Approximating the Held-Karp Bound for Metric TSP in Nearly Linear Work and Polylog Depth

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THE HEBREW UNIVERSITY OF JERUSALEM

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Work-Depth Model

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Sequential computation

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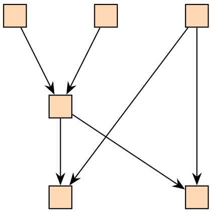

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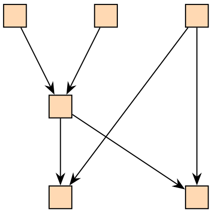

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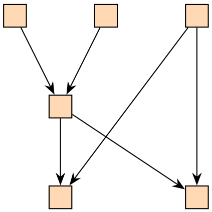

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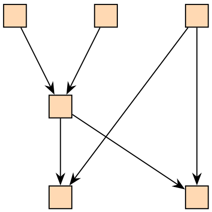

- Work = Total number of operations
- Depth = Length of a longest chain of dependent operations
- Fast parallel algorithm = Nearly linear work and polylog depth.

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- **APX**-hard [Lampis '14]
- $3/2$ approximation [Christofides '76] [Serdyukov '78]
- $3/2 - 10^{-36}$ approximation [Karlin, Klein, Oveis Gharan '22]

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- Metric TSP on $(G, c) \equiv$ TSP on the metric completion $(\hat{G}, \hat{c})$

$$\hat{G} = \text{Complete graph on } V$$
$$\hat{c}_{uv} = \text{Shortest path length between } u \text{ and } v \text{ in } G$$

Subtour Elimination LP

[Dantzig, Fulkerson, Johnson '54]

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\min\ & \hat{c}^\top x \\
\text{s.t.}\ & \sum_{v} x_{uv} = 2 && \forall\, u \in V \\
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Conjecture: The LP integrality gap is at most $4/3$ [Goemans '95].

2-ECSM LP

- LP relaxation of the 2-edge-connected spanning multisubgraph problem:

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  - **Multiplicative weight update (MWU)**

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Main Result [KWY '25]

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Framework: Width-independent epoch-based MWU.

[Garg, Könemann '07] [Fleischer '00] [Luby, Nisan '93] [Young '01]

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- Terminate when $\|w\|_\infty$ is big.

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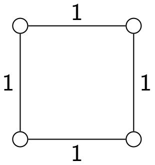

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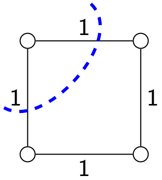

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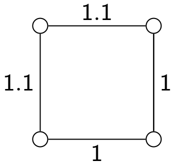

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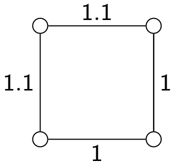

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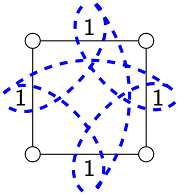

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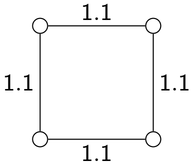

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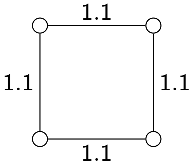

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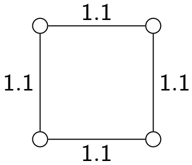

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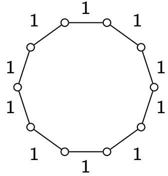
$\implies \tilde{O}(1/\varepsilon^4)$ iterations because $|\mathcal{C}^*| = O(n^2)$ for cuts.

Core-Sequence

- Parallel MWU can incur $\Omega(n^2)$ work.



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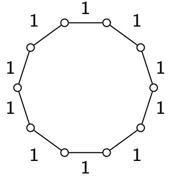

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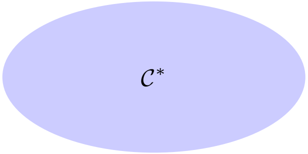

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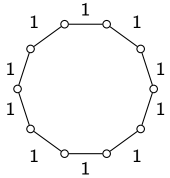

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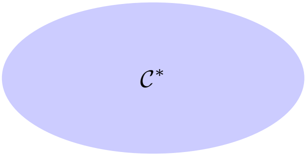

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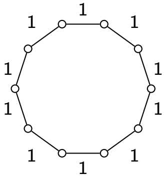

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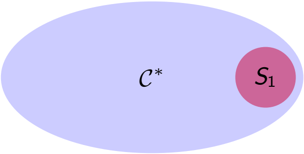

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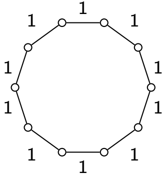

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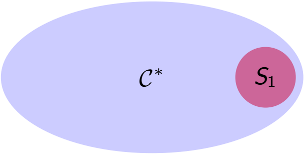

