Arbitrage and Trading in Financial Markets

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Abstract

The paper "Arbitrage concepts under trading restrictions in discrete-time financial markets" explores relationships between concepts of arbitrage and trading strategies in a financial market with risky and risk-free assets in discrete time.[1] Expanding on the notion of *classical no-arbitrage*, this paper characterizes trading strategies under *no-arbitrage of the first kind*. In both one-period and multi-period markets, this paper shows that no-arbitrage of the first kind allows for the existence of an optimal trading strategy, numeraire portfolio, and a risk-neutral measure, three key concepts in financial markets and trading. In our report, we present in detail the proofs of existence for the three concepts named above in the one-period setting of the paper's model.

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[1] Fontana, C. and Runggaldier, W. J. (2020). "Arbitrage concepts under trading restrictions in discrete-time financial markets." *Journal of Mathematical Economics* (forthcoming).

1 The single-period setting

We reproduce the single-period setting as delineated in the paper, with minor changes in notation. There is a financial market consisting of n risky assets and a risk-free asset with unit price. Asset prices are denoted by $S_t = (S_t^1, \dots, S_t^n)^\top$ for $t = 0, 1$. Returns on assets are specified by an a random vector $R = (R^1, \dots, R^n)^\top \in \mathbb{R}^n$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfies $R^i \geq -1$ for $i = 1, \dots, n$ almost surely. Let $\mathcal{S}$ be the support of R, and let $\mathcal{L}$ be the smallest subspace of $\mathbb{R}^n$ that contains $\mathcal{S}$.

A portfolio, or trading strategy as the paper calls it, is specified by $\pi = (\pi^1, \dots, \pi^n) \in \mathbb{R}^n$, where π^i is the fraction of wealth invested in asset i and the remaining fraction $1 - \sum_{i=1}^n \pi^i$ is invested in the risk-free asset. The value of the portfolio π at time t with initial wealth v is denoted by $V_t(\pi, v)$. If the initial wealth is 1, then we will just write $V_t(\pi)$. Value evolves over time in this market as follows: $V_0(\pi, v) = v$ and $V_1(\pi, v) = v(1 + \langle \pi, R \rangle)$.

A portfolio or trading strategy π is *admissible* if $V_1(\pi) \geq 0$ almost surely, equivalent to $\langle \pi, R \rangle \geq -1$ almost surely. We denote the set of all admissible portfolios with Θ_+. We can impose additional trading restrictions so long as the set of all restricted portfolios, denoted by Θ_0, is closed and convex. A portfolio or trading strategy is *allowed* if it is admissible and satisfies all additional restrictions. We denote the set of all allowed portfolios with Θ. Note that $\Theta = \Theta_+ \cap \Theta_0$. Some restrictions the paper includes are:

- No short-selling: $\Theta_0 = \mathbb{R}^n_+$.
- No short-selling nor borrowing: $\Theta_0 = \Delta^n$ where $\Delta^n = \{\pi \in \mathbb{R}^n_+ : \sum_{i=1}^n \pi^i \leq 1\}$.
- A limit on borrowing: $\Theta_0 = \{\pi \in \mathbb{R}^n : \sum_{i=1}^n \pi^i \leq c\}$ for some $c \geq 1$.
- Intervals for each asset: $\Theta_0 = \bigtimes_{i=1}^n [\alpha_i, \beta_i]$ for some $\alpha_i < \beta_i$ for $i = 1, \dots, n$.

Note that for any $\pi \in \Theta$ and $\lambda \in \mathcal{L}^\perp$, we have $\langle \pi, R \rangle = \langle \pi + \lambda, R \rangle$ almost surely, since $\langle \lambda, R \rangle$ $= 0$ almost surely, so we always include $\mathcal{L}^\perp \in \Theta_0$ as convention. Note that $\mathcal{L} \in \Theta$ as well. We also define $\widehat{\Theta} = \{\pi \in \mathbb{R}^n : \alpha\pi \in \Theta \ ^\forall \alpha > 0\}$.

The set of arbitrage opportunities is defined by $\mathcal{A} = \{\pi \in \mathbb{R}^n \backslash \mathcal{L}^\perp : \langle \pi, z \rangle > 0 \ ^\forall z \in \mathcal{S}\}$. Under classical no-arbitrage, denoted by NA, we have $\mathcal{A} = \varnothing$. Intuitively, this notion simply implies that a portfolio $\pi \in \mathcal{A}$ will generate positive returns almost surely.

For a random variable $\xi \in \mathcal{M}^+(\Omega, \mathcal{F})$, we first define the *super-hedging value* $v(\xi)$ as $v(\xi) = \inf\{v > 0 : {}^\exists \pi \in \Theta \text{ such that } v(1 + \langle \pi, R(\omega) \rangle) \geq \xi(\omega) \text{ almost surely}\}$. We refer to a random variable $\xi \in \mathcal{M}^+(\Omega, \mathcal{F})$ with $v(\xi) = 0$ and $\mathbb{P}(\{\xi > 0\}) > 0$ as an *arbitrage of the first kind*. Intuitively, an arbitrage of the first kind that nonnegative returns can be generated from an arbitrarily small amount of initial wealth almost surely. Under no-arbitrage of the first kind, denoted by NA(1), $v(\xi) = 0$ implies that $\xi = 0$ almost surely. The NA(1) condition is weaker than the NA condition, as will be shown later on.

2 Arbitrage and trading in one period

2.1 Basic characterizations

Theorem 2.1. (Proposition 2.2 in the paper) The following are equivalent:

(i) NA(1) holds;

(ii) $\mathcal{A} \cap \widehat{\Theta} = \varnothing$;

(iii) $\widehat{\Theta} = \mathcal{L}^{\perp}$;

(iv) $\Theta \cap \mathcal{L}$ is compact.

Proof. We first show that (i) $\implies$ (ii). By way of contradiction, suppose that NA(1) holds and there exists $\pi \in \mathcal{A} \cap \widehat{\Theta}$. Let $\xi = \langle \pi, R \rangle$. Since $\pi \in \mathcal{A}$, we have $\xi \geq 0$ almost surely, so $\xi \in \mathcal{M}^{+}(\Omega, \mathcal{F})$. Since $\pi \in \mathcal{A}$, we have $\pi \notin \mathcal{L}^{\perp}$, so $\xi \neq 0$ and thus $\xi > 0$ almost surely. Then ξ is an arbitrage of the first kind. However, since $\pi \in \widehat{\Theta}$, we have $\alpha\pi \in \Theta$ and $\alpha(1 + \langle \alpha\pi, R \rangle)$ $= \alpha + \langle \pi, R \rangle > \langle \pi, R \rangle = \xi$, so $v(\xi) = 0$. Since $\xi \neq 0$ almost surely, we have a contradiction.

