

The Path Integral: Continuum Representation and Connections

The discretized path integral formula for the partition function, which represents a quantum particle as a classical ring polymer, can be expressed in a continuous form.

By defining an "imaginary time" variable, u , that runs from 0 to $\hbar\beta$, the sum over the discrete beads of the polymer becomes an integral over this continuous path. In this limit, the effective potential of the ring polymer transforms into the integral of a **Lagrangian in Euclidean time** ($L_E = T + V$). The total integral is known as the **action**, S_E .

The quantum partition function can then be written as a **functional integral** over all possible closed paths $\underline{R}(u)$ that start and end at the same point:

$$Z = \oint \mathcal{D}\underline{R} e^{-S_E[\underline{R}]/\hbar}$$

This means the partition function is a sum over all possible trajectories, where each path is weighted by its action.

The Wick Rotation: Connecting Statistical Mechanics and Quantum Dynamics

There is a profound connection between the path integral for statistical mechanics (which describes systems at a finite temperature, $T > 0$ K) and the path integral for time-dependent quantum mechanics (at $T = 0$ K). This connection is made through a mathematical trick called a **Wick rotation**.

By substituting imaginary time with real time ($u = i\tau$), the Euclidean action (S_E) transforms back into the familiar **action** in real time ($S = \int L d\tau$), where the Lagrangian is the usual $L = T - V$.

This reveals that the statistical density matrix, $\rho(\underline{R}'', \underline{R}', \hbar\beta)$, is mathematically equivalent to the quantum mechanical **propagator**, which evolves a system in time. The key insight is:

Inverse temperature ($\hbar\beta$) in statistical mechanics is equivalent to imaginary time in quantum mechanics.

The partition function, Z , is simply the trace of this Wick-rotated quantum propagator, which means we sum over all paths that start and end at the same point.

The Classical Limit

The path integral formulation correctly recovers classical mechanics in the high-temperature limit ($T \rightarrow \infty$, so $\beta \rightarrow 0$).

- In this limit, the imaginary time interval $\hbar\beta$ becomes very small.
- The ring polymer (the path) does not have enough "time" to fluctuate far from its starting point. It remains highly localized.
- As a result, the potential energy along the short path is nearly constant.
- The path integral then simplifies into the product of the classical potential energy term and the partition function for a free quantum particle. This product is exactly the **classical canonical partition function**.

Quantum Phenomena in the Path Integral Picture

The classical isomorphism provides intuitive, visual interpretations of quantum effects:

- **Zero-Point Energy (ZPE):** Classically, at $T=0K$, a particle would sit motionless at the bottom of a potential well ($\Delta x^2 = 0$). In the path integral picture, the harmonic springs connecting the beads of the ring polymer prevent it from collapsing to a single point. This inherent stiffness and non-compressibility of the polymer means it always has a finite size, even at zero temperature. This finite spatial distribution ($\Delta x^2 > 0$) is the manifestation of zero-point energy.
- **Tunneling:** In a system like a double-well potential, a classical particle would be trapped in one well. Quantum tunneling is represented by a ring polymer configuration that is **delocalized across the barrier**, with some beads residing in the left well and some in the right. Such a configuration, where the path makes an excursion under the barrier, allows the "particle" to have a non-zero probability of being found on the other side of a classically forbidden region.