Fudan Mathematics Study Group Entrance Exam

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These problems should be quite challenging, so take your time to explore them and don't get discouraged. Here are some concepts that might be useful in solving these problems:

- Cauchy's functional equation
- Fermat's little theorem

Make sure to write down your thoughts on the problems, as this may earn you additional points. Write your proofs carefully and neatly. If your handwriting is difficult to read, it will be considered that you did not answer the problem.

A1

Given real numbers a and b with $a > b$, such that a and b are roots of the equation $x^2 - x - 1 = 0$, find $a^6 + b^6$.

A2

Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that for all rational numbers x, y, we have

$$f(x + y) = f(x) + f(y) - 2xy.$$

C1

A and B each toss n fair coins, and let X and Y denote the number of heads obtained by A and B, respectively. Assuming the coin tosses are independent experiments. When $n = 5$, what is the probability that $X = Y$?

C2

There are 6 people in the room, and each pair of people either knows each other or doesn't know each other. Prove that there must be at least three people who all know each other or all don't know each other.

G1

In an acute triangle ABC, line L is the perpendicular bisector of BC, and line M is the angle bisector of $\angle ABC$. They intersect at point P. If $\angle A = 60°$ and $\angle ACP = 24°$, what is the measure of $\angle ABP$?

G2

Let ω be the circumcircle of triangle ABC. Choose a point P on the arc BC of the circumcircle (not containing A). Draw perpendiculars from P to AB, AC, and BC, which intersect AB, AC, and BC at points D, E, and F respectively. Prove that the points D, E, and F are collinear.

N1

Find the remainder when 2^{2024} is divided by 13

N2

Prove that for any primes p, q,

$$p^{q^2-q+1} + q^{p^2-p+1} - p - q$$

is never a square.