A Strongly Polynomial Algorithm for Linear Programs with At Most 2 Nonzero Entries per Row or Column

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Joint work with

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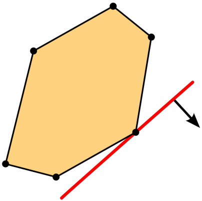
Talk Overview

1. Linear Program (LP)
   - Polynomial vs Strongly Polynomial Algorithms
2. LPs with ≤ 2 variables per Inequality
3. Minimum Cost Generalized Flow
4. A Strongly Polynomial Interior Point Method

Linear Program (LP)

Primal:

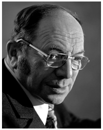
$$\begin{aligned} \min\ & c^\top x \\ \text{s.t.}\ & Ax = b \\ & x \geq \mathbf{0} \end{aligned}$$

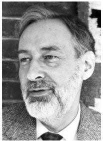


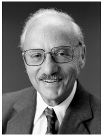
Dual:

$$\begin{aligned} \max\ & b^\top y \\ \text{s.t.}\ & A^\top y \leq c \end{aligned}$$

- Introduced by [Kantorovich '39] [Hitchcock '41] [Koopmans '42] [Dantzig '47].



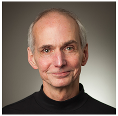




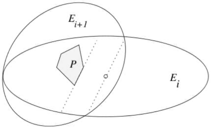
LP Algorithms

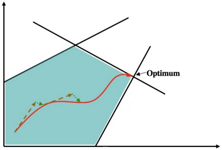
Input: $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^m$. Total bit length L.

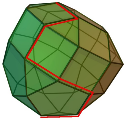
Def: A polynomial algorithm runs in $\mathrm{poly}(m, n, L)$ time.



- Polynomial algorithms for LP:
  - Ellipsoid method [Khachiyan '79]
  - Interior point method [Karmarkar '84] [Renegar '88]







- Simplex method [Dantzig '47]
  - Not known to be polynomial, but efficient in practice.

Strongly Polynomial

Input: $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^m$. Total bit length L.

Def: An algorithm is strongly polynomial if it uses

1. $\text{poly}(m, n)$ elementary arithmetic operations $(+, -, \times, \div, <?)$, and
2. $\text{poly}(m, n, L)$ space.

Smale's 9th Problem [Megiddo '83]

Is there a strongly polynomial algorithm for linear programming?

The Zoo of LP Subclasses

General LP $\equiv$ LP with ≤ 3 variables per inequality

Strongly polynomial (as of today) (known before 2024)

Combinatorial LP:

- Shortest path
- Bipartite matching
- Maximum flow
- Minimum cost flow

- LP feasibility with ≤ 2 variables per inequality
- Discounted MDP
- Maximum generalized flow
- LP with ≤ 2 variables per inequality

- Undiscounted MDP
- LP with ≤ 2 variables per inequality

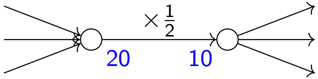
2-Variables-per-Inequality LP

- [Hochbaum '04] Every 2-variables-per-inequality LP can be reduced to

$$\begin{aligned} &\min\ c^\top x \\ &\text{s.t.} \sum_{e\in\delta^{\text{in}}(v)} \gamma_e x_e - \sum_{e\in\delta^{\text{out}}(v)} x_e = b_v \quad \forall v \in V \\ &\qquad\qquad\qquad\qquad\qquad\qquad x \geq \mathbf{0} \end{aligned}$$

Interpretation: Given directed graph $G = (V, E)$, node demands $b \in \mathbb{R}^V$, arc costs $c \in \mathbb{R}^E$ and gain factors $\gamma \in \mathbb{R}^E_{>0}$,

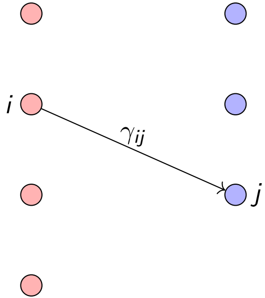
Find a **minimum cost generalized flow** satisfying all node demands.



Models leaky pipes, currency exchange etc.

Example: Production with Different Machines

• Variant of a problem proposed by Kantorovich in his 1939 paper introducing Linear Programming.



M: machines P: parts

• Machine i can produce γ_{ij} units of part j in one day at cost c_{ij}.

• Daily demand d_j for part j.

$$\begin{aligned} \min \quad & \sum_{i \in M, j \in P} c_{ij} x_{ij} \\ \text{s. t.} \quad & \sum_{j \in P} x_{ij} \leq 1 \qquad \forall i \in M \\ & \sum_{i \in M} \gamma_{ij} x_{ij} \geq d_j \quad \forall j \in P \\ & x \geq \mathbf{0} \end{aligned}$$

Previous Algorithms for Generalized Flow

- Algorithms for **primal** feasibility:
  - Polynomial [Goldberg, Plotkin, Tardos '91]
  - Strongly polynomial [Végh '13] [Olver, Végh '20]
- Algorithms for **dual** feasibility:
  - Polynomial [Aspvall, Shiloach '80]
  - Strongly polynomial [Megiddo '83] [Cohen, Megiddo '94] [Hochbaum, Naor '94] [Dadush, K, Natura, Végh '21] [Karczmarz '22]
- Algorithms for optimization:
  - Polynomial [Wayne '02]

Main Result

Theorem [Dadush, K, Natura, Olver, Végh '24]

There is a strongly polynomial algorithm for the minimum cost generalized flow problem, and consequently, for LPs with at most 2 variables per inequality.