We then show that (ii) $\implies$ (iii). Since $\mathcal{L}^{\perp} \in \Theta$ is a linear subspace, $\mathcal{L}^{\perp} \subseteq \widehat{\Theta}$ as well. Now suppose that $\pi \in \widehat{\Theta}$. Then $\langle \alpha\pi, R \rangle = \alpha \langle \pi, R \rangle \geq 0$ for all $\alpha > 0$, so $\langle \pi, R \rangle \geq 0$ almost surely. From (ii), $\pi \notin \mathcal{A}$, so $\langle \pi, R \rangle = 0$ almost surely and thus $\pi \in \mathcal{L}^{\perp}$. Thus, $\widehat{\Theta} = \mathcal{L}^{\perp}$.

We then show that (iii) $\implies$ (iv). Note that $0 \in \mathcal{L}$ by definition and $0 \in \mathcal{L}^{\perp} \subseteq \Theta$, where 0 is the zero vector in $\mathbb{R}^n$, so $\Theta \cap \mathcal{L}$ is nonempty. Note that $\Theta \cap \mathcal{L}$ is also closed and convex by definitions. A theorem from convex analysis states that $\Theta \cap \mathcal{L}$ is bounded if and only if its recession cone $\widehat{\Theta \cap \mathcal{L}} = \{\pi \in \Theta \cap \mathcal{L} : \alpha\pi \in \Theta \cap \mathcal{L} \ \forall\, \alpha > 0\} = \{0\}$.[2] Since $\widehat{\Theta \cap \mathcal{L}}$ $\widehat{\Theta} \cap \mathcal{L} = \mathcal{L}^{\perp} \cap \mathcal{L} = \{0\}$ from (iii), the set $\Theta \cap \mathcal{L}$ is bounded and thus compact.

We lastly show that (iv) $\implies$ (i). By way of contradiction, suppose that there exists $\xi \in \mathcal{M}^{+}(\Omega, \mathcal{F})$ with $\mathbb{P}(\{\xi > 0\}) > 0$, and suppose that for all $n \in \mathbb{N}$, there exists $\pi_n \in \Theta$ such that $n^{-1}(1 + \langle \pi_n, R \rangle) \geq \xi$ almost surely. Then $1 + \langle \mathrm{proj}_{\mathcal{L}}(\pi_n), R \rangle \geq n\xi$ almost surely for all $n \in \mathbb{N}$. Since $\mathbb{P}(\{\xi > 0\}) > 0$ almost surely and $\mathrm{proj}_{\mathcal{L}}(\pi_n) \in \Theta \cap \mathcal{L}$, it follows that $\Theta \cap \mathcal{L}$ cannot be bounded and thus not compact, so we have a contradiction. $\square$

This provides some insight regarding the implications of no-arbitrage of the first kind. If NA(1) holds, then there does not exist any classical arbitrage opportunities that can be arbitrarily scaled, showing that NA(1) is weaker than NA. Additionally, all arbitrarily scalable allowed strategies are “redundant” in the sense that they do not generate returns, since $\langle \pi, R \rangle = 0$ for any $\pi \in \mathcal{L}^{\perp}$. Finally, the set of non-redundant allowed strategies is bounded and thus compact, which is useful for showing the existence of an optimal strategy.

[2] Rockafellar, T. (1970). *Convex Analysis.* Princeton University Press, Princeton (NJ).

2.2 Market viability

We now incorporate investor preferences, represented by a utility function. The investor chooses the strategy that maximizes expected utility. Let $\mathcal{U}$ be the set of all *random utility functions*, i.e. functions $U : \Omega \times \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ such that $U(\cdot, x)$ is $\mathcal{F}$-measurable and bounded below for all $x > 0$, and $U(\omega, \cdot)$ is continuous, strictly increasing, and concave almost surely. The following theorem shows market viability, or the existence of an optimal portfolio under no-arbitrage of the first kind.

Lemma 2.2.1. Let $\{f_n\}_{n\in\mathbb{N}} \in \mathcal{M}(\Omega, \mathcal{F})$. If there exists $g \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $f_n \leq g$ for all g, then $\limsup_{n\to\infty} \mathbb{E}[f_n] \leq \mathbb{E}\left[\limsup_{n\to\infty} f_n\right]$.

Proof. Note that $\{g - f_n\}_{n\in\mathbb{N}} \in \mathcal{M}^+(\Omega, \mathcal{F})$. Then by Fatou's Lemma and linearity,

$$\mathbb{E}\left[\liminf_{n\to\infty}(g - f_n)\right] \leq \liminf_{n\to\infty} \mathbb{E}[g - f_n]$$

$$\mathbb{E}[g] - \mathbb{E}\left[\liminf_{n\to\infty} f_n\right] \leq \liminf_{n\to\infty}(\mathbb{E}[g] - \mathbb{E}[f_n])$$

$$\mathbb{E}[g] - \mathbb{E}\left[\limsup_{n\to\infty} f_n\right] \leq \mathbb{E}[g] - \mathbb{E}\left[\liminf_{n\to\infty} f_n\right] \leq \liminf_{n\to\infty}(\mathbb{E}[g] - \mathbb{E}[f_n]) \leq \limsup_{n\to\infty}(\mathbb{E}[g] - \mathbb{E}[f_n])$$

Then $\mathbb{E}[g] - \mathbb{E}\left[\limsup_{n\to\infty} f_n\right] \leq \limsup_{n\to\infty}(\mathbb{E}[g] - \mathbb{E}[f_n]) = \mathbb{E}[g] - \limsup_{n\to\infty} \mathbb{E}[f_n]$. Note that $\mathbb{E}[g] < \infty$ by assumption, subtracting $\mathbb{E}[g]$ and multiplying by -1 on both sides gives $\limsup_{n\to\infty} \mathbb{E}[f_n] \leq \mathbb{E}\left[\limsup_{n\to\infty} f_n\right]$, the desired result. $\square$

Lemma 2.2.2. Let $U \in \mathcal{U}$ such that $U(1) \geq 0$. Then for all $x > 0$ and $\lambda \geq 1$, we have that $U^+(\lambda x) \leq 2\lambda(U^+(x) + U(2))$.