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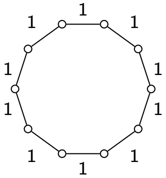

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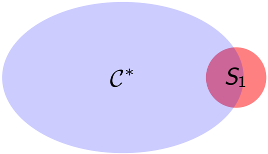

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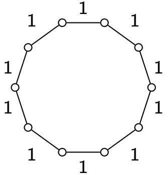

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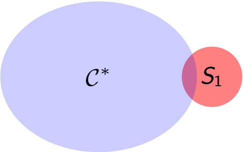

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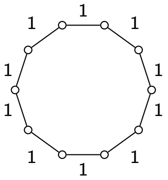

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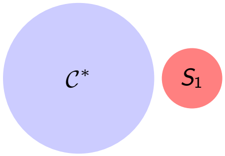

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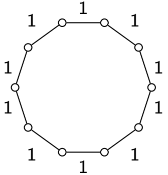

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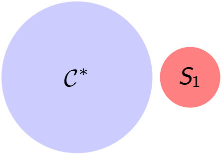

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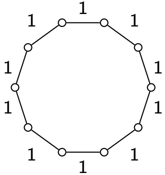

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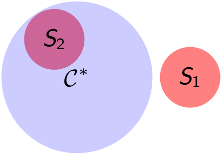

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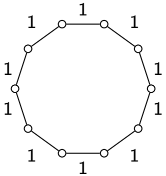

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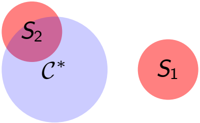

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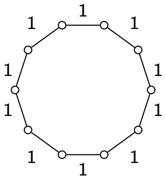

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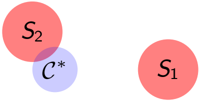

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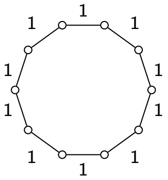

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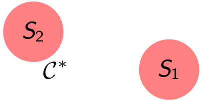

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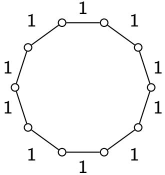

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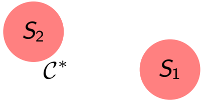

Definition

The sequence $\mathcal{S} = (S_1, \ldots, S_\ell)$ of representative sets is called a core-sequence of the epoch.

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Theorem [KWY '25]

If MWU uses a core-sequence of length $\leq \ell$ with sets of size $\leq k$ in every epoch, then the number of iterations is

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- Tradeoff between ℓ and k.

Core-Sequence for 2-ECSM LP

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Finding a Good Core-Sequence

- $\tilde{O}(1)$ sets
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Fix a spanning tree T of G. A cut C k-respects T if $|C \cap E(T)| = k$.

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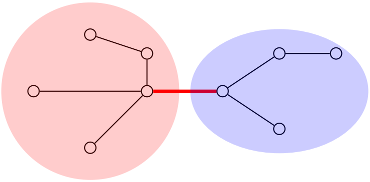

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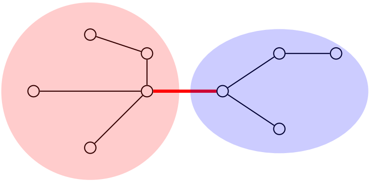

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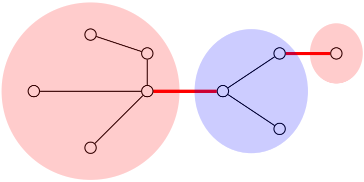

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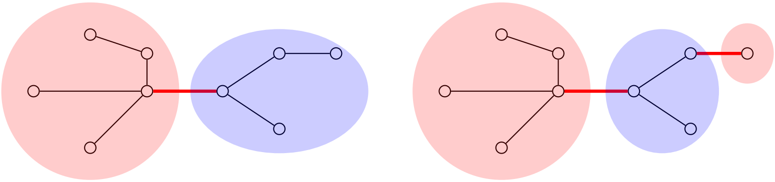

1-respecting cut

2-respecting cut

Theorem [Karger '00]

There exists a family $\mathcal{T}$ of $O(\log n)$ spanning trees such that w.h.p., every cut in $\mathcal{C}^*$ 1- or 2-respects some $T \in \mathcal{T}$.

Reducing to a Tree

- For every tree $T \in \mathcal{T}$, let

$$\mathcal{C}^*_T := \{C \in \mathcal{C}^* : |C \cap E(T)| \leq 2\}.$$

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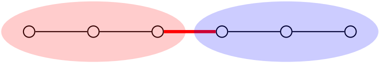

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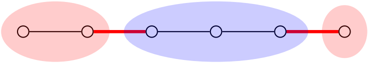

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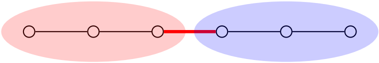

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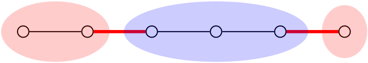
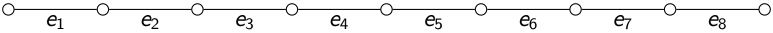


2-respecting cut

- $\mathcal{C}_T^*$ can still be as large as $O(n^2)$.

