Information Disclosure in Dynamic Innovation Contests

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Abstract

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1 Introduction

- Kaggle[1] ...
- Meta-kaggle Risdal and Bozsolik (2022).

1.1 Literature Review

This paper focuses on the two players innovation contest with a continuous time where the players' relative position is public information throughout the game. This is closely related to tug-or-war contest, which, to our knowledge, was first formally given by Harris and Vickers (1987) as a one-dimensional simplification of the multi-stage R&D race. The output processes are model by Brownian motions drifted with effort inputs, which is followed by Budd et al. (1993) who model the state of a dynamic competition of two innovative duopoly firms by a Brownian motion drifted by the effort gap, and solve the equilibrium approximately. Furthermore, Moscarini and Smith (2007) model the tug-of-war state as the gap of the two outputs directly, and draw an analytical equilibrium of the pure strategies.

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Information disclosure in contest - Bimpikis et al. (2019).

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Closest paper - Ryvkin (2022).

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2 The Model

We assume two players, i and j, compete for a prize $\theta > 0$. Winner gets the prize and loser gets nothing. The contest starts at time zero. At every time $t \geq 0$, the representative player i chooses an effort level $q_{i,t}$ and burdens a quadratic cost $C_i(q_{i,t}) = c_i q_{i,t}^2/2$, with a lower c_i corresponding to higher ability. The output of player i denoted by $x_{i,t}$ follows

$$dx_{i,t} = q_{i,t}dt + \sigma_i dW_{i,t} \tag{1}$$

[1] https://www.kaggle.com

Here $W_{i,t}$ is a standard Brownian motion and $\sigma_i > 0$ measures her production risk. Similar to the discrete time model, we assume the output level $x_{i,t}$ is only known by the game designer but not the two players. Denoted by $y_t \equiv x_{i,t} - x_{j,t}$ the output gap of two players at time t. The dynamic of y_t is given by

$$dy_t = (q_{i,t} - q_{j,t})dt + \sigma dW_t \tag{2}$$

where $\sigma^2 = \sigma_i^2 + \sigma_j^2$ and $\sigma dW_t = \sigma_i dW_{i,t} - \sigma_j dW_{j,t}$.

At any time t, the game holder emits a *public* signal of the real output gap y_t. The signal is ambiguous and the game holder controls the ambiguity. The dynamic of signal is

$$dZ_t = y_{i,t}dt + \frac{dB_t}{\sqrt{\lambda}} \tag{3}$$

where B_t is standard Brownian motion independent with $(W_{i,t})$ and $(W_{j,t})$, and the parameter λ is set by the game holder to control the precision of signal. The larger the λ, the more accurate the signal would be. The information set of both players at time $t \geq 0$ is $I_t \equiv \{Z_s : 0 \leq s \leq t\}$. Player i estimates the unknown output gap y_t based on the information set I_t. Let $\tilde{y}_t \equiv E(y_{i,t}|I_t)$ be the estimated output gap and $S_t \equiv E[(\tilde{y}_{i,t} - y_{i,t})^2|I_t]$ be the estimation variance. According to Chapter 1.2 of Bensoussan (1992), *Kalman-Bucy filter* gives the dynamics of $\tilde{y}_t$ and S_t,

$$d\tilde{y}_t = (q_{i,t} - q_{j,t})dt + \lambda S_t(dZ_t - \tilde{y}_t dt) \tag{4}$$

$$\frac{dS_t}{dt} = \sigma^2 - \lambda S_t^2 \tag{5}$$

Hence, the conditional distribution $y_t|I_t \sim \mathcal{N}(\tilde{y}_t, S_t|I_t)$ is fully captured by the mean $\tilde{y}_t$ and variance S_t. If $\lambda = 0$, we have $S_t = S_0 + \sigma^2 t$, i.e., the estimation variance is increasing in time linearly. If $\lambda > 0$, the solution of (5) is

$$S_t = \begin{cases} \bar{S} \cdot \tanh\left\{t \cdot \sigma\sqrt{\lambda} + \tanh^{-1}\left(S_0/\bar{S}\right)\right\} & \text{if } S_0 < \bar{S} \\ \bar{S} & \text{if } S_0 = \bar{S} \\ \bar{S} \cdot \coth\left\{t \cdot \sigma\sqrt{\lambda} + \coth^{-1}\left(S_0/\bar{S}\right)\right\} & \text{if } S_0 > \bar{S} \end{cases} \tag{6}$$