Proof. Suppose that $x \geq 2$. By monotonicity and concavity,

$$\begin{aligned} U^+(\lambda x) = U(\lambda x) &\leq U(x) + U'(x)(\lambda x - x) \\ &\leq U(x) + \frac{U(x) - U(1)}{x - 1} \cdot x(\lambda - 1) \\ &\leq U(x) + 2(\lambda - 1)(U(x) - U(1)) \leq 2\lambda \cdot U(x) \end{aligned}$$

Now suppose that $x < 2$. By monotonicity,

$$U^+(\lambda x) \leq U(2\lambda) \leq 2\lambda \cdot U(2)$$

Putting everything together, we have

$$U^+(\lambda x) \leq 2\lambda \cdot \max\{U(x), U(2)\} \leq 2\lambda(U^+(x) + U(2)) \quad \square$$

Theorem 2.2. (Theorem 2.5 in the paper) NA(1) holds if and only if for all random utility functions $U \in \mathcal{U}$ such that $\sup_{\pi\in\Theta} \mathbb{E}[U^+(V_1(\pi))] < \infty$, there exists a $\pi^\star \in \Theta \cap \mathcal{L}$ such that $\mathbb{E}[U(V_1(\pi^\star))] = \sup_{\pi\in\Theta} \mathbb{E}[U(V_1(\pi))]$.

Proof. Forward direction: Suppose that NA(1) holds. Since $\langle \pi, R\rangle = \langle \mathrm{proj}_{\mathcal{L}}(\pi), R\rangle$ for any $\pi \in \Theta$, maximizing $\mathbb{E}[U(V_1(\pi))]$ over Θ is equivalent to maximizing $\mathbb{E}[U(V_1(\pi))]$ over $\Theta \cap \mathcal{L}$. Define a function $u : \Theta \cap \mathcal{L} \to \mathbb{R} : \pi \mapsto \mathbb{E}[U(V_1(\pi))]$. Since NA(1) holds, $\Theta \cap \mathcal{L}$ is compact by Theorem 2.1, so it suffices to show that u is upper semi-continuous. Since $\Theta \cap \mathcal{L}$ is bounded, there exists a polyhedron $\mathcal{K} \subseteq \mathrm{span}(\Theta \cap \mathcal{L})$ such that $\Theta \cap \mathcal{L} \subseteq \mathcal{K}$. Since $V_1(\pi) = \langle \pi, R\rangle$ is a linear function in the π_i, the optimal solution when $V_1(\pi)$ is maximized over $\mathcal{K}$ lies on a vertex of $\mathcal{K}$. Let $p_1, \ldots, p_k$ be the vertices of $\mathcal{K}$. Then for all $\pi \in \Theta \in \mathcal{L}$

$$\langle \pi, R\rangle \leq \max_{i\in\{1,\ldots,k\}} \langle p_i, R\rangle$$

By the monotonicity of U, for all $\pi \in \Theta \in \mathcal{L}$

$$U^+(1 + \langle \pi, R\rangle) \leq \sum_{i=1}^{k} U^+(1 + \langle p_i, R\rangle)$$

Let $\zeta = \sum_{i=1}^{k} U^+(1 + \langle p_i, R\rangle)$. We would like to show that $\mathbb{E}[\zeta] < \infty$. Since U is bounded below, we can let $U(1) \geq 0$ without loss of generality. Let $\pi \in \Theta \cap \mathcal{L}$ and for each i, pick $\varepsilon_i \in (0, 1]$ such that $\pi + \varepsilon_i(p_i - \pi) \in \Theta \cap \mathcal{L}$. By concavity and Lemma 2.2.2, for each i,

$$\begin{aligned}
U^+(1 + \langle p_i, R\rangle) &= U^+(1 + \langle \pi, R\rangle + \langle p_i - \pi, R\rangle) \\
&\leq 2(U^+(1 + \langle \pi, R\rangle + \langle p_i - \pi, R\rangle) + U(2)) \\
&\leq \frac{2}{\varepsilon_i}(U^+(\varepsilon_i(1 + \langle \pi, R\rangle) + \varepsilon_i\langle p_i - \pi, R\rangle) + U(2)) \\
&= \frac{2}{\varepsilon_i}(U^+(\varepsilon_i + \langle \varepsilon_i\pi + \varepsilon_i(p_i - \pi), R\rangle) + U(2)) \\
&\leq \frac{2}{\varepsilon_i}(U^+(1 + \langle \pi + \varepsilon_i(p_i - \pi), R\rangle) + U(2))
\end{aligned}$$

If $\sup_{\pi\in\Theta} \mathbb{E}[U^+(V_1(\pi)] < \infty$, then $\mathbb{E}[U^+(1 + \langle \pi + \varepsilon_i(p_i - \pi), R\rangle)] < \infty$ and $\mathbb{E}[U(2)] < \infty$ since $\pi + \varepsilon_i(p_i - \pi) \in \Theta \cap \mathcal{L}$. Then $\mathbb{E}[\zeta] < \infty$, so $\zeta \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{\pi_n\}_{n\in\mathbb{N}} \in \Theta \cap \mathcal{L}$ be a sequence of portfolios converging to a $\pi \in \Theta \cap \mathcal{L}$. Note that $u(\pi_n) = \mathbb{E}[U(1 + \langle \pi_n, R\rangle)] \leq \zeta$ almost surely for all n. By Lemma 2.2.1 and the continuity of U,

$$\limsup_{n\to\infty} u(\pi_n) = \limsup_{n\to\infty} \mathbb{E}[U(1 + \langle \pi_n, R\rangle)] \leq \mathbb{E}\left[\limsup U(1 + \langle \pi_n, R\rangle)\right] = u(\pi)$$

showing that u is indeed upper semi-continuous. Then $u(\pi) = \mathbb{E}[V_1(\pi)]$ attains a maximum on $\Theta \cap \mathcal{L}$, so there exists an optimal portfolio $\pi^\star$ such that $u(\pi^\star) \geq u(\pi)$ for all $\pi \in \Theta \cap \mathcal{L}$.

