Approximating the Held-Karp Bound for Metric TSP in Nearly Linear Work and Polylog Depth

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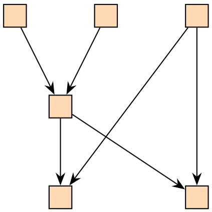
Work-Depth Model

Sequential computation

Parallel computation



- Work = Total number of operations
- Depth = Length of a longest chain of dependent operations
- Fast parallel algorithm = Nearly linear work and polylog depth.

Metric TSP

- Given an undirected graph $G = (V, E)$ with edge costs $c \in \mathbb{R}^m_{\geq 0}$,

TSP: Find a minimum-cost Hamiltonian cycle in G.

- Inapproximable

Metric TSP: Find a minimum-cost spanning tour in G.

- **APX**-hard [Lampis '14]
- $3/2$ approximation [Christofides '76] [Serdyukov '78]
- $3/2 - 10^{-34}$ approximation [Karlin, Klein, Oveis Gharan '22] [Gurvits, Klein, Leake '24]

- Metric TSP on $(G, c) \equiv$ TSP on the metric completion $(\hat{G}, \hat{c})$

$$\hat{G} = \text{Complete graph on } V$$
$$\hat{c}_{uv} = \text{Shortest path length between } u \text{ and } v \text{ in } G$$

Subtour Elimination LP

[Dantzig, Fulkerson, Johnson '54]

$$
\begin{aligned}
\min\ & \hat{c}^\top x \\
\text{s. t.}\ & \sum_{v} x_{uv} = 2 && \forall\, u \in V \\
& \sum_{u \in S, v \notin S} x_{uv} \geq 2 && \forall\, \emptyset \subsetneq S \subsetneq V \\
& x_{uv} \geq 0 && \forall\, u, v \in V
\end{aligned}
$$

- Used in many approximation/exact algorithms for TSP.
- The LP optimal value coincides with the Held–Karp bound.

Conjecture: The LP integrality gap is at most $4/3$ [Goemans '95].

2-ECSM LP

- LP relaxation of the 2-edge-connected spanning multisubgraph problem:

$$\begin{aligned} \min\ & c^\top x \\ \text{s. t.}\ & \sum_{e \in \delta_G(S)} x_e \geq 2 \qquad \forall\, \emptyset \subsetneq S \subsetneq V \\ & x_e \geq 0 \qquad \forall\, e \in E. \end{aligned}$$

Fact: Subtour LP optimal value = 2-ECSM LP optimal value.

[Cunningham '90] [Goemans, Bertsimas '93]

- Methods for solving the LP:
  - **Ellipsoid:** separation oracle is min cut
  - **Held–Karp bound/heuristic:** iterate over 1-trees
  - **Multiplicative weight update (MWU)**

Solving the LP via MWU

- FPTAS which returns a $(1+\varepsilon)$-approximate solution.
- Sequential algorithms:
  - $\tilde{O}(n^4/\varepsilon^2)$ [Plotkin, Shmoys, Tardos '95]
  - $\tilde{O}(m^2/\varepsilon^2)$ [Garg, Khandekar '02]
  - $\tilde{O}(m/\varepsilon^2)$ [Chekuri, Quanrud '17]

Main Result [KWY '25]

Parallel algorithm that runs in $\tilde{O}(m/\varepsilon^4)$ work and $\tilde{O}(1/\varepsilon^4)$ depth.

Framework: Width-independent epoch-based MWU.

[Garg, Könemann '07] [Fleischer '00] [Luby, Nisan '93] [Young '01]

Epoch-Based MWU

- Initialize edge weights as $w = 1/c$.
- Given a fixed lower bound λ on the mincut value, define

$$\mathcal{C}^* := \{C \text{ cut } : w(C) < (1+\varepsilon)\lambda\}.$$

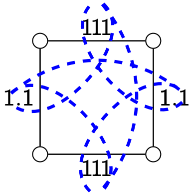
$$\left.\begin{array}{l} \textbf{While } \mathcal{C}^* \neq \emptyset\text{:} \\ \quad \text{1. Select cut(s) from } \mathcal{C}^*. \\ \quad \text{2. Multiplicatively increase } w \text{ along these cuts.} \end{array}\right\} \text{ an epoch}$$

- $\lambda \leftarrow \lambda(1+\varepsilon)$ and a new epoch begins.
- Terminate when $\|w\|_\infty$ is big.