- The algorithm is the interior point method by [Allamigeon, Dadush, Loho, Natura, Végh '22].
- What we'll need for this talk:
  1. Interior point method
  2. Straight line complexity

Central Path

• For each $\mu > 0$, there exists a unique optimal solution $x^{\mathrm{cp}}(\mu)$ to

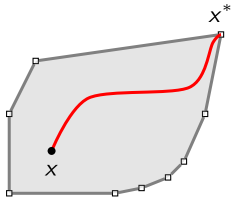
$$\begin{aligned} &\min \ c^\top x - \mu \sum_{i=1}^{n} \log(x_i) \\ &\text{s.t.} \ Ax = b. \end{aligned}$$

Def: The central path is the curve

$$\{x^{\mathrm{cp}}(\mu) : \mu > 0\}.$$

• As $\mu \to 0$, $x^{\mathrm{cp}}(\mu)$ converges to an optimal solution x^* of the LP.

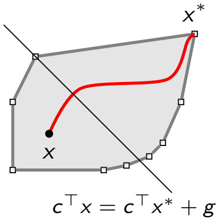
• Interior Point Method (IPM): Walk down the central path with geometrically decreasing μ.



Max Central Path

- Let us reparameterize x^{cp} by the optimality gap:

$$c^\top x^{\text{cp}}(g) = c^\top x^* + g \qquad \forall g \geq 0.$$



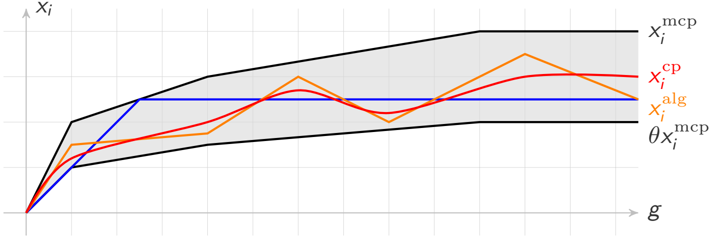
- For every $g \geq 0$ and $i \in [m]$, define

$$\begin{aligned} x_i^{\text{mcp}}(g) := \max\ & x_i \\ \text{s.t.}\ & x \text{ feasible} \\ & \text{optimality gap} \leq g. \end{aligned}$$

Def: The max central path is the curve $\{x^{\text{mcp}}(g) : g \geq 0\}$,

Theorem: $\frac{1}{2m} x^{\text{mcp}} \leq x^{\text{cp}} \leq x^{\text{mcp}}$.

Straight Line Complexity



- IPM generates a piecewise-affine curve x^{alg} near the central path

$$\theta x^{\mathrm{mcp}} \leq x^{\mathrm{alg}} \leq x^{\mathrm{mcp}}.$$

Def: The straight line complexity of x_i^{mcp}, $\mathrm{SLC}_\theta(x_i^{\mathrm{mcp}})$, is the minimum number of pieces of a piecewise-affine function h such that

$$\theta x_i^{\mathrm{mcp}} \leq h \leq x_i^{\mathrm{mcp}}.$$

Straight Line Complexity

- $\#$ iterations required by any IPM is at least

$$\max_{i \in [m]} \mathsf{SLC}_\theta(x_i^{\mathrm{mcp}}).$$

Theorem [Allamigeon, Dadush, Loho, Natura, Végh '22]

There is an interior point method which solves LP in

$$O\left(\sqrt{m}\log\left(\frac{m}{\theta}\right)\sum_{i=1}^{m}\mathsf{SLC}_\theta(x_i^{\mathrm{mcp}})\right)$$

iterations.

Main Result

Theorem [Dadush, K, Natura, Olver, Végh '24]

For the minimum-cost generalized flow problem on $G = (V, E)$ with n nodes and m arcs,

$$\mathsf{SLC}_{\frac{1}{m}}(x_e^{\mathrm{mcp}}) = O(mn \log n) \qquad \forall e \in E.$$

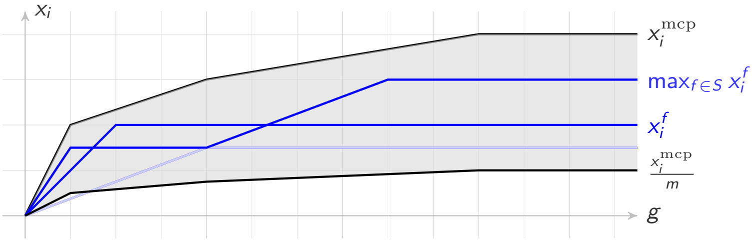
Theorem [Dadush, K, Natura, Olver, Végh '24]

There is a strongly polynomial algorithm for the minimum cost generalized flow problem.

Upper Bounding the SLC

- Every $f \in \ker(A)$ with $c^\top f > 0$ induces a line segment in P:

$$x^f(g) := x^* + \frac{g}{c^\top f} f.$$



Strategy: Find $S \subseteq \ker(A)$ such that $\max_{f \in S} x_i^f \geq \frac{x_i^{\mathrm{mcp}}}{m}$.

$$\implies \mathrm{SLC}_{\frac{1}{m}}(x_i^{\mathrm{mcp}}) \leq 2|S|.$$

Conclusion

- Strongly polynomial algorithm for LPs with ≤ 2 variables per inequality.

- Also strongly polynomial for $\min\{c^\top x : Ax = b, x \geq \mathbf{0}\}$ when A has at most 2 nonzero entries per column.

- [Allamigeon, Benchimol, Gaubert, Joswig '18] There exist LPs with

$$\mathsf{SLC}_\theta(x_i^{\mathrm{mcp}}) = 2^{\Omega(m)}.$$

- Future directions:
  - Undiscounted MDP: strongly polynomial solvability/straight line complexity open.
  - Faster strongly polynomial algorithm for minimum cost generalized flow.

Thank you!