Backward direction: by way of contradiction, suppose that NA(1) does not hold and let $\pi^\star \in \Theta \cap \mathcal{L}$ maximize $\mathbb{E}[U(V_1(\pi)]$ over $\Theta \cap \mathcal{L}$. Since NA(1) does not hold, by Theorem 2.1, there exists $\pi \in \mathcal{A} \cap \widehat{\Theta}$. Note that $\pi^\star + \pi \in \Theta \cap \mathcal{L}$ and $\mathbb{E}[U(V_1(\pi^\star + \pi))] > \mathbb{E}[U(V_1(\pi^\star)]$, so $\pi^\star$ is not optimal, which is a contradiction. $\square$

2.3 Numéraire portfolio

Deflators can be viewed as normalizers for the expected value of a portfolio. A *deflator* is a random variable $Z \in \mathcal{M}^+(\Omega, \mathcal{F})$ with $\mathbb{P}[Z > 0] > 0$ such that $\mathbb{E}[ZV_1(\pi)] \leq 1$ for all $\pi \in \Theta$. The set of all deflators is denoted by $\mathcal{D}$. A special case of a deflator is a *numéraire portfolio* $\rho \in \Theta$ such that $\mathbb{E}[V_1(\pi)/V_1(\rho)] \leq 1$ for all $\pi \in \Theta$, or equivalently, such that $1/V_1(\rho)$ is a deflator. The following theorem shows the existence of deflators and of numéraire portfolio under no-arbitrage of the first kind.

Theorem 2.3. (Theorem 2.9 in the paper) The following are equivalent:

(i) NA(1) holds;

(ii) $\mathcal{D} \neq \varnothing$;

(iii) there exists a numéraire portfolio $\rho \in \Theta$.

Proof. We first show that (i) $\implies$ (iii). Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions such that $f_k : \mathbb{R}^n \to (0, 1]$ and $\mathbb{E}[\log(1 + \|R\|) \cdot f_k(R)] < \infty$ for all $k \in \mathbb{N}$, and $f_k \nearrow 1$ and $n \to \infty$. For each $k \in \mathbb{N}$, define a function $U_k : \Omega \times (0, \infty) \to \mathbb{R} \cup \{-\infty\} : (\omega, x) \mapsto \log(x) \cdot f_k(R(\omega))$ with $U_n(\omega, 0) = \lim_{x\to 0^+} U_k(\omega, x) = -\infty$.

For fixed $\omega \in \Omega$, $U_k(\omega, \cdot) = c \log(x)$ where $c = f_k(R(\omega))$ is fixed, so U_k is continuous, strictly increasing, and concave for all $\omega \in \Omega$. For fixed $x \in (0, \infty)$, $U_k(\cdot, x) = c \cdot f_k(R(\omega))$ where $c = \log(x)$ is fixed, so U_k is $\mathcal{F}$-measurable as R is $\mathcal{F}$-measurable and U_k is bounded below as $U_k(\omega, 0) = -\infty$ and $U(\omega, \cdot)$ is strictly increasing. Then $U_k \in \mathcal{U}$ for all $k \in \mathbb{N}$ and

$$\begin{aligned}
\mathbb{E}[U_k^+(1 + \langle \pi, R\rangle)] &= \mathbb{E}[(\log(1 + \langle \pi, R\rangle) \cdot f_k(R))^+] \\
&\leq \mathbb{E}[\log(1 + \|\pi\|\|R\|) \cdot f_k(R)] \\
&\leq \mathbb{E}[(\log(\|\pi\|\|R\|) - 1) \cdot f_k(R)] \\
&\leq \mathbb{E}[\log \|\pi\| \cdot f_k(R)] + \mathbb{E}[\log \|R\| \cdot f_k(R)] - \mathbb{E}[f_k(R)] \\
&\leq \log \|\pi\| + \mathbb{E}[\log(1 + \|R\|) \cdot f_k(R)] \\
&\leq \|\pi\| + \mathbb{E}[\log(1 + \|R\|) \cdot f_k(R)] < \infty
\end{aligned}$$

where we use the Cauchy-Schwarz inequality, the inequality $\log x \leq x - 1$ for $x > 0$ and the fact that $\mathbb{E}[f_k(R)] \leq 1$. Then $U_k(V_1(\pi))$ is bounded for all $\pi \in \Theta$ and $k \in \mathbb{N}$.

Since NA(1) holds, $\Theta \cap \mathcal{L}$ is bounded by Theorem 2.1. For each $k \in \mathbb{N}$, there exists some $\rho_k^\star \in \Theta \cap \mathcal{L}$ that maximizes U_k over $\Theta \cap \mathcal{L}$ by Theorem 2.2. For some $\pi \in \Theta$ and $\varepsilon \in (0,1)$, let $\pi^\varepsilon = \varepsilon\pi + (1-\varepsilon)\rho_k^\star$. By the optimality of $\rho_k^\star$ and the inequality $\log x \geq (x-1)/x$,

$$\begin{aligned}
\frac{1}{\varepsilon} \cdot \mathbb{E}[U_k(1 + \langle \rho_k^\star, R\rangle)] &\geq \frac{1}{\varepsilon} \cdot \mathbb{E}[U_k(1 + \langle \pi^\varepsilon, R\rangle)] \\
0 &\geq \frac{1}{\varepsilon}\left(\mathbb{E}[U_k(1 + \langle \pi^\varepsilon, R\rangle)] - \mathbb{E}[U_k(1 + \langle \rho_k^\star, R\rangle)]\right) \\
&\geq \frac{1}{\varepsilon}\left(\mathbb{E}[U_k(V_1(\pi^\varepsilon))] - \mathbb{E}[U_k(V_1(\rho_k^\star))]\right) \\
&= \frac{1}{\varepsilon} \cdot \mathbb{E}\left[\log(V_1(\pi^\varepsilon)) \cdot f_k(R) - \log(V_1(\rho_k^\star)) \cdot f_k(R)\right] \\
&= \frac{1}{\varepsilon} \cdot \mathbb{E}\left[\log\left(\frac{V_1(\pi^\varepsilon)}{V_1(\rho_k^\star)}\right) \cdot f_k(R)\right] \\
&\geq \frac{1}{\varepsilon} \cdot \mathbb{E}\left[\frac{\varepsilon\langle \pi - \rho_k^\star, R\rangle}{1 + \langle \rho_k^\star, R\rangle - \varepsilon\langle \pi - \rho^\star - k\rangle} \cdot f_k(R)\right] \\
&= \mathbb{E}\left[\frac{\langle \pi - \rho_k^\star, R\rangle}{1 + \langle \rho_k^\star, R\rangle - \varepsilon\langle \pi - \rho^\star - k\rangle} \cdot f_k(R)\right]
\end{aligned}$$

Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a sequence of real numbers given by $\varepsilon_j = 1/2j$ and let $\{g_j\}_{j\in\mathbb{N}}$ be a sequence of functions given by

$$g_j = \mathbb{E}\left[\frac{\langle \pi - \rho_k^\star, R\rangle}{1 + \langle \rho_k^\star, R\rangle - \varepsilon_j\langle \pi - \rho_k^\star\rangle} \cdot f_k(R)\right]$$

Note that for all $\varepsilon \in (0, 1/2)$, $y > 0$, and $x \geq -y$,

$$\frac{x}{y + \varepsilon x} \geq \frac{x}{y + x/2} \geq -2$$

Then $g_k \geq -2$ for all $k \in \mathbb{N}$. Since $g_k \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ for all $k \in \mathbb{N}$, by Fatou's Lemma,

$$\mathbb{E}\left[\liminf_{j\to\infty} g_j\right] = \mathbb{E}\left[\frac{\langle \pi - \rho_k^\star, R\rangle}{1 + \langle \rho_k^\star, R\rangle} \cdot f_k(R)\right] \leq \liminf_{j\to\infty} \mathbb{E}[g_j] \leq 0 \text{ for all } k \in \mathbb{N}$$

$$\implies \mathbb{E}\left[\frac{\langle \pi - \rho_k^\star, R\rangle}{1 + \langle \rho_k^\star, R\rangle} \cdot f_k(R)\right] \leq 0 \text{ for all } k \in \mathbb{N}$$

By Theorem 2.1, $\Theta \cap \mathcal{L}$ is compact, so the sequence of portfolios $\{\rho_k^\star\}_{k\in\mathbb{N}}$ converges to some $\rho^\star \in \Theta \cap \mathcal{L}$ as $n \to \infty$. Let $\{h_k\}_{k\in\mathbb{N}}$ be a sequence of functions given by

$$h_k = \mathbb{E}\left[\frac{\langle \pi - \rho_k^\star, R\rangle}{1 + \langle \rho_k^\star, R\rangle} \cdot f_k(R)\right]$$

Note that $\langle \pi - \rho_k^\star, R\rangle/(1 + \langle \rho_k^\star, R\rangle) \geq -1$ almost surely for all $k \in \mathbb{N}$ by assumption, and that $f_n \to 1$ as $n \to \infty$. Then by Fatou's Lemma,

$$
\begin{aligned}
\mathbb{E}\left[\liminf_{k\to\infty} h_k\right] = \mathbb{E}\left[\frac{\langle \pi - \rho^\star, R\rangle}{1+\langle \rho^\star, R\rangle}\right] &\le \liminf_{k\to\infty} \mathbb{E}[h_k] \le 0 \text{ for all } k \in \mathbb{N} \\
\mathbb{E}\left[\frac{\langle \pi - \rho^\star, R\rangle}{1+\langle \rho^\star, R\rangle}\right] &\le 0 \\
\mathbb{E}\left[\frac{1+\langle \pi, R\rangle - (1+\langle \rho^\star, R\rangle)}{1+\langle \rho^\star, R\rangle}\right] &\le 0 \\
\mathbb{E}\left[\frac{1+\langle \pi, R\rangle}{1+\langle \rho^\star, R\rangle}\right] &\le \mathbb{E}\left[\frac{1+\langle \rho^\star, R\rangle}{1+\langle \rho^\star, R\rangle}\right] = 1 \\
\mathbb{E}[V_1(\pi)/V_1(\rho^\star)] &\le 1
\end{aligned}
$$

so $\rho^\star$ is a numéraire portfolio.

We then show that (iii) $\implies$ (ii). This is clear since $\rho^\star \in \mathcal{D}$ by definition.

We lastly show that (ii) $\implies$ (i). Let $Z \in \mathcal{D}$ and $\xi \in \mathcal{M}^+(\Omega, \mathcal{F})$ with $\mathbb{P}[\xi > 0] > 0$ such that for every $k \in \mathbb{N}$, there exists $\pi_k \in \Theta$ such that $V_1(\pi_k, 1/k) \ge \xi$ almost surely. Then

$$
\mathbb{E}[Z\xi] \le \mathbb{E}[Z \cdot V_1(\pi_k, 1/k)] = \frac{1}{k} \cdot \mathbb{E}[Z \cdot V_1(\pi_k)] \le \frac{1}{k} \text{for all } k \in \mathbb{N}
$$

by definition. Since $Z > 0$ almost surely, taking $n \to \infty$ means that $\xi = 0$ almost surely, showing that NA(1) holds. □

2.4 Risk-neutral measure

In financial theory, the Fundamental Theorem of Asset Pricing states that a market is arbitrage-free if and only if there exists a risk-neutral probability measure that is equivalent to the original probability measure (i.e. both measures agree on events with measure zero). Below, we state an analogous theorem under classical no-arbitrage.

Theorem 2.4. (Theorem 2.12 in the paper) Assuming that cone(Θ) is closed, then NA holds if and only if there exists a probability measure $\mathbb{P}^\star \sim \mathbb{P}$ such that $\mathbb{E}_{\mathbb{P}^\star}[V_1(\pi)] \le 1$ for all $\pi \in \text{cone}(\Theta)$.

Proof. Forward direction: suppose that NA holds and let $C = \text{cone}(\Theta) \cap \mathcal{L}$. Note that $\mathcal{A} \cap \Theta = \varnothing$ if and only if $\mathcal{A} \cap \text{cone}(\Theta) = \varnothing$ if and only if $\mathcal{A} \cap C = \varnothing$. Let $\mathbb{P}'$ be an equivalent probability measure specified by the Radon-Nikodym derivative

$$
\frac{d\mathbb{P}'}{d\mathbb{P}} = \frac{\exp\left(-\|R\|^2\right)}{\mathbb{E}\left[\exp\left(-\|R\|^2\right)\right]}
$$

Define a function $f : \Omega \times C \to \mathbb{R} : (\omega, \pi) \mapsto \mathbb{E}_{\mathbb{P}'}[\exp(-1 - \langle \pi, R(\omega)\rangle)]$.