Epoch-Based MWU

While $\mathcal{C}^* \neq \emptyset$:

1. Select cut(s) from $\mathcal{C}^*$.
2. Multiplicatively increase $w^{(t)}$ along these cuts.



Sequential MWU: Select one cut from $\mathcal{C}^*$

$\implies \tilde{O}(m/\varepsilon^2)$ iterations [Garg, Könemann '07] [Fleischer '00].

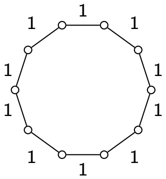
Parallel MWU: Select all cuts from $\mathcal{C}^*$

$\implies \tilde{O}(\log(|\mathcal{C}^*|)/\varepsilon^4)$ iterations [Luby, Nisan '93] [Young '01].

$\implies \tilde{O}(1/\varepsilon^4)$ iterations because $|\mathcal{C}^*| = O(n^2)$ for cuts.

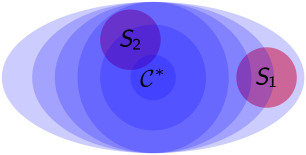
Core-Sequence

- Parallel MWU can incur $\Omega(n^2)$ work.



New Selection Rule:

1. Fix a representative set $S \subseteq \mathcal{C}^*$.
2. In every iteration, select $S \cap \mathcal{C}^*$ as long as it is nonempty.
3. Repeat Steps 1 and 2 until $\mathcal{C}^* = \emptyset$.



Definition

The sequence $\mathcal{S} = (S_1, \ldots, S_\ell)$ of representative sets is called a core-sequence of the epoch.

Core-Sequence

- Special cases:
  - $\mathcal{S} = (S_1, \ldots, S_\ell)$ where $|S_i| = 1$ for all $i \in [\ell] \implies$ sequential MWU.
  - $\mathcal{S} = (\mathcal{C}^*) \implies$ parallel MWU.

Theorem [KWY '25]

If MWU uses a core-sequence of length $\leq \ell$ with sets of size $\leq k$ in every epoch, then the number of iterations is

$$\tilde{O}\left(\frac{\ell \log(k)}{\varepsilon^4}\right).$$

- Tradeoff between ℓ and k.

Core-Sequence for 2-ECSM LP

Theorem [KWY '25]

For 2-ECSM LP, every epoch has a core-sequence of length $\tilde{O}(1)$, in which every set has size $\tilde{O}(n)$.

- Despite $|\mathcal{C}^*| = O(n^2)$, only need to select $\tilde{O}(n)$ of them!

Theorem [KWY '25]

There is a parallel FPTAS for the Held–Karp bound that runs in $\tilde{O}(m/\varepsilon^4)$ work and $\tilde{O}(1/\varepsilon^4)$ depth.

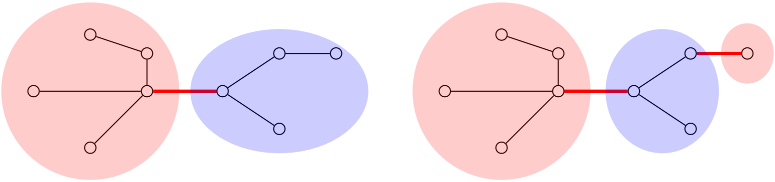
Finding a Good Core-Sequence

- $\tilde{O}(1)$ sets
- Every set has size $\tilde{O}(n)$

Tree Packing

Definition

Fix a spanning tree T of G. A cut C k-respects T if $|C \cap E(T)| = k$.



