let  $C_i = C \cap \{\tau = s_i\}$ . Then  $C_i \in \mathcal{F}_{s_i}$  by PROPOSITION 5.6 (iii). We have

$$\begin{aligned} \mathbb{E}[f({}^\tau B_{t_1}, \dots, {}^\tau B_{t_m}); C] &= \sum_{i=1}^{\infty} \mathbb{E}[f(B_{t_1+s_i} - B_{s_i}, \dots, B_{t_m+s_i} - B_{s_i}); C_i] \\ &= \sum_{i=1}^{\infty} \mathbb{P}[C_i] \mathbb{E}[f(B_{t_1+s_i} - B_{s_i}, \dots, B_{t_m+s_i} - B_{s_i})] \\ &= \sum_{i=1}^{\infty} \mathbb{P}[C_i] \mathbb{E}[f(B_{t_1}, \dots, B_{t_m})] \\ &= \mathbb{P}[C] \mathbb{E}[f(B_{t_1}, \dots, B_{t_m})]. \end{aligned}$$

Thus (5.1) holds for discrete stopping times.

For a general finite stopping time  $\tau$ , we approximate it by the decreasing sequence of stopping times

$$\tau_n = \frac{[2^n \tau] + 1}{2^n}.$$

The strong Markov property holds for each  $\tau_n$ . Now suppose that  $C \in \mathcal{F}_\tau$ . Since  $\tau \leq \tau_n$ , we have  $\mathcal{F}_\tau \subset \mathcal{F}_{\tau_n}$  (PROPOSITION 5.6 (i)), hence  $C \in \mathcal{F}_{\tau_n}$ . We have

$$\mathbb{E}[f({}^\tau B_{t_1}, \dots, {}^\tau B_{t_m}); C] = \mathbb{P}[C] \mathbb{E}[f(B_{t_1}, \dots, B_{t_m})].$$

We have  ${}^{\tau_n} B_{t_j} \rightarrow {}^\tau B_{t_j}$  because Brownian motion has continuous sample paths. Therefore we can use the dominated convergence theorem to take limit on the left side and obtained the desired equality (5.1).  $\square$

The strong Markov property of Brownian motion now follows easily from the equality proved in the above proposition.

**THEOREM 5.9.** *Let  $B$  be a Brownian motion with respect to a filtration  $\mathcal{F}_*$  and  $\tau$  a finite stopping time with respect to the same filtration. Then the shifted process  ${}^\tau B = \{B_{t+\tau} - B_\tau, t \geq 0\}$  is a Brownian motion independent of the  $\sigma$ -algebra  $\mathcal{F}_\tau$ .*

**PROOF.** Everything is contained in the equality (5.1). Random variables of the form  $f({}^\tau B_{t_1}, \dots, {}^\tau B_{t_m})$  is dense in the space of integrable random variables measurable with respect to the process  ${}^\tau B$ . The fact that the right side of the equality is a product shows that  $\mathcal{F}_\tau$  is independent of  ${}^\tau B$ . If we take  $C = \Omega$  then the equality says that  ${}^\tau B$  has the same finite dimensional marginal distributions as  $B$ , which implies that  ${}^\tau B$  is a Brownian motion.  $\square$

**REMARK 5.10.** An equivalent statement of the strong Markov property is as follows. Let  $B$  be a Brownian motion and  $\tau$  a finite stopping time, both with respect to a given filtration  $\mathcal{F}_*$ . Then the process  $\{B_{t+\tau}, t \geq 0\}$  is a Brownian motion with respect to the filtration  $\mathcal{F}_{\tau+*} = \{\mathcal{F}_{t+\tau}, t \geq 0\}$ . This is a very useful restatement in view of future generalization of the strong

Markov property to a general diffusion process obtained from solving a stochastic differential equation.