Specifically, $\bar{S} = \sigma/\sqrt{\lambda}$ when $\lambda > 0$ and $\bar{S} = \infty$ when $\lambda = 0$. Please refer to Appendix A for the derivations. Figure 1 shows the evolution of S_t in time: estimation variance S_t converges to *steady state* $\bar{S}$ as time goes by regardless of the starting estimation variance. For simplicity, we henceforth assume that $S_0 = \bar{S}$, hence $S_t \equiv \bar{S}$.

Following Ryvkin (2022), let's consider a dynamic contest with a given fixed deadline. Suppose the contest is terminated when time $t = T > 0$. Since the variance $\bar{S}$ is fixed as assumed above, the state of the game is fully characterized by a tuple $(\tilde{y}_t, t)$. At any time $0 \leq t < T$, player i optimizes her effort level $q_{i,\tau}$ in the remaining contest period $\tau \in [t, T)$ according to the

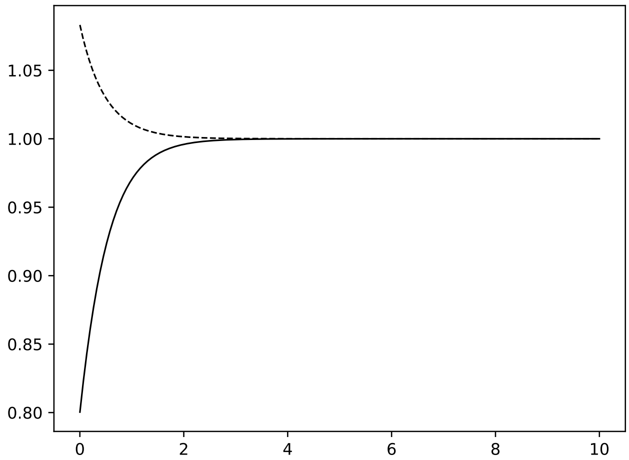

Figure 1: The evolution of S_t in time t, given that $\lambda = 1$ and $\sigma = 1$.

following optimization problem,

$$V^i(\tilde{y}_t, t; q_{j,t}, \Theta_i) = \max_{\{q_{i,\tau}\}_{\tau=t}^{T}} \mathbb{E}\left(\theta \cdot 1_{\tilde{y}_T > 0} - \int_t^T C_i(q_{i,\tau}) d\tau \middle| I_t\right) \tag{7}$$

where $\Theta \equiv \{\theta, \lambda, \sigma, c_i, c_j\}$, subject to constraints (4), (5) and $q_{i,\tau} \geq 0$ for all $\tau \in [t, T)$. The optimization problem for player j is just symmetric to that of player i as $V^j(\tilde{y}_t, t) = V^i(-\tilde{y}_t, t)$. The corresponding Hamilton-Jacobi-Bellman (HJB) equation for player i is

$$0 = \max_{q_{i,t} \geq 0} \left[-\frac{c_i q_i^2}{2} + V_y^i \cdot (q_{i,t} - q_{j,t}) + V_t^i + \frac{V_{yy}^i}{2} \lambda \bar{S}^2 \right]$$

Under the assumption of inner solution, we plug into the first order conditions $q_{i,t} = V_y^i / c_i$ and $q_{j,t} = -V_y^j / c_j$, we have the system of equations

$$\frac{1}{2c_i}(V_y^i)^2 + \frac{1}{c_j} V_y^i V_y^j + V_t^i + V_{yy}^i \frac{\lambda \bar{S}}{2} = 0$$
$$\frac{1}{2c_j}(V_y^j)^2 + \frac{1}{c_i} V_y^j V_y^i + V_t^j + V_{yy}^j \frac{\lambda \bar{S}}{2} = 0$$

subject to boundary conditions $V^i(-\infty, t) = 0$, $V^i(+\infty, t) = \theta$, $V^j(-\infty, t) = \theta$, $V^j(+\infty, t) = 0$, $V^i(\tilde{y}_T, T) = \theta \cdot 1_{\tilde{y}_T > 0}$ and $V^j(\tilde{y}_T, T) = \theta \cdot 1_{\tilde{y}_T < 0}$.

