Optimal Stopping in a Dynamic Auction

ECON 417

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1 Introduction

In this project, we investigate a simple dynamic auction with computational methods such as dynamic programming, Gaussian quadrature, and Monte Carlo methods. To briefly describe the auction, a seller has one indivisible item is up for auction among N agents over T periods where the agents have values that evolve stochastically and dynamically. Specifically, we are interested in maximizing the expected revenue of an incentive-compatible auction. Results from static mechanism design have been refined for the dynamic setting to give a concise expression for expected revenue such that the selling price of the item does not have to be calculated explicitly. Taking the optimal expected revenue as our value function, we can compute the value function at time zero by iterating backwards from time T and taking expected values of future value functions over a grid. We differentiate the cases of one agent, two agents, and many agents on two periods and many periods, and explore comparative statics, such as changing the value distributions, discount factors, and value process parameters, in the case of one agent in two periods.

2 The Model

2.1 A Dynamic Auction

This auction is based off of Section 5.2 in Bergemann and Välimäki (2019). There are N agents indexed $i = 1, \ldots, N$ and $T + 1$ periods indexed $t = 0, 1, \ldots, T$ with one indivisible item up for auction, meaning that the seller must decide if and when to sell it, and whom to sell it to. This means that the set of feasible allocations at time t is $\mathcal{X}_t = \{0, e_1, \ldots, e_N\}$, where e_i is the i-th basis vector in $\mathbb{R}^N$, if $x_s = 0$ for all $s < t$, and $\mathcal{X}_t = \{0\}$ if otherwise. Agent i has an initial type, or value on the item, $\theta_{i,0} \sim G_i$ which evolves according to

$$\theta_{i,t} = \gamma\theta_{i,t-1} + \varepsilon_{i,t}$$

where $\gamma \in [0, 1]$ and $\varepsilon_{i,t} \sim H_i$. Define $\Theta_i = \text{supp}\{\theta_{i,t}\}_{t=0}^{T} = [\underline{\theta}_i, \overline{\theta}_i]$ and $\Theta = \bigtimes_{i=1}^{N} \Theta_i$. We assume that G_i and H_i are public knowledge, but the $\theta_{i,t}$ are private at all times.

2.2 Static Mechanisms

Recalling the static setting, a direct mechanism (x, p) describes a Bayesian game where agents simultaneously submit sealed bids $b = (b_1, \ldots, b_N) \in \mathbb{R}^N$. Given the bids, the mechanism then decides an allocation of the item(s) and payments for each agent with the allocation rule $x : \mathbb{R}^N \to \mathcal{X}$ and payment rule $p : \mathbb{R}^N \to \mathbb{R}^N$. Then an agent's (pure) bidding strategy in the static setting is a map $b_i : \Theta_i \to \mathbb{R}$ from type to bid, and an agent's utility is

$$u_i(b) = \theta_i \cdot x_i(b) - p_i(b)$$

An agent has a truthful bidding strategy if $b_i(\theta_i) = \theta_i$ for all $\theta_i \in \Theta_i$, and a mechanism is Bayesian incentive-compatible (BIC) if each agent truthfully bidding is a Bayesian-Nash equilibrium, i.e. truthful bidding maximizes the expected utility of agent i given that agent i knows their type θ_i and the distribution of $\theta_{-i}|\theta_i$.

2.3 Dynamic Mechanisms

In the Bayesian game that corresponds to a dynamic mechanism, agents simultaneously submit sealed bids $b_t = (b_{1,t}, \ldots, b_{N,t})$ at each period t. Given the bids at period t, the mechanism then decides an allocation x_t of the item(s) and payments p_t for each agent for period t. The bids and outcomes then become public knowledge in the form of a history $h_t = \{(b_s, x_s, p_s)\}_{s<t} \in H_t$. By convention, $H_0 = \varnothing$.

A dynamic mechanism $\{(x_t, p_t)\}_{t=0}^T \equiv (x, p)$ is formally specified by a periodic allocation rule $x_t : \mathbb{R}^N \times H_t \to \mathcal{X}_t$ and a periodic payment rule $p_t : \mathbb{R}^N \times H_t \to \mathbb{R}^N$ for each period t. Then an agent's periodic bidding strategy in the dynamic setting is a map $b_{i,t} : \Theta_i \times H_t \to \mathbb{R}$ from type and history to bid, and an agent's utility under a discount factor $\delta \in (0, 1)$ is

$$u_{i,t}(b_t) = \theta_{i,t} \cdot x_{i,t}(b_t) - p_{i,t}(b_t)$$

$$u_i(b_0, b_1, \ldots, b_T) = \sum_{t=0}^{T} \delta^t \cdot u_{i,t}(b_t)$$

Here, an agent has a truthful periodic bidding strategy if $b_{i,t}(\theta_{i,t}, h_t) = \theta_{i,t}$ for any $\theta_{i,t} \in \Theta_i$ and $h_t \in H_t$, and a mechanism is periodic Bayesian incentive-compatible (pBIC) if truthful bidding is a perfect Bayesian-Nash equilibrium, i.e. truthful bidding at each period maximizes the expected utility of agent i knows their type $\theta_{i,t}$ and the distribution of $\theta_{-i,t}|\theta_{i,t}$. Note that it is possible to know the latter since G_{-i}, H_{-i}, and h_t are public knowledge, and h_t reveals the past types of all agents under truthful bidding.

2.4 Revenue Maximization

An important result of Myerson (1981) is the payment equivalence theorem, which states that for any monotone allocation rule x, there is a unique payment rule p that makes (x, p) BIC. In the dynamic setting, Bergemann and Välimäki (2019) show that it can be applied to the periodic allocation rule as

$$p_{i,t}(\theta) = \theta_{i,t} \cdot x_{i,t}(\theta_t) - \int_{\underline{\theta}}^{\theta_i} x_{i,t}(\eta_{i,t}, \theta_{-i,t})\, d\eta_{i,t} \tag{2.4.1}$$