Let $\pi \in C$ and $\{\pi_k\}_{k\in\mathbb{N}}$ be a sequence of portfolios converging to π. Let $\{f(\omega,\pi_k)\}_{k\in\mathbb{N}}$ be a sequence of functions and note that $f(\omega,\pi_k) \to f(\omega,\pi)$ as $n \to \infty$. By Fatou's Lemma,

$$\mathbb{E}_{\mathbb{P}'}\left[\liminf_{k\to\infty} f(\omega,\pi_k)\right] = \mathbb{E}_{\mathbb{P}'}[f(\omega,\pi)] = f(\omega,\pi) \le \liminf_{k\to\infty} \mathbb{E}_{\mathbb{P}'}[f(\omega,\pi_k)] \le \liminf_{k\to\infty} f(\omega,\pi_k)$$

so $f(\omega,\pi)$ is lower semi-continuous. Note that the choice of $\omega \in \Omega$ does not affect f, so we will omit the ω from here. Since C is closed and $f(\cdot,\pi)$ is convex, a theorem from convex analysis implies that f has a minimizer $\pi^\star \in C$ if it has no directions of recession in common with C, so we would like to show that

$$\lim_{\gamma\to\infty} \frac{f(\gamma\pi)}{\gamma} > 0 \text{ for all } \pi \in C\backslash\{0\}$$

By way of contradiction, suppose that there exists a $\pi \in C\backslash\{0\}$ such that $\lim_{\gamma\to\infty} f(\gamma\pi)/\gamma \le 0$. Applying Fatou's Lemma again, we have

$$\begin{aligned} 0 \ge \lim_{\gamma\to\infty} \frac{f(\gamma\pi)}{\gamma} &\ge \mathbb{E}_{\mathbb{P}'}\left[\liminf_{\gamma\to\infty} \frac{\exp(-1-\gamma\langle\pi,R\rangle)}{\gamma}\right] \\ &\ge \mathbb{E}_{\mathbb{P}'}\left[\liminf_{\gamma\to\infty} \frac{\exp(-1-\gamma\langle\pi,R\rangle)}{\gamma} \cdot \mathbb{1}\{\langle\pi,R\rangle < 0\}\right] \end{aligned}$$

Since $\exp(-1-\gamma\langle\pi,R\rangle) > 0$ always, it follows that $\langle\pi,R\rangle \ge 0$ almost surely, so $\pi \in \mathcal{A}$. However, since $\pi \in \mathcal{L}$ as well, we have a contradiction under NA. Then there exists some $\pi^\star \in C$ such that $f(\pi^\star) \le f(\pi)$ for all $\pi \in C$.

Let $g_i : \Omega \times \mathbb{R} \to \mathbb{R} : (\omega, \pi_i; \pi_{-i}) \mapsto \exp(-1-\langle\pi,R\rangle)$. Note that $g_i \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P}')$ since

$$\begin{aligned} \mathbb{E}_{\mathbb{P}'}\left[|g_i|\right] = \mathbb{E}_{\mathbb{P}'}[\exp(-1-\langle\pi,R\rangle)] &= \int_\Omega \exp(-1-\langle\pi,R\rangle)\, d\mathbb{P}' \\ &= \int_\Omega \exp(-1-\langle\pi,R\rangle) \cdot \frac{d\mathbb{P}'}{d\mathbb{P}}\, d\mathbb{P} \\ &= \int_\Omega \exp(-1-\langle\pi,R\rangle) \cdot \frac{\exp\left(-\|R\|^2\right)}{\mathbb{E}\left[\exp(-\|R\|)^2\right]}\, d\mathbb{P} \\ &= \frac{1}{\mathbb{E}\left[\exp\left(-\|R\|^2\right)\right]} \int_\Omega \exp\left(-1-\langle\pi,R\rangle-\|R\|^2\right) d\mathbb{P} \\ &= \frac{\mathbb{E}\left[\exp\left(-1-\langle\pi,R\rangle-\|R\|^2\right)\right]}{\mathbb{E}\left[\exp\left(-\|R\|^2\right)\right]} \le 1 \end{aligned}$$

where the last line follows since $1+\langle\pi,R\rangle \ge 0$ almost surely, by assumption. Note that

$$\frac{\partial}{\partial\pi_i} g(\omega,\pi_i) = -R_i(\omega) \cdot \exp(-1-\langle\pi,R(\omega)\rangle) \le |R_i(\omega)|$$

since $\exp(-x) \le 1$ for $x \ge 0$ and $1+\langle\pi,R\rangle \ge 0$ almost surely, by assumption. Then

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}'}[|R_i|] = \int_\Omega |R_i| \, d\mathbb{P}' &= \int_\Omega |R_i| \cdot \frac{d\mathbb{P}'}{d\mathbb{P}} \, d\mathbb{P} \\
&= \int_\Omega |R_i| \cdot \frac{\exp\bigl(-\|R\|^2\bigr)}{\mathbb{E}\left[\exp(-\|R\|)^2\right]} \, d\mathbb{P} \\
&\leq \int_\Omega \frac{\|R\| \cdot \exp\bigl(-\|R\|^2\bigr)}{\mathbb{E}\left[\exp\bigl(-|R|^2\bigr)\right]} \, d\mathbb{P} \\
&= \frac{1}{\mathbb{E}\left[\exp\bigl(-\|R\|^2\bigr)\right]} \int_\Omega \|R\| \cdot \exp\bigl(-\|R\|^2\bigr) \, d\mathbb{P} \\
&= \frac{1}{\mathbb{E}\left[\exp\bigl(-\|R\|^2\bigr)\right]} \int_\Omega \|R\| \cdot \exp\bigl(-\|R\|^2\bigr) \cdot \frac{d\mathbb{P}}{d\lambda} \, d\lambda \\
&\leq \frac{1}{\mathbb{E}\left[\exp\bigl(-\|R\|^2\bigr)\right]} \int_\Omega \frac{d\mathbb{P}}{d\lambda} \, d\lambda = \frac{1}{\mathbb{E}\left[\exp\bigl(-\|R\|^2\bigr)\right]} < \infty
\end{aligned}
$$

where λ is the Lebesgue measure and the last lines follow since $x \cdot \exp(-x^2) \leq 1$ for all $x \geq 0$ and $d\mathbb{P}/d\lambda$ is a density. Then $|R_i| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}')$.