## 6. Applications of the strong Markov property

In this section we discuss several properties of Brownian motion which can be proved by strong Markov property. Recall that this property claims that  ${}^{\tau}B = \{B_{t+\tau} - B_{\tau}, t \geq 0\}$  is a Brownian motion independent of  $\mathcal{F}_{\tau}$ . This property is often used in the following form. Let  $X$  and  $Y$  be two random variables measurable with respect to  ${}^{\tau}B$  and  $\mathcal{F}_{\tau}$ , respectively. Then the joint distribution of  $(X, Y)$  must be the product measure of the marginal distributions of  $X$  and  $Y$ . This allows us to use Fubini's theorem to calculate the expected value of a function of the two variables:

$$\begin{aligned}\mathbb{E}f(X, Y) &= \int_{E_1 \times E_2} f(x, y) \mathbb{P}(X \in dx) \mathbb{P}(Y \in dy) \\ &= \mathbb{E}[\mathbb{E}[f(x, Y)]|_{x=X}] \\ &= \mathbb{E}[\mathbb{E}[f(X, y)]|_{y=Y}].\end{aligned}$$

**PROPOSITION 6.1.** *Let  $M_t = \max_{0 \leq s \leq t} B_s$  be the maximum process of a Brownian motion from 0. Then  $M_t$  and  $|B_t|$  have the same distribution.*

**PROOF.** Let  $b > 0$  and

$$\tau_b = \inf \{t > 0 : B_t = b\}$$

be the first passage time of Brownian motion to  $b$ . It is a finite stopping time. We compute the probability  $\mathbb{P}[B_t \geq b, \tau_b \leq t]$  by the strong Markov property. First note that  $\{B_t = b\}$  implies  $\{\tau_b \leq t\}$ , therefore this probability is just  $\mathbb{P}\{B_t \geq b\}$ . Now let

$$W_t = B_{t+\tau_b} - B_{\tau_b} = B_{t+\tau_b} - b$$

the shifted process. Then the probability we wanted to compute can be written as  $\mathbb{P}[W_{t-\tau_b} \geq 0, \tau_b \leq t]$ . It is in the form of the expected value of a function of two random variables  $W$  and  $\tau_b \in \mathcal{F}_{\tau_b}$ , which are independent by the strong Markov property. By Fubini's theorem, we can compute this expectation by assuming one of the variables to be fixed first. In the present case we can regard  $\tau_b$  as a constant and compute the probability that  $W_{t-s} \geq 0$ , which is of course equal to  $1/2$  because  $W$  is a Brownian motion. It follows that

$$\mathbb{P}[B_t \geq b] = \mathbb{P}[B_t \geq b, \tau_b \leq t] = \frac{1}{2} \mathbb{P}[\tau_b \leq t].$$

On the other hand, we have  $\{\tau_b \leq t\} = \{M_t \geq b\}$ . Therefore,

$$\mathbb{P}[M_t \geq b] = 2\mathbb{P}[B_t \geq b] = \mathbb{P}[|B_t| \geq b].$$

This shows that  $M_t$  and  $|B_t|$  have the same distribution.  $\square$

We can calculate the density function of the first passage time  $\tau_b$ . From the above result we have

$$\mathbb{P}\{\tau_b \leq t\} = \frac{2}{\sqrt{2\pi t}} \int_b^\infty e^{-x^2/2t} dx.$$

Differentiating with respect to  $t$  and integrating by parts we obtain the density function

$$p_{\tau_b}(t) = \frac{b}{\sqrt{2\pi t^3}} e^{-b^2/2t} dt.$$

**THEOREM 6.2.** *Let  $B$  be a Brownian motion from 0 and  $\tau$  a stopping time of  $B$ , both with respect to a given filtration  $\mathcal{F}_*$ . Let  $W$  be another Brownian motion from 0 independent of  $\mathcal{F}_\tau$ . Define a new process  $Z$  by*

$$Z_t = \begin{cases} B_t & \text{if } t \leq \tau, \\ W_{t-\tau} + B_\tau & \text{if } t > \tau. \end{cases}$$