2-Respecting Cuts



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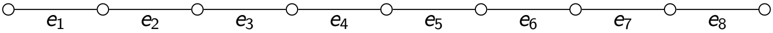

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Fix a root $r \in V$. A 2-respecting cut $\{e_i, e_j\}$ crosses r if r lies on the subpath between e_i and e_j.

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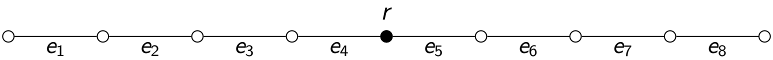

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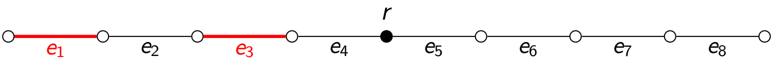

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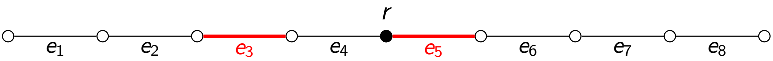

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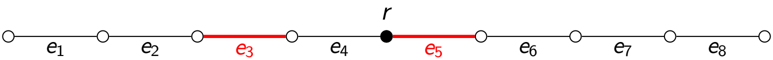

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***r*-Crossing Lemma**

If every 2-respecting cut in $\mathcal{C}_T^*$ crosses r, then $|\mathcal{C}_T^*| = O(n)$.

Proof of r-Crossing Lemma

- Let k and ℓ be the number of edges to the left and right of r respectively.

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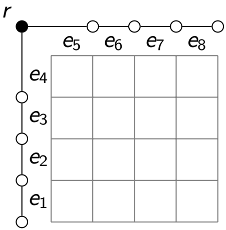

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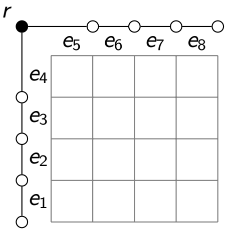

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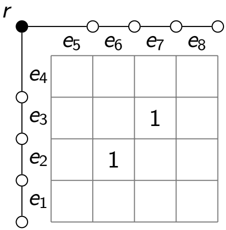

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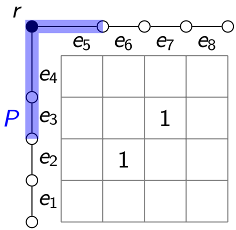

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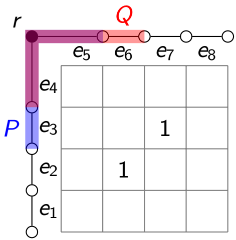

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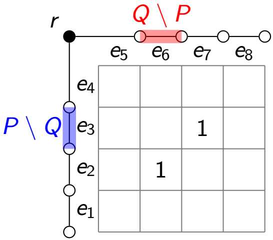

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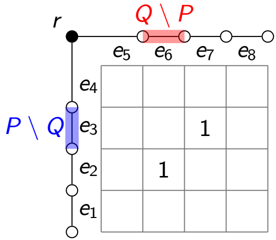

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$\implies$ $P \setminus Q$ or $Q \setminus P$ belongs to $\mathcal{C}_T^*$. $\square$

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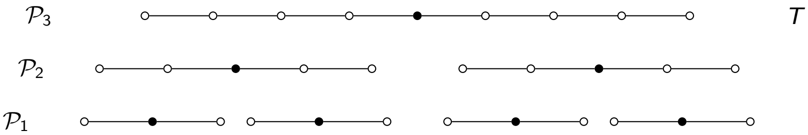

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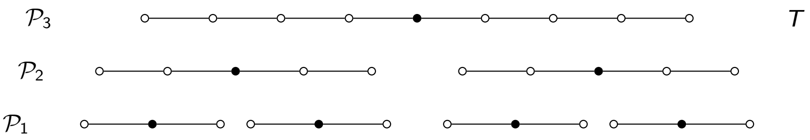

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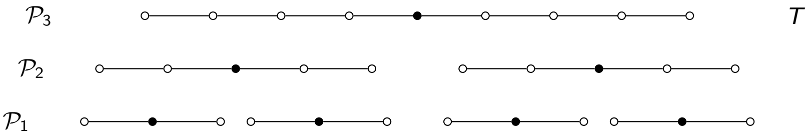

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Thank You!