Since $g_i \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\partial g(\omega, \pi_i)/\partial \pi_i$ is defined for all $(\omega, \pi_i) \in \Omega \times \mathbb{R}$, is $\mathcal{F}$-measurable, and is bounded by $|R_i| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}')$, we can differentiate under the integral, i.e.

$$
\begin{aligned}
\frac{\partial}{\partial \pi_i} f(\pi) = \frac{\partial}{\partial \pi_i} \mathbb{E}[\exp(-1 - \langle \pi, R \rangle)] &= \frac{\partial}{\partial \pi_i} \mathbb{E}[g_i] \\
&= \frac{\partial}{\partial \pi_i} \int_\Omega g_i \, d\mathbb{P}' \\
&= \int_\Omega \frac{\partial}{\partial \pi_i} g_i \, d\mathbb{P}' \\
&= \int_\Omega -R_i \cdot \exp(-1 - \langle \pi, R \rangle) \, d\mathbb{P}' \\
&= \mathbb{E}_{\mathbb{P}'}[-R_i \cdot \exp(-1 - \langle \pi, R \rangle)]
\end{aligned}
$$

Repeating this logic for $i = 1, \ldots, n$, for any $\pi \in C$, we have

$$
\nabla f(\pi) = \mathbb{E}_{\mathbb{P}'}[-R \cdot \exp(-1 - \langle \pi, R \rangle)]
$$

By a theorem from convex optimization, for the minimizer $\pi^\star$ and any other $\pi \in C$, we have

$$
\begin{aligned}
0 \geq \langle \pi, -\nabla f(\pi^\star) \rangle &= \langle \pi, \mathbb{E}_{\mathbb{P}'}[R \cdot \exp(-1 - \langle \pi^\star, R \rangle)] \rangle \\
&= \mathbb{E}_{\mathbb{P}'}[\exp(-1 - \langle \pi^\star, R \rangle) \cdot \langle \pi, R \rangle] \\
&= \mathbb{E}_{\mathbb{P}'}[\exp(-V_1(\pi^\star)) \cdot \langle \pi, R \rangle] \\
&= \int_\Omega \exp(-V_1(\pi^\star)) \cdot \langle \pi, R \rangle \, d\mathbb{P}'
\end{aligned}
$$

$$
\begin{aligned}
&= \int_\Omega \exp(-V_1(\pi^\star)) \cdot \langle \pi, R\rangle \cdot \frac{d\mathbb{P}'}{d\mathbb{P}}\, d\mathbb{P} \\
&= \int_\Omega \exp(-V_1(\pi^\star)) \cdot \langle \pi, R\rangle \cdot \frac{\exp\left(-\|R\|^2\right)}{\mathbb{E}\left[\exp\left(-\|R\|^2\right)\right]}\, d\mathbb{P} \\
&= \frac{1}{\mathbb{E}\left[\exp\left(-\|R\|^2\right)\right]} \int_\Omega \exp\left(-V_1(\pi^\star) - \|R\|^2\right) \cdot \langle \pi, R\rangle\, d\mathbb{P} \\
&= \frac{1}{\mathbb{E}\left[\exp\left(-\|R\|^2\right)\right]} \cdot \mathbb{E}\left[\exp\left(-V_1(\pi^\star) - \|R\|^2\right)\langle \pi, R\rangle\right]
\end{aligned}
$$

Since $\mathbb{E}\left[\exp\left(-\|R\|^2\right)\right] > 0$, it follows that $\mathbb{E}\left[\exp\left(-V_1(\pi^\star) - \|R\|^2\right)\langle \pi, R\rangle\right] \leq 0$. Let $\mathbb{P}^\star$ be an equivalent probability measure specified by the Radon-Nikodym derivative

$$
\frac{d\mathbb{P}^\star}{d\mathbb{P}} = \frac{\exp(-V_1(\pi^\star)) - \|R\|^2}{\mathbb{E}\left[\exp(-V_1(\pi^\star)) - \|R\|^2\right]}
$$

Then for any $\pi \in C$ and hence for any $\pi \in \operatorname{cone}(\Theta)$,

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}^\star}[V_1(\pi)] &= \int_\Omega V_1(\pi)\, d\mathbb{P}^\star \\
&= \int_\Omega V_1(\pi) \cdot \frac{d\mathbb{P}^\star}{d\mathbb{P}}\, d\mathbb{P} \\
&= \int_\Omega V_1(\pi) \cdot \frac{\exp\left(-V_1(\pi^\star) - \|R\|^2\right)}{\mathbb{E}\left[\exp\left(-V_1(\pi^\star) - \|R\|^2\right)\right]}\, d\mathbb{P} \\
&= \frac{1}{\mathbb{E}\left[\exp\left(-V_1(\pi^\star) - \|R\|^2\right)\right]} \cdot \int_\Omega (1 + \langle \pi, R\rangle) \cdot \exp\left(-V_1(\pi^\star) - \|R\|^2\right) d\mathbb{P} \\
&= \frac{1}{\mathbb{E}\left[\exp\left(-V_1(\pi^\star) - \|R\|^2\right)\right]} \cdot \int_\Omega \exp\left(-V_1(\pi^\star) - \|R\|^2\right) d\mathbb{P} \\
&\qquad + \frac{1}{\mathbb{E}\left[\exp\left(-V_1(\pi^\star) - \|R\|^2\right)\right]} \cdot \int_\Omega \exp\left(-V_1(\pi^\star) - \|R\|^2\right) \cdot \langle \pi, R\rangle\, d\mathbb{P} \\
&= 1 + \frac{\mathbb{E}\left[\exp\left(-V_1(\pi^\star) - \|R\|^2\right) \cdot \langle \pi, R\rangle\right]}{\mathbb{E}\left[\exp\left(-V_1(\pi^\star) - \|R\|^2\right)\right]} \leq 1
\end{aligned}
$$

where the last line follows since $\mathbb{E}\left[\exp\left(-V_1(\pi^\star) - \|R\|^2\right)\right] > 0$ since $\exp(-x) > 0$ for all x and from above, $\mathbb{E}\left[\exp\left(-V_1(\pi^\star) - \|R\|^2\right) \cdot \langle \pi, R\rangle\right] \leq 0$.