Define the revenue of a dynamic mechanism (x, p) as

$$R(x, p, \theta) = \sum_{t=0}^{T} \sum_{i=1}^{N} \delta^t \cdot p_{i,t}(\theta_t)$$

assuming that the agents and the seller share a common discount factor $\delta \in (0, 1)$. Consider the maximization problem

$$\max_{x \in \mathcal{X}} \mathbb{E}_0[R(x, p, \theta)]$$

where $\mathcal{X} = \bigtimes_{t=0}^{T} \mathcal{X}_t$ subject to the constraints in (2.4.1). The $\mathbb{E}_0$ operator indicates that the expectation is taken after the agents submit their (truthful) bids at time 0. Bergemann and Välimäki (2019) show that if the process $\{\theta_t\}_{t=0}^{T}$ follows an AR(1) process with persistence parameter γ, then

$$\mathbb{E}_0[R(x, p, \theta)] = \mathbb{E}_0\left[\sum_{t=0}^{T} \sum_{i=1}^{N} \delta^t \cdot \left(\theta_{i,t} - \frac{1 - G_i(\theta_{i,0})}{g_i(\theta_{i,0})} \cdot \gamma^t\right) \cdot x_{i,t}(\theta_t)\right] \tag{2.4.2}$$

once again noting that the selling price of the item does not have to be explicitly calculated. Note that (2.4.2) is similar to the revenue equivalence theorem of Myerson (1981), which states that for a static mechanism,

$$\mathbb{E}[R(x, p, \theta)] = \mathbb{E}\left[\sum_{i=1}^{N} p_i(\theta)\right] = \mathbb{E}\left[\left(\theta_i - \frac{1 - G_i(\theta_i)}{g_i(\theta_i)}\right) \cdot x_i(\theta)\right]$$

Interpreting the inverse hazard function of agent i's type at time 0 as "information rent" charged to the seller by the agent, we see that this distortion approaches zero in the dynamic setting as time goes on. However, waiting may not be optimal as the payments are discounted. The revenue-optimal mechanism, i.e. the pBIC mechanism that maximizes expected revenue, is given by an allocation rule $x^\star = (x_1^\star, \ldots, x_T^\star)$ that maximizes (2.4.2) and a payment rule $p^\star$ that satisfies (2.4.1).

3 Methodology

3.1 Overview

We now outline methods to compute $\mathbb{E}_0[R(x^\star, p^\star, \theta)]$ and $x^\star$ for various choices of agents N and periods T. We define our value function $V_t : \Theta \to \mathbb{R}$ at time t to be the undiscounted value of the optimal expected revenue given types θ_t and that the item has not yet been sold. Let $\mathcal{X}^{(t)} = \{x \in \mathcal{X} : x_s = 0, s < t\}$. Then we can write

$$V_t(\theta_t) = \max_{x \in \mathcal{X}^{(t)}} \mathbb{E}_t \left[\sum_{\tau=t}^{T} \sum_{i=1}^{N} \delta^{\tau - t} \cdot \left(\theta_{i,\tau} - \frac{1 - G_i(\theta_{i,0})}{g_i(\theta_{i,0})} \cdot \gamma^t \right) \cdot x_{i,\tau}(\theta_\tau) \right]$$

Note that at time T, the feasible allocations in $\mathcal{X}^{(T)}$ are to sell the item to one of the agents or to not sell the item. Then to maximize $V_T(\theta_T)$, it follows that the item should be sold to the agent with the highest value of

$$\theta_{i,T} - \frac{1 - G_i(\theta_{i,0})}{g_i(\theta_{i,0})} \cdot \gamma^T$$

if it is positive, and to not sell the item if it is negative. Then

$$V_T(\theta_T) = \max \left\{ \max_{i=1,\ldots,N} \left\{ \theta_{i,T} - \frac{1 - G_i(\theta_{i,0})}{g_i(\theta_{i,0})} \cdot \gamma^T \right\}, 0 \right\}$$

At time $T-1$, the feasible allocations in $\mathcal{X}^{(T-1)}$ are to sell the item to one of the agents at time $T-1$, to sell the item to one of the agents at time T, or to not sell the item. Note that the maximum value of $V_{T-1}(\theta_{T-1})$ from selling the item at time $T-1$ is

$$\max_{i=1,\ldots,N} \left\{ \theta_{i,T-1} - \frac{1 - G_i(\theta_{i,0})}{g_i(\theta_{i,0})} \cdot \gamma^{T-1} \right\}$$

and from selling the item at time T is $\delta \cdot \mathbb{E}_{T-1}[V_T(\theta_T)]$, which is nonnegative. Then

$$V_{T-1}(\theta_{T-1}) = \max \left\{ \max_{i=1,\ldots,N} \left\{ \theta_{i,T-1} - \frac{1 - G_i(\theta_{i,0})}{g_i(\theta_{i,0})} \cdot \gamma^{T-1} \right\}, \delta \cdot \mathbb{E}_{T-1}[V_T(\theta_T)] \right\}$$

Repeating this logic, for any $t < T$, we have

$$V_t(\theta_t) = \max \left\{ \max_{i=1,\ldots,N} \left\{ \theta_{i,t} - \frac{1 - G_i(\theta_{i,0})}{g_i(\theta_{i,0})} \cdot \gamma^t \right\}, \delta \cdot \mathbb{E}_t[V_{t+1}(\theta_{t+1})] \right\}$$

Iterating backwards to $V_0(\theta_0)$, the optimal expected revenue is then given by $\mathbb{E}[V_0(\theta_0)]$.

The optimal allocation rule $x^\star$ can be framed as the decision rule that solves the following optimal stopping problem: at time t, the seller either sells the item to one of the agents or waits for a new type vector θ_{t+1}. Based on the value functions, the optimal allocation rule can be described as follows: for $t < T$, if

$$\max_{i=1,\dots,N} \left\{ \theta_{i,t} - \frac{1 - G_i(\theta_{i,0})}{g_i(\theta_{i,0})} \cdot \gamma^t \right\} \geq \delta \cdot \mathbb{E}_t[V_{t+1}(\theta_{t+1})]$$

sell the item to agent $i^\star$ where

$$i^\star = \operatorname*{argmax}_{i=1,\dots,N} \left\{ \theta_{i,t} - \frac{1 - G_i(\theta_{i,0})}{g_i(\theta_{i,0})} \cdot \gamma^t \right\}$$

and wait for θ_{t+1} if otherwise. If $t = T$, if

$$\max_{i=1,\dots,N} \left\{ \theta_{i,T} - \frac{1 - G_i(\theta_{i,0})}{g_i(\theta_{i,0})} \cdot \gamma^T \right\} \geq 0$$

sell the item to agent $i^\star$ where

$$i^\star = \operatorname*{argmax}_{i=1,\dots,N} \left\{ \theta_{i,T} - \frac{1 - G_i(\theta_{i,0})}{g_i(\theta_{i,0})} \cdot \gamma^t \right\}$$

and do not sell the item at all if otherwise.

Once all value functions $\{V_t(\theta_t)\}_{t=0}^T$ are computed over a reasonable grid given auction and type parameters, then we can simulate the process $\{\theta_t\}_{t=0}^T$ and find optimal allocation rules on a per-simulation basis. These allocation rules can be specified by $(i^\star, t^\star)$ where $i^\star$ is the agent that receives the item and $t^\star$ is the period in which the item is sold. We can then estimate the probability distributions of $i^\star$ and $t^\star$ by simulating the auction for a large number of times.