1-respecting cut

2-respecting cut

Theorem [Karger '00]

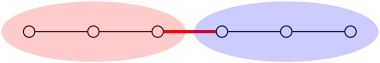
There exists a family $\mathcal{T}$ of $O(\log n)$ spanning trees such that w.h.p., every cut in $\mathcal{C}^*$ 1- or 2-respects some $T \in \mathcal{T}$.

Reducing to a Tree

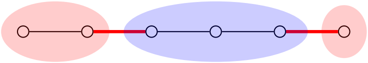
- For every tree $T \in \mathcal{T}$, let

$$\mathcal{C}^*_T := \{C \in \mathcal{C}^* : |C \cap E(T)| \leq 2\}.$$

- By Karger's Theorem, it suffices to find a good core-sequence for $\mathcal{C}^*_T$.
- For simplicity, assume that T is a path.



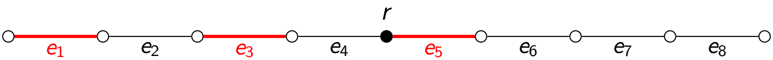
1-respecting cut



2-respecting cut

- $\mathcal{C}^*_T$ can still be as large as $O(n^2)$.

2-Respecting Cuts



Definition

Fix a root $r \in V$. A 2-respecting cut $\{e_i, e_j\}$ crosses r if r lies on the subpath between e_i and e_j.

***r*-Crossing Lemma**

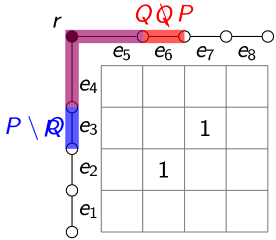
If every 2-respecting cut in $\mathcal{C}_T^*$ crosses r, then $|\mathcal{C}_T^*| = O(n)$.

Proof of r-Crossing Lemma

- Let k and ℓ be the number of edges to the left and right of r respectively.

- Define a matrix $M \in \{0,1\}^{k\times\ell}$ as

$$M_{e_i,e_j} := \begin{cases} 1, & \text{if } \{e_i, e_j\} \in \mathcal{C}_T^* \\ 0, & \text{otherwise.} \end{cases}$$



Claim: Every anti-diagonal of M has at most one 1.

- Suppose there are two 1's in some anti-diagonal.

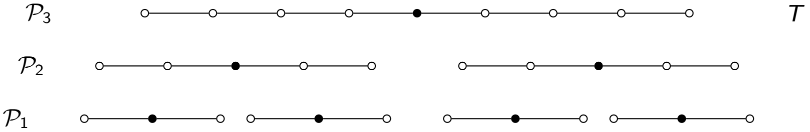
$$2(1+\varepsilon)\lambda > w(\delta_G(P)) + w(\delta_G(Q)) \geq w(\delta_G(P\setminus Q)) + w(\delta_G(Q\setminus P))$$

$\implies$ $P \setminus Q$ or $Q \setminus P$ belongs to $\mathcal{C}_T^*$. $\square$

Good Core-Sequence for $\mathcal{C}_T^*$

- First set in the core-sequence

$$S_0 := \{C \in \mathcal{C}_T^* : |C \cap E(T)| = 1\}.$$



- For $i \geq 1$, decompose T into paths $\mathcal{P}_i$ of length 2^i. Set

$$S_i := \{C \in \mathcal{C}_T^* : |C \cap E(P)| = 2 \text{ for some } P \in \mathcal{P}_i\}.$$

- By r-crossing lemma, $|S_i| = O(n)$ for all i.

Conclusion

- Introduced core-sequence as a new selection rule for MWU.
- Parallel FPTAS that runs in nearly linear work and polylog depth for
  - Held–Karp bound and k-ECSM LP
  - k-ECSS LP
- Future directions:
  - Apply core-sequence to other implicit packing/covering LPs
  - Better dependence on ε
  - Extension to streaming/distributed models

Thank You!