Backward direction: suppose that there exists an probability measure $\mathbb{P}^\star \sim \mathbb{P}$ such that $\mathbb{E}_{\mathbb{P}^\star}[V_1(\pi)] \leq 1$ for all $\pi \in \operatorname{cone}(\Theta)$. Then for every $\pi \in \Theta$, we have $\mathbb{E}_{\mathbb{P}^\star}[\langle \pi, R\rangle] \leq 0$. By way of contradiction, suppose that there exists a $\pi \in \mathcal{A} \cap \Theta$. Then $\langle \pi, R\rangle > 0$ almost surely $\mathbb{P}$. Since $\mathbb{P}^\star \sim \mathbb{P}$, we have $\mathbb{P}^\star[\langle \pi, R\rangle \leq 0] = \mathbb{P}[\langle \pi, R\rangle \leq 0] = 0$, so we must have $\mathbb{E}_{\mathbb{P}^\star}[\langle \pi, R\rangle] > 0$, a contradiction. Thus, $\mathcal{A} \cap \Theta = \varnothing$. $\square$

The requirement that cone(Θ) is closed is quite necessary for Theorem 2.4, as the following counterexample based off another paper shows.[3]

Example 2.5. Consider a three-asset binomial model with $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{F} = 2^\Omega$, $d = 3$, $R_1(\omega_1) = (1, 0, 0)$ and $R_1(\omega_2) = (0, 1, 0)$. Let the allowed set be $\Theta = \{x \in \mathbb{R}^3 : x_1 < 0, x_2 > 0, x_3 = 0\}$. Note that Θ is convex but not closed, and that $\Theta = \text{cone}(\Theta)$. For any $\pi \in \Theta$, we have $\langle \pi, R_1(\omega_1)\rangle < 0$, so $\mathcal{A} \cap \Theta = \varnothing$ and thus NA holds. Let $\mathbb{P}^\star$ be a probability measure specified by $p_1^\star = \mathbb{P}^\star[\omega_1]$ and $p_2^\star = \mathbb{P}^\star[\omega_2]$. Then for any $\pi \in \text{cone}(\Theta)$,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^\star}[V_1(\pi)] = 1 + \mathbb{E}_{\mathbb{P}^\star}[\langle \pi, R\rangle] &= 1 + p_1^\star \pi_1 + p_2^\star \pi_2 \\ &= 1 + p_1^\star \pi_1 + (1 - p_1^\star)\pi_2 = (1 + \pi_2) - p_1^\star(\pi_1 - \pi_2)\end{aligned}$$

Given $p_1^\star$ and π_1, solving $\mathbb{E}_{\mathbb{P}^\star}[V_1(\pi)] > 1$ gives

$$\pi_2 > -\frac{p_1^\star \pi_1}{1 - p_1^\star}$$

Since $\pi_1 < 0$, $-p_1^\star \pi_1/(1 - p_1^\star) > 0$, so there exists a valid choice of π such that $\mathbb{E}_{\mathbb{P}^\star}[V_1(\pi)] > 1$. Thus, there cannot exist a $\mathbb{P}^\star$ such that $\mathbb{E}_{\mathbb{P}^\star}[V_1(\pi)] \le 1$ for all $\pi \in C = \Theta$ in this case. $\square$

We now provide an example of a financial market where NA holds, showing the existence of a risk-neutral measure, numéraire portfolio, and optimal portfolio.

Example 2.6. Let $d = 1$, $\Theta = [0, \alpha]$ for $\alpha < 1/2$ and $R = \exp(X) - 1$ where $X \sim \mathcal{N}(0, 1)$. Clearly, NA holds and Θ is compact. Consider a probability measure $\mathbb{P}^\star$ specified by

$$\frac{d\mathbb{P}^\star}{d\mathbb{P}} = \exp(-X - 1/2)$$

Then for any $\theta \in [0, 1]$,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^\star}[V_1(\theta)] = \mathbb{E}_{\mathbb{P}^\star}[1 + \theta R] &= 1 + \theta \cdot \mathbb{E}[R] \\ &= 1 + \theta \int_\Omega R \cdot \frac{d\mathbb{P}^\star}{d\mathbb{P}} \cdot \frac{d\mathbb{P}}{d\lambda}\, d\lambda \\ &= 1 + \theta \int_{\mathbb{R}} (\exp(x) - 1)(\exp(-x - 1/2)) \left(\frac{\exp(-x^2/2)}{\sqrt{2\pi}}\right) dx \\ &= 1 - \theta \cdot \frac{\sqrt{e} - 1}{\sqrt{e}} \le 1\end{aligned}$$

so $\mathbb{P}^\star$ satisfies $\mathbb{E}_{\mathbb{P}^\star}[V_1(\pi)] \le 1$ for all $\pi \in \Theta$.

[3] Rokhlin, D. B. (2005). "An extended version of the Daland-Morton-Willinger Theorem under portfolio constraints." *Theory of Probability and Its Applications* 49 (3), pp. 429-443.

Note that R is log-normally distributed and that the function $\varphi(r) = (1+\theta r)/(1+\alpha r)$ is convex for $0 \leq \theta \leq \alpha < 1/2$. Then choosing $\eta = \alpha$ gives

$$\mathbb{E}\left[\frac{V_1(\theta)}{V_1(\eta)}\right] = \mathbb{E}\left[\frac{1+\theta R}{1+\alpha R}\right] \leq \frac{1+\theta \cdot \mathbb{E}[R]}{1+\alpha \cdot \mathbb{E}[R]} = \frac{1+\theta(\sqrt{e}-1)}{1+\alpha(\sqrt{e}-1)} \leq 1$$

noting that $\theta \leq \alpha$ and $\sqrt{e}-1 \geq 0$, so $\rho = \alpha$ is a numéraire portfolio for this market. Since $\mathbb{E}[V_1(\theta)] = 1 + \theta \cdot \mathbb{E}[R]$ and $\mathbb{E}[R] \geq 0$, choosing $\rho = \alpha$ gives the optimal portfolio as well. $\square$

3 Conclusion

This report has summarized key theorems in mathematical finance under the framework of no-arbitrage of the first kind. We show some basic characterizations of the set of allowed strategies Θ, the existence of an optimal portfolio, and the existence of numéraire portfolios under no-arbitrage of the first kind. Since this arbitrage concept is weaker than classical no-arbitrage, it is insightful that these properties still hold. We conclude by showing the fundamental theorem of asset pricing under no-arbitrage and a counterexample to that theorem when one of the assumptions is relaxed. Finally, a toy example illustrating the results in section 2 is presented. It is noteworthy how these theorems of mathematical finance have a rigorous backing in measure-theoretic probability (e.g. Fatou's Lemma, Radon-Nikodym derivatives) and convex optimization.

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