Define the value of selling at time t as a function $V_t^S(\theta_t) : \Theta \to \mathbb{R}$ given by

$$V_t^S(\theta_t) = \max_{i=1,\dots,N} \left\{ \theta_{i,t} - \frac{1 - G_i(\theta_{i,0})}{g_i(\theta_{i,0})} \cdot \gamma^t \right\}$$

Then it is simple to compute $V_t^S(\theta_t)$ no matter the N and T, but the difficulty of computing $\mathbb{E}_t[V_{t+1}(\theta_{t+1})]$ varies depending on the N and T.

In this project, we will assume that all G_i and H_i are in the Normal family of distributions, which simplifies some computations. Since we require that each agent's type space Θ_i be closed and bounded, for each i, we take $\underline{\theta}_i$ sufficiently small and $\overline{\theta}_i$ sufficiently large such that truncating G_i and H_i to the interval $[\underline{\theta}_i, \overline{\theta}_i]$ has a negligible effect on the density of $\theta_{i,t}$ for all t, allowing the computations to be accurate and consistent with the theory.

3.2 One or Two Agents and Two Periods

When $N \in \{1, 2\}$ and $T = 1$, we can compute $V_1(\theta_1) = \max\{V_1^S(\theta_1), 0\}$ for any input θ_1. For any input θ_0, we compute $V_0^S(\theta_0)$ easily and compute $\delta \cdot \mathbb{E}_0[V_1(\theta_1)]$ with Gauss-Hermite quadrature. Let $\{(\alpha_k, \omega_k)\}_{k=1}^M$ be the abscissae and weights for Gauss-Hermite quadrature with M sample points, i.e. the α_k are the roots of the M-th Hermite polynomial H_M and

$$\omega_k = \frac{n! \cdot 2^{n-1}\sqrt{\pi}}{n^2 \cdot (H_{n-1}(\alpha_k))^2}$$

Then the expected value of a function $f(X)$ where $X \sim \mathcal{N}(\mu, \sigma^2)$ is approximately

$$\mathbb{E}[f(\theta)] \approx \frac{1}{\sqrt{\pi}} \sum_{k=1}^{M} \omega_k \cdot f\left(\mu + \alpha_k\sqrt{2\sigma^2}\right)$$

and the expected value of a bivariate function $f(X, Y)$ where $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$, and X and Y are independent is approximately

$$\mathbb{E}[f(\theta_1, \theta_2)] \approx \frac{1}{\pi} \sum_{k=1}^{M} \sum_{l=1}^{M} \omega_k\omega_l \cdot f\left(\mu_1 + \alpha_k\sqrt{2\sigma_1^2}, \mu_1 + \alpha_l\sqrt{2\sigma_2^2}\right)$$

Note that if $H_i = \mathcal{N}(\mu_i, \sigma_i^2)$, then the conditional distribution of $\theta_{i,1}$ given $\theta_{i,0}$ is

$$\theta_{i,1}|\theta_{i,0} \sim \mathcal{N}(\mu_i + \theta_{i,0}, \sigma_i^2)$$

so given any input θ_0, we can easily compute $\mathbb{E}_0[V_1(\theta_1)] \equiv \mathbb{E}[V_1(\theta_1)|\theta_0]$ with Gauss-Hermite quadrature. For example, if $N = 1$ and $H_1 = \mathcal{N}(\mu, \sigma^2)$, then

$$\mathbb{E}[V_1(\theta_1)|\theta_0] \approx \frac{1}{\sqrt{\pi}} \sum_{k=1}^{M} \omega_k \cdot V_1\left(\mu + \theta_0 + \alpha_k\sqrt{2\sigma^2}\right)$$

We then compute $\mathbb{E}[V_0(\theta_0)]$ with Gauss-Hermite quadrature (since the G_i are known) or with the trapezoidal rule given a grid of θ_0 values.

3.3 Many Agents and Two Periods

Similarly to Section 3.2, it is not difficult to compute $V_1(\theta_1)$ given θ_1 and $V_0^S(\theta_0)$ given θ_0, so it remains to compute $\mathbb{E}_0[V_1(\theta_1)]$. Note that as the number of agents increases linearly, the number of computations required for Gauss-Hermite quadrature increases exponentially.

To avoid the curse of dimensionality here, we use the randomized method of Monte Carlo integration. To approximate the expected value of a function $f(X)$, draw $X_1, \ldots, X_n$ from the distribution of X. Then

$$\mathbb{E}[f(X)] \approx \frac{1}{n} \sum_{j=1}^{n} f(X_i)$$

by the Strong Law of Large Numbers. In our case, we draw $\eta_1, \ldots, \eta_n$ from the distribution of $\theta_{i,1}|\theta_{i,0}$, i.e. $\mathcal{N}(\mu_i + \theta_{i,0}, \sigma_i^2)$, evaluate V_1 at each draw, and average, so

$$\mathbb{E}[V_1(\theta_1)|\theta_0] \approx \frac{1}{n} \sum_{j=1}^{n} V_1(\eta_j)$$

This provides a cruder, but quicker method to approximate $\mathbb{E}_0[V_1(\theta_1)]$, especially when the number of agents N is large.

3.4 One or Two Agents and Many Periods

When $T > 2$, the general approach is to compute $V_T(\theta_T)$ on a grid, use its interpolated and extrapolated values to compute $V_{T-1}(\theta_{T-1})$, and repeat. Specifically, on a grid of possible θ_t values, we compute $V_0(\theta_0)$ given an input θ_0 as follows:

- Create an array to store the values of all value functions at the grid points. Compute $V_T(\theta_T)$ on the grid.
- For $t = T - 1, \ldots, 1$:
  - Obtain linear interpolations and extrapolations of $V_{t+1}(\theta_{t+1})$. In our setting, the value functions have hockey-stick shapes, so using linear interpolations and extrapolations is suitable.
  - For each grid point $\tilde{\theta}_t$, compute the value if the seller were to sell the item to an agent at $\tilde{\theta}_t$ and $\mathbb{E}[V_{t+1}(\theta_{t+1})|\tilde{\theta}_t]$, the expected value if the seller were to wait at $\tilde{\theta}_t$, using Gauss-Hermite quadrature with the interpolated and extrapolated values. Store the maximum of the two in the array.
- Obtain linear interpolations and extrapolations of $V_1(\theta_1)$. Compute the value if the seller were to sell the item to an agent at θ_0 and $\mathbb{E}[V_1(\theta_1)|\theta_0]$. The maximum of the two is $V_0(\theta_0)$.

Similarly to section 3.2, we can then compute $\mathbb{E}[V_0(\theta_0)]$ with Gauss-Hermite quadrature or with the trapezoidal rule given a grid of θ_0 values.

3.5 Many Agents and Many Periods

As the amount of agents N increases, the curse of dimensionality applies to the method described in Section 3.4. It is also impractical to use the method described in Section 3.2, as a simulations would be needed for every point at each period, again resulting in the curse of dimensionality. One possible quicker method to approximate $\mathbb{E}[V_0(\theta_0)]$ is to simulate type paths for each agent over all periods and select

$$\max_{t=0,1,\ldots,T}\left\{\max_{i=1,\ldots,N}\left\{\delta^t\cdot\left(\theta_{i,t}-\frac{1-G_i(\theta_{i,0})}{g_i(\theta_{i,0})}\cdot\gamma^t\right)\right\}\right\}$$

as the value for that simulation, then average such values over a large number of simulations. However, this method selects the ex-post, rather than interim, optimal revenue, so it overestimates the actual $\mathbb{E}[V_0(\theta_0)]$. We will compare the performance and accuracy of this method with other methods in tractable cases.

4 Results

4.1 One Agent and Two Periods

We set $\gamma = 0.9$, $\delta = 0.95$, $G \sim \mathcal{N}(1, 0.2)$, $H \sim (0, 0.2)$, and $M = 21$. Using the method described in Section 3.2, the plot of the optimal expected revenue on θ_0 is shown below:

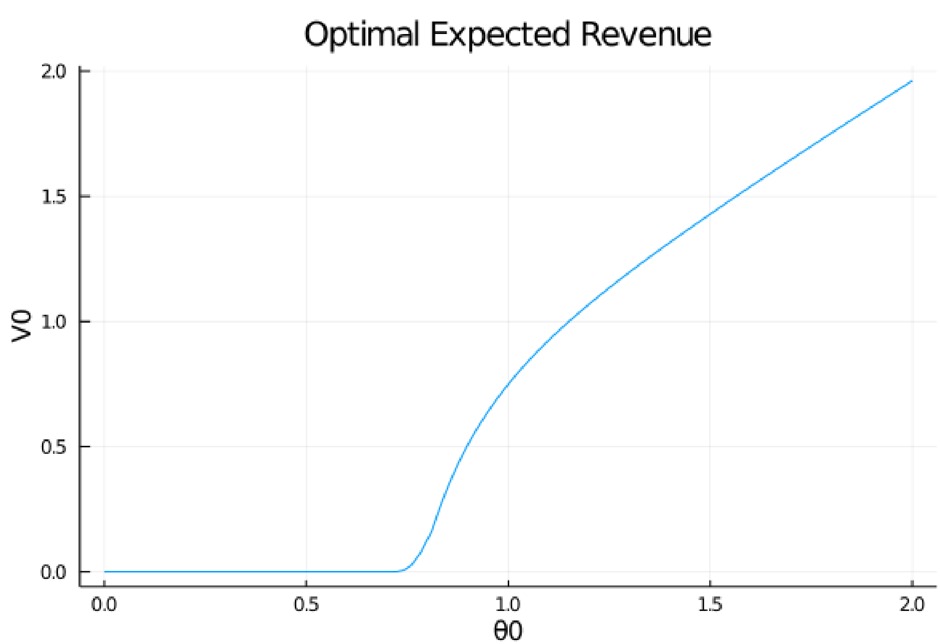

Figure 4.1.1. $V_0(\theta_0)$ for $N = 1$ and $T = 1$.

There appears to be a threshold $\theta_0^\star$ such that the optimal expected revenue for any $\theta_0 \leq \theta_0^\star$ is zero. For any $\theta_0 > \theta_0^\star$, it appears that $V_0(\theta_0)$ is increasing in θ_0. Taking the expected value of $V_0(\theta_0)$ with $\theta_0 \sim G$, we have that $\mathbb{E}[V_0(\theta_0)] = 0.6794$ with Gauss-Hermite quadrature and $\mathbb{E}[V_0(\theta_0)] = 0.6768$ with the trapezoidal rule.

We then simulate 1×10^6 auctions with these parameters and outcomes based on $V_0(\theta_0)$ and $V_1(\theta_1)$ and show the results:

Figure 4.1.2. The distribution of revenue for simulated auctions according to the optimal allocation rule, subsetted by selling time.

Monte Carlo integration of the simulated revenues gives $\mathbb{E}[V_0(\theta_0)] = 0.6763$, which is consistent with the result of the dynamic programming approach. The item was sold at $t = 1$ or not sold at all for 17.3159% of the auctions. The revenue from auctions where the item was sold at $t = 1$ appears to be much lower than that from auctions where the item was sold at $t = 0$, suggesting that the seller waits only if the initial value θ_0 is quite low, and sells immediately if otherwise.

We now investigate comparative statics, specifically the comparative statics of $\mathbb{E}[V_0(\theta_0)]$ and the time of selling. We will denote the proportion of simulations where the item is sold in period t with S_t, and the proportion of simulations where the item is not sold with T_{-1}. In this section, the parameters of interest are γ, δ, μ_G, σ_G, μ_H, and σ_H. For each parameter, we choose a reasonable grid to vary the parameter along, and calculate the objects of interest, i.e. $\mathbb{E}[V_0(\theta_0)]$ and (T_0, T_1, T_{-1}), while keeping all other parameters at their original values. The results are shown on the following page:

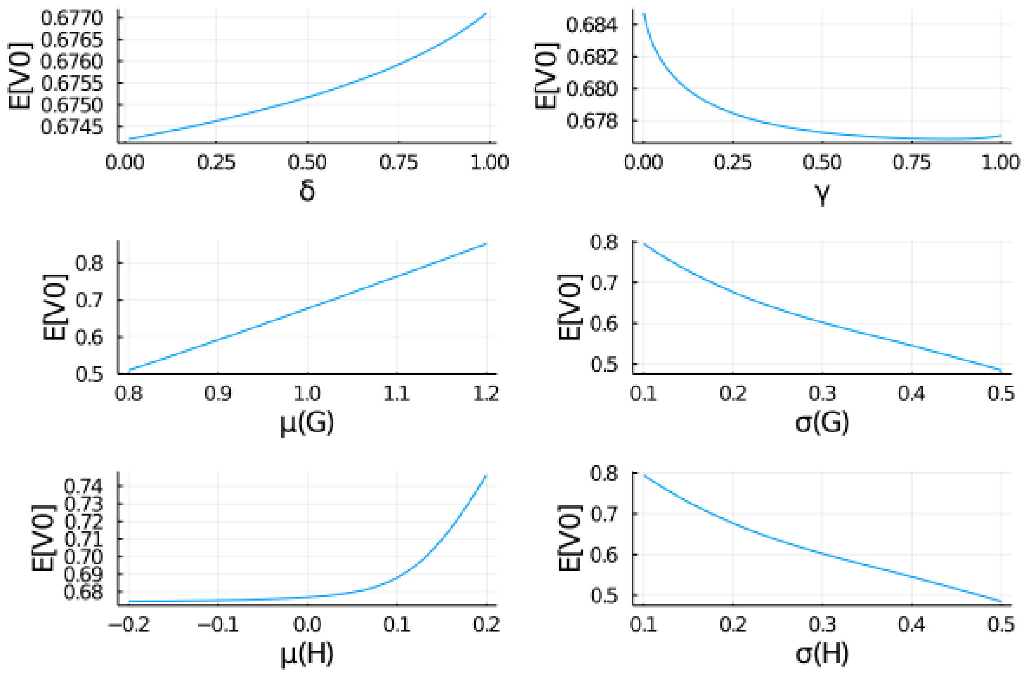

Figure 4.1.3. Comparative statics of $\mathbb{E}[V_0(\theta_0)]$ for γ, δ, μ_G, σ_G, μ_H, and σ_H.

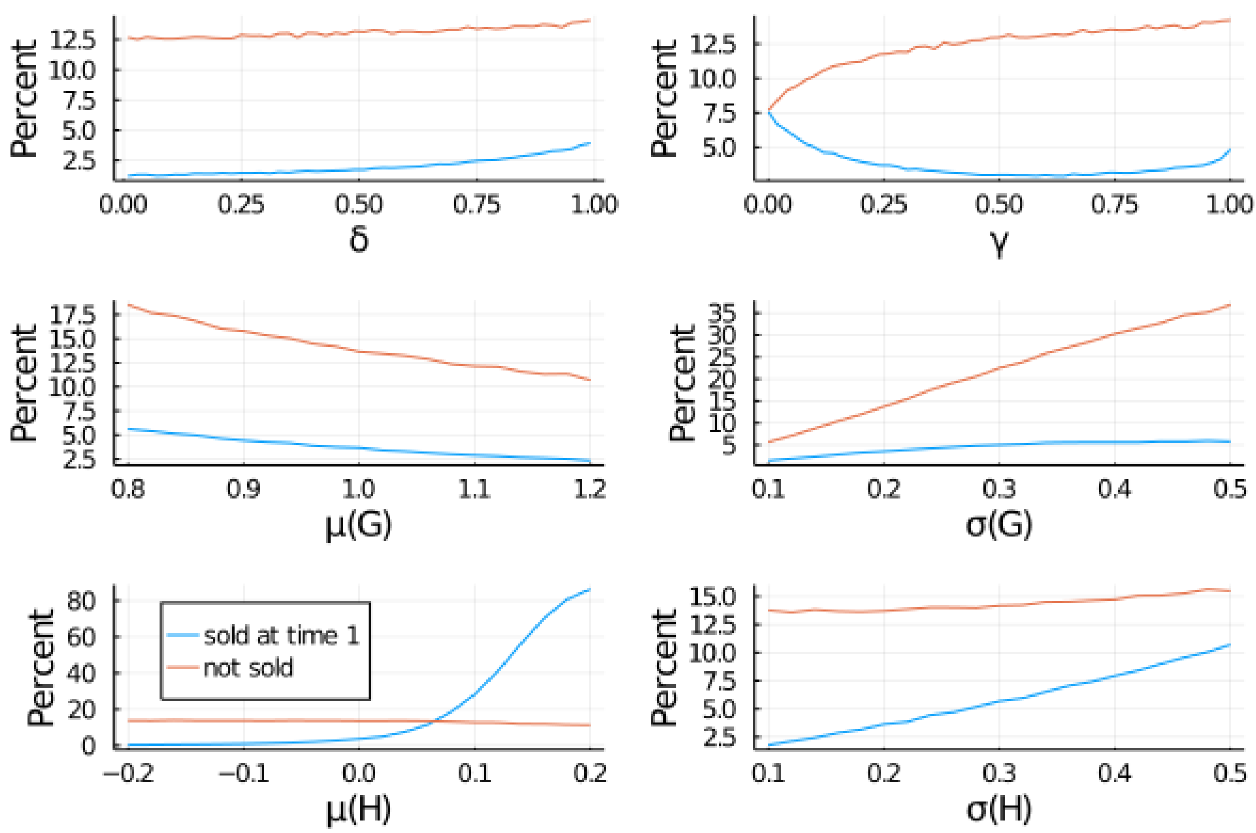

Figure 4.1.4. Comparative statics of T_1 and T_{-1} for γ, δ, μ_G, σ_G, μ_H, and σ_H.

We see that $\mathbb{E}[V_0(\theta_0)]$ is increasing in δ and generally decreasing in γ, although the change in $\mathbb{E}[V_0(\theta_0)]$ is relatively small. Increasing μ_G or μ_H results in an increase in $\mathbb{E}[V_0(\theta_0)]$, as intuition would suggest. However, increasing σ_G or σ_H, which introduces uncertainty, results in a relatively large decrease in $\mathbb{E}[V_0(\theta_0)]$. The plots in Figure 4.1.4 are quite interesting and reveal more about the effect of each parameter on the auction model. For most parameters, T_1 and T_{-1} move in tandem but at different rates. For example, increasing σ_G increases T_{-1} at a higher rate than T_1, while increasing σ_H increases T_1 at higher rate than T_{-1}. Notably, T_1 and T_{-1} move in opposite directions to changes in γ, and T_1 increases greatly while T_{-1} does not change much in response to a change in μ_H.

4.2 Two Agents and Two Periods

We set $\gamma = 0.9$, $\delta = 0.95$, $G_i = \mathcal{N}(1, 0.2)$ and $H_i = \mathcal{N}(0, 0.2)$ for $i = 1, 2$ and $M = 21$. The plot of the optimal expected revenue on θ_0 is shown below:

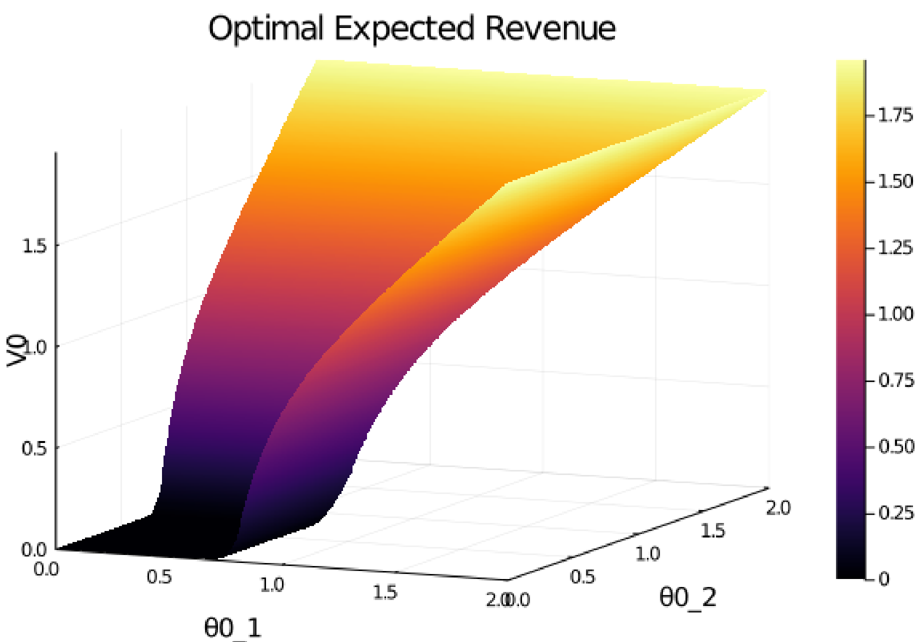

Figure 4.2.1. $V_0(\theta_0)$ for $N = 2$ and $T = 1$.

Similar to the $N = 1$ case, there appears to be thresholds $\theta^\star_{1,0}$ and $\theta^\star_{2,0}$ such that the optimal expected revenue for any θ_0 such that $\theta_{1,0} < \theta^\star_{1,0}$ and $\theta_{2,0} < \theta^\star_{2,0}$ is zero. Taking the expected value of $V_0(\theta_0)$ with $\theta_0 \sim G_1 \otimes G_2$, the product distribution of G_1 and G_2 since $\theta_{1,0}$ and $\theta_{2,0}$ are independent, we have that $\mathbb{E}[V_0(\theta_0)] = 0.8965$ with Gauss-Hermite quadrature and $\mathbb{E}[V_0(\theta_0)] = 0.9001$ with the trapezoidal rule.

We then simulate 2×10^t auctions with these parameters and outcomes based on $V_0(\theta_0)$ and $V_1(\theta_1)$ and show the results, including the agent that the item is sold to now:

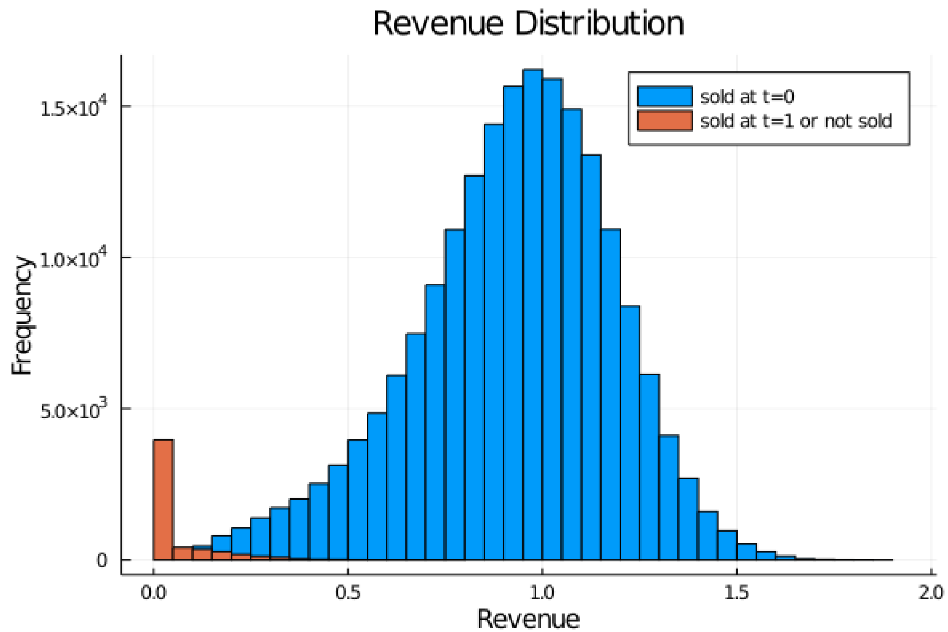

Figure 4.2.2. The distribution of revenue for simulated auctions according to the optimal allocation rule, subsetted by selling time.

The mean of the simulated revenues is $\mathbb{E}[V_0(\theta_0)] = 0.8994$, which is consistent with the dynamic programming solution. The item was sold at $t = 1$ or not sold at all in 2.7555% of the simulated auctions, a sharp decrease compared to the $N = 1$ case, and was sold to agent 1 in 48.955% of the auctions, which is unsurprising as the agents are symmetric.

For comparative statics, we explore how the distribution of agent 1 affects the probability of agent 1 receiving the item, denoted by N_1, by varying μ_{G_1}, σ_{G_1}, μ_{H_1}, and σ_{H_1}:

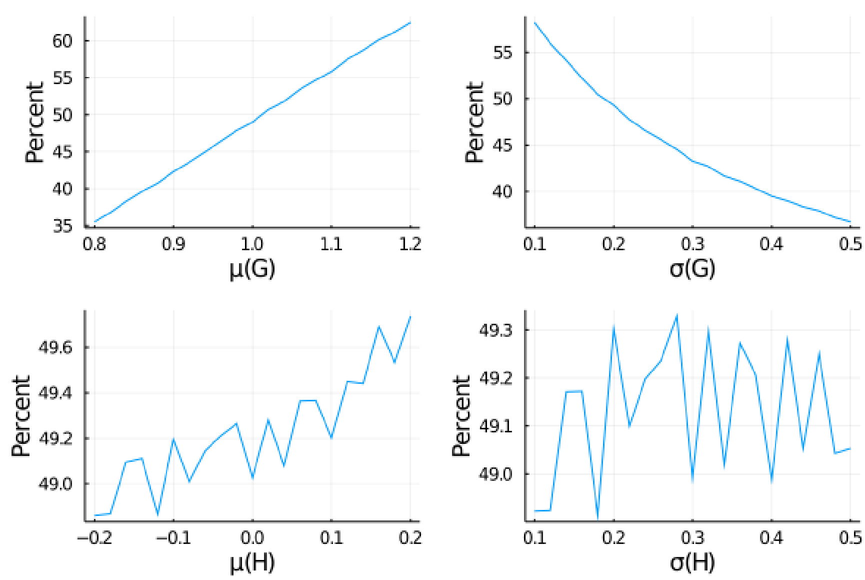

Figure 4.2.3. Comparative statics of N_1 for μ_{G_1}, σ_{G_1}, μ_{H_1}, and σ_{H_1}.

It appears that changes in the parameters of G_1 affect N_1 much more than changes in the parameters of H_1, with N_1 increasing in μ_{G_1} and decreasing in σ_{G_1}. At 2×10^5 simulations, the plots for μ_{H_1} and σ_{H_1} are still jagged, suggesting a large variation when the parameters are within the grids used.

4.3 Many Agents and Two Periods

We use the same parameters in the previous sections and compute $\mathbb{E}_0[V_0(\theta_0)]$ using the method in Section 3.3, now varying the number of agents N, who have the same distribution $G_i \sim \mathcal{N}(1, 0.2)$ and $H_i \sim \mathcal{N}(0, 0.2)$, and have types independent of each other. The results are shown below:

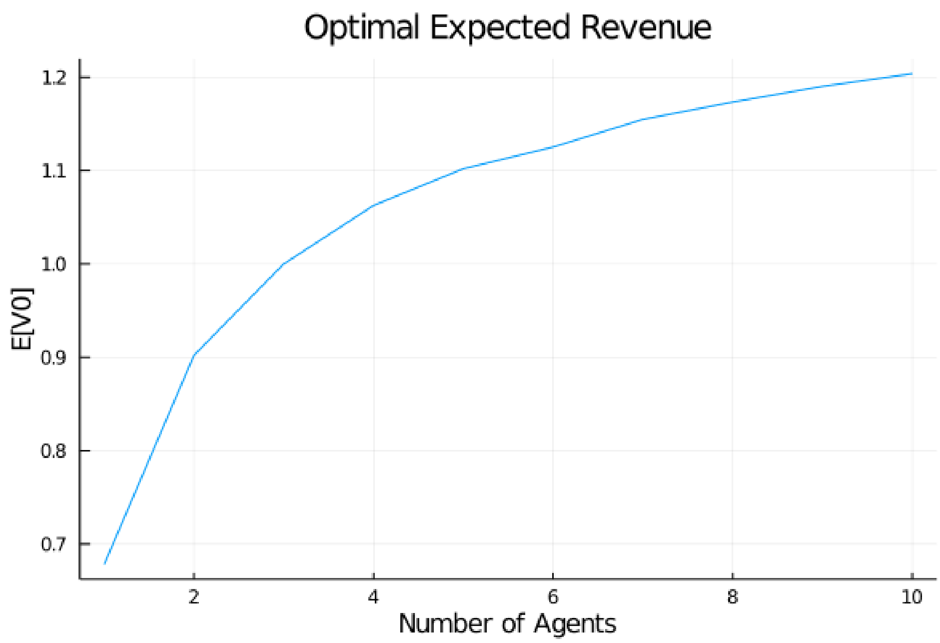

Figure 4.3.1. Comparative statics of $\mathbb{E}_0[V_0(\theta_0)]$ for N.

We see that $\mathbb{E}_0[V_0(\theta_0)]$ is increasing in N, although the rate of increase is decreasing in N.

4.4 One or Two Agents and Many Periods

We use the same parameters in the previous sections and compute $\mathbb{E}_0[V_0(\theta_0)]$ using the method in Section 3.4, taking $N = 1$ or 2 and varying T. We first verified that the results for $T = 1$ were consistent with those in Section 4.1 and 4.2. The approach of dynamic programming on a grid allows us the see $V_0(\theta_0)$ for different T, which is illustrated on the next page. For aesthetic purposes, we use $\sigma_G = \sigma_H = 0.3$ only for this example:

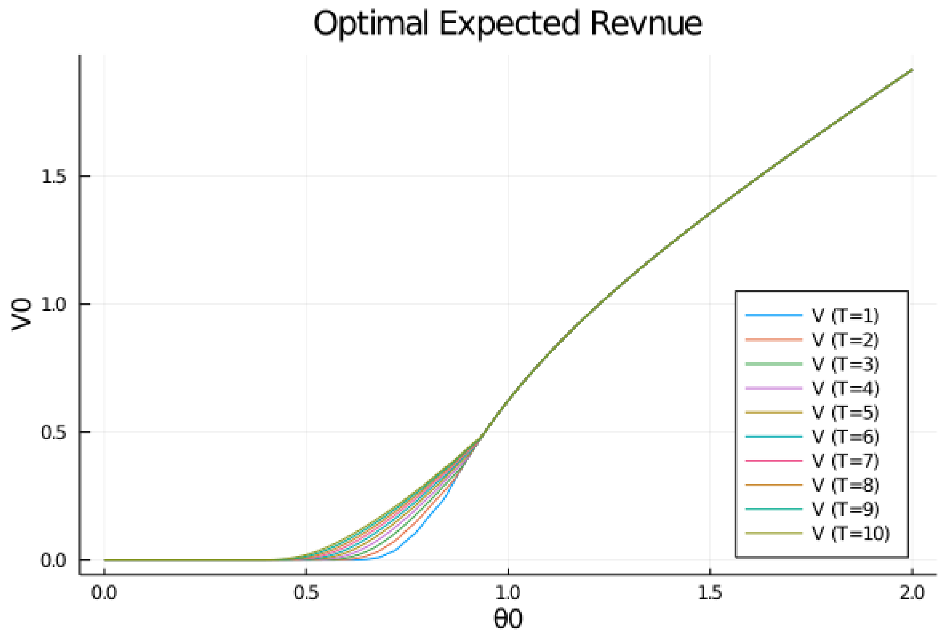

Figure 4.4.1. $V_0(\theta_0)$ for $N = 1$ and $T = 1, \ldots, 10$.

There appears to be a threshold $\theta_0^\star$ such that $V_0(\theta_0)$ is the same for all $\theta_0 > \theta_0^\star$ for any number of time periods T. The most change in $V_0(\theta_0)$ over different T appears to be in the are below $\theta_0^\star$, where optimal expected revenue given θ_0 increases over time as there is more chance for the agent's type to increase. Note that this reasoning is similar to how the value of an option is increasing in time. We now compute $\mathbb{E}_0[V_0(\theta_0)]$ for $N = 1$ and varying T, showing the results below:

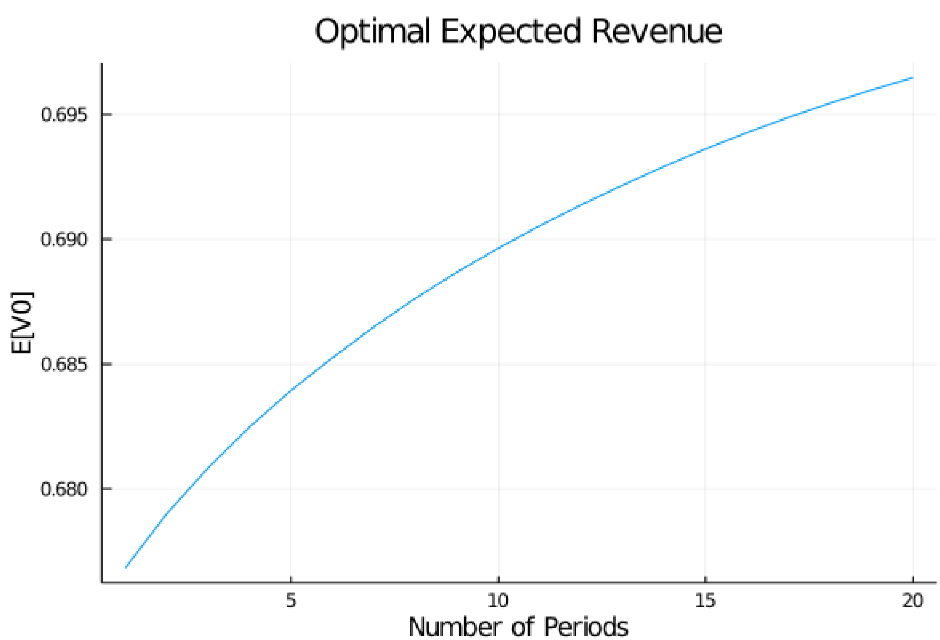

Figure 4.4.2. Comparative statics of $\mathbb{E}_0[V_0(\theta_0)]$ for T.

4.5 Many Agents and Many Periods

We now compute $\mathbb{E}_0[V_0(\theta_0)]$ using our approximate method in Secion 3.5, which we will call "Ex-Post Monte Carlo (EPMC)" as it involves Monte Carlo integration of ex-post optimal revenues. We explore its accuracy for tractable cases compared to the dynamic programming (DP) computed values and the approximate $\mathbb{E}_0[V_0(\theta_0]$ values:

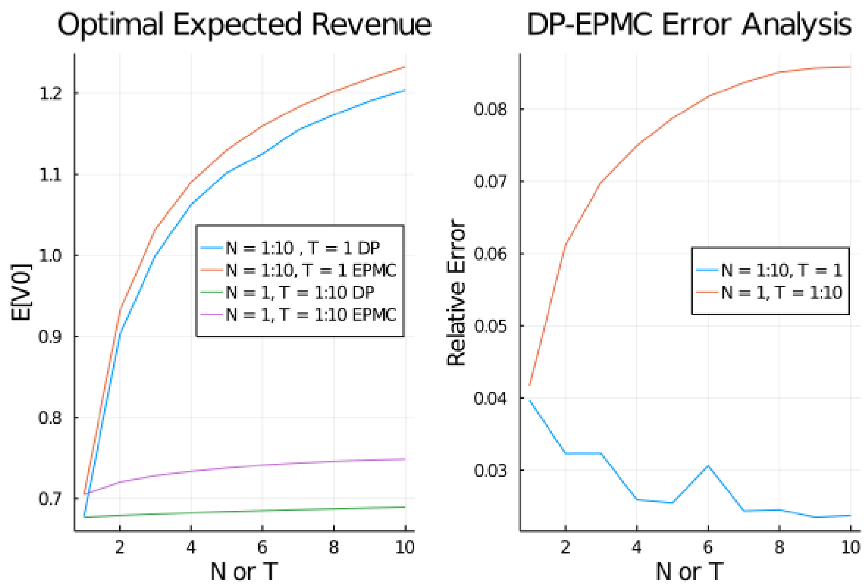

Figure 4.5.1. Comparative statics and error analysis of $\mathbb{E}_0[V_0(\theta_0)]$ for N and T using both DP and EPMC methods.

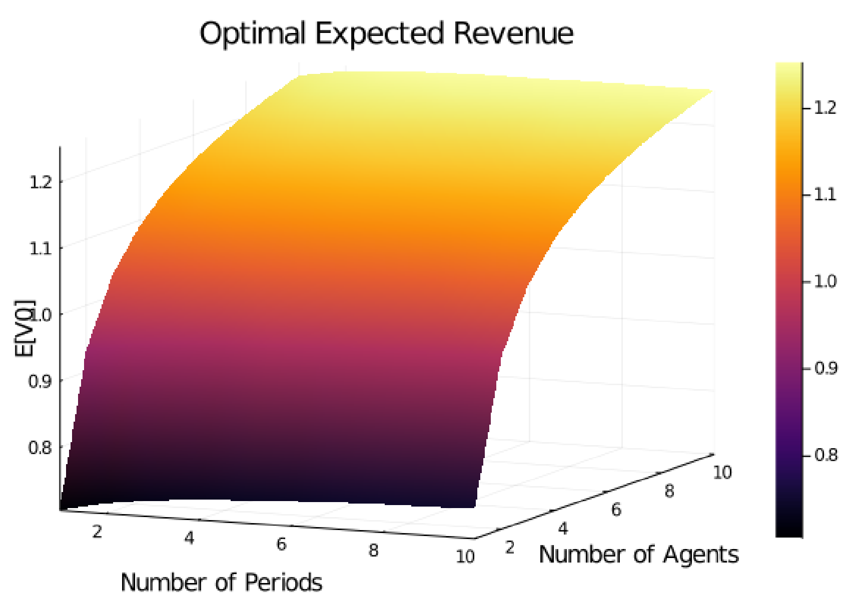

Figure 4.5.2. Comparative statics of $\mathbb{E}_0[V_0(\theta_0)]$ for (N, T) using the EPMC method.

We see that the change of $\mathbb{E}_0[V_0(\theta_0)]$ is much more sensitive in N than in T. The relative error between both methods seems to be increasing in T as N is held constant, but decreasing in N as T is held constant. This suggests that the EPMC method is more accurate for higher values of N.

5 Conclusion

We have explored the problem of calculating the optimal expected revenue of a simple dynamic auction where the agents bid to maximize their expected utility. Recent innovations in dynamic mechanism design applied to this setting yield an expression for the optimal expected revenue that can be computed through dynamic programming and numerical integration techniques. The difficulty of computation depends on the number of agents N and number of time periods T, and we have developed various methods with varying speeds and accuracies for different N and T. Additionally, we have used the optimal decision rule implied from the dynamic programming formulation of this problem to simulate auctions with explicit outcomes, i.e. when and to whom the item is sold. For both the optimal expected revenue and auction metrics, we have performed comparative statics that reveal more insight about how each of the auction parameters contributes to the results and outcomes. An area for further exploration is in the many agents and many periods case, where one could perform more extensive error analysis, find bounds on the error, or develop more accurate yet tractable approximation algorithms. All in all, computational methods including dynamic programming are quite useful in the rapidly developing field of dynamic mechanism design.

References

Bergemann, Dirk, and Juuso Välimäