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3.3 BALANCED SEARCH TREES

- ▶ 2-3 search trees
- ▶ red-black BSTs
- ▶ B-trees

Symbol table review

implementation	worst-case cost (after N inserts)			average case (after N random inserts)			ordered iteration?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	N	N	N	N/2	N	N/2	no	equals()
binary search (ordered array)	$\lg N$	N	N	$\lg N$	N/2	N/2	yes	compareTo()
BST	N	N	N	$1.39 \lg N$	$1.39 \lg N$?	yes	compareTo()
goal	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	yes	compareTo()

Challenge. Guarantee performance.

This lecture. 2-3 trees, left-leaning red-black BSTs, B-trees.

Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

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3.3 BALANCED SEARCH TREES

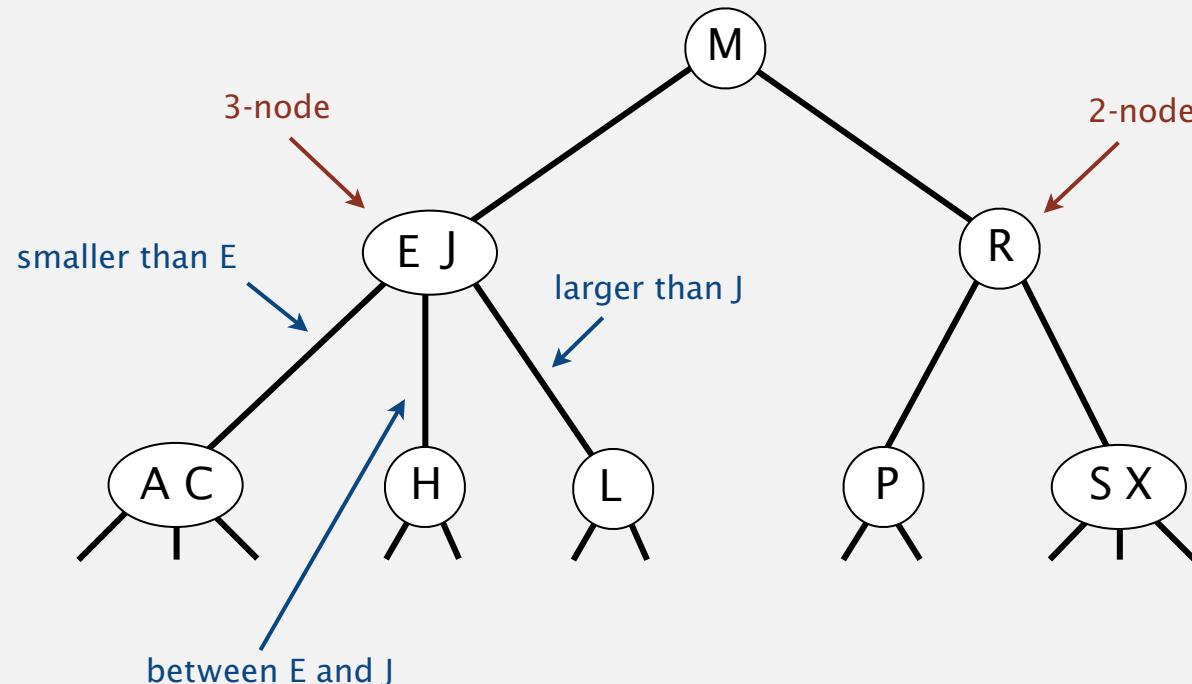
- ▶ 2-3 search trees
- ▶ red-black BSTs
- ▶ B-trees

2-3 tree

Allow 1 or 2 keys per node.

- 2-node: one key, two children.
- 3-node: two keys, three children.

Symmetric order. Inorder traversal yields keys in ascending order.



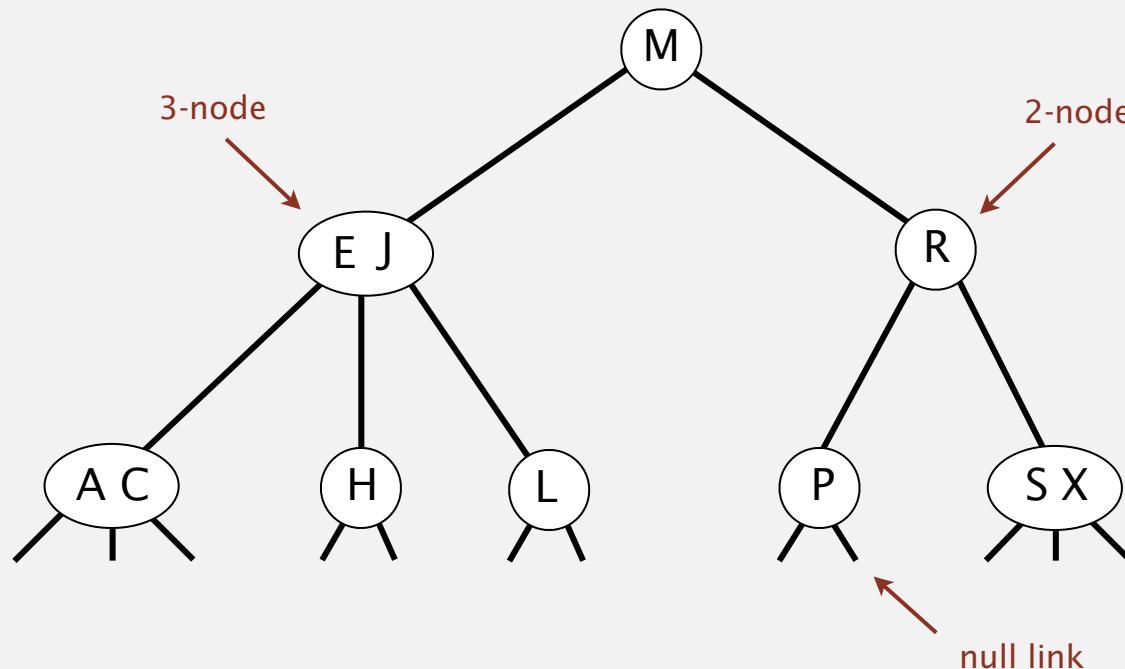
2-3 tree

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Symmetric order. Inorder traversal yields keys in ascending order.

Perfect balance. Every path from root to null link has same length.



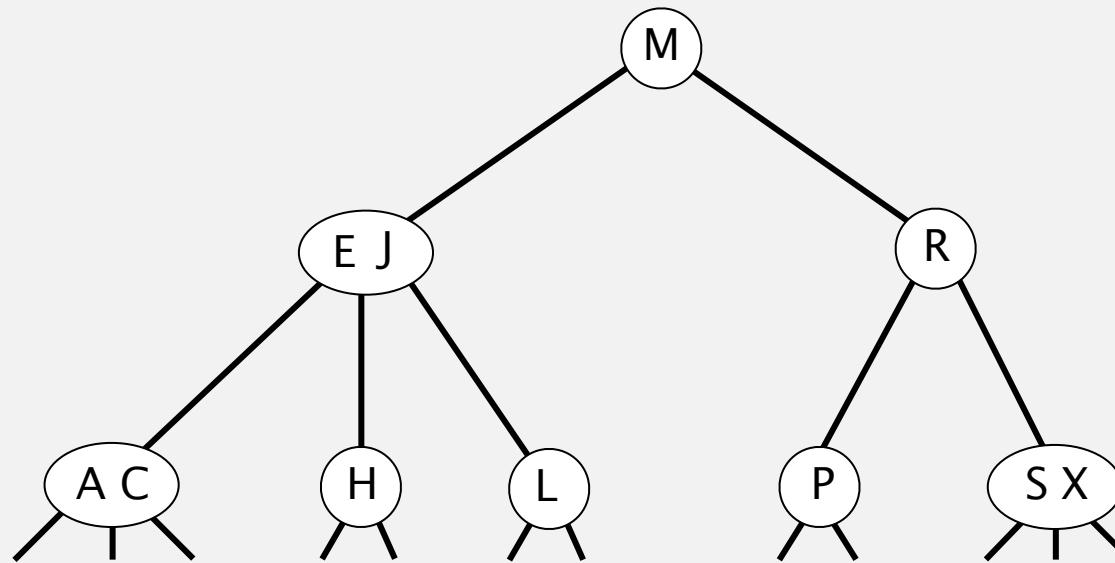
2-3 tree demo

Search.

- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).



search for H

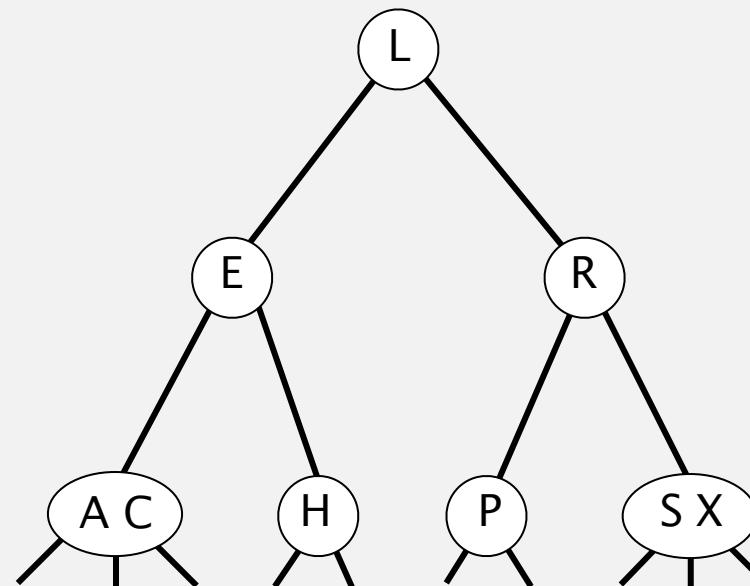


2-3 tree demo

Insertion into a 3-node at bottom.

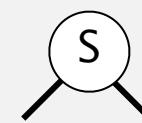
- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

insert L



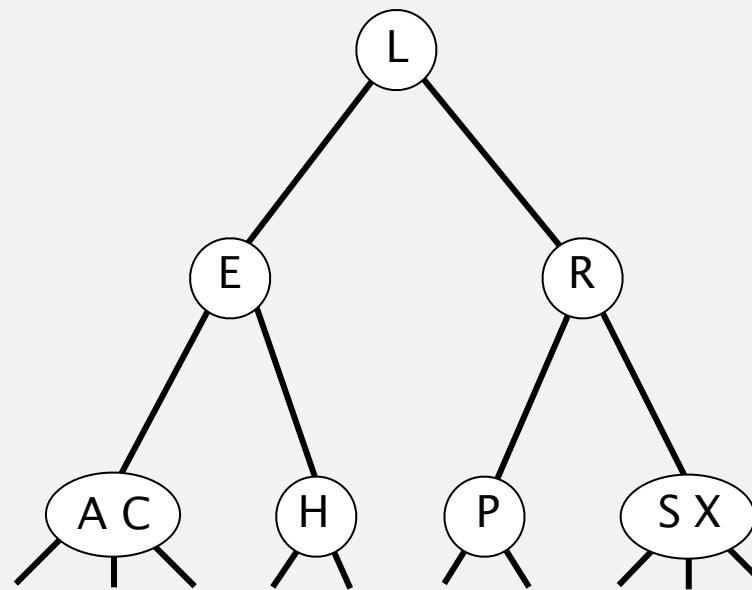
2-3 tree construction demo

insert S



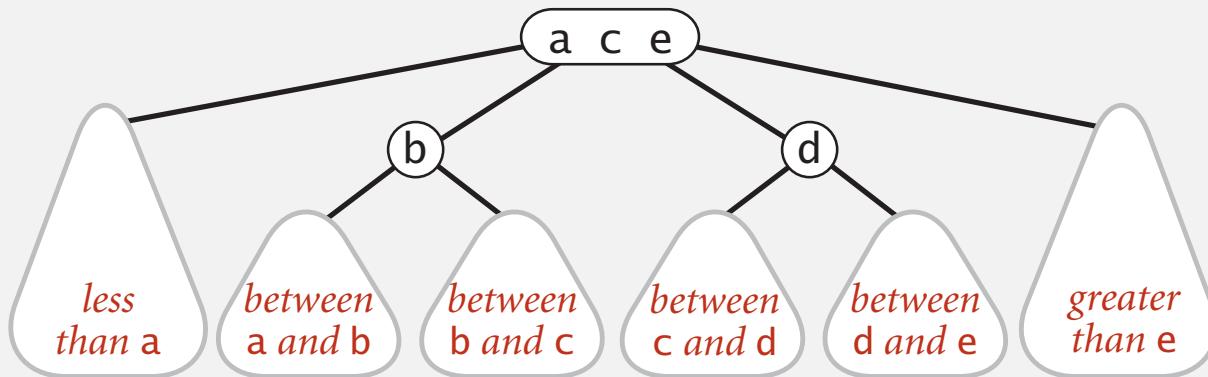
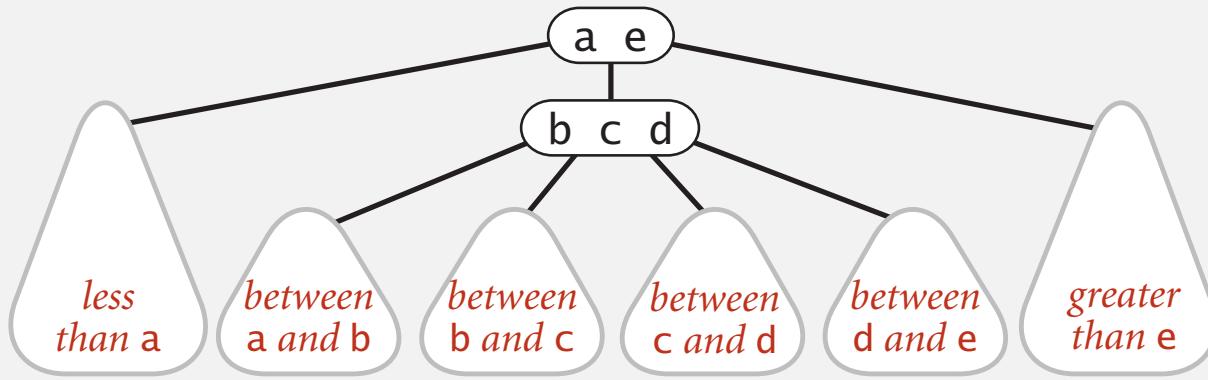
2-3 tree construction demo

2-3 tree



Local transformations in a 2-3 tree

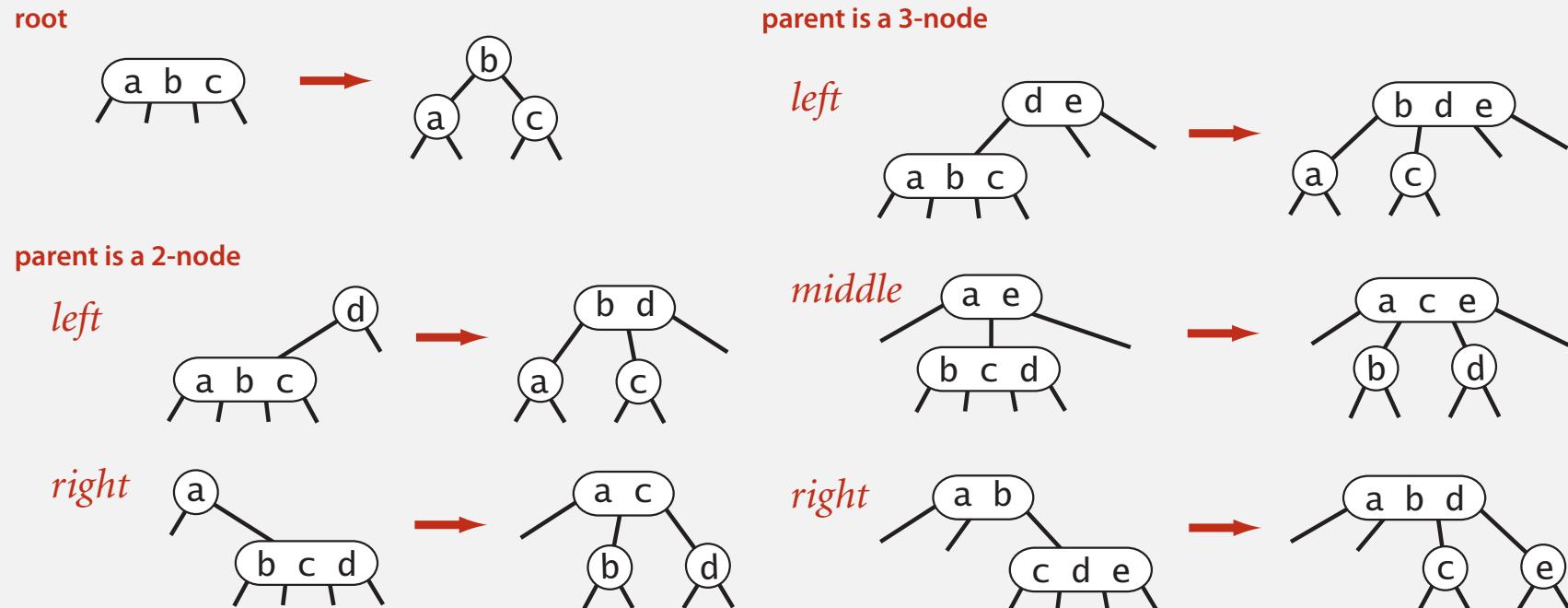
Splitting a 4-node is a **local** transformation: constant number of operations.



Global properties in a 2-3 tree

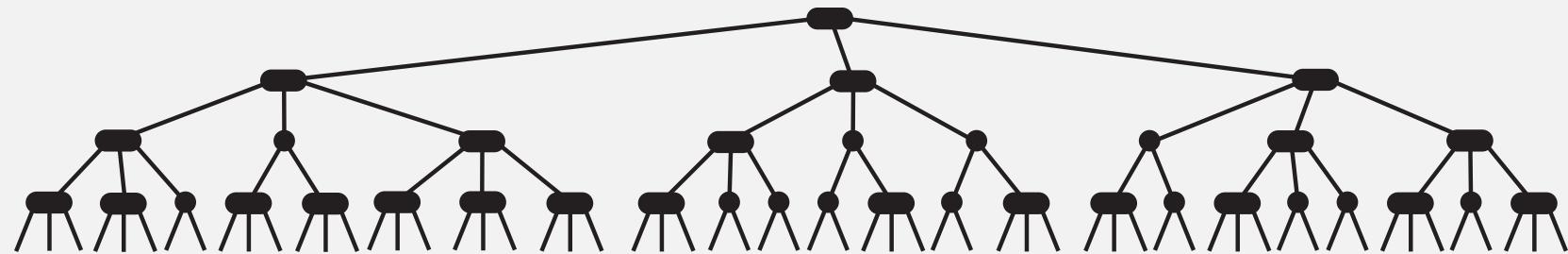
Invariants. Maintains symmetric order and perfect balance.

Pf. Each transformation maintains symmetric order and perfect balance.



2-3 tree: performance

Perfect balance. Every path from root to null link has same length.

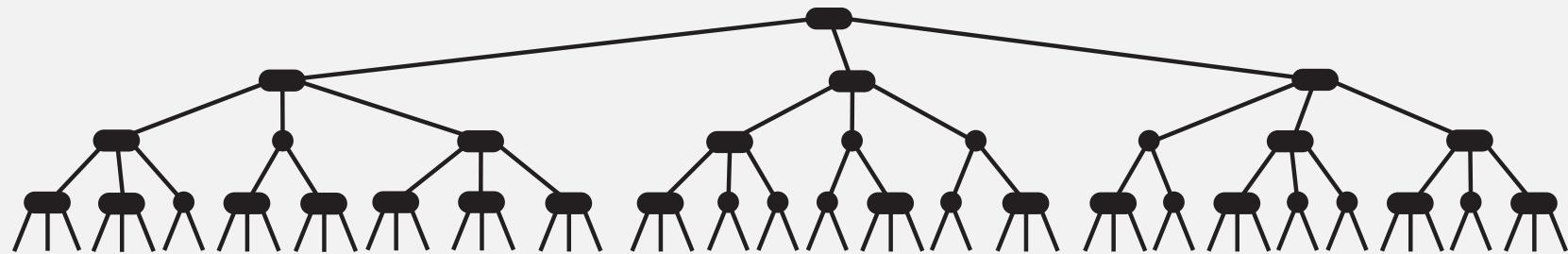


Tree height.

- Worst case:
- Best case:

2-3 tree: performance

Perfect balance. Every path from root to null link has same length.



Tree height.

- Worst case: $\lg N$. [all 2-nodes]
- Best case: $\log_3 N \approx .631 \lg N$. [all 3-nodes]
- Between 12 and 20 for a million nodes.
- Between 18 and 30 for a billion nodes.

Guaranteed **logarithmic** performance for search and insert.

ST implementations: summary

implementation	worst-case cost (after N inserts)			average case (after N random inserts)			ordered iteration?	key interface
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sequential search (unordered list)	N	N	N	N/2	N	N/2	no	equals()
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BST	N	N	N	$1.39 \lg N$	$1.39 \lg N$?	yes	compareTo()
2-3 tree	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	yes	compareTo()



 constants depend upon implementation

2-3 tree: implementation?

Direct implementation is complicated, because:

- Maintaining multiple node types is cumbersome.
- Need multiple compares to move down tree.
- Need to move back up the tree to split 4-nodes.
- Large number of cases for splitting.

Bottom line. Could do it, but there's a better way.

Algorithms

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Algorithms

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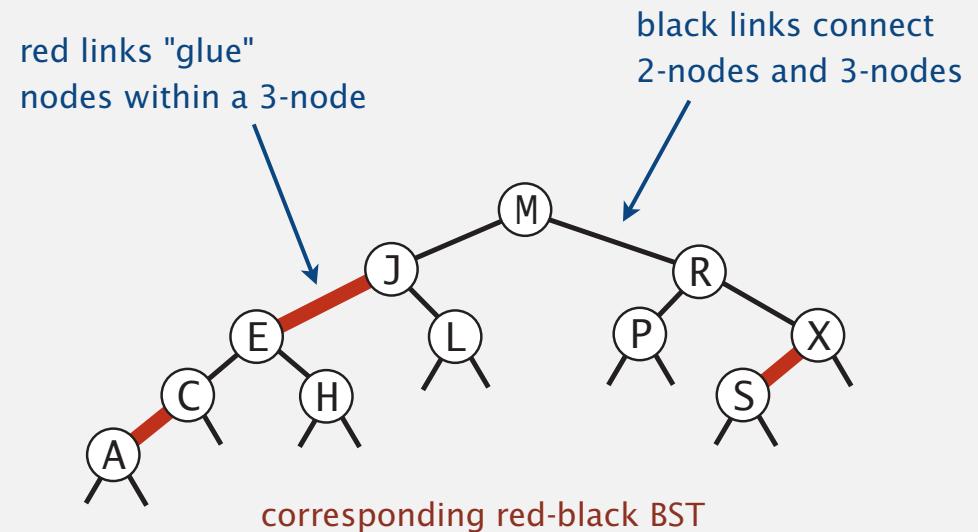
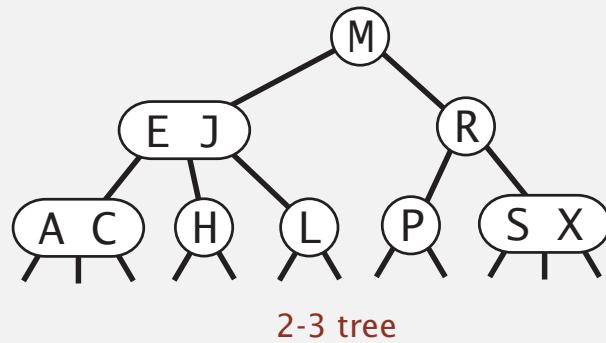
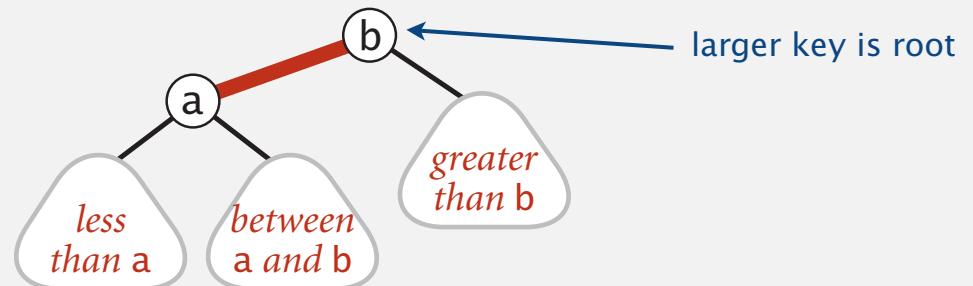
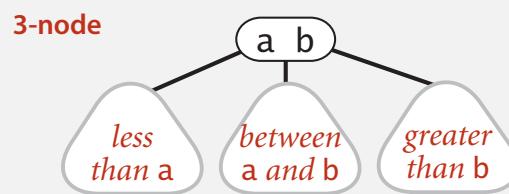
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3.3 BALANCED SEARCH TREES

- ▶ *2-3 search trees*
- ▶ *red-black BSTs*
- ▶ *B-trees*

Left-leaning red-black BSTs (Guibas-Sedgewick 1979 and Sedgewick 2007)

1. Represent 2–3 tree as a BST.
2. Use "internal" left-leaning links as "glue" for 3-nodes.

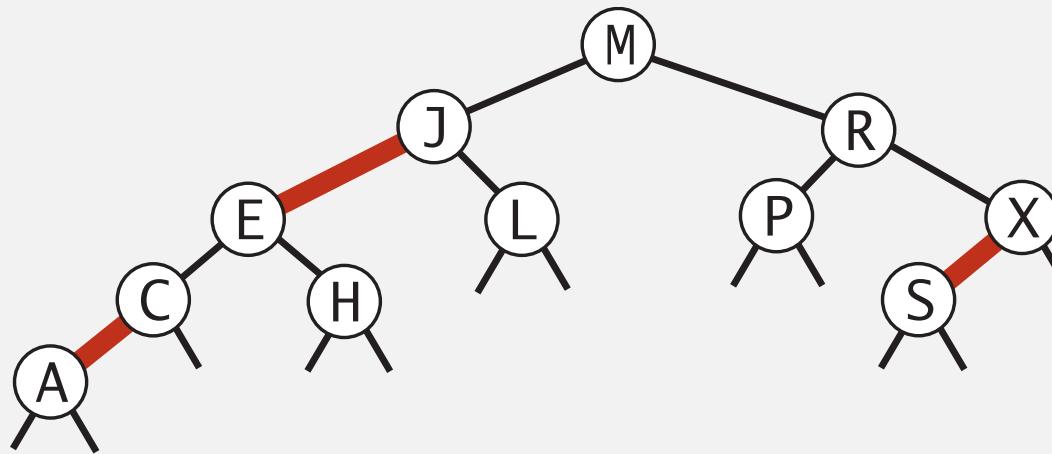


An equivalent definition

A BST such that:

- No node has two red links connected to it.
- Every path from root to null link has the same number of black links.
- Red links lean left.

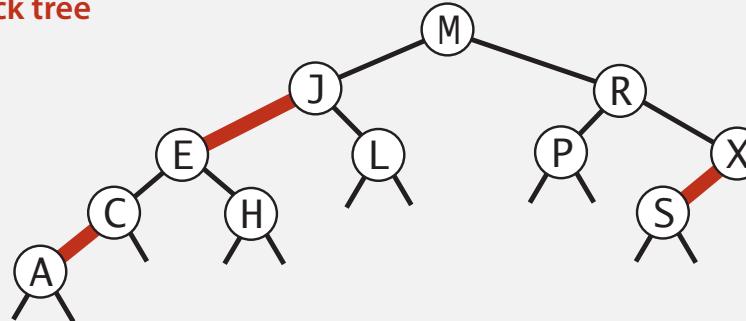
"perfect black balance"



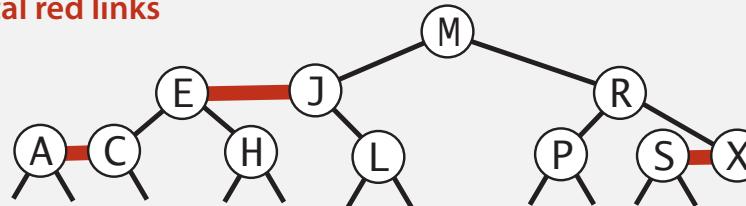
Left-leaning red-black BSTs: 1-1 correspondence with 2-3 trees

Key property. 1–1 correspondence between 2–3 and LLRB.

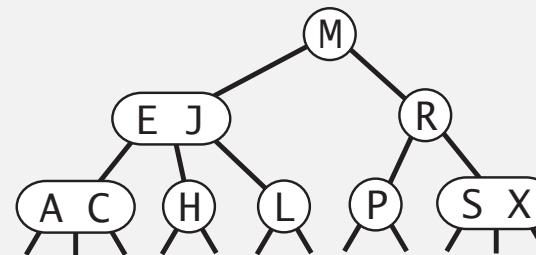
red–black tree



horizontal red links



2-3 tree

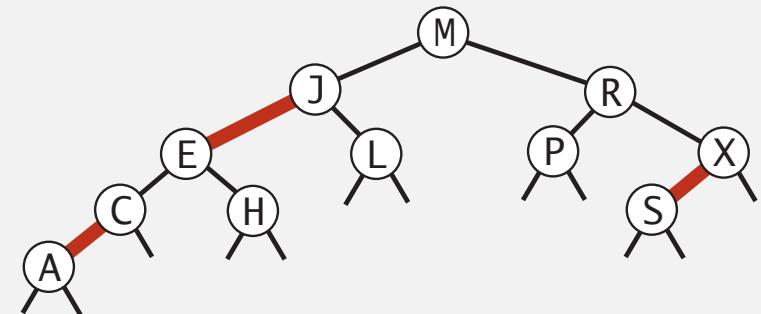


Search implementation for red-black BSTs

Observation. Search is the same as for elementary BST (ignore color).

but runs faster
because of better balance

```
public Val get(Key key)
{
    Node x = root;
    while (x != null)
    {
        int cmp = key.compareTo(x.key);
        if      (cmp < 0) x = x.left;
        else if (cmp > 0) x = x.right;
        else if (cmp == 0) return x.val;
    }
    return null;
}
```



Remark. Most other ops (e.g., floor, iteration, selection) are also identical.

Red-black BST representation

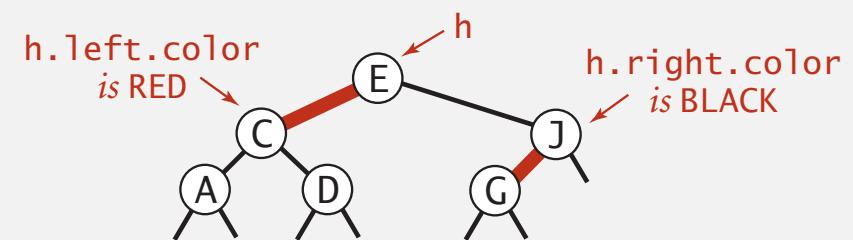
Each node is pointed to by precisely one link (from its parent) \Rightarrow
can encode color of links in nodes.

```
private static final boolean RED  = true;
private static final boolean BLACK = false;
```

```
private class Node
{
    Key key;
    Value val;
    Node left, right;
    boolean color; // color of parent link
}
```

```
private boolean isRed(Node x)
{
    if (x == null) return false;
    return x.color == RED;
}
```

null links are black

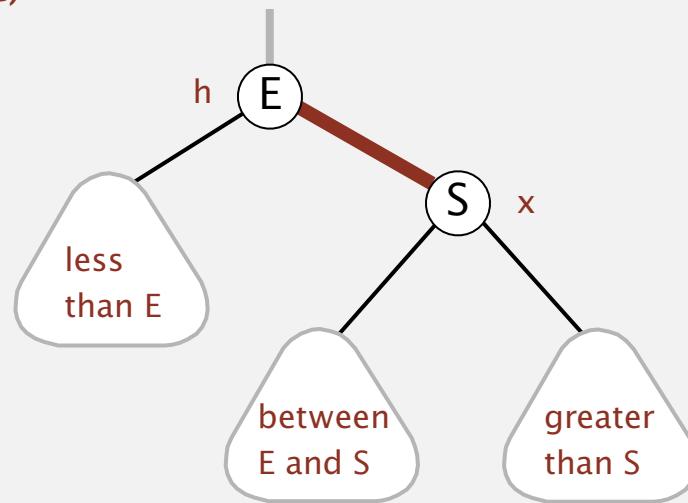


Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

rotate E left

(before)



```
private Node rotateLeft(Node h)
{
    assert isRed(h.right);
    Node x = h.right;
    h.right = x.left;
    x.left = h;
    x.color = h.color;
    h.color = RED;
    return x;
}
```

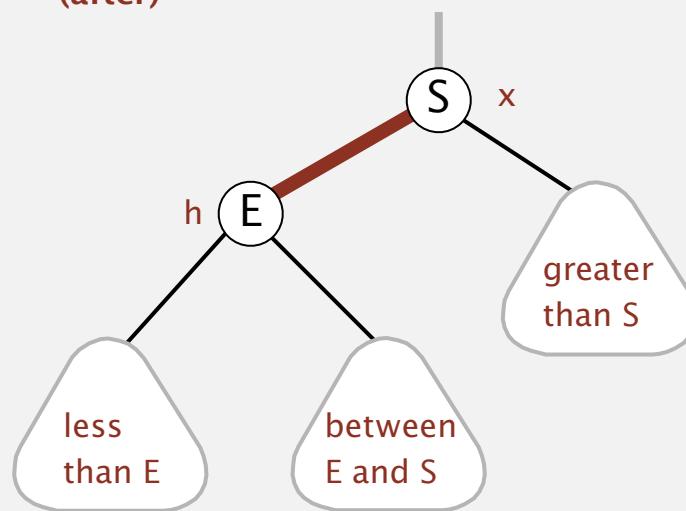
Invariants. Maintains symmetric order and perfect black balance.

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    h.right = x.left;
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    x.color = h.color;
    h.color = RED;
    return x;
}
```

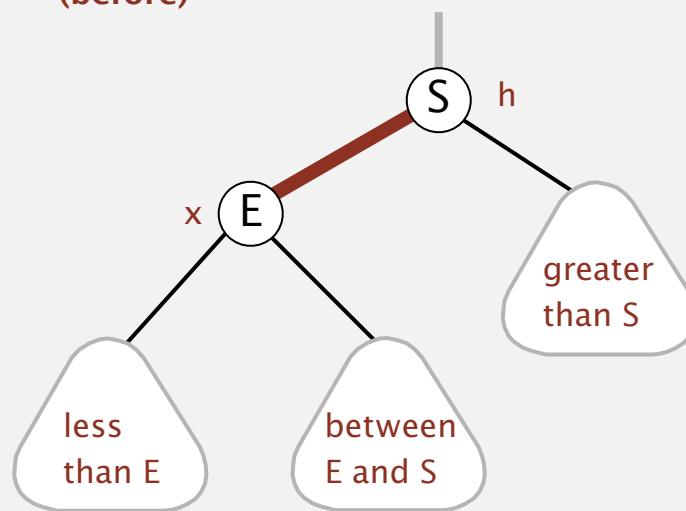
Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

rotate S right

(before)



```
private Node rotateRight(Node h)
{
    assert isRed(h.left);
    Node x = h.left;
    h.left = x.right;
    x.right = h;
    x.color = h.color;
    h.color = RED;
    return x;
}
```

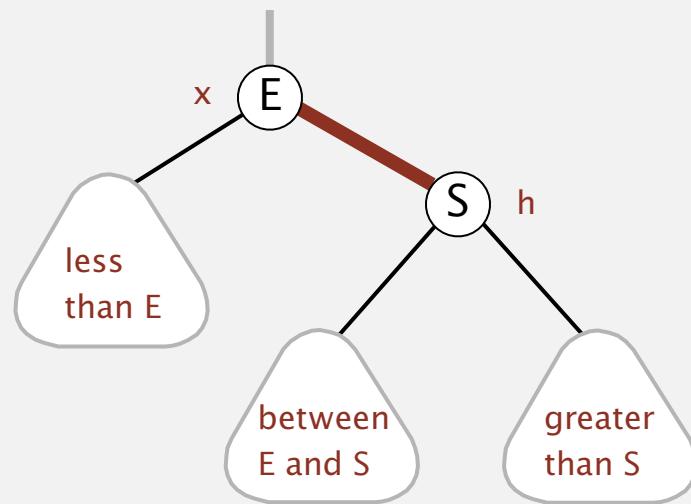
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Elementary red-black BST operations

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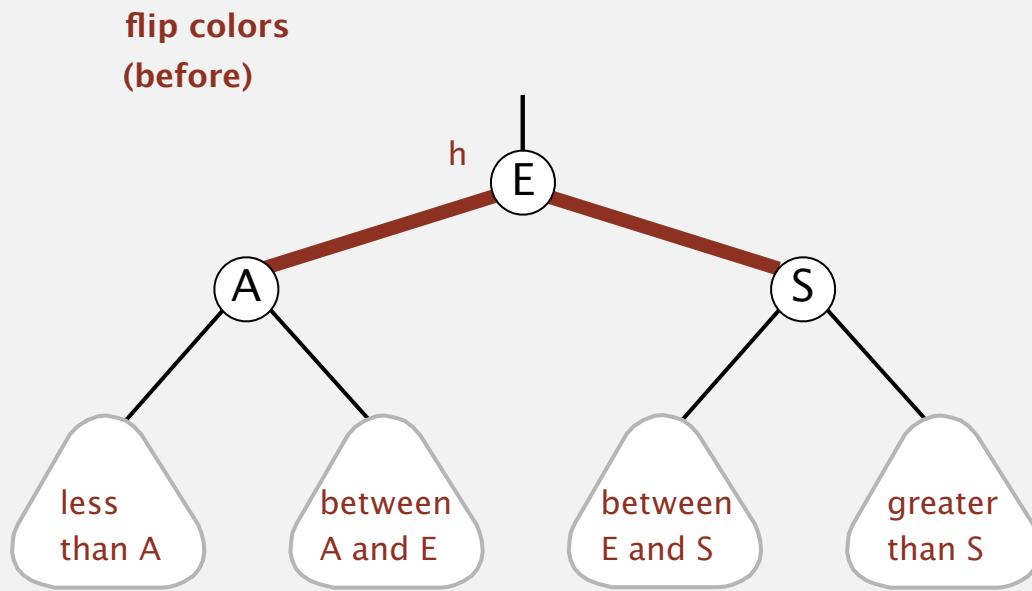


```
private Node rotateRight(Node h)
{
    assert isRed(h.left);
    Node x = h.left;
    h.left = x.right;
    x.right = h;
    x.color = h.color;
    h.color = RED;
    return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

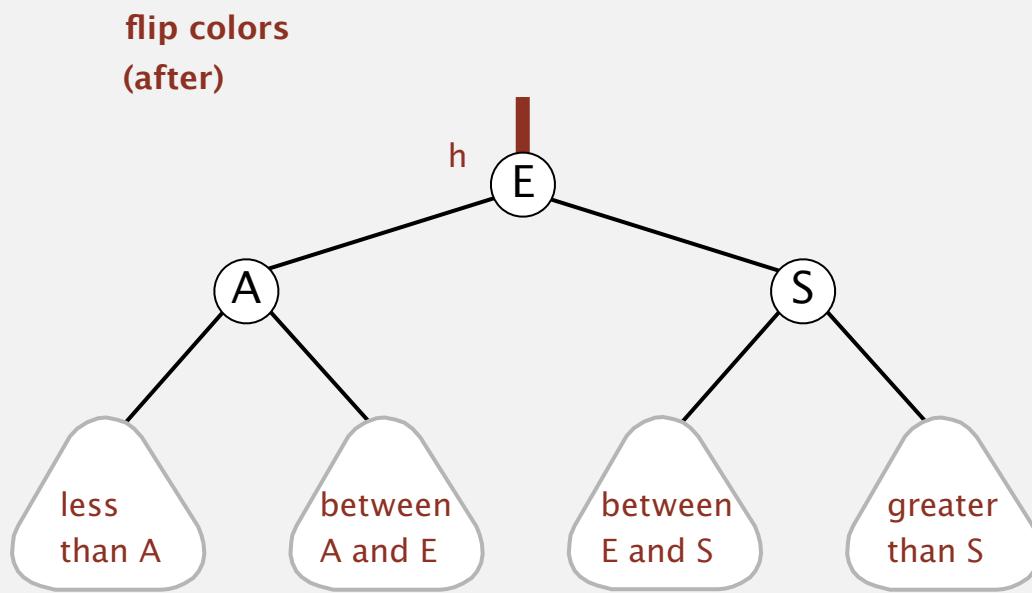


```
private void flipColors(Node h)
{
    assert !isRed(h);
    assert isRed(h.left);
    assert isRed(h.right);
    h.color = RED;
    h.left.color = BLACK;
    h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

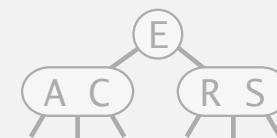
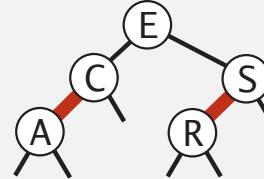
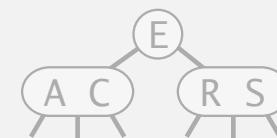
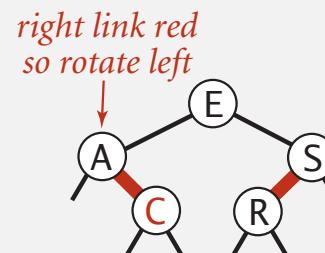
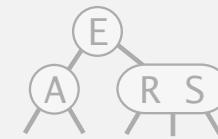
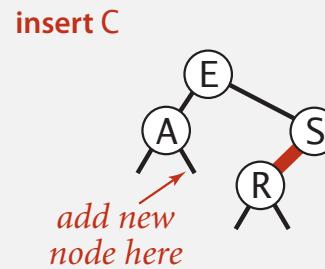


```
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    assert isRed(h.left);
    assert isRed(h.right);
    h.color = RED;
    h.left.color = BLACK;
    h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

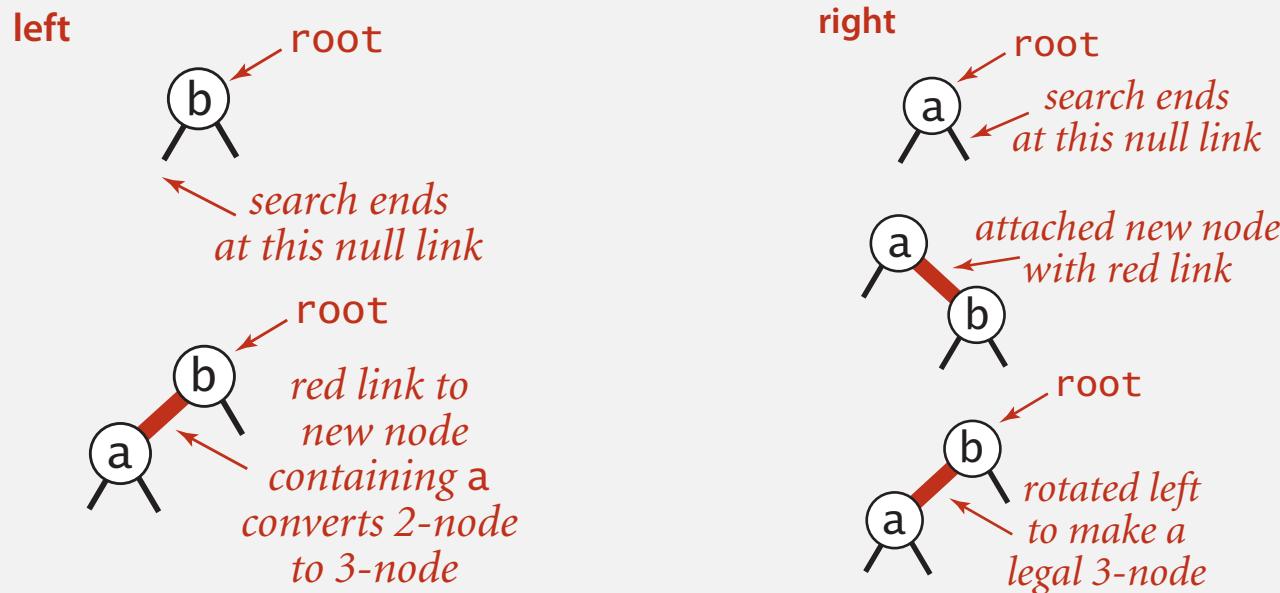
Insertion in a LLRB tree: overview

Basic strategy. Maintain 1-1 correspondence with 2-3 trees by applying elementary red-black BST operations.



Insertion in a LLRB tree

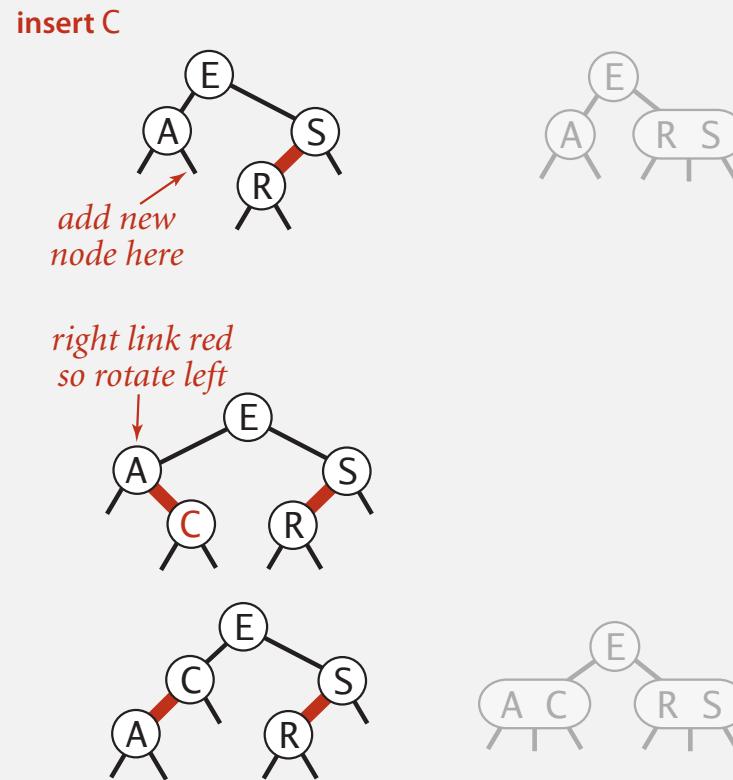
Warmup 1. Insert into a tree with exactly 1 node.



Insertion in a LLRB tree

Case 1. Insert into a 2-node at the bottom.

- Do standard BST insert; color new link red.
- If new red link is a right link, rotate left.



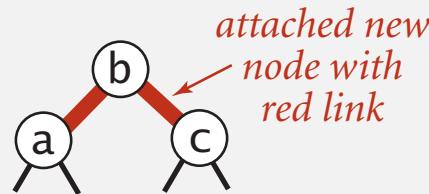
Insertion in a LLRB tree

Warmup 2. Insert into a tree with exactly 2 nodes.

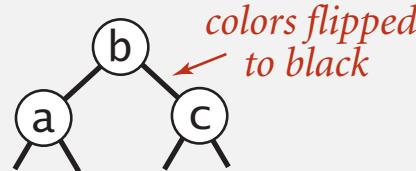
larger



search ends
at this
null link

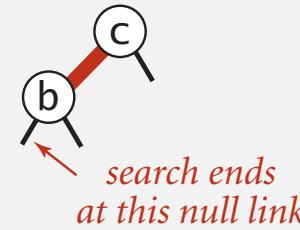


attached new
node with
red link

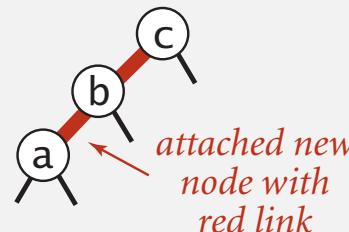


colors flipped
to black

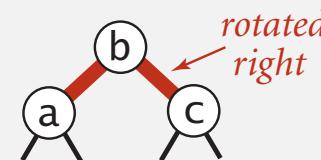
smaller



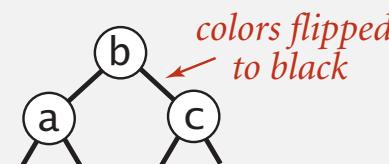
search ends
at this null link



attached new
node with
red link

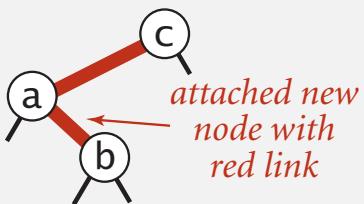


rotated
right

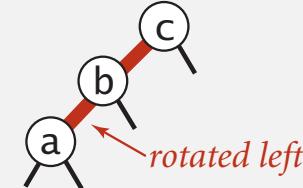


colors flipped
to black

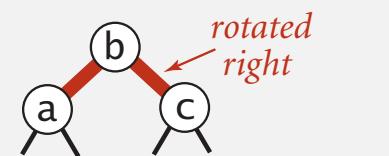
between



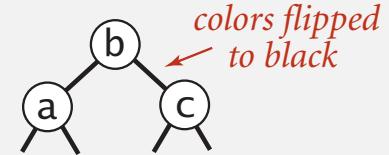
attached new
node with
red link



rotated left



rotated right

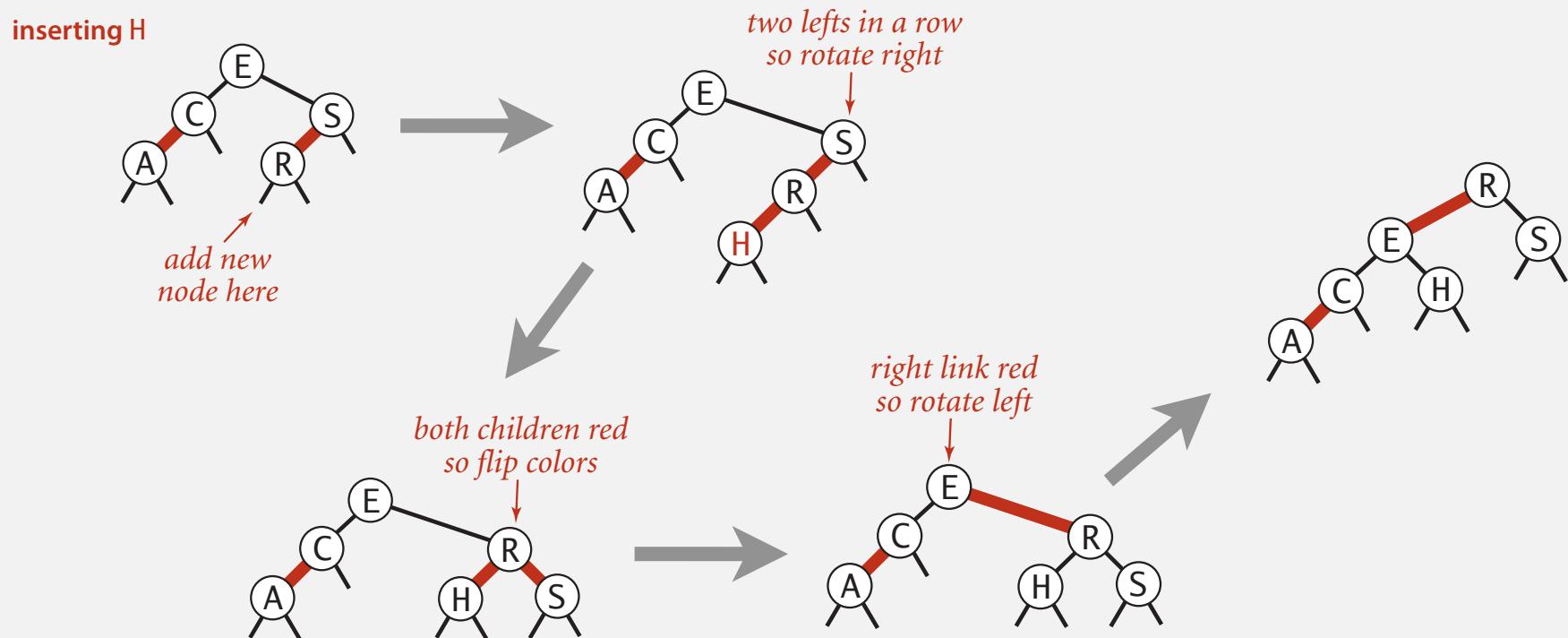


colors flipped
to black

Insertion in a LLRB tree

Case 2. Insert into a 3-node at the bottom.

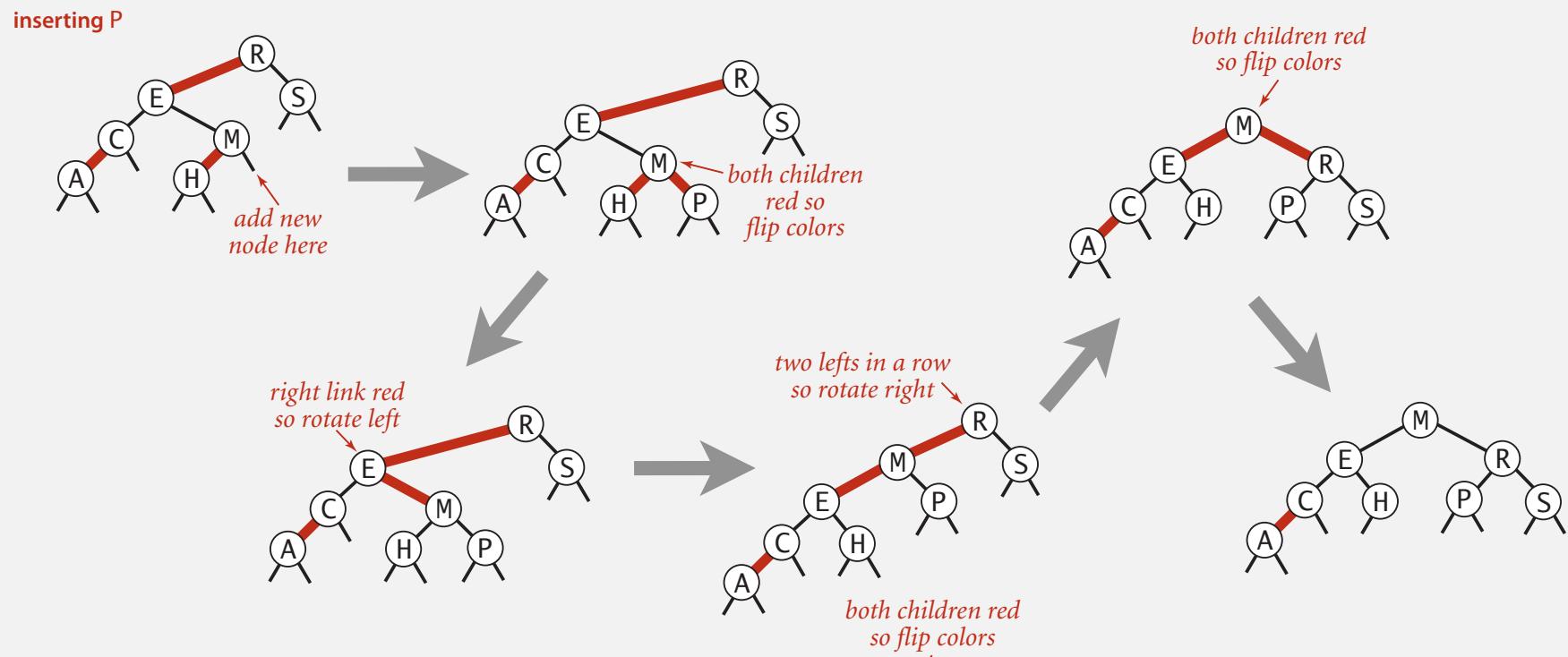
- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).



Insertion in a LLRB tree: passing red links up the tree

Case 2. Insert into a 3-node at the bottom.

- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).
- Repeat case 1 or case 2 up the tree (if needed).



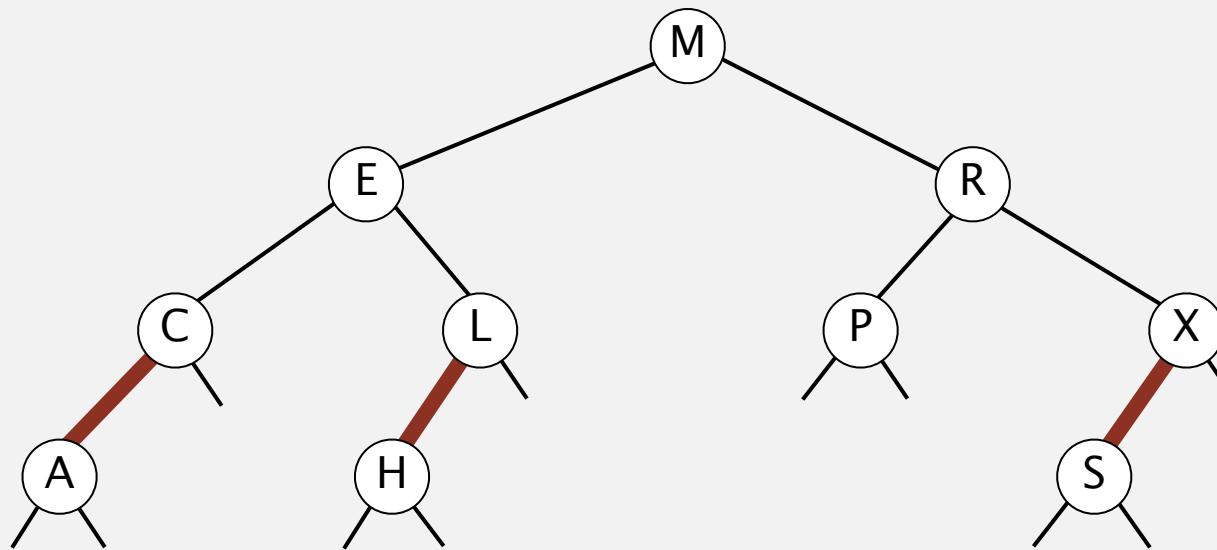
Red-black BST construction demo

insert S



Red-black BST construction demo

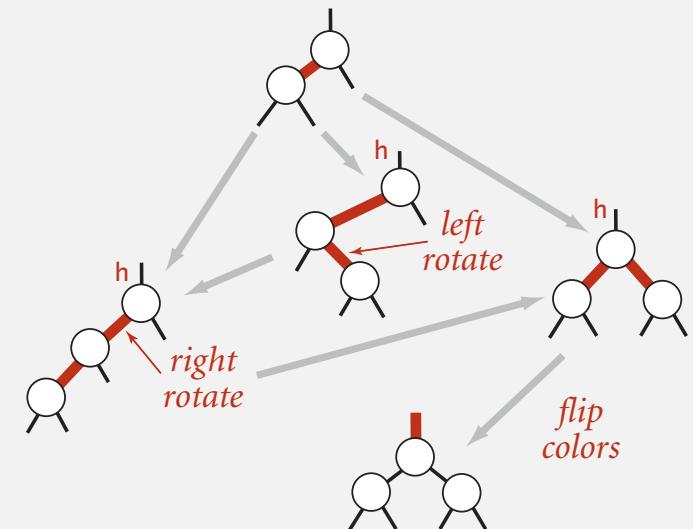
red-black BST



Insertion in a LLRB tree: Java implementation

Same code for all cases.

- Right child red, left child black: **rotate left**.
- Left child, left-left grandchild red: **rotate right**.
- Both children red: **flip colors**.



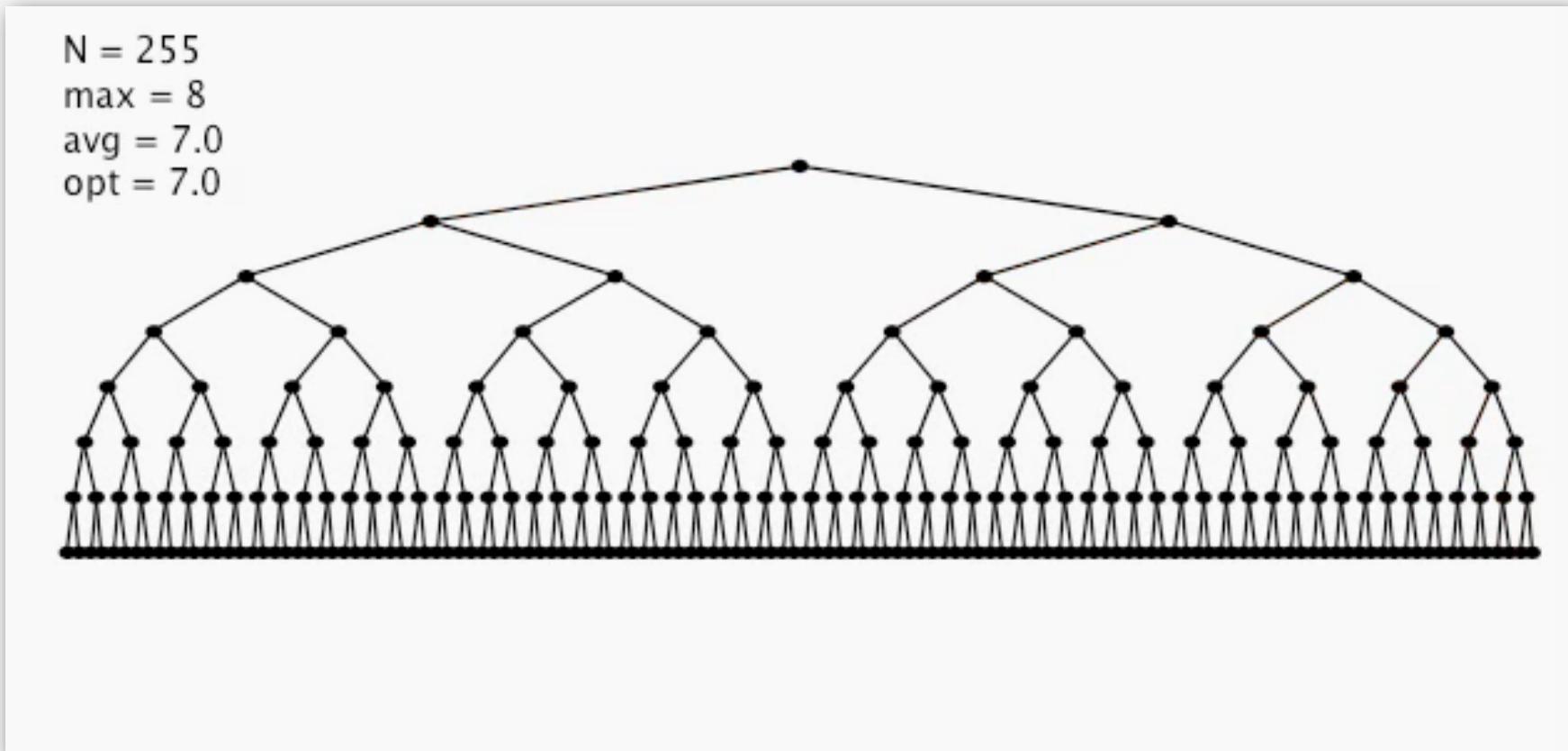
```
private Node put(Node h, Key key, Value val)
{
    if (h == null) return new Node(key, val, RED);           ← insert at bottom
    int cmp = key.compareTo(h.key);                           (and color it red)
    if      (cmp < 0) h.left  = put(h.left,  key, val);
    else if (cmp > 0) h.right = put(h.right, key, val);
    else if (cmp == 0) h.val  = val;

    if (isRed(h.right) && !isRed(h.left))     h = rotateLeft(h);   ← lean left
    if (isRed(h.left)  && isRed(h.left.left)) h = rotateRight(h);  ← balance 4-node
    if (isRed(h.left)  && isRed(h.right))       flipColors(h);    ← split 4-node

    return h;
}
```

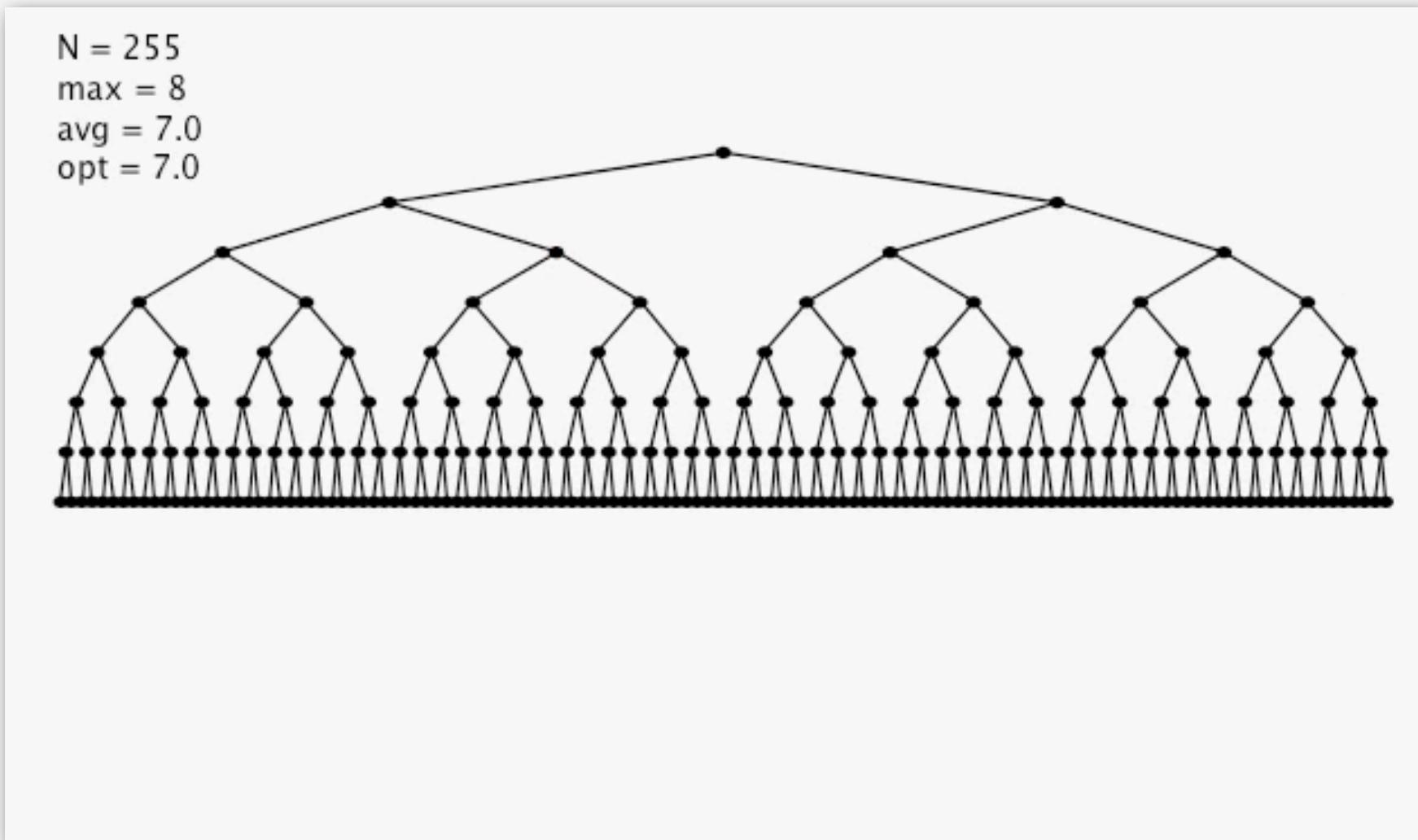
only a few extra lines of code provides near-perfect balance

Insertion in a LLRB tree: visualization

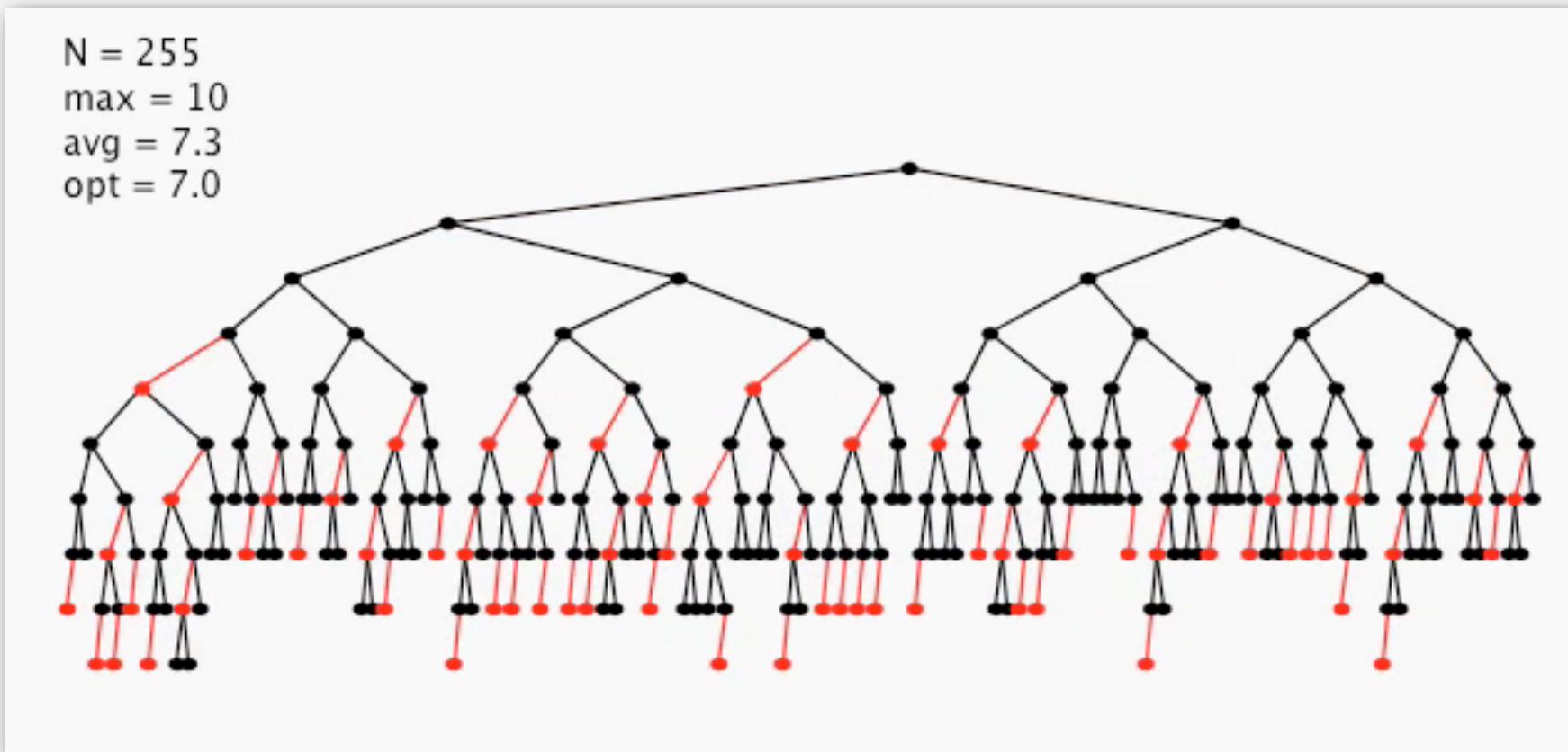


255 insertions in ascending order

Insertion in a LLRB tree: visualization



Insertion in a LLRB tree: visualization



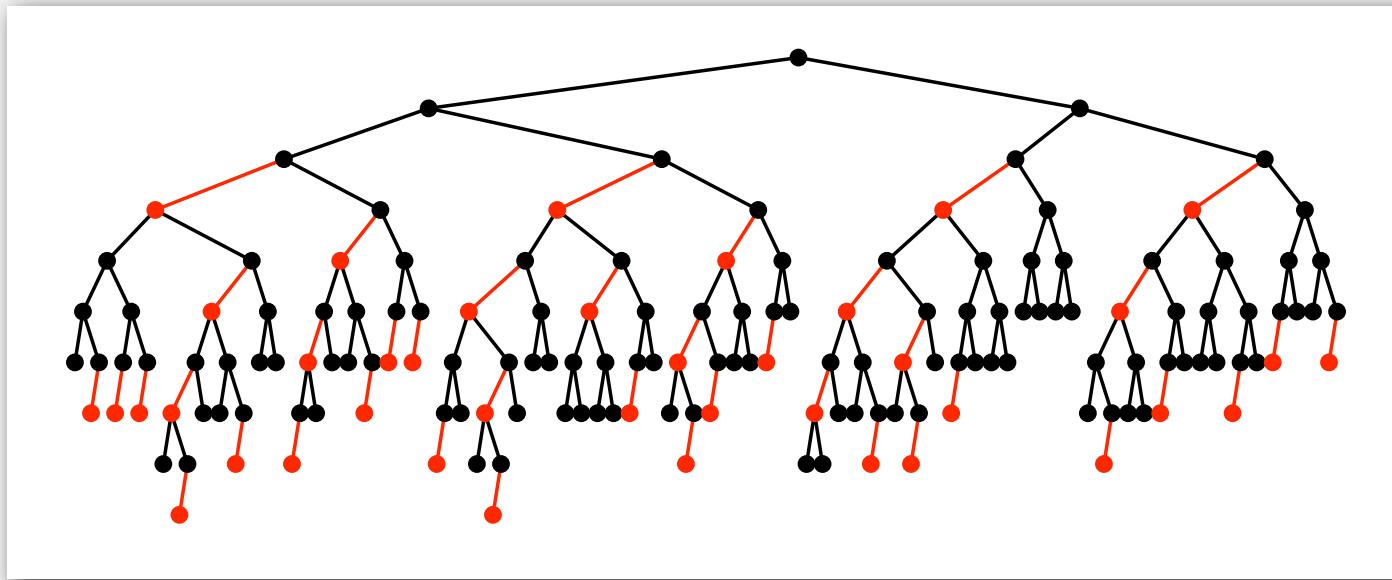
255 random insertions

Balance in LLRB trees

Proposition. Height of tree is $\leq 2 \lg N$ in the worst case.