The Nash equilibrium is summarized in the following lemma:

Lemma 1 (Ryvkin 2022)**.** *In the Markov perfect equilibrium, the players' efforts in state* $(\tilde{y}_t, t) \in$

$\mathbb{R} \times [0, T)$ *are given by*

$$m_i(\tilde{y}_t, t) = \frac{e^{-z_t^2/2}\sqrt{\lambda\bar{S}}}{2\sqrt{2\pi(T-t)}} \cdot [\gamma(\rho_1) + \gamma(\rho_2)] \left[1 - \rho(z_t)^2\right] [1 + \rho(z_t)]$$
$$m_j(\tilde{y}_t, t) = \frac{e^{-z_t^2/2}\sqrt{\lambda\bar{S}}}{2\sqrt{2\pi(T-t)}} \cdot [\gamma(\rho_1) + \gamma(\rho_2)] \left[1 - \rho(z_t)^2\right] [1 - \rho(z_t)]$$

where $z_t = \tilde{y}_t / \sqrt{\lambda\bar{S}(T-t)}$, $\rho(z_t) = \gamma^{-1}\left(\Phi(z_t)\left[\gamma(\rho_1) + \gamma(\rho_2)\right] - \gamma(\rho_2)\right)$ *and*

$$\gamma(u) = \frac{u}{1-u^2} + \frac{1}{2}\ln\frac{1+u}{1-u},$$

$$\rho_1 = \frac{e^{w_1} + e^{-w_2} - 2}{e^{w_1} - e^{-w_2}}, \quad \rho_2 = \frac{e^{w_2} + e^{-w_1} - 2}{e^{w_2} - e^{-w_1}}, \quad w_i = \frac{\theta}{\lambda\bar{S}c_i}.$$

We include a simplified version of the proof in the appendix.

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3 Structural Estimation

References

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A Solve S_t in Equation (5)

If S is in steady state $dS/dt = 0 \Leftrightarrow S = \bar{S} \equiv \sigma/\sqrt{\lambda}$. If S is not in steady state, i.e. $S \neq \bar{S}$, we first isolate the two variables and get

$$\frac{dS}{\sigma - \lambda S^2} = dt$$

Then, we take the integral on both sides

$$t = \int \frac{dS}{\sigma - \lambda S^2} = \frac{1}{\sigma\sqrt{\lambda}} \int \frac{dS\sqrt{\lambda}/\sigma}{1 - (S\sqrt{\lambda}/\sigma)^2} \equiv \frac{1}{\sigma\sqrt{\lambda}} \int \frac{du}{1 - u^2}$$

where $u = S\sqrt{\lambda}/\sigma = S/\bar{S}$. Hence,

$$\sigma\sqrt{\lambda} \cdot t = \begin{cases} \tanh^{-1}(u) - K_1, & \text{if } |u| < 1 \\ \coth^{-1}(u) - K_2, & \text{if } |u| > 1 \end{cases} = \begin{cases} \tanh^{-1}(S/\bar{S}) - K_1, & \text{if } S < \bar{S} \\ \coth^{-1}(S/\bar{S}) - K_2, & \text{if } S > \bar{S} \end{cases}$$

Thus, we conclude the non-steady state case that

$$S = \begin{cases} \bar{S} \cdot \tanh(\sigma\sqrt{\lambda} \cdot t + K_1), & \text{if } S < \bar{S} \\ \bar{S} \cdot \coth(\sigma\sqrt{\lambda} \cdot t + K_2), & \text{if } S > \bar{S} \end{cases}$$

Finally, we determine the constants K_1, K_2 by the initial condition S_0 and have

$$K_1 = \tanh^{-1}(S_0/\bar{S})$$
$$K_2 = \coth^{-1}(S_0/\bar{S})$$