*Then  $Z$  is a Brownian motion from 0.*

**PROOF.** By the strong Markov property  ${}^\tau B$  is a Brownian motion independent of  $(B^\tau, \tau)$ , where  $B^\tau = \{B_{t \wedge \tau}, t \geq 0\}$  is the Brownian motion stopped at  $\tau$ . By the assumption the triples  $(W, B^\tau, \tau)$  and  $({}^\tau B, B^\tau, \tau)$  have the same distribution. The processes  $Z$  and  $B$  are constructed from the two triples in the same way. More precisely, there is a measurable function  $F$  such that  $Z = F(W, B^\tau, \tau)$  and  $B = F({}^\tau B, B^\tau, \tau)$ . This shows that  $Z$  and  $B$  must have the same distribution, which implies that  $Z$  is a Brownian motion.  $\square$

**COROLLARY 6.3.** (*André's reflection principle*) *Let  $\tau$  be a finite stopping time of a Brownian motion  $B$  from 0. Let  $Z$  be the process obtained from  $B$  by reflecting with respect to  $B_\tau$  after the time  $\tau$ :*

$$Z_t = \begin{cases} B_t & \text{if } t \leq \tau, \\ 2B_\tau - B_t & \text{if } t > \tau. \end{cases}$$

*Then  $Z$  is a Brownian motion.*

**PROOF.**  $-{}^\tau B$ , like  ${}^\tau B$  itself, is also independent of  $\mathcal{F}_\tau$ . We can then take  $W = -{}^\tau B$  in the theorem.  $\square$

We use the reflection principle to compute the joint distribution of  $B_t$  and  $M_t = \max_{0 \leq s \leq t} B_s$ .

**PROPOSITION 6.4.** *The joint distribution of  $M_t$  and  $B_t$  is given by*

$$\mathbb{P}[B_t \in da, M_t \in db] = \left(\frac{2}{\pi t^3}\right)^{1/2} (2b - a) e^{-(2b-a)^2/2t} da db$$

*for  $a < b$  and  $b > 0$ .*



PROOF. Let  $Z$  be the Brownian motion obtained  $B$  by reflecting its path after the first passage time  $\tau_b$ . For clarity we introduce the notations  $\tau_b^B$  and  $\tau_b^Z$  for the first passage times to  $b$  of the Brownian motions  $B$  and  $Z$ . Of course these two times are identical. For  $a < b$  and  $b > 0$ , we have

$$\begin{aligned}\mathbb{P}[B_t \leq a, M_t \geq b] &= \mathbb{P}[B_t \leq a, \tau_b^B \leq t] \\ &= \mathbb{P}[Z_t \leq a, \tau_b^Z \leq t] \\ &= \mathbb{P}[B_t \geq 2b - a, \tau_b^B \leq t] \\ &= \mathbb{P}[B_t \geq 2b - a].\end{aligned}$$

Let's explain these steps. In the first step we have used the identity

$$\{M_t \geq b\} = \{\tau_b^B \leq t\}.$$

In the second step we have simply replaced  $B$  by  $Z$  because they are both Brownian motions by the reflection principle. In the third step we have replaced  $\tau_b^Z$  by  $\tau_b^B$  because they are identical, and we have used  $\{Z_t < a\} = \{B_t > 2b - a\}$ . This holds because  $Z_t = 2b - B_t$  if  $t > \tau_b^B$  by the definition of  $Z$ . In the fourth step we have used the implication

$$\{B_t > 2b - a\} \subset \{B_t > b\} \subset \{\tau_t^B \leq t\},$$

which follows from the assumption that  $b > a$ . Now we have

$$\mathbb{P}[B_t \leq a, M_t \geq b] = \mathbb{P}[B_t \geq 2b - a] = \frac{1}{\sqrt{2\pi t}} \int_{2b-a}^{\infty} e^{-x^2/2t} dx.$$

Differentiating with respect to  $a$  and then with respect to  $b$  we obtain the joint density function of  $B_t$  and  $M_t$ .  $\square$

COROLLARY 6.5.  $M_t - B_t$  and  $|B_t|$  has the same distribution for each fixed  $t$ .

PROOF. Integrate the joint density function  $p_{M_t, B_t}(b, a)$  on the region  $b - a \geq c$ .  $\square$

REMARK 6.6. It is a perhaps a curious observation that  $M_t - B_t$  and  $M_t$  have the same distribution, since both are distributed as  $|B_t|$ . The process  $|B| = \{|B_t|, t \geq 0\}$  is called a reflecting Brownian motion. Reflecting Brownian motion will be discussed in great detail later. Among other things we will show the much stronger result that the processes  $\{M_t - B_t, t \geq 0\}$  and  $\{|B_t|, t \geq 0\}$  have the same distribution.

## 7. Quadratic variation of Brownian motion

In this section we study Brownian motion from the point of view of martingale theory. We know that Brownian motion has continuous sample paths; in fact we have shown that sample paths are Hölder continuous for any exponent  $\alpha < 1/2$ . On the other hand, it can be shown that the sample

paths are nowhere differentiable. Thus they cannot be functions of bounded variation. Nevertheless they have finite quadratic variations. We have shown that both  $B_t$  and  $B_t^2 - t$  are continuous martingales. This implies that the quadratic variation process  $\langle B, B \rangle_t = t$ .

Fix a time  $t > 0$  and consider a partition

$$\Delta : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$$

of the interval  $[0, t]$ . Denote  $|\Delta| = \max_{1 \leq i \leq n} (t_i - t_{i-1})$ , the length of the longest intervals. We write  $\Delta_1 \subset \Delta_2$  and say that  $\Delta_2$  is a refinement of  $\Delta_1$  if every partition point in  $\Delta_1$  is also a partition point of  $\Delta_2$ , or equivalently, every subinterval in  $\Delta_1$  is a union of subintervals in  $\Delta_2$ .

The quadratic variation of Brownian motion  $B$  along a partition  $\Delta$  is defined to be

$$Q(B; \Delta) = \sum_{i=1}^n |B_{t_i} - B_{t_{i-1}}|^2.$$

THEOREM 7.1. *We have*

$$\lim_{|\Delta| \rightarrow 0} \mathbb{E} \left| \sum_{i=1}^n |B_{t_i} - B_{t_{i-1}}|^2 - t \right| = 0.$$

PROOF. We have

$$\mathbb{E} |B_t|^2 = t \quad \text{and} \quad \mathbb{E} |B_t^2 - t|^2 = 2t^2.$$

Let  $\Delta B_i = B_{t_i} - B_{t_{i-1}}$  and  $\Delta t_i = t_i - t_{i-1}$ . Then the expression under the limit is equal to

$$\sum_{i,j=1}^n \mathbb{E} [ (|\Delta B_i|^2 - \Delta t_i) (|\Delta B_j|^2 - \Delta t_j) ].$$

In the last sum, the off diagonal terms vanish because the increments are independent and each factor has zero expectation. For the diagonal term  $i = j$  the expectation is equal to  $2|\Delta t_i|^2$ . Hence

$$\mathbb{E} \left| \sum_{i=1}^n |B_{t_i} - B_{t_{i-1}}|^2 - t \right| \leq 2t|\Delta|.$$

The theorem follows immediately.  $\square$

REMARK 7.2. The existence of a bounded quadratic variation of a continuous function implies that it cannot be of bounded variation itself. Therefore almost surely Brownian motion paths have unbounded variation on any time interval.

REMARK 7.3. The above argument can be carried without much change to a general continuous martingale. Furthermore, if  $|\Delta| \rightarrow 0$  sufficiently fast, then the quadratic variation along the partition  $\Delta$  converges almost surely to  $t$ . This can be proved by using the Borel-Cantelli lemma.

Finally we state another much deeper almost sure version of the above theorem. See Freedman [4] for a proof.

**THEOREM 7.4.** *The quadratic variations  $Q(B; \Delta_n)$  of Brownian motion  $B$  along a refining sequence of partitions  $\Delta_1 \subset \Delta_2 \subset \dots$  is a reverse martingale and hence converges almost surely. If  $|\Delta_n| \rightarrow 0$ , then  $\lim_{n \rightarrow \infty} Q(B; \Delta_n) = t$  almost surely.*

**PROOF.** See Freedman [4]. □

## 8. Second assignment

**EXERCISE 2.1.** Let  $B$  be a Brownian motion. For any  $0 \leq r < s < t$  and Borel sets  $C, D$  on  $\mathbb{R}^1$ ,

$$\mathbb{P}[B_r \in C, B_t \in D | B_s] = \mathbb{P}[B_r \in C | B_s] \mathbb{P}[B_t \in D | B_s].$$

This says that given the present, the past is independent of the future.

**EXERCISE 2.2.** Let  $\{Z_n, n \geq 0\}$  be an i.i.d. sequence with the standard distribution  $N(0, 1)$ . Define

$$B_t = tZ_0 + \frac{\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{Z_n}{n} \sin n\pi t, \quad 0 \leq t \leq 1.$$

Show that  $B$  is a Brownian motion.

**EXERCISE 2.3.** Let  $a \in \mathbb{R}$  and  $\tau_a = \inf\{t \geq 0 : B_t = a\}$  be the first passage time of  $a$ . Show that  $\tau_a$  is a finite stopping time, that is,  $\mathbb{P}\{\tau_a < \infty\} = 1$ .

**EXERCISE 2.4.** Let  $Z = \{t \in \mathbb{R}_+ : B_t = 0\}$  be the zero set of the Brownian motion  $B$ . Show that with probability one the point  $t = 0$  is not an isolated point of  $Z$ .

**EXERCISE 2.5.** Use the joint density function of  $M_t = \max_{0 \leq s \leq t} B_s$  and  $B_t$  to show that  $M_t - B_t$  and  $|B_t|$  has the same distribution.

**EXERCISE 2.6.** Let  $B$  be a Brownian motion. Show that

$$M_t = B_t^3 - 3 \int_0^t B_s ds$$

is a martingale.

**EXERCISE 2.7.** Suppose that  $\{\tau_n\}$  is a decreasing sequence of stopping times on a filtered measurable space  $(\Omega, \mathcal{F}_*)$  such that for each  $\omega \in \Omega$  there is a natural number  $N(\omega)$  such that  $\tau_n(\omega) = \tau_{n+1}(\omega)$  for all  $n \geq N(\omega)$ . Show that  $\tau = \lim_{n \rightarrow \infty} \tau_n$  is a stopping time.

**EXERCISE 2.8.** Suppose that the filtration  $\mathcal{F}_*$  is right continuous, i.e.,  $\mathcal{F}_t = \mathcal{F}_{t+}$ . Let  $\tau$  be a finite stopping time. Let  $\{\tau_n\}$  be a decreasing sequence of finite stopping times such that  $\tau_n \downarrow \tau$ . Show that  $\bigcap_{n=1}^{\infty} \mathcal{F}_{\tau_n} = \mathcal{F}_{\tau}$ .

EXERCISE 2.9. Let  $B$  be a standard Brownian motion and  $\mathcal{F}_*^B$  its filtration. Define a new filtration  $\mathcal{G}_*$  by  $\mathcal{G}_t = \sigma\{\mathcal{F}_t^B, B_1\}$ . Then  $B$  is no longer a martingale respect to  $\mathcal{G}_*$ . Let  $X_t = B_t - tB_1$ . Show that

$$W_t = X_t + \int_0^t \frac{X_s}{1-s} ds$$

is a martingale with respect to  $\mathcal{G}_*$ . Thus the Doob-Meyer decomposition of  $B$  with respect to the filtration  $\mathcal{G}_*$  is

$$B_t = W_t + tB_1 - \int_0^t \frac{B_s - sB_1}{1-s} ds.$$

We can show later by Lévy's criterion that  $W$  is in fact a Brownian motion.

EXERCISE 2.10. For each  $x \in \mathbb{R}$  we use  $\mathbb{P}_x$  to denote the law of Brownian motion starting from  $x$ . Let  $a < x < b$  and let  $\tau_a$  and  $\tau_b$  be the first passage times of Brownian motion to  $a$  and  $b$  respectively. Show that

$$\mathbb{P}_x\{\tau_a < \tau_b\} = \frac{b-x}{b-a}.$$