Pf.

- Every path from root to null link has same number of black links.
- Never two red links in-a-row.



Property. Height of tree is $\sim 1.00 \lg N$ in typical applications.

ST implementations: summary

implementation	worst-case cost (after N inserts)			average case (after N random inserts)			ordered iteration?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	N	N	N	N/2	N	N/2	no	equals()
binary search (ordered array)	$\lg N$	N	N	$\lg N$	N/2	N/2	yes	compareTo()
BST	N	N	N	$1.39 \lg N$	$1.39 \lg N$?	yes	compareTo()
2-3 tree	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	yes	compareTo()
red-black BST	$2 \lg N$	$2 \lg N$	$2 \lg N$	$1.00 \lg N^*$	$1.00 \lg N^*$	$1.00 \lg N^*$	yes	compareTo()

* exact value of coefficient unknown but extremely close to 1

War story: why red-black?

Xerox PARC innovations. [1970s]

- Alto.
- GUI.
- Ethernet.
- Smalltalk.
- InterPress.
- Laser printing.
- Bitmapped display.
- WYSIWYG text editor.
- ...

XEROX®

PARC



Xerox Alto

A DICHROMATIC FRAMEWORK FOR BALANCED TREES

Leo J. Guibas
*Xerox Palo Alto Research Center,
Palo Alto, California, and
Carnegie-Mellon University*

and

Robert Sedgewick*
*Program in Computer Science
Brown University
Providence, R. I.*

ABSTRACT

In this paper we present a uniform framework for the implementation and study of balanced tree algorithms. We show how to imbed in this

the way down towards a leaf. As we will see, this has a number of significant advantages over the older methods. We shall examine a number of variations on a common theme and exhibit full implementations which are notable for their brevity. One implementation is examined carefully, and some properties about its

War story: red-black BSTs

Telephone company contracted with database provider to build real-time database to store customer information.

Database implementation.

- Red-black BST search and insert; Hibbard deletion.
- Exceeding height limit of 80 triggered error-recovery process.

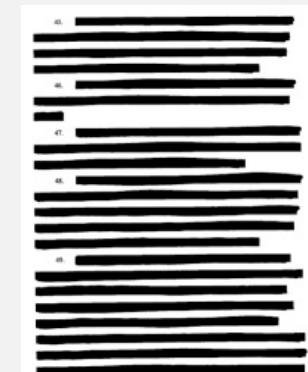
allows for up to 2^{40} keys

Extended telephone service outage.

- Main cause = height bounded exceeded!
- Telephone company sues database provider.
- Legal testimony:

“If implemented properly, the height of a red-black BST with N keys is at most $2 \lg N$. ” — expert witness

Hibbard deletion
was the problem



Algorithms

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3.3 BALANCED SEARCH TREES

- ▶ *2-3 search trees*
- ▶ *red-black BSTs*
- ▶ *B-trees*

Algorithms

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3.3 BALANCED SEARCH TREES

- ▶ *2-3 search trees*
- ▶ *red-black BSTs*
- ▶ *B-trees*

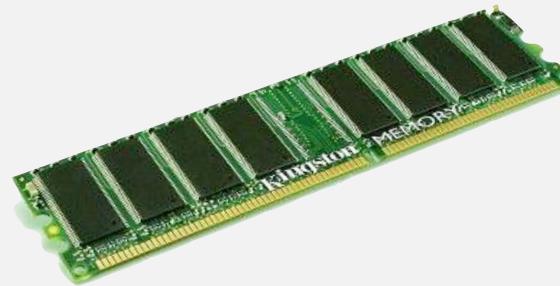
File system model

Page. Contiguous block of data (e.g., a file or 4,096-byte chunk).

Probe. First access to a page (e.g., from disk to memory).



slow



fast

Property. Time required for a probe is much larger than time to access data within a page.

Cost model. Number of probes.

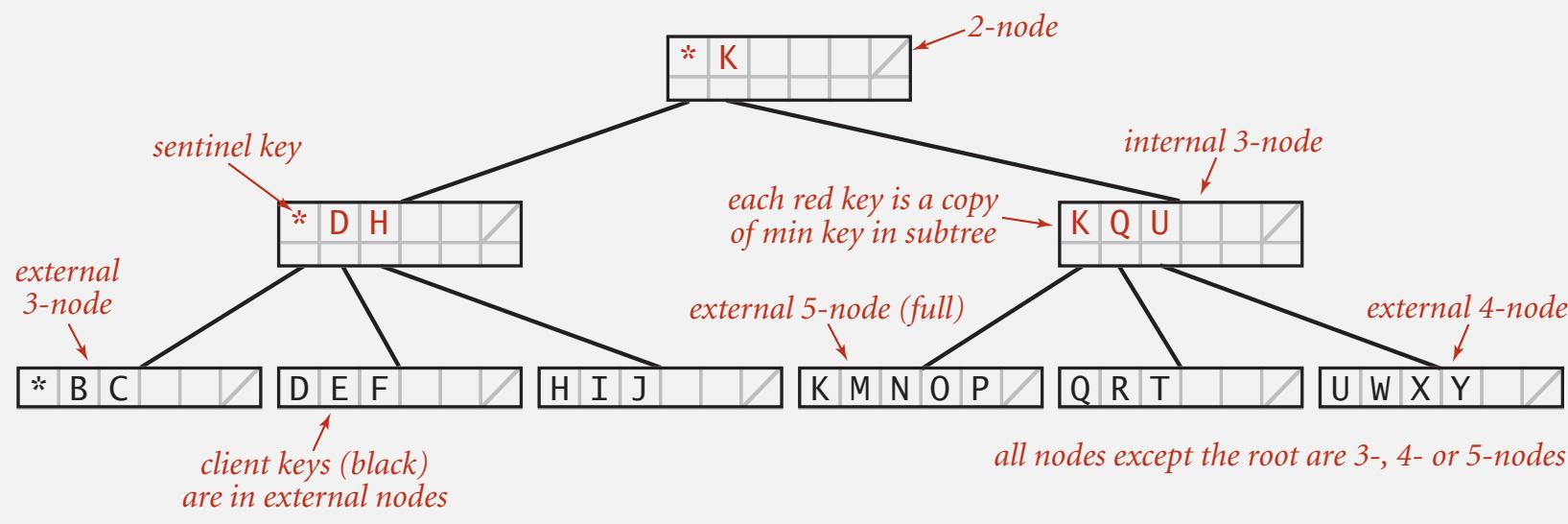
Goal. Access data using minimum number of probes.

B-trees (Bayer-McCreight, 1972)

B-tree. Generalize 2-3 trees by allowing up to $M - 1$ key-link pairs per node.

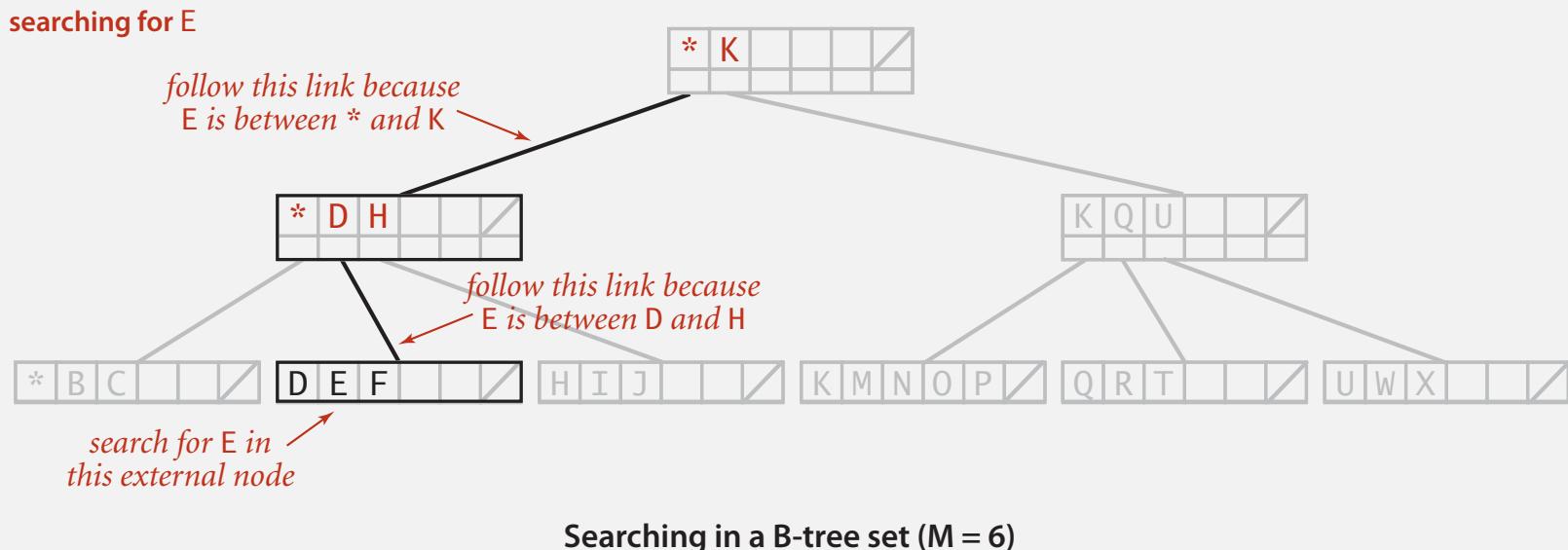
- At least 2 key-link pairs at root.
- At least $M/2$ key-link pairs in other nodes.
- External nodes contain client keys.
- Internal nodes contain copies of keys to guide search.

choose M as large as possible so that M links fit in a page, e.g., $M = 1024$



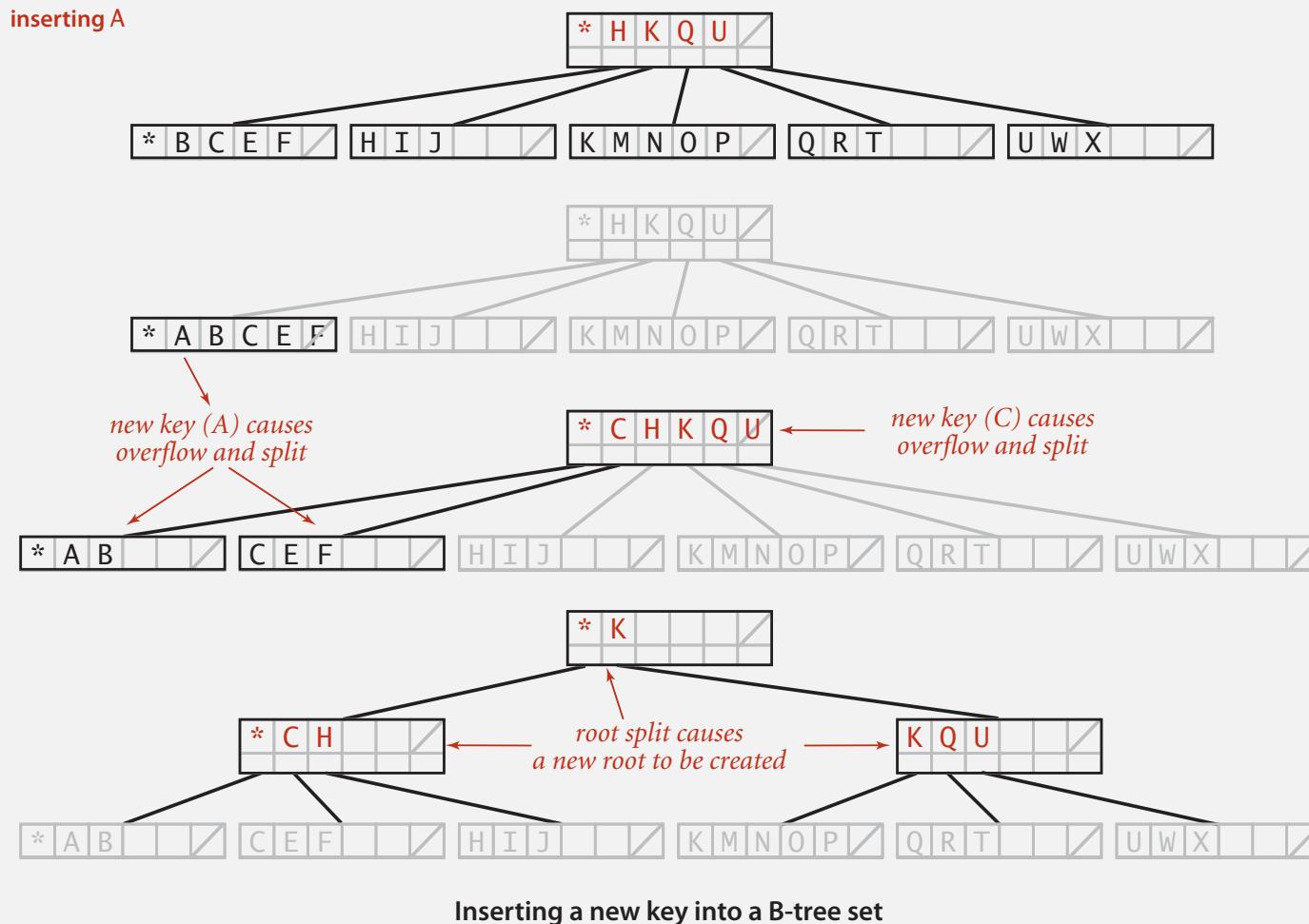
Searching in a B-tree

- Start at root.
- Find interval for search key and take corresponding link.
- Search terminates in external node.



Insertion in a B-tree

- Search for new key.
- Insert at bottom.
- Split nodes with M key-link pairs on the way up the tree.



Balance in B-tree

Proposition. A search or an insertion in a B-tree of order M with N keys requires between $\log_{M-1} N$ and $\log_{M/2} N$ probes.

Pf. All internal nodes (besides root) have between $M/2$ and $M - 1$ links.

In practice. Number of probes is at most 4. $\leftarrow M = 1024; N = 62 \text{ billion}$
 $\log_{M/2} N \leq 4$

Optimization. Always keep root page in memory.

Building a large B tree



Balanced trees in the wild

Red-black trees are widely used as system symbol tables.

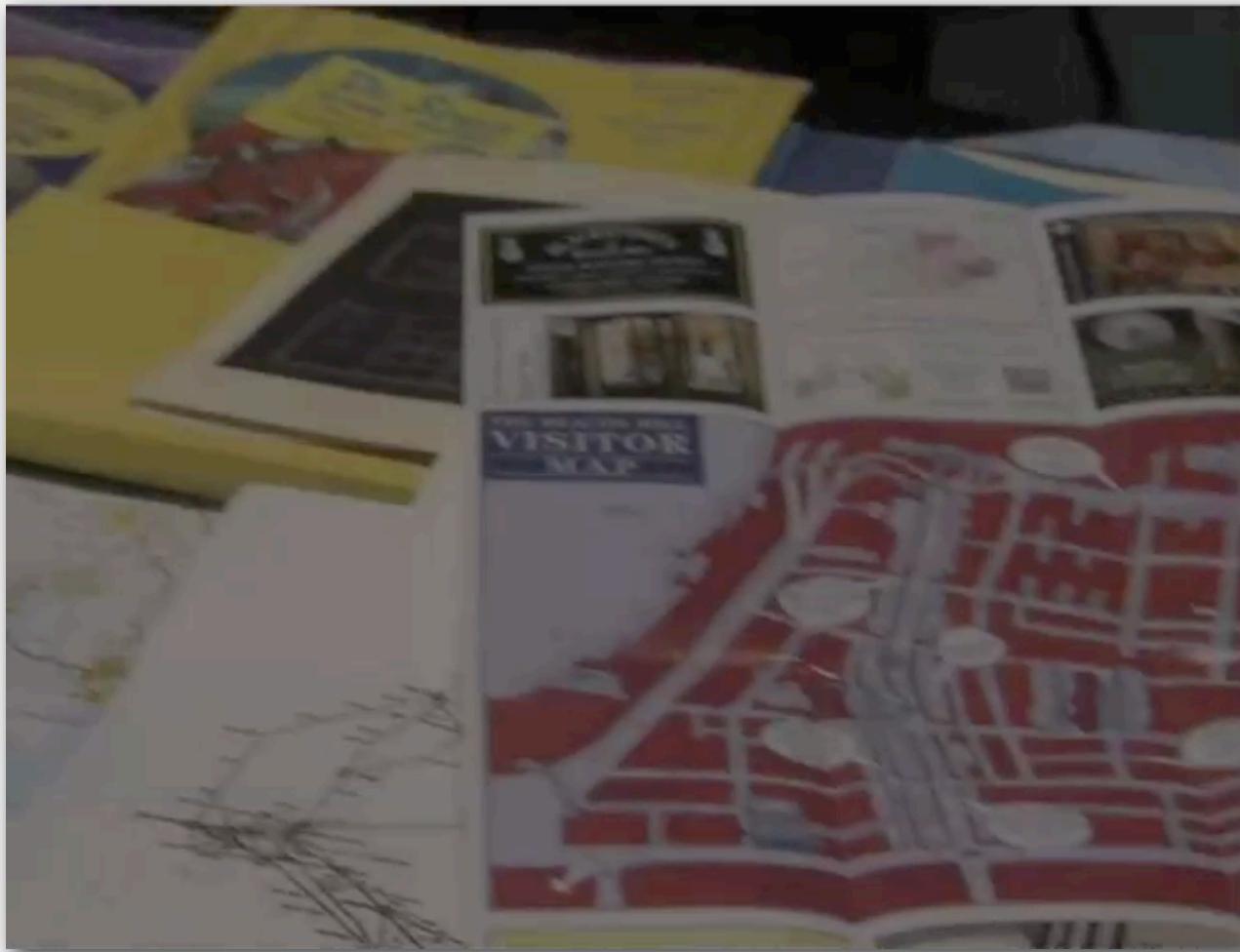
- Java: `java.util.TreeMap`, `java.util.TreeSet`.
- C++ STL: `map`, `multimap`, `multiset`.
- Linux kernel: completely fair scheduler, `linux/rbtree.h`.
- Emacs: conservative stack scanning.

B-tree variants. B+ tree, B*tree, B# tree, ...

B-trees (and variants) are widely used for file systems and databases.

- Windows: NTFS.
- Mac: HFS, HFS+.
- Linux: ReiserFS, XFS, Ext3FS, JFS.
- Databases: ORACLE, DB2, INGRES, SQL, PostgreSQL.

Red-black BSTs in the wild



*Common sense. Sixth sense.
Together they're the
FBI's newest team.*

Red-black BSTs in the wild

ACT FOUR

FADE IN:

48 INT. FBI HQ - NIGHT

48

Antonio is at THE COMPUTER as Jess explains herself to Nicole and Pollock. The CONFERENCE TABLE is covered with OPEN REFERENCE BOOKS, TOURIST GUIDES, MAPS and REAMS OF PRINTOUTS.

JESS

It was the red door again.

POLLOCK

I thought the red door was the storage container.

JESS

But it wasn't red anymore. It was black.

ANTONIO

So red turning to black means... what?

POLLOCK

Budget deficits? Red ink, black ink?

NICOLE

Yes. I'm sure that's what it is. But maybe we should come up with a couple other options, just in case.

Antonio refers to his COMPUTER SCREEN, which is filled with mathematical equations.

ANTONIO

It could be an algorithm from a binary search tree. A red-black tree tracks every simple path from a node to a descendant leaf with the same number of black nodes.

JESS

Does that help you with girls?

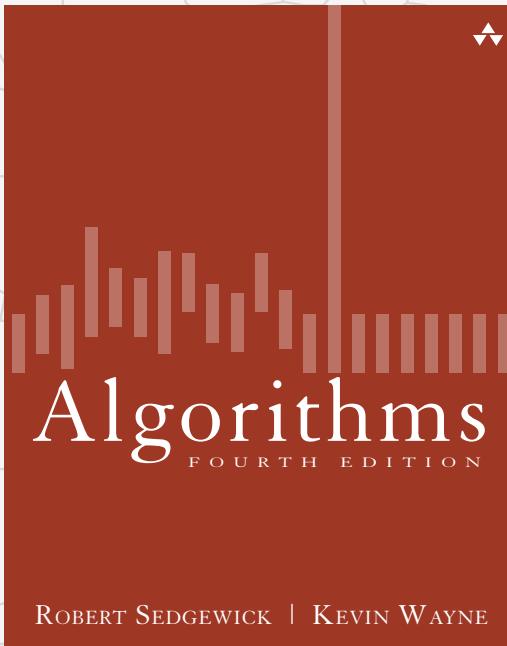
Algorithms

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3.3 BALANCED SEARCH TREES

- ▶ *2-3 search trees*
- ▶ *red-black BSTs*
- ▶ *B-trees*



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3.3 BALANCED SEARCH TREES

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Algorithms

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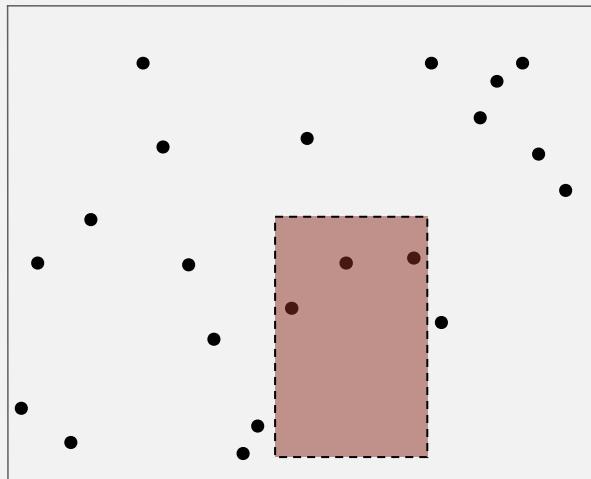
<http://algs4.cs.princeton.edu>

GEOMETRIC APPLICATIONS OF BSTs

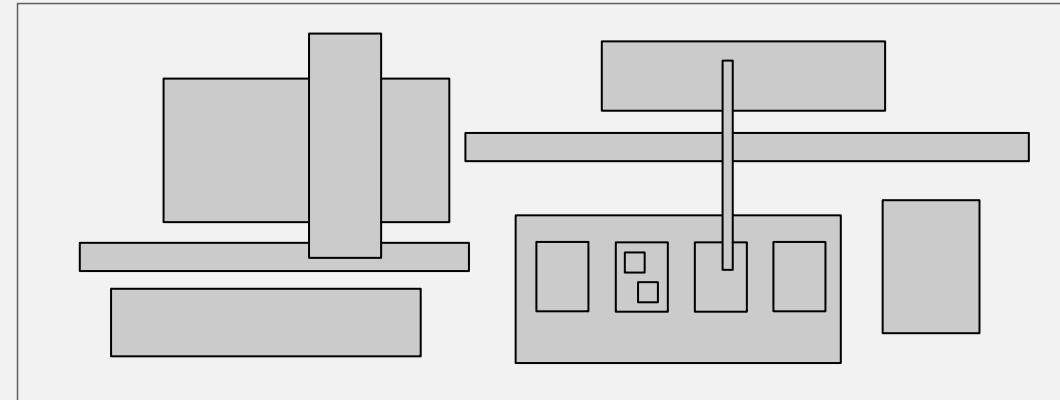
- ▶ *1d range search*
- ▶ *line segment intersection*
- ▶ *kd trees*
- ▶ *interval search trees*
- ▶ *rectangle intersection*

Overview

This lecture. Intersections among **geometric objects**.



2d orthogonal range search



orthogonal rectangle intersection

Applications. CAD, games, movies, virtual reality, databases, GIS,

Efficient solutions. **Binary search trees** (and extensions).

Algorithms

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GEOMETRIC APPLICATIONS OF BSTs

- ▶ *1d range search*
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- ▶ *rectangle intersection*

1d range search

Extension of ordered symbol table.

- Insert key-value pair.
- Search for key k .
- Delete key k .
- Range search: find all keys between k_1 and k_2 .
- Range count: number of keys between k_1 and k_2 .

Application. Database queries.

Geometric interpretation.

- Keys are point on a line.
- Find/count points in a given 1d interval.

• • • • [• • •] • • • •

insert B	B
insert D	B D
insert A	A B D
insert I	A B D I
insert H	A B D H I
insert F	A B D F H I
insert P	A B D F H I P
count G to K	2
search G to K	H I

1d range search: elementary implementations

Unordered list. Fast insert, slow range search.

Ordered array. Slow insert, binary search for k_1 and k_2 to do range search.

order of growth of running time for 1d range search

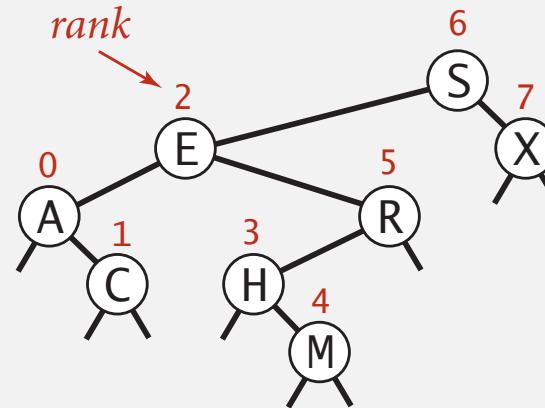
data structure	insert	range count	range search
unordered list	1	N	N
ordered array	N	$\log N$	$R + \log N$
goal	$\log N$	$\log N$	$R + \log N$

N = number of keys

R = number of keys that match

1d range count: BST implementation

1d range count. How many keys between l_o and h_i ?



```
public int size(Key lo, Key hi)
{
    if (contains(hi)) return rank(hi) - rank(lo) + 1;
    else                  return rank(hi) - rank(lo);
}
```

number of keys < h_i

Proposition. Running time proportional to $\log N$.

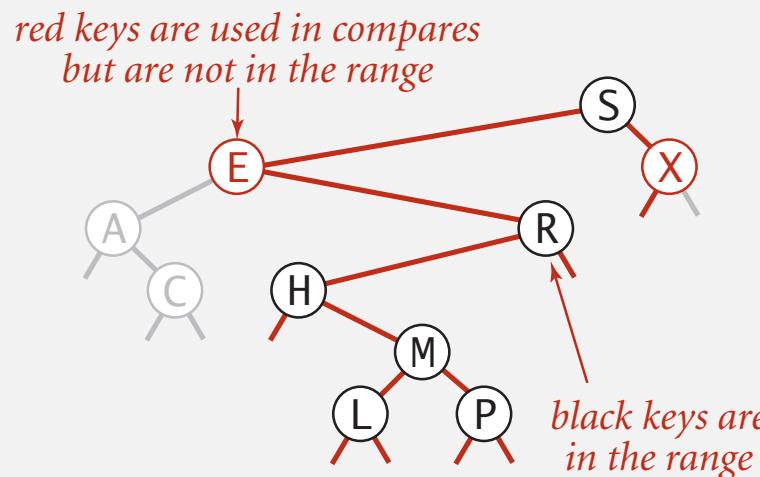
Pf. Nodes examined = search path to l_o + search path to h_i .

1d range search: BST implementation

1d range search. Find all keys between l_0 and h_i .

- Recursively find all keys in left subtree (if any could fall in range).
- Check key in current node.
- Recursively find all keys in right subtree (if any could fall in range).

searching in the range $[F .. T]$



Proposition. Running time proportional to $R + \log N$.

Pf. Nodes examined = search path to l_0 + search path to h_i + matches.

Algorithms

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GEOMETRIC APPLICATIONS OF BSTs

- ▶ *1d range search*
- ▶ *line segment intersection*
- ▶ *kd trees*
- ▶ *interval search trees*
- ▶ *rectangle intersection*

Algorithms

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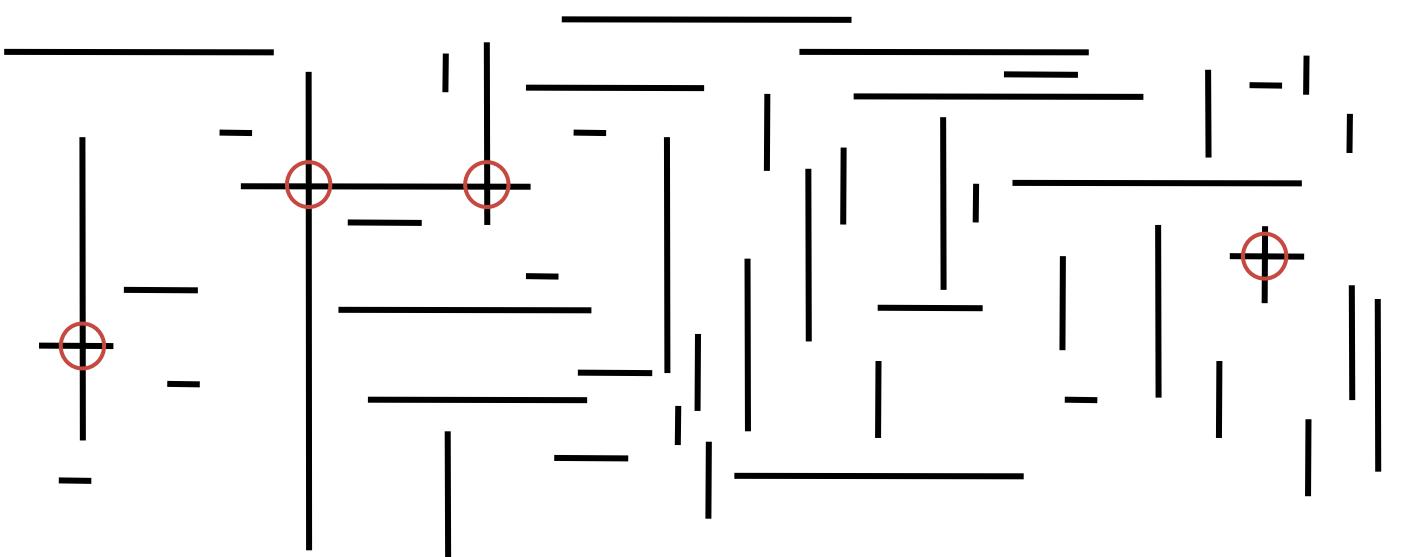
<http://algs4.cs.princeton.edu>

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Orthogonal line segment intersection

Given N horizontal and vertical line segments, find all intersections.



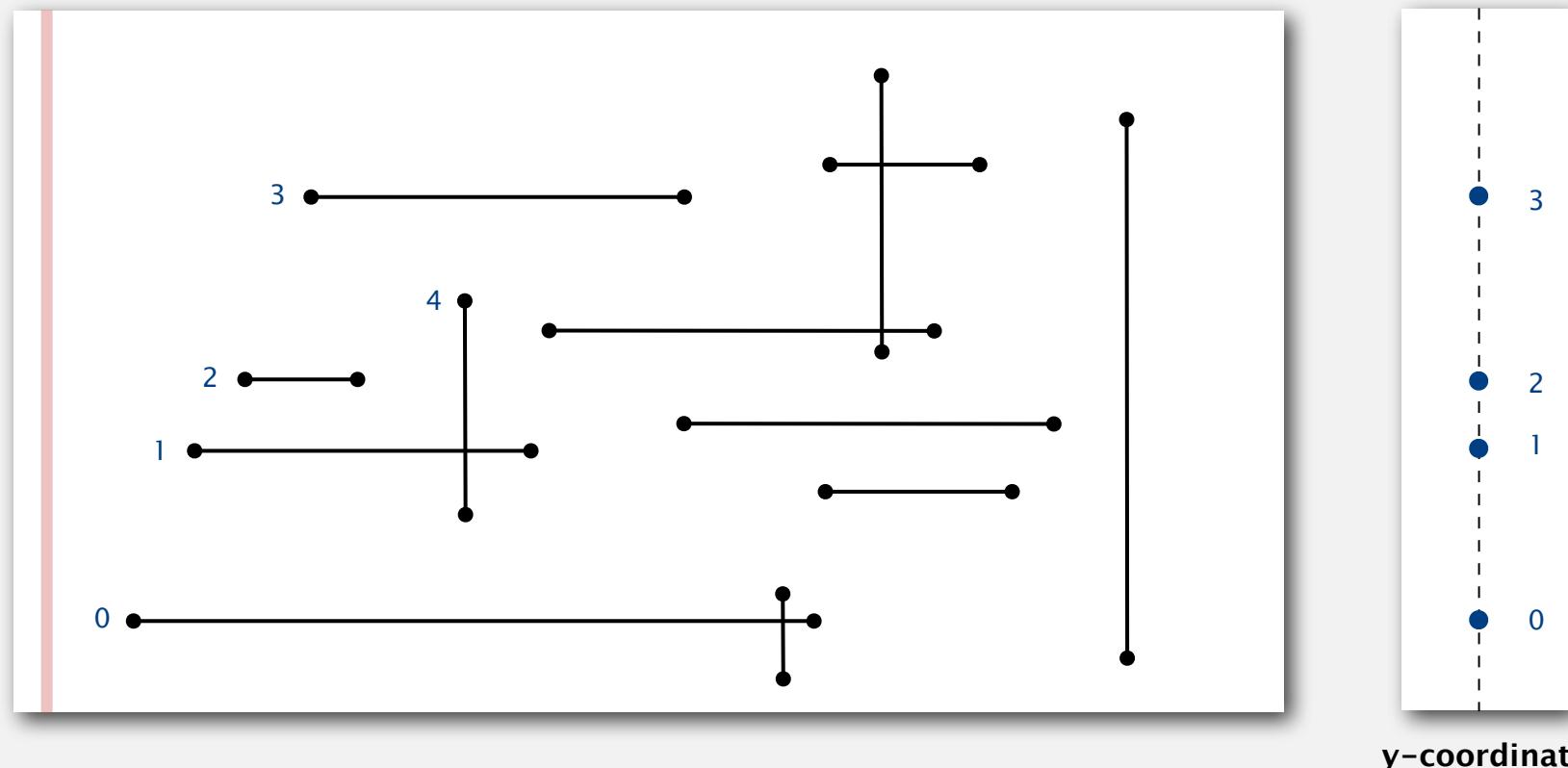
Quadratic algorithm. Check all pairs of line segments for intersection.

Nondegeneracy assumption. All x - and y -coordinates are distinct.

Orthogonal line segment intersection: sweep-line algorithm

Sweep vertical line from left to right.

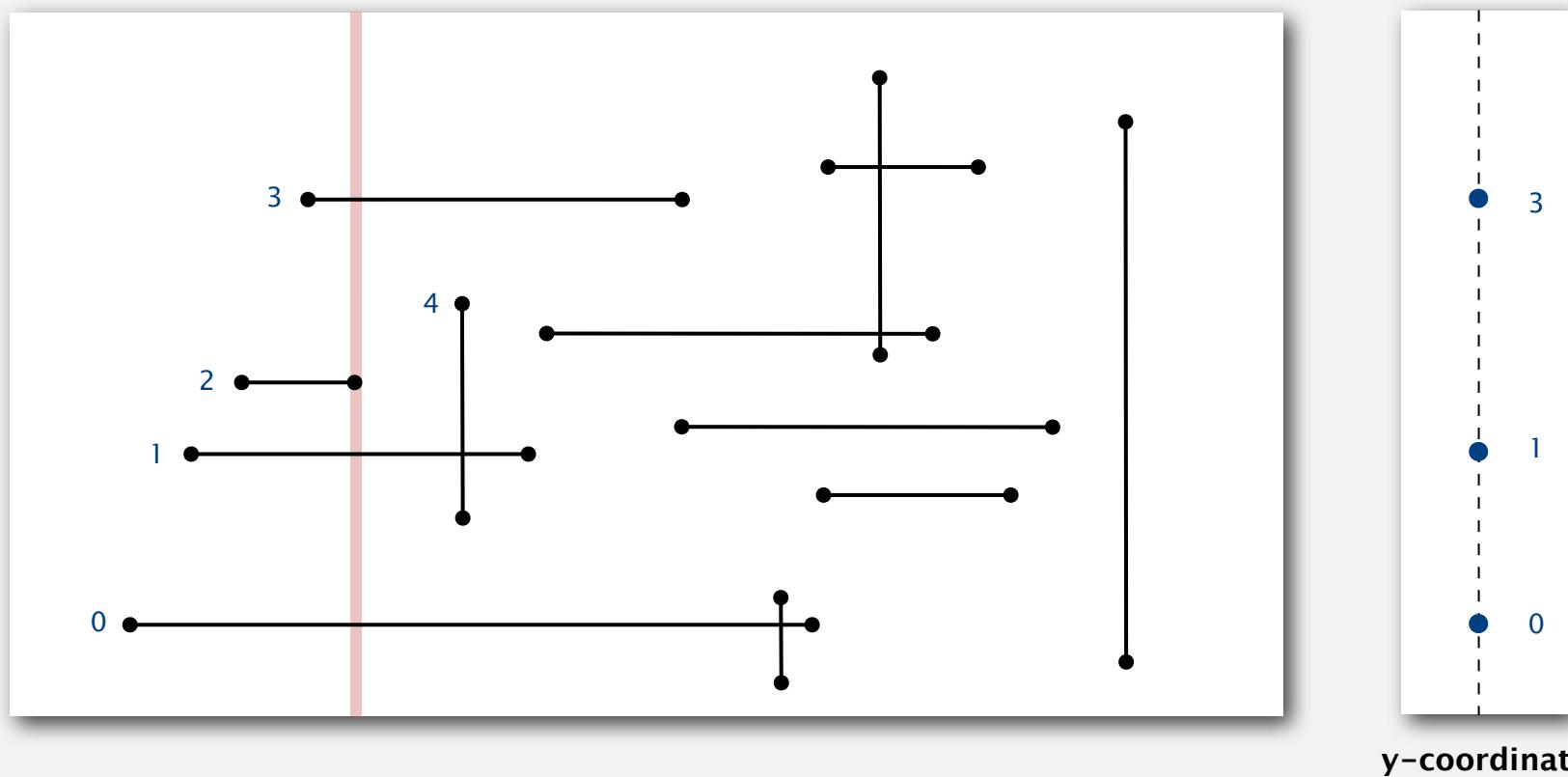
- x -coordinates define events.
- h -segment (left endpoint): insert y -coordinate into BST.



Orthogonal line segment intersection: sweep-line algorithm

Sweep vertical line from left to right.

- x -coordinates define events.
- h -segment (left endpoint): insert y -coordinate into BST.
- h -segment (right endpoint): remove y -coordinate from BST.

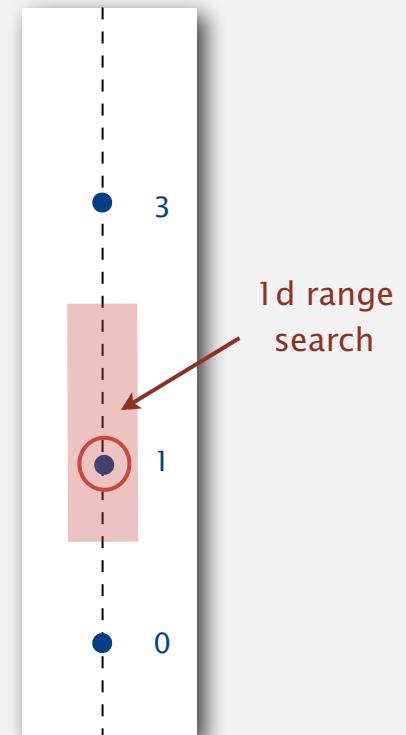
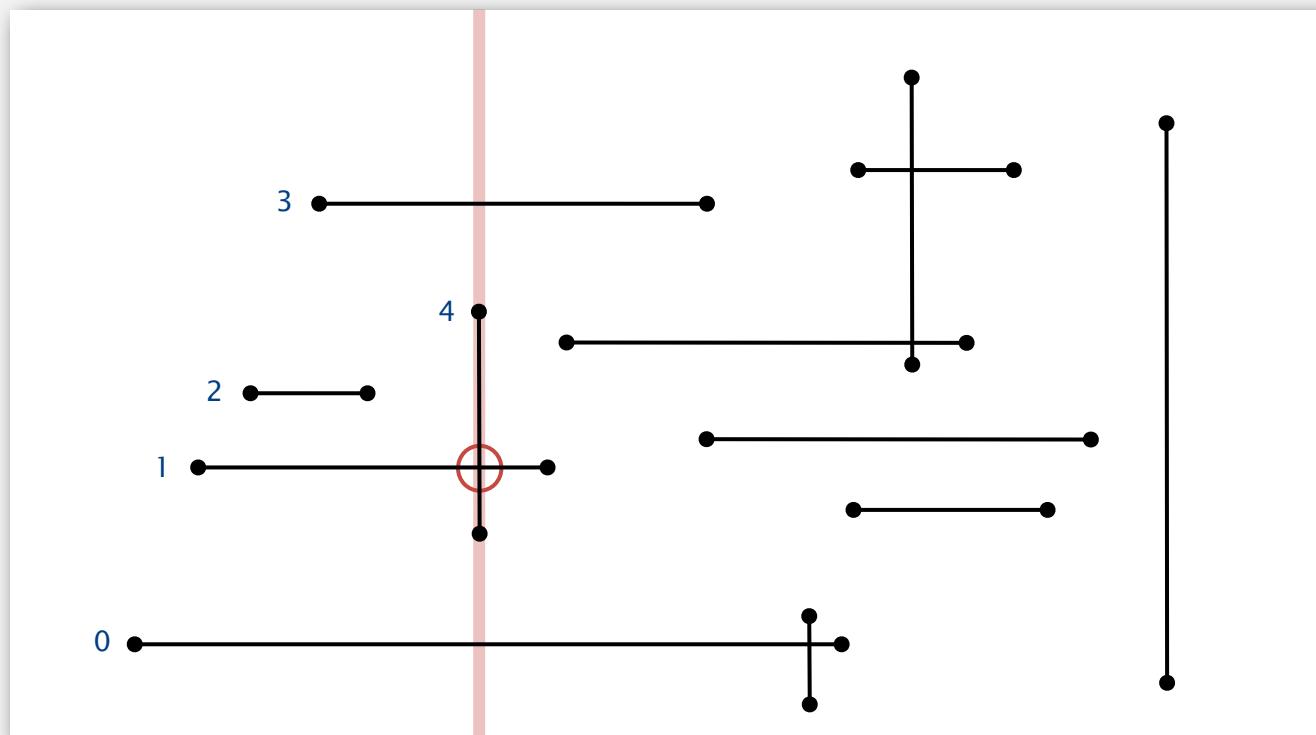


y-coordinates

Orthogonal line segment intersection: sweep-line algorithm

Sweep vertical line from left to right.

- x -coordinates define events.
- h -segment (left endpoint): insert y -coordinate into BST.
- h -segment (right endpoint): remove y -coordinate from BST.
- v -segment: range search for interval of y -endpoints.



y -coordinates

Orthogonal line segment intersection: sweep-line analysis

Proposition. The sweep-line algorithm takes time proportional to $N \log N + R$ to find all R intersections among N orthogonal line segments.

Pf.

- Put x -coordinates on a PQ (or sort). $\leftarrow N \log N$
- Insert y -coordinates into BST. $\leftarrow N \log N$
- Delete y -coordinates from BST. $\leftarrow N \log N$
- Range searches in BST. $\leftarrow N \log N + R$

Bottom line. Sweep line reduces 2d orthogonal line segment intersection search to 1d range search.

Algorithms

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GEOMETRIC APPLICATIONS OF BSTs

- ▶ *1d range search*
- ▶ *line segment intersection*
- ▶ *kd trees*
- ▶ *interval search trees*
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Algorithms

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GEOMETRIC APPLICATIONS OF BSTs

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- ▶ *interval search trees*
- ▶ *rectangle intersection*

2-d orthogonal range search

Extension of ordered symbol-table to 2d keys.

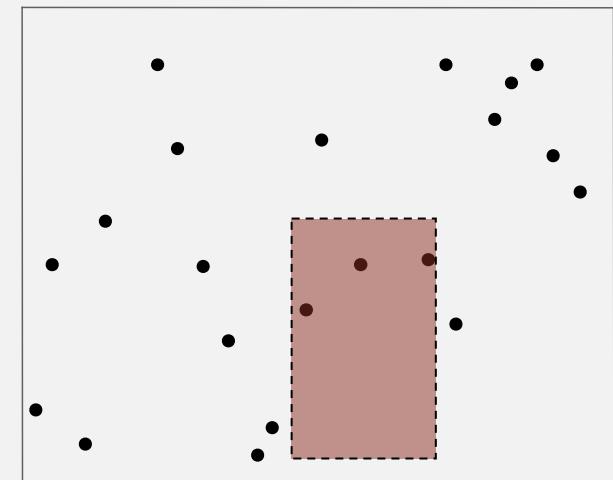
- Insert a 2d key.
- Delete a 2d key.
- Search for a 2d key.
- Range search: find all keys that lie in a 2d range.
- Range count: number of keys that lie in a 2d range.

Applications. Networking, circuit design, databases, ...

Geometric interpretation.

- Keys are point in the plane.
- Find/count points in a given *h-v* rectangle

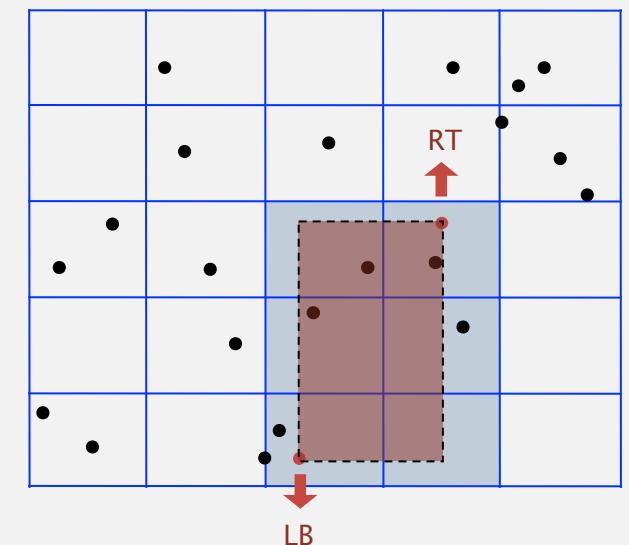
↑
rectangle is axis-aligned



2d orthogonal range search: grid implementation

Grid implementation.

- Divide space into M -by- M grid of squares.
- Create list of points contained in each square.
- Use 2d array to directly index relevant square.
- Insert: add (x, y) to list for corresponding square.
- Range search: examine only squares that intersect 2d range query.



2d orthogonal range search: grid implementation analysis

Space-time tradeoff.

- Space: $M^2 + N$.
- Time: $1 + N/M^2$ per square examined, on average.

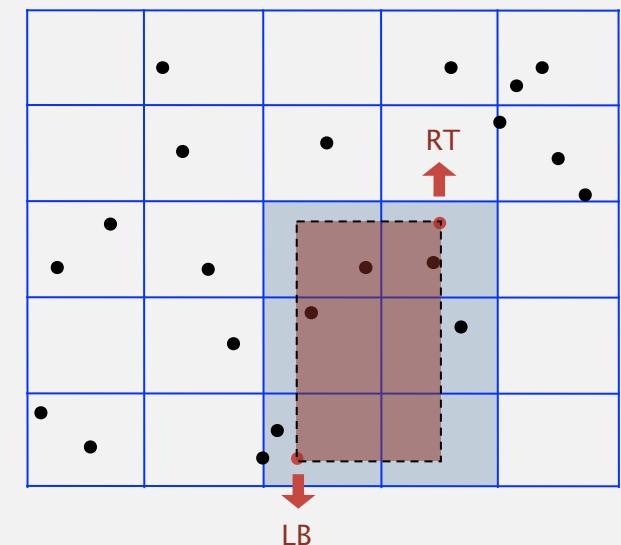
Choose grid square size to tune performance.

- Too small: wastes space.
- Too large: too many points per square.
- Rule of thumb: \sqrt{N} -by- \sqrt{N} grid.

Running time. [if points are evenly distributed]

- Initialize data structure: N .
- Insert point: 1.
- Range search: 1 per point in range.

choose $M \sim \sqrt{N}$

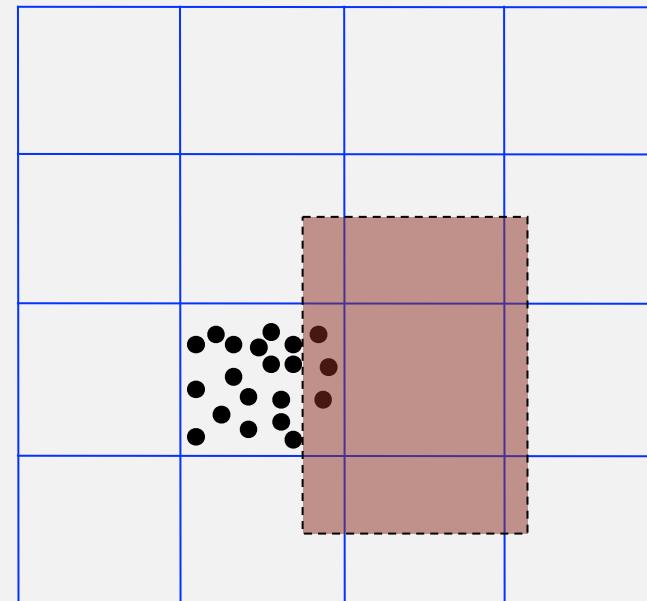


Clustering

Grid implementation. Fast, simple solution for evenly-distributed points.

Problem. Clustering a well-known phenomenon in geometric data.

- Lists are too long, even though average length is short.
- Need data structure that adapts gracefully to data.



Clustering

Grid implementation. Fast, simple solution for evenly-distributed points.

Problem. Clustering a well-known phenomenon in geometric data.

Ex. USA map data.



13,000 points, 1000 grid squares



half the squares are empty

half the points are
in 10% of the squares

Space-partitioning trees

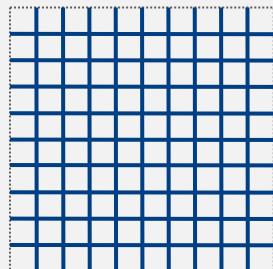
Use a **tree** to represent a recursive subdivision of 2d space.

Grid. Divide space uniformly into squares.

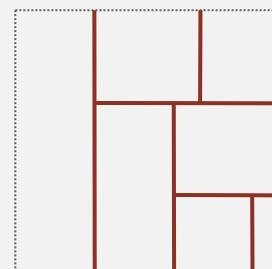
2d tree. Recursively divide space into two halfplanes.

Quadtree. Recursively divide space into four quadrants.

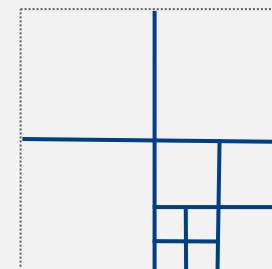
BSP tree. Recursively divide space into two regions.



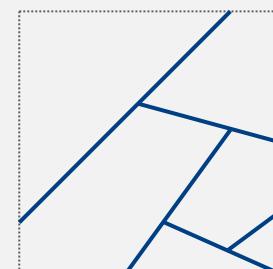
Grid



2d tree



Quadtree

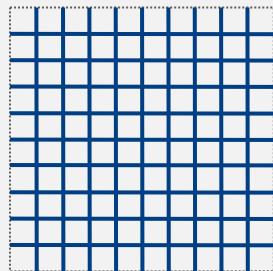


BSP tree

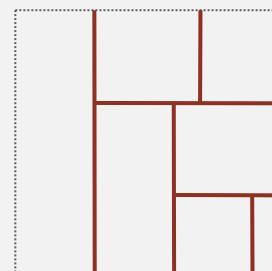
Space-partitioning trees: applications

Applications.

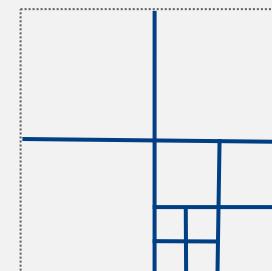
- Ray tracing.
- **2d range search.**
- Flight simulators.
- N-body simulation.
- Collision detection.
- Astronomical databases.
- **Nearest neighbor search.**
- Adaptive mesh generation.
- Accelerate rendering in Doom.
- Hidden surface removal and shadow casting.



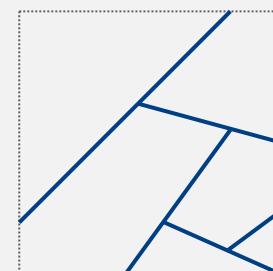
Grid



2d tree



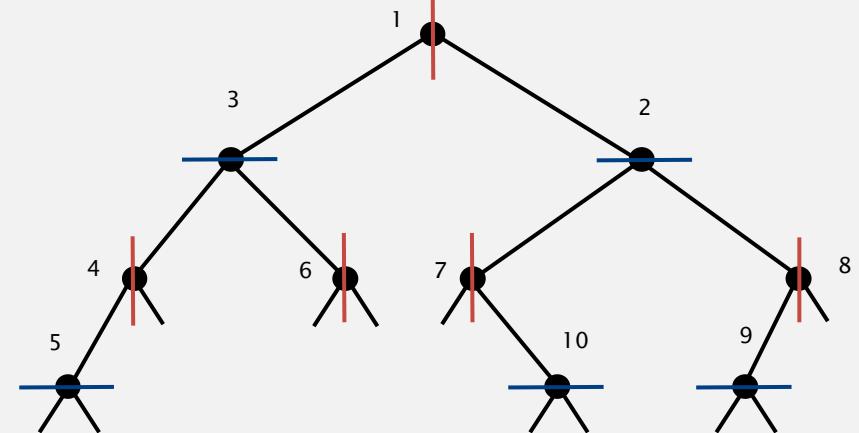
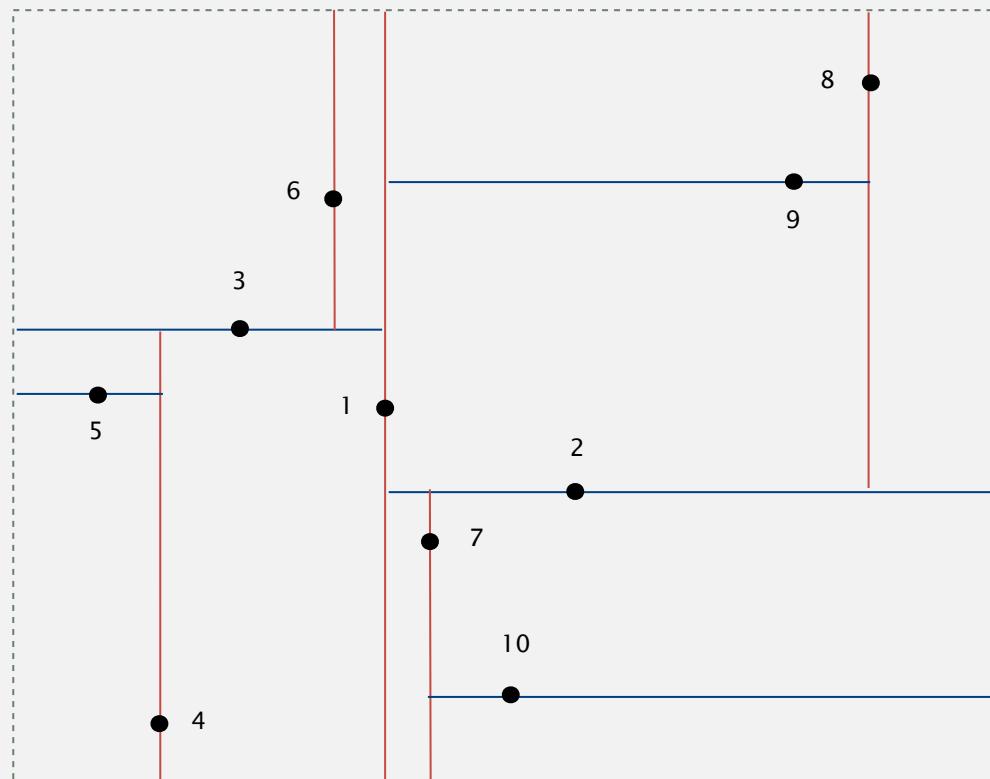
Quadtree



BSP tree

2d tree construction

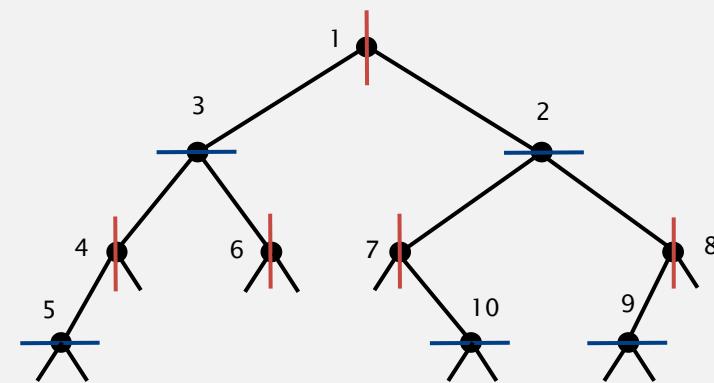
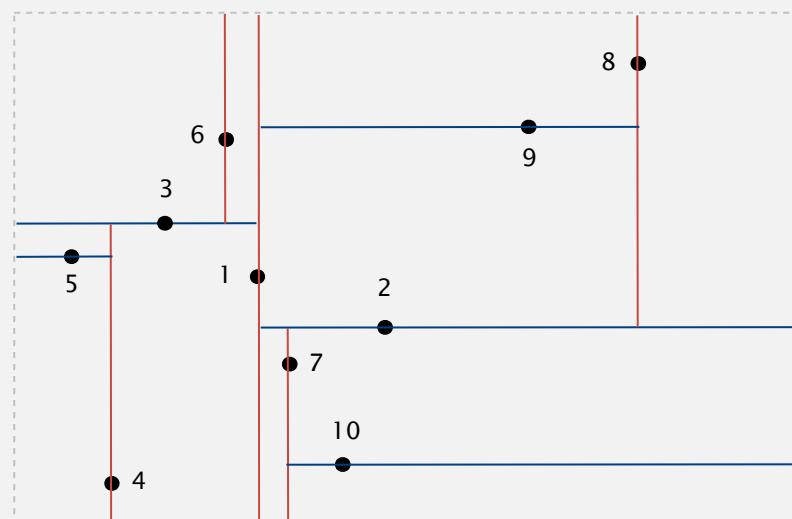
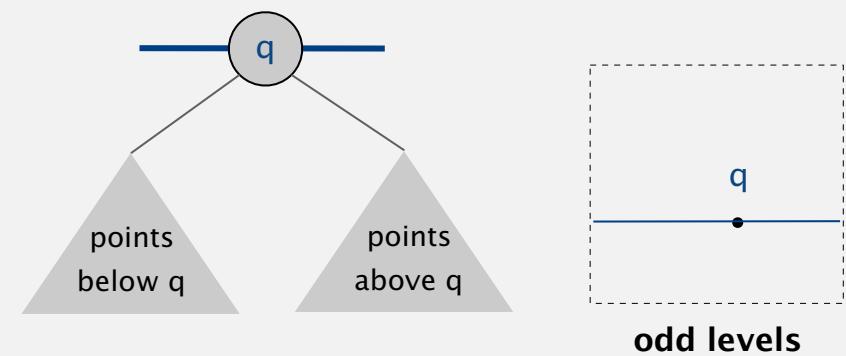
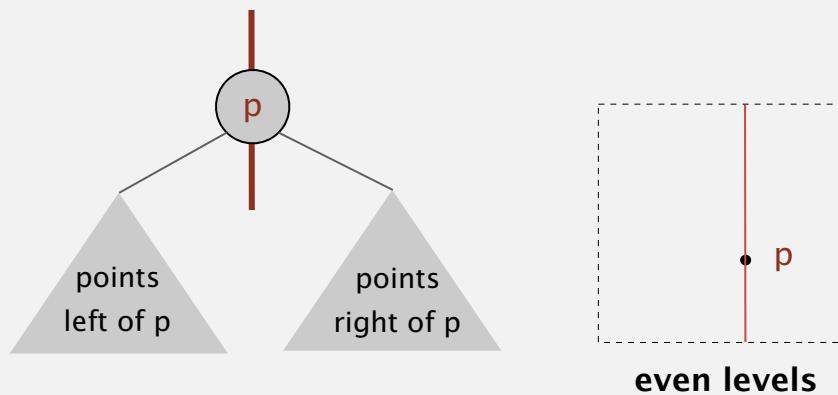
Recursively partition plane into two halfplanes.



2d tree implementation

Data structure. BST, but alternate using x - and y -coordinates as key.

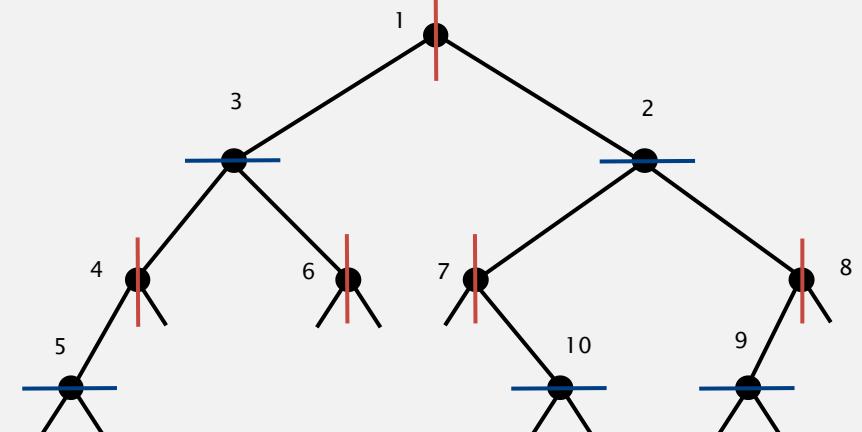
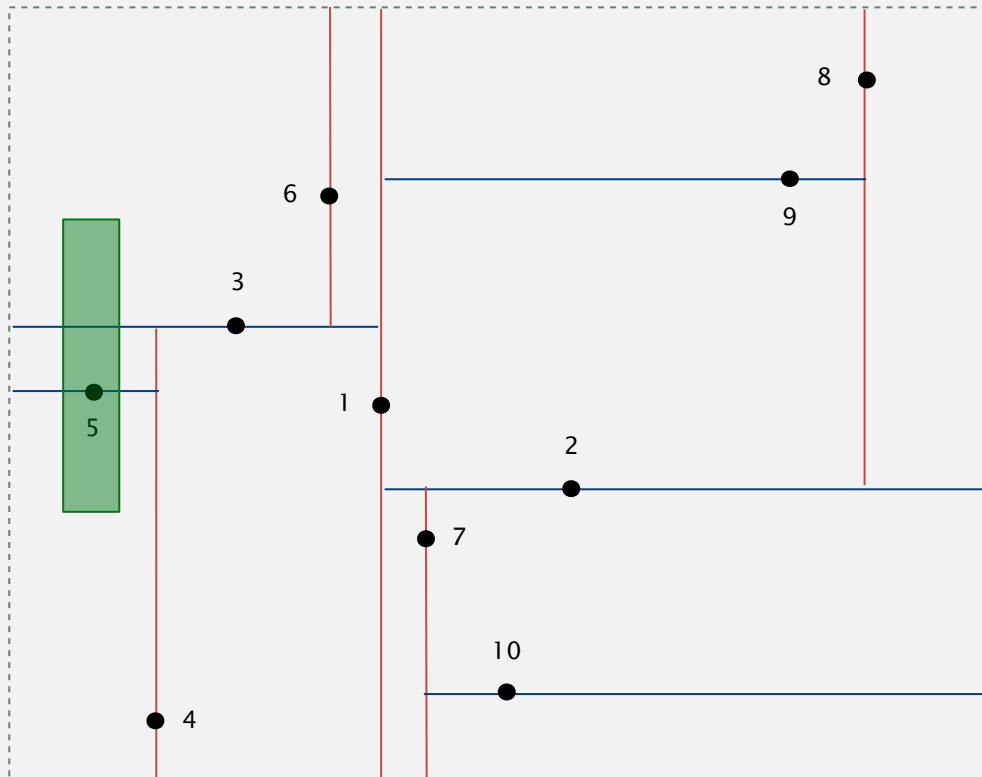
- Search gives rectangle containing point.
- Insert further subdivides the plane.



Range search in a 2d tree demo

Goal. Find all points in a query axis-aligned rectangle.

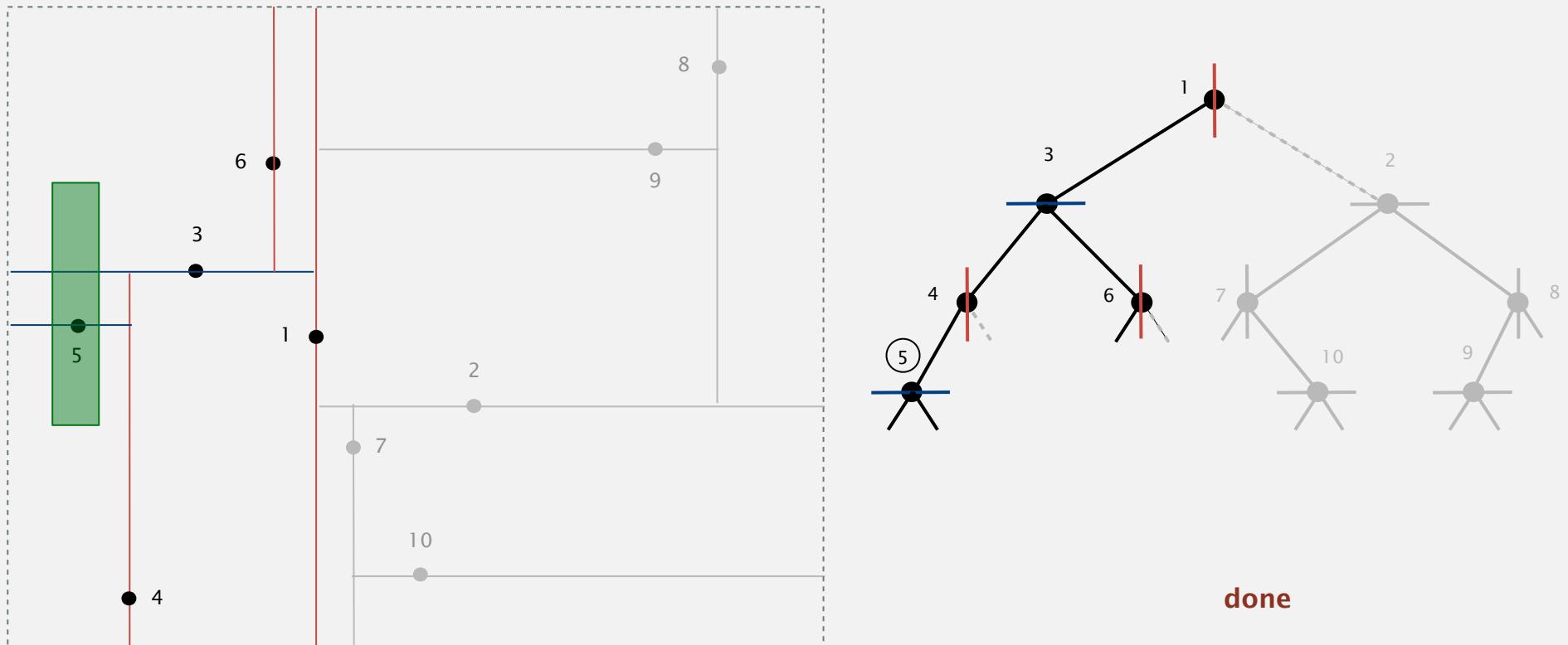
- Check if point in node lies in given rectangle.
- Recursively search left/bottom (if any could fall in rectangle).
- Recursively search right/top (if any could fall in rectangle).



Range search in a 2d tree demo

Goal. Find all points in a query axis-aligned rectangle.

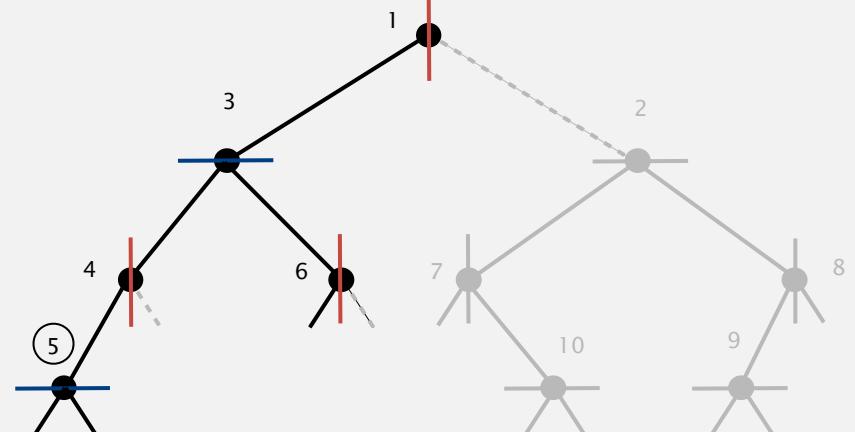
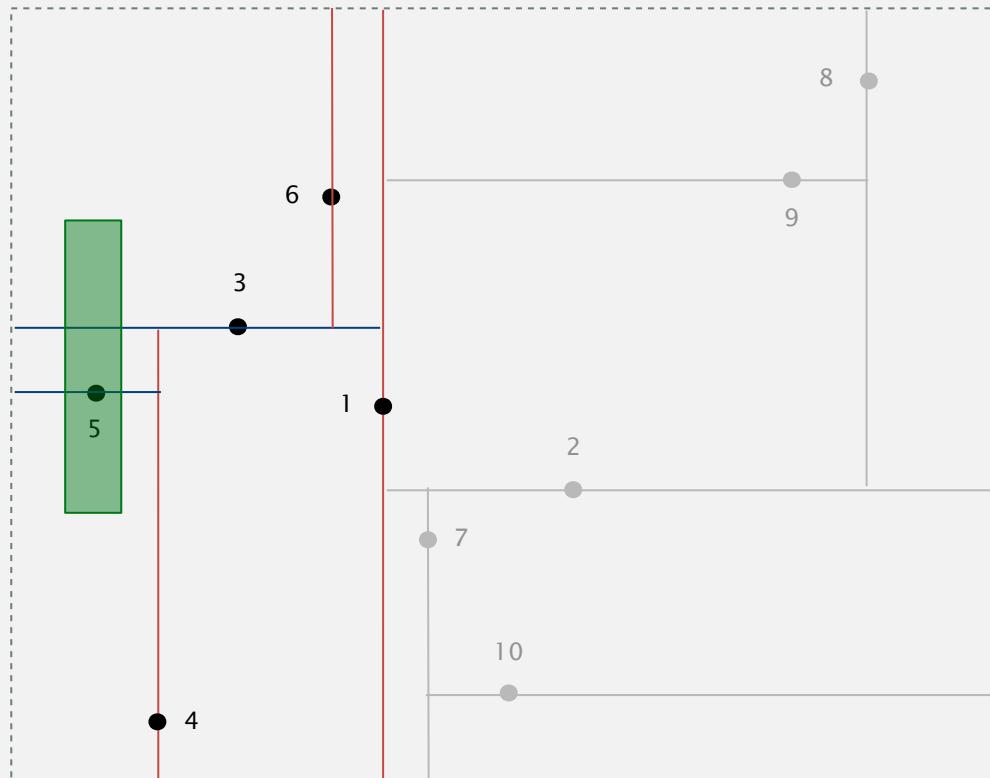
- Check if point in node lies in given rectangle.
- Recursively search left/bottom (if any could fall in rectangle).
- Recursively search right/top (if any could fall in rectangle).



Range search in a 2d tree analysis

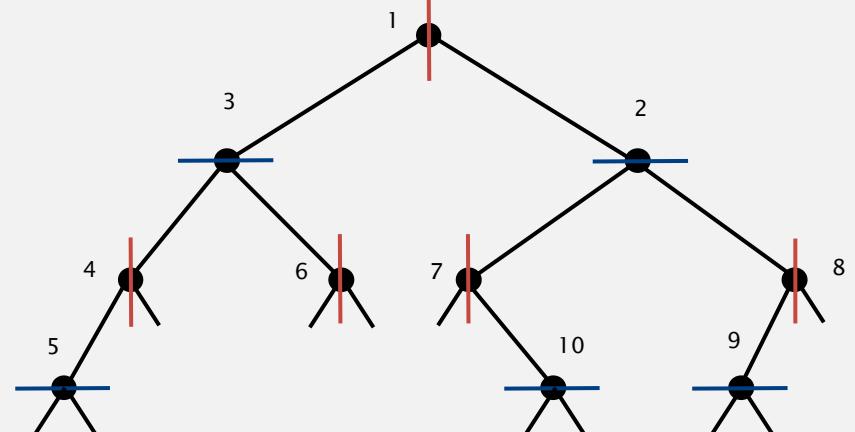
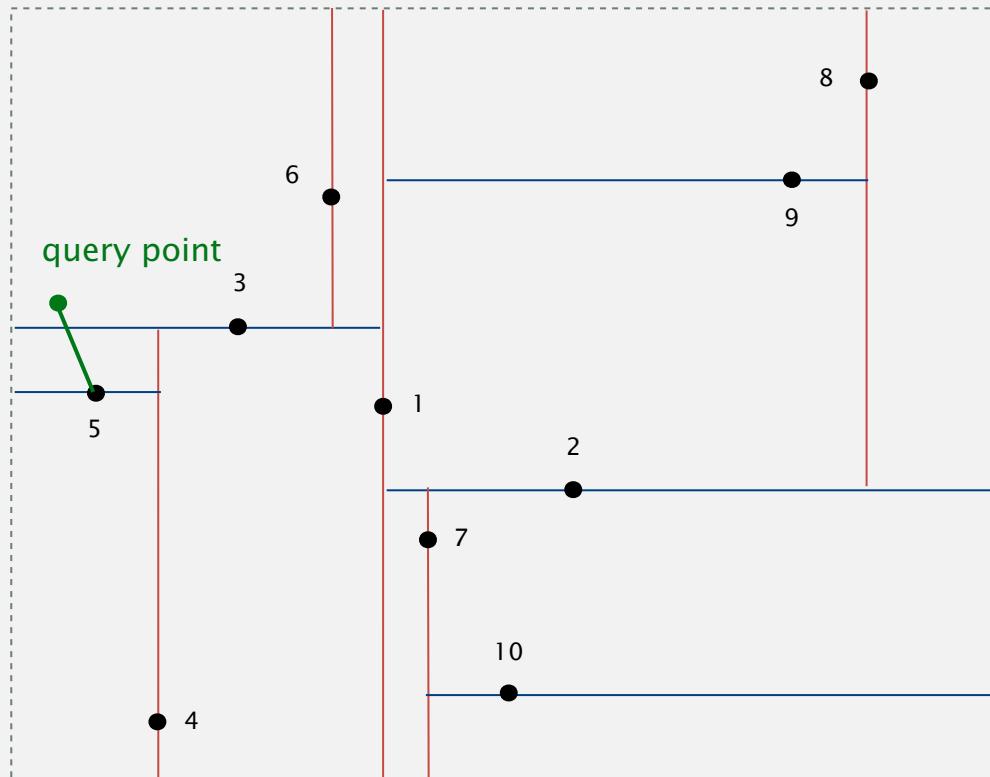
Typical case. $R + \log N$.

Worst case (assuming tree is balanced). $R + \sqrt{N}$.



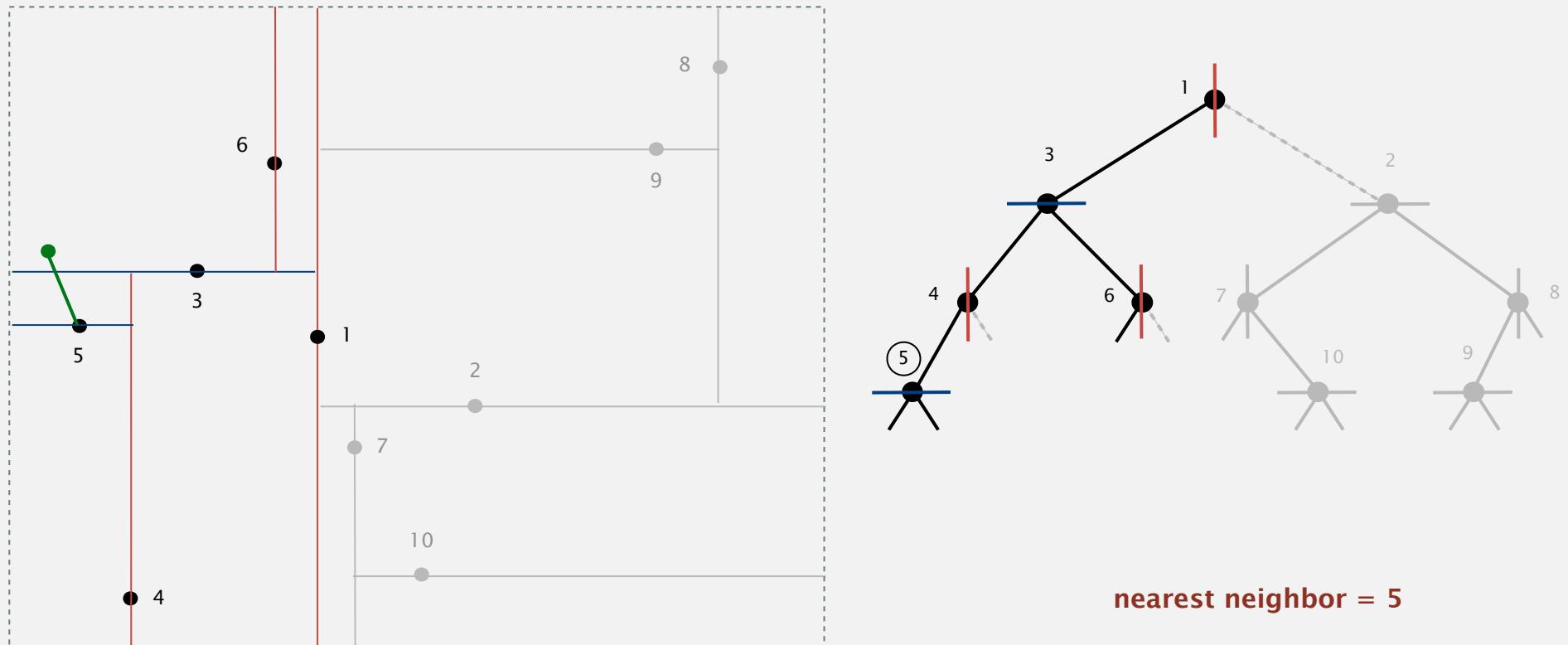
Nearest neighbor search in a 2d tree demo

Goal. Find closest point to query point.



Nearest neighbor search in a 2d tree demo

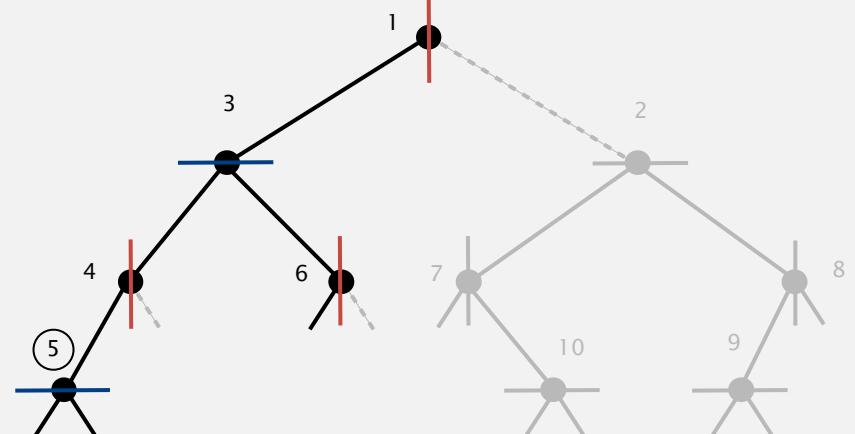
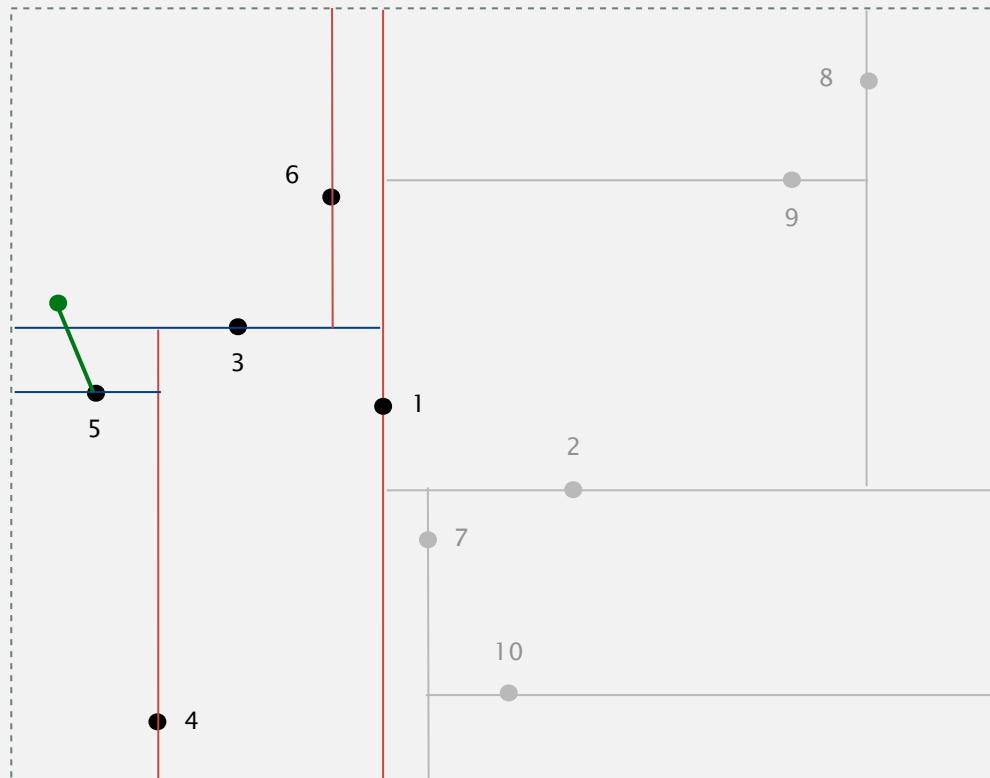
- Check distance from point in node to query point.
- Recursively search left/bottom (if it could contain a closer point).
- Recursively search right/top (if it could contain a closer point).
- Organize method so that it begins by searching for query point.



Nearest neighbor search in a 2d tree analysis

Typical case. $\log N$.

Worst case (even if tree is balanced). N .



nearest neighbor = 5

Flocking birds

Q. What "natural algorithm" do starlings, migrating geese, starlings, cranes, bait balls of fish, and flashing fireflies use to flock?

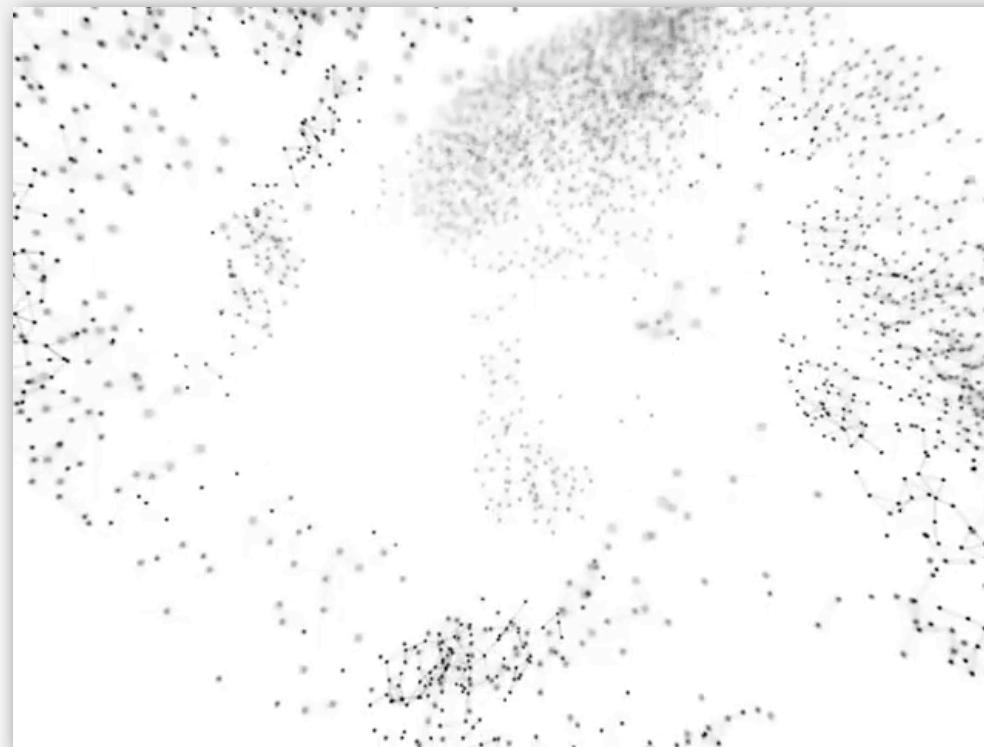


<http://www.youtube.com/watch?v=XH-groCeKbE>

Flocking boids [Craig Reynolds, 1986]

Boids. Three simple rules lead to complex emergent flocking behavior:

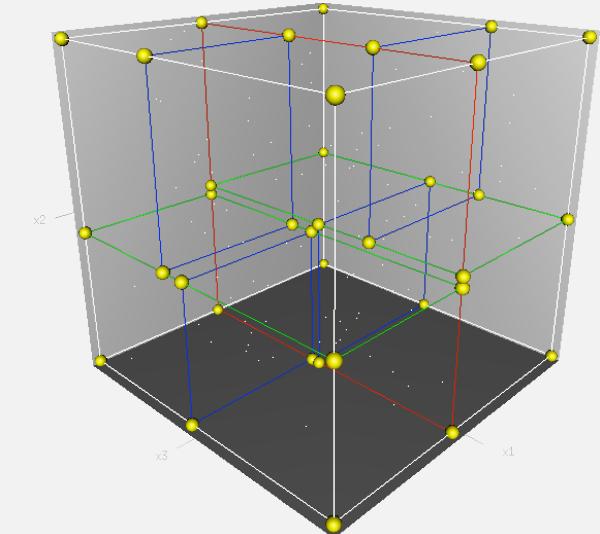
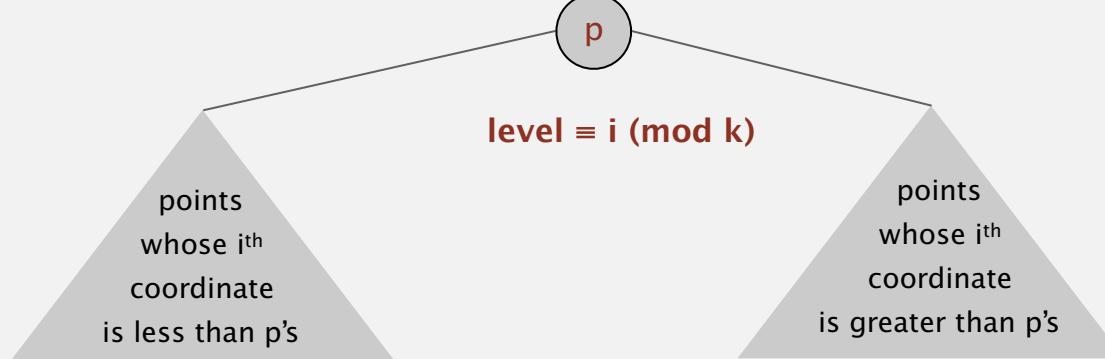
- Collision avoidance: point away from **k nearest** boids.
- Flock centering: point towards the center of mass of **k nearest** boids.
- Velocity matching: update velocity to the average of **k nearest** boids.



Kd tree

Kd tree. Recursively partition k -dimensional space into 2 halfspaces.

Implementation. BST, but cycle through dimensions ala 2d trees.



Efficient, simple data structure for processing k -dimensional data.

- Widely used.
- Adapts well to high-dimensional and clustered data.
- Discovered by an undergrad in an algorithms class!



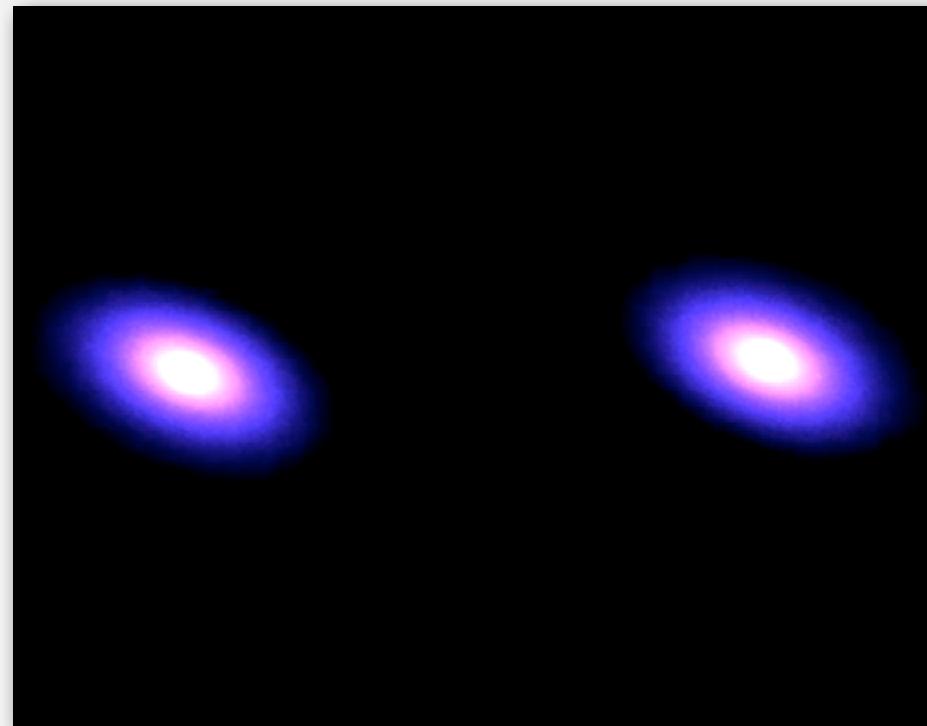
Jon Bentley

N-body simulation

Goal. Simulate the motion of N particles, mutually affected by gravity.

Brute force. For each pair of particles, compute force: $F = \frac{G m_1 m_2}{r^2}$

Running time. Time per step is N^2 .

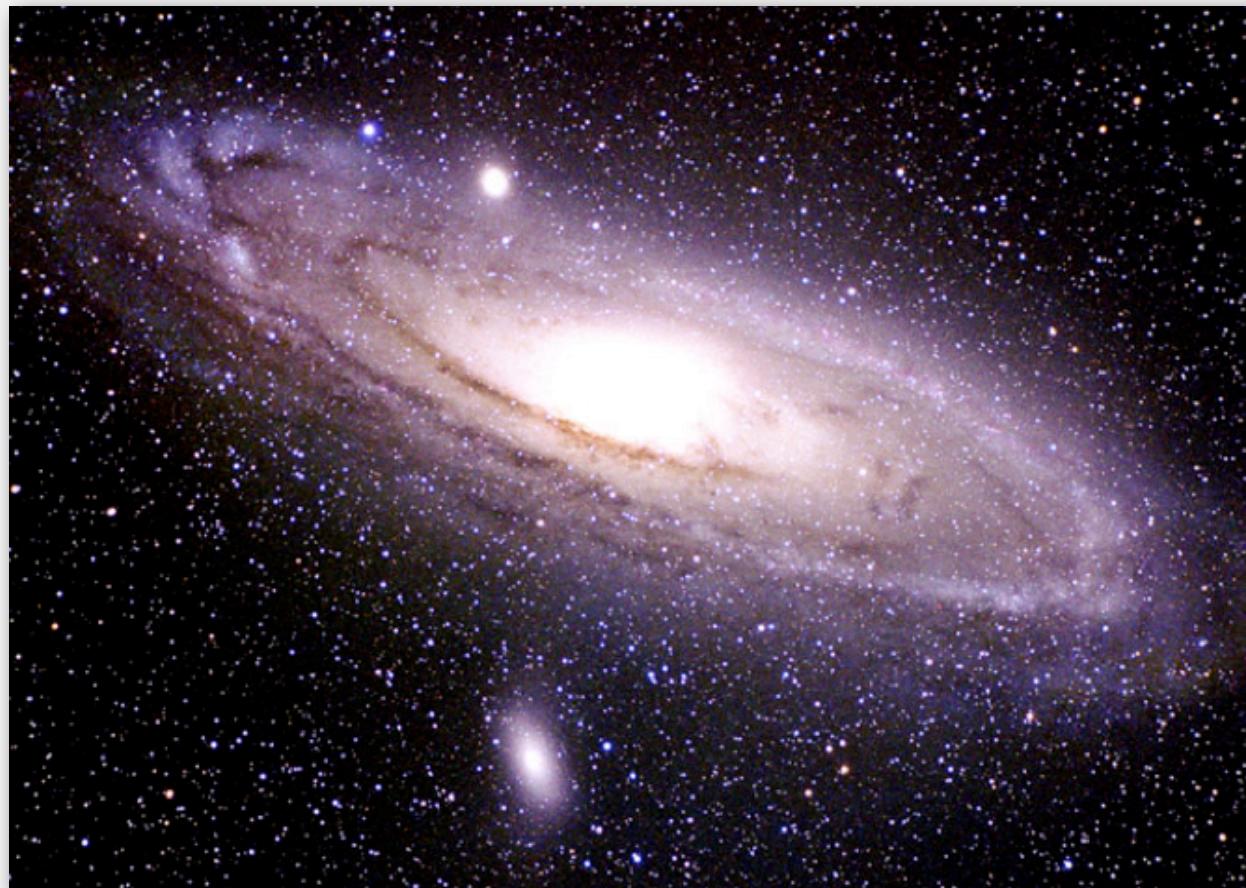


http://www.youtube.com/watch?v=ua7YlN4eL_w

Appel's algorithm for N-body simulation

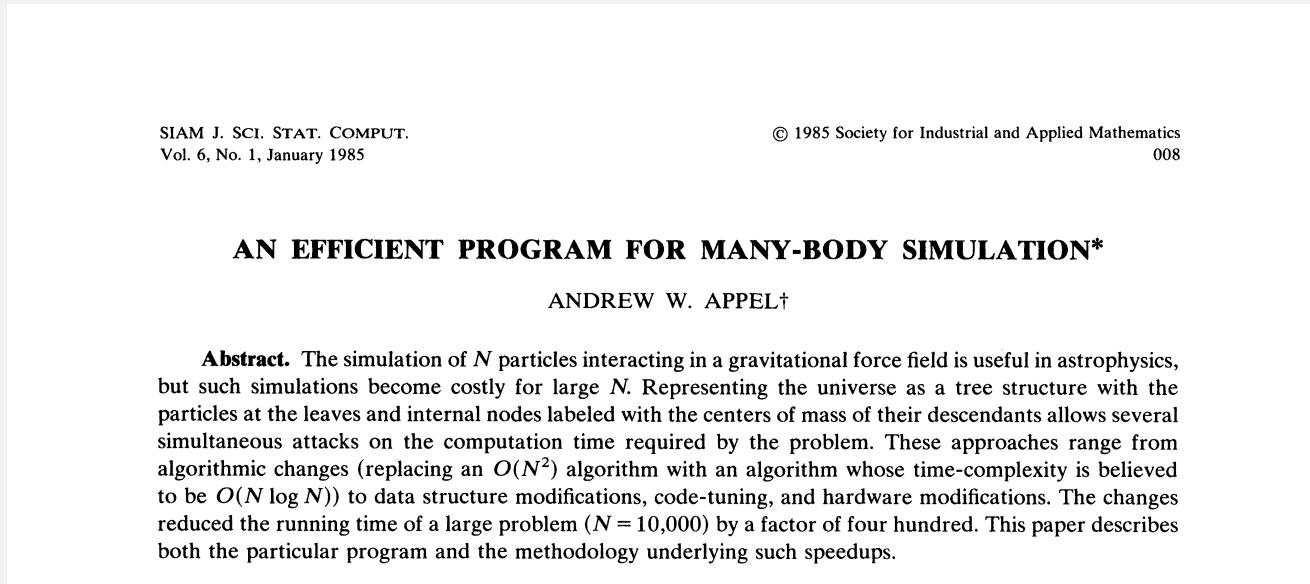
Key idea. Suppose particle is far, far away from cluster of particles.

- Treat cluster of particles as a single aggregate particle.
- Compute force between particle and **center of mass** of aggregate.



Appel's algorithm for N-body simulation

- Build 3d-tree with N particles as nodes.
- Store center-of-mass of subtree in each node.
- To compute total force acting on a particle, traverse tree, but stop as soon as distance from particle to subdivision is sufficiently large.



Impact. Running time per step is $N \log N \Rightarrow$ enables new research.

Algorithms

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GEOMETRIC APPLICATIONS OF BSTs

- ▶ *1d range search*
- ▶ *line segment intersection*
- ▶ *kd trees*
- ▶ *interval search trees*
- ▶ *rectangle intersection*

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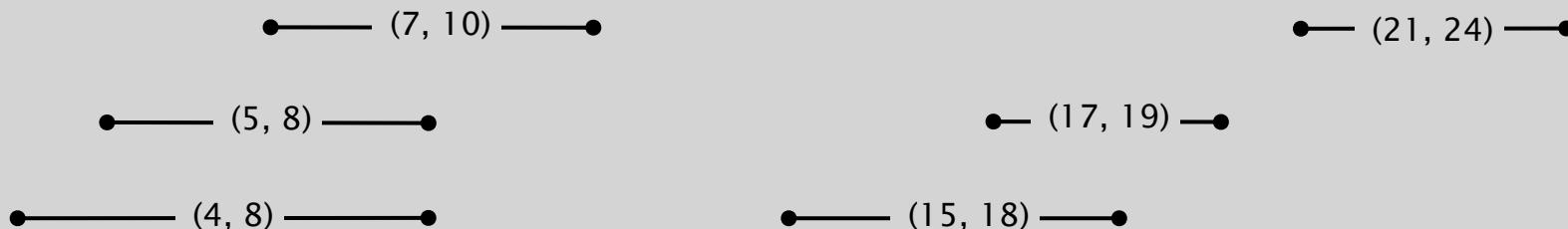
1d interval search

1d interval search. Data structure to hold set of (overlapping) intervals.

- Insert an interval (lo, hi).
- Search for an interval (lo, hi).
- Delete an interval (lo, hi).
- **Interval intersection query:** given an interval (lo, hi), find all intervals (or one interval) in data structure that intersects (lo, hi).

Q. Which intervals intersect ($9, 16$)?

A. ($7, 10$) and ($15, 18$).



1d interval search API

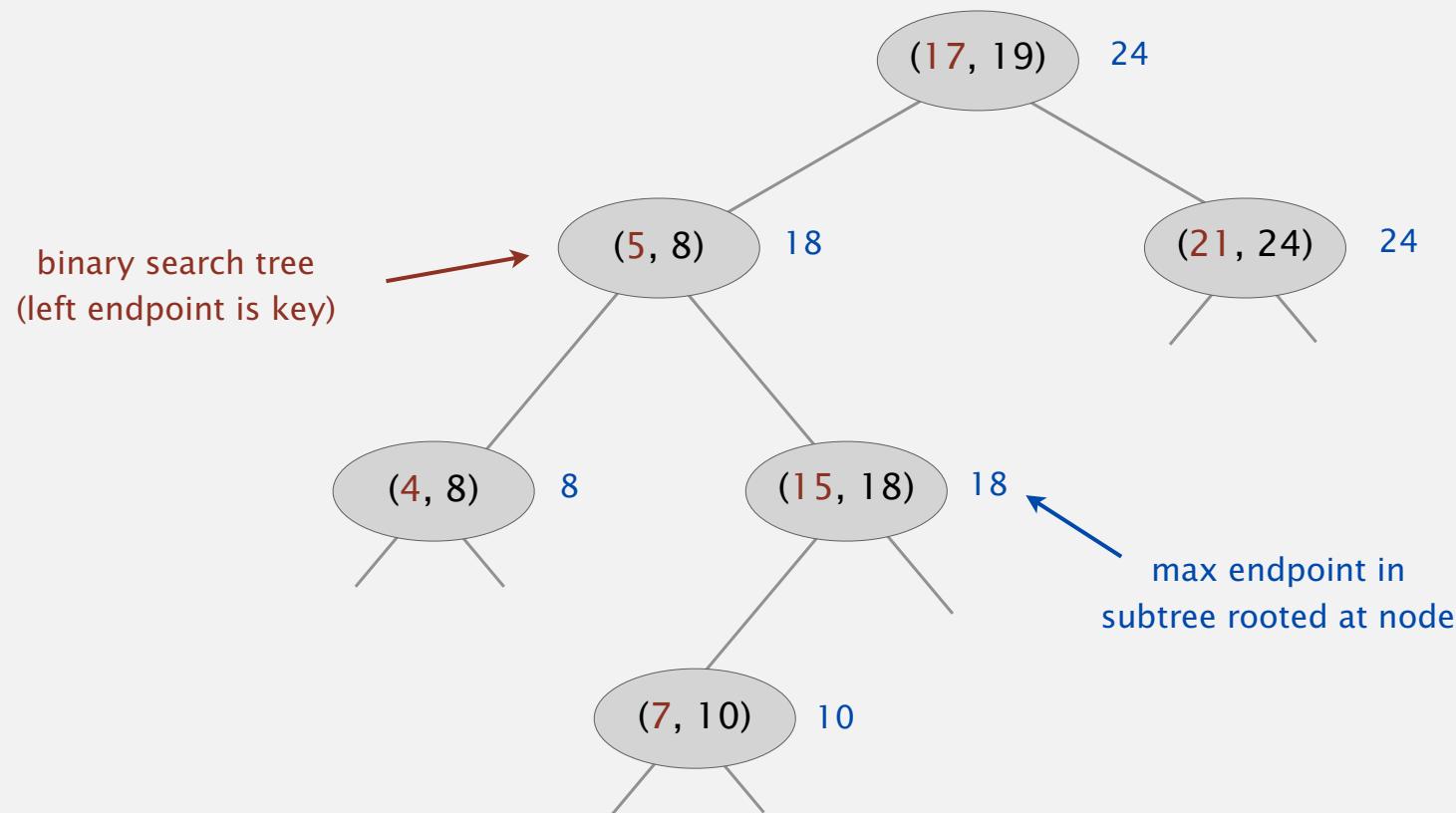
public class IntervalST<Key extends Comparable<Key>, Value>	
IntervalST()	<i>create interval search tree</i>
void put(Key lo, Key hi, Value val)	<i>put interval-value pair into ST</i>
Value get(Key lo, Key hi)	<i>value paired with given interval</i>
void delete(Key lo, Key hi)	<i>delete the given interval</i>
Iterable<Value> intersects(Key lo, Key hi)	<i>all intervals that intersect the given interval</i>

Nondegeneracy assumption. No two intervals have the same left endpoint.

Interval search trees

Create BST, where each node stores an interval (lo, hi).

- Use left endpoint as BST **key**.
- Store **max endpoint** in subtree rooted at node.



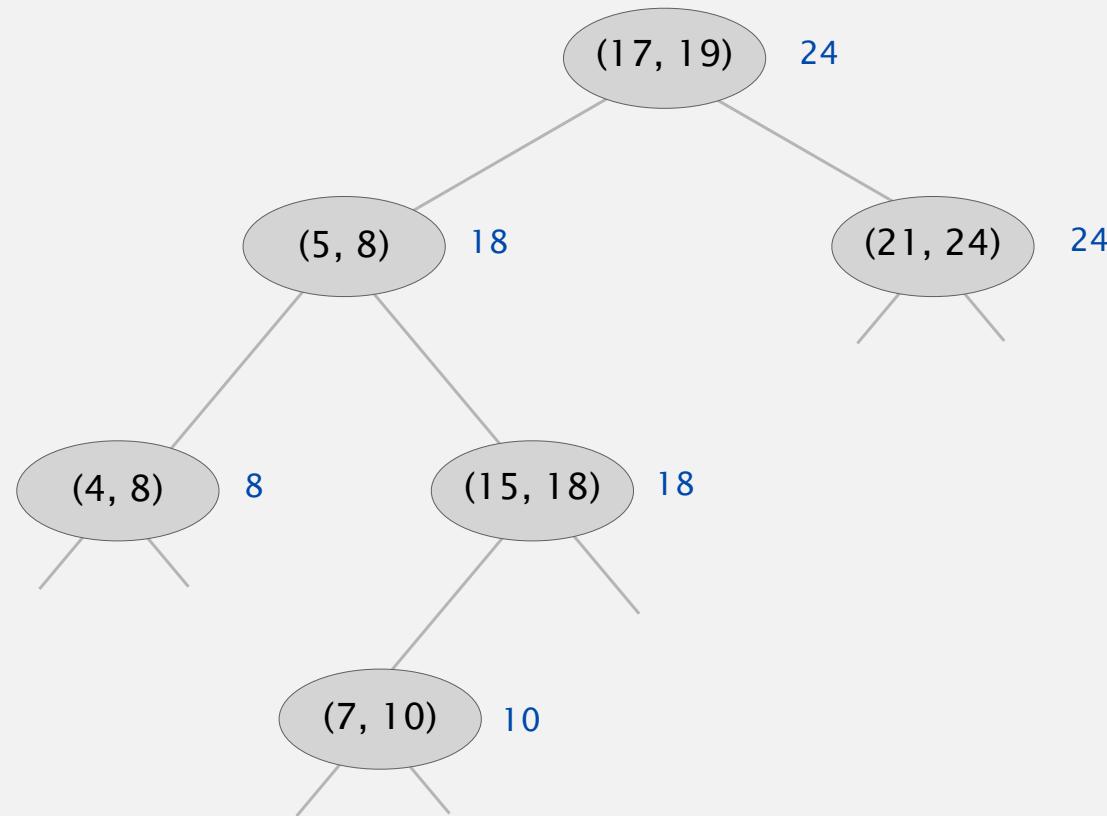
Interval search tree demo

To insert an interval (lo, hi) :

- Insert into BST, using lo as the key.
- Update max in each node on search path.



insert interval (16, 22)

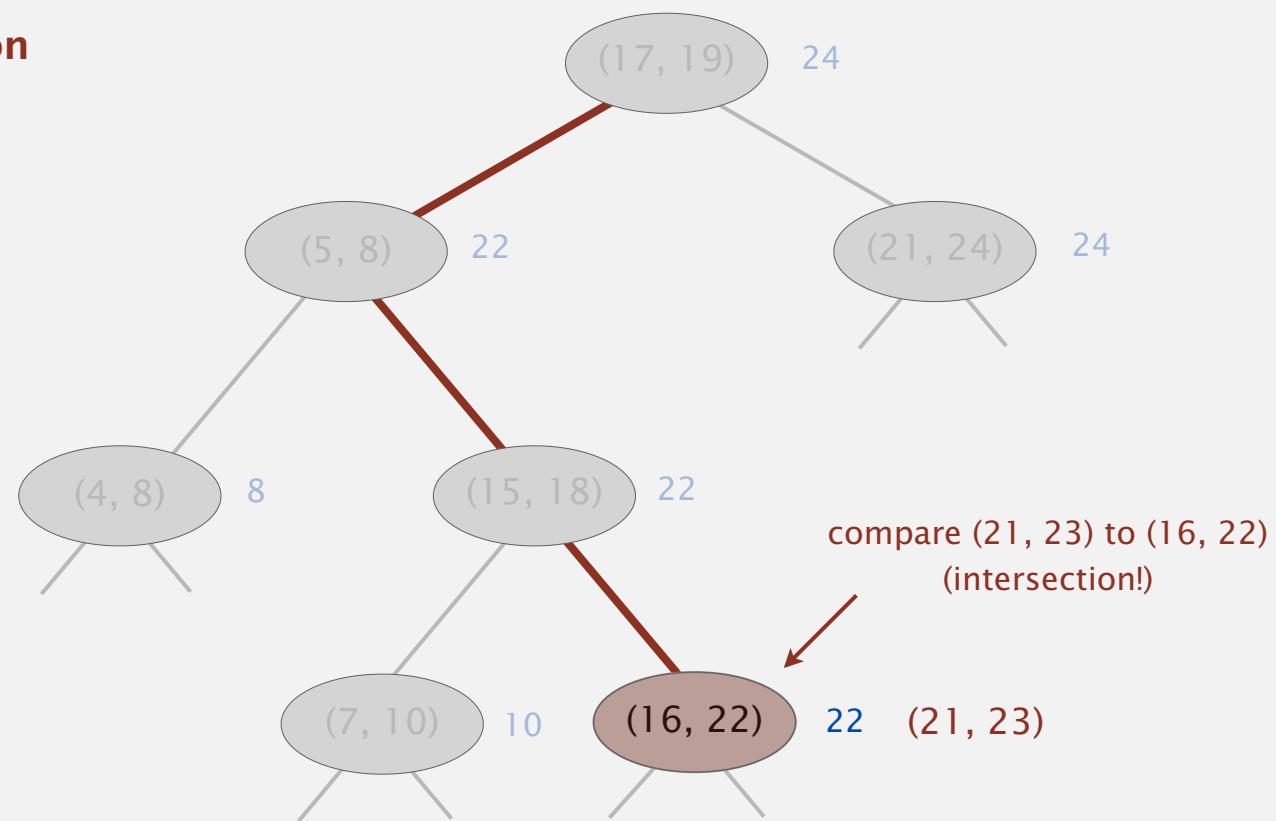


Interval search tree demo

To search for any one interval that intersects query interval (lo, hi):

- If interval in node intersects query interval, return it.
- Else if left subtree is null, go right.
- Else if max endpoint in left subtree is less than lo , go right.
- Else go left.

interval intersection
search for (21, 23)



Search for an intersecting interval implementation

To search for any one interval that intersects query interval (lo, hi):

- If interval in node intersects query interval, return it.
- Else if left subtree is null, go right.
- Else if max endpoint in left subtree is less than lo , go right.
- Else go left.

```
Node x = root;
while (x != null)
{
    if      (x.interval.intersects(lo, hi)) return x.interval;
    else if (x.left == null)                  x = x.right;
    else if (x.left.max < lo)                x = x.right;
    else                                      x = x.left;
}
return null;
```

Search for an intersecting interval analysis

To search for any one interval that intersects query interval (lo, hi):

- If interval in node intersects query interval, return it.
- Else if left subtree is null, go right.
- Else if max endpoint in left subtree is less than lo , go right.
- Else go left.

Case 1. If search goes **right**, then no intersection in left.

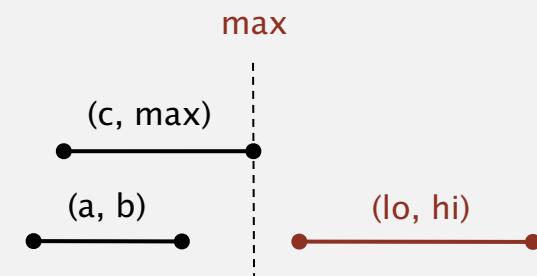
Pf. Suppose search goes right and left subtree is non empty.

- Max endpoint max in left subtree is less than lo .
- For any interval (a, b) in left subtree of x ,

we have $b \leq max < lo$.

definition of max

reason for going right



left subtree of x

right subtree of x

Search for an intersecting interval analysis

To search for any one interval that intersects query interval (lo, hi) :

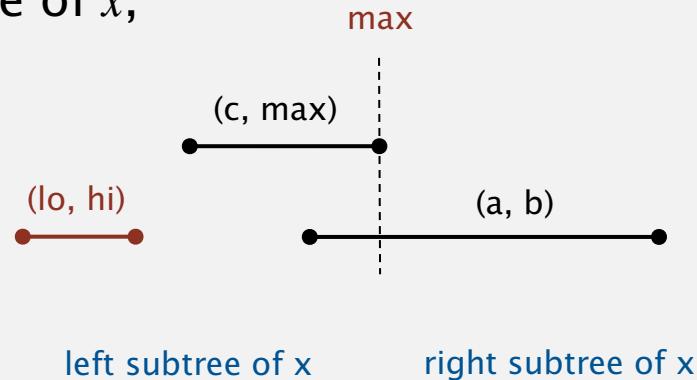
- If interval in node intersects query interval, return it.
- Else if left subtree is null, go right.
- Else if max endpoint in left subtree is less than lo , go right.
- Else go left.

Case 2. If search goes **left**, then there is either an intersection in left subtree or no intersections in either.

Pf. Suppose no intersection in left.

- Since went left, we have $lo \leq max$.
- Then for any interval (a, b) in right subtree of x ,
 $hi < c \leq a \Rightarrow$ no intersection in right.

no intersections in left subtree intervals sorted by left endpoint



Interval search tree: analysis

Implementation. Use a red-black BST to guarantee performance.

easy to maintain auxiliary information
using $\log N$ extra work per op

operation	brute	interval search tree	best in theory
insert interval	1	$\log N$	$\log N$
find interval	N	$\log N$	$\log N$
delete interval	N	$\log N$	$\log N$
find any one interval that intersects (lo, hi)	N	$\log N$	$\log N$
find all intervals that intersects (lo, hi)	N	$R \log N$	$R + \log N$

order of growth of running time for N intervals

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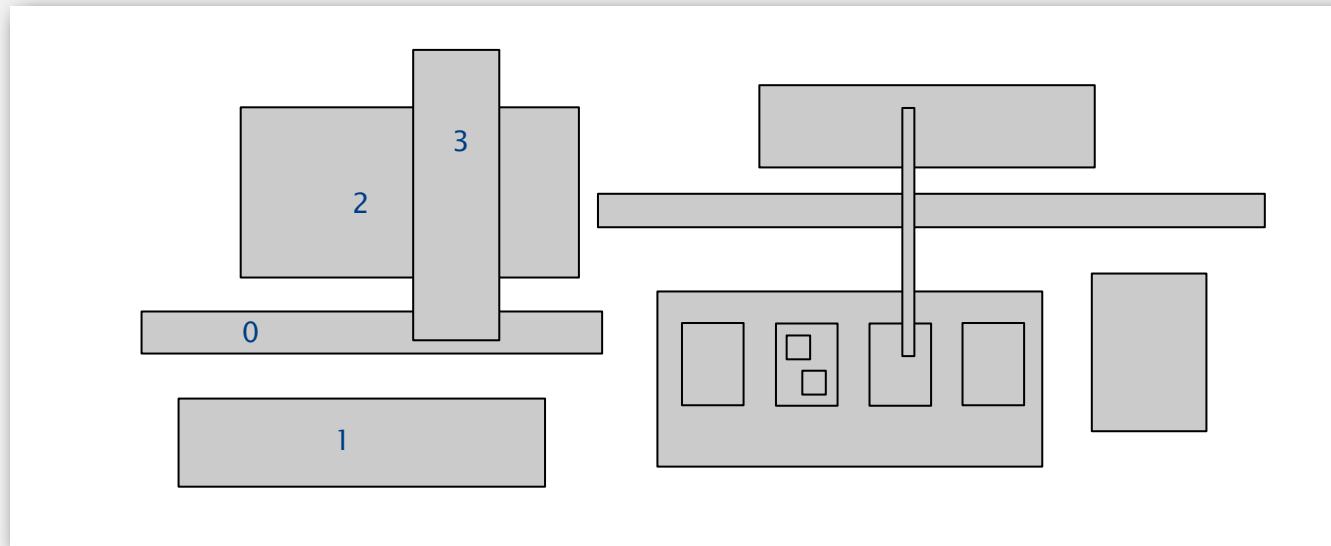
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Orthogonal rectangle intersection

Goal. Find all intersections among a set of N orthogonal rectangles.

Quadratic algorithm. Check all pairs of rectangles for intersection.



Non-degeneracy assumption. All x - and y -coordinates are distinct.

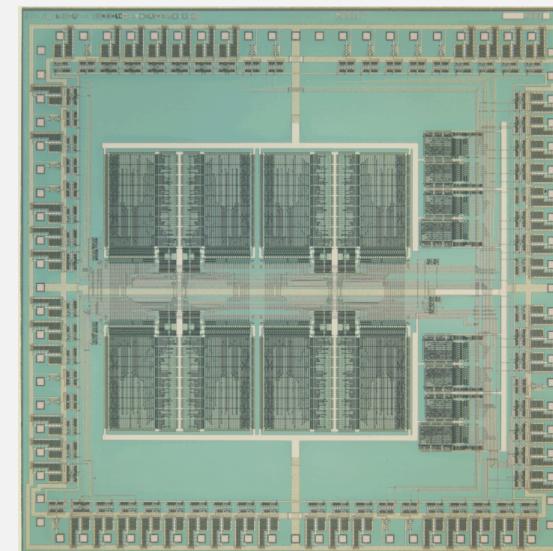
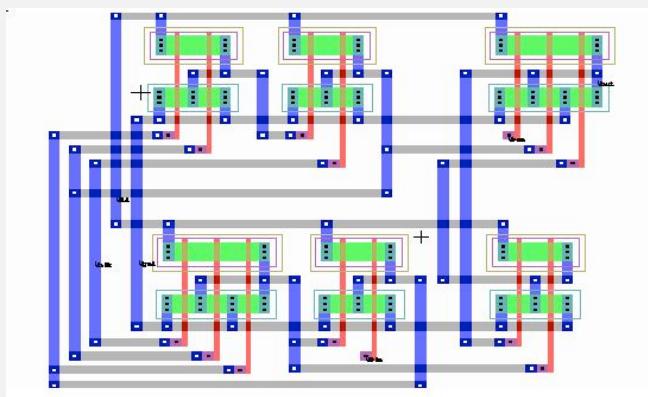
Microprocessors and geometry

Early 1970s. microprocessor design became a **geometric** problem.

- Very Large Scale Integration (VLSI).
- Computer-Aided Design (CAD).

Design-rule checking.

- Certain wires cannot intersect.
- Certain spacing needed between different types of wires.
- Debugging = orthogonal rectangle intersection search.



Algorithms and Moore's law

"Moore's law." Processing power doubles every 18 months.

- $197x$: check N rectangles.
- $197(x+1.5)$: check $2N$ rectangles on a $2x$ -faster computer.



Gordon Moore

Bootstrapping. We get to use the faster computer for bigger circuits.

But bootstrapping is not enough if using a quadratic algorithm:

- $197x$: takes M days.
- $197(x+1.5)$: takes $(4M)/2 = 2M$ days. (!)

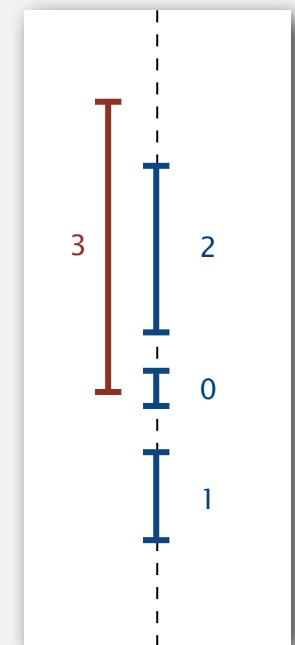
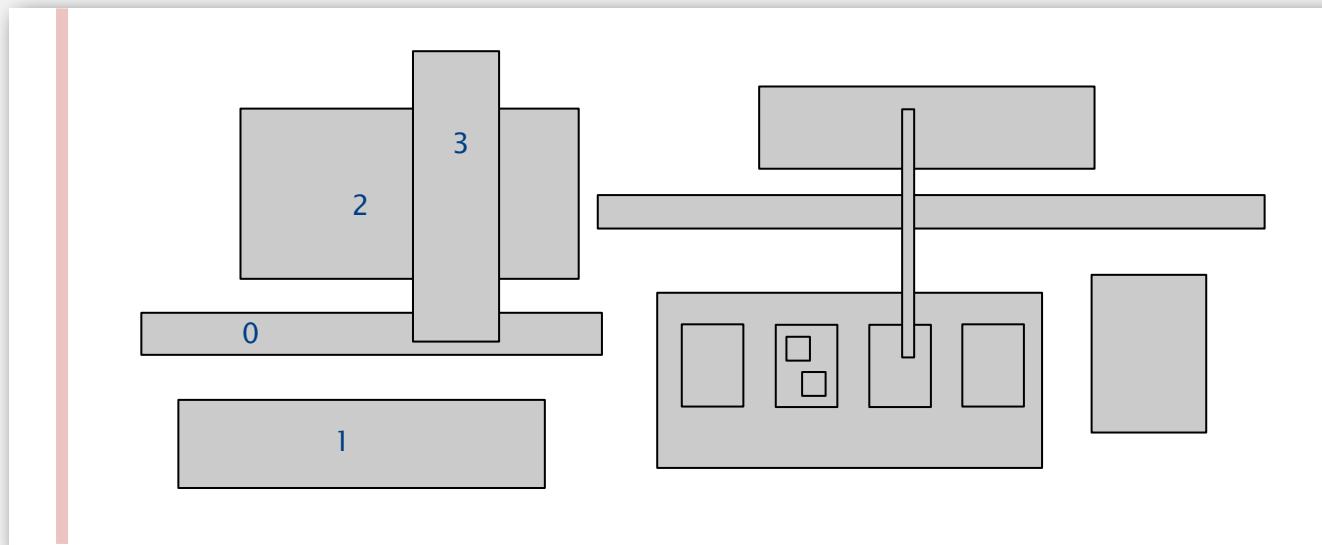


Bottom line. Linearithmic algorithm is **necessary** to sustain Moore's Law.

Orthogonal rectangle intersection: sweep-line algorithm

Sweep vertical line from left to right.

- x -coordinates of left and right endpoints define events.
- Maintain set of rectangles that intersect the sweep line in an interval search tree (using y -intervals of rectangle).
- Left endpoint: interval search for y -interval of rectangle; insert y -interval.
- Right endpoint: remove y -interval.



y-coordinates

Orthogonal rectangle intersection: sweep-line analysis

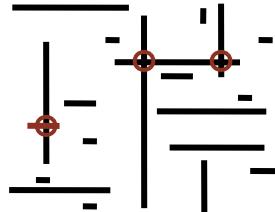
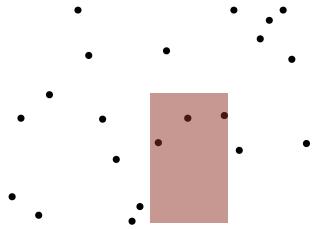
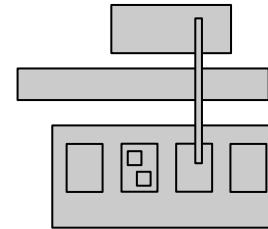
Proposition. Sweep line algorithm takes time proportional to $N \log N + R \log N$ to find R intersections among a set of N rectangles.

Pf.

- Put x -coordinates on a PQ (or sort). $\leftarrow N \log N$
- Insert y -intervals into ST. $\leftarrow N \log N$
- Delete y -intervals from ST. $\leftarrow N \log N$
- Interval searches for y -intervals. $\leftarrow N \log N + R \log N$

Bottom line. Sweep line reduces 2d orthogonal rectangle intersection search to 1d interval search.

Geometric applications of BSTs

problem	example	solution
1d range search	BST
2d orthogonal line segment intersection		sweep line reduces to 1d range search
kd range search		kd tree
1d interval search		interval search tree
2d orthogonal rectangle intersection		sweep line reduces to 1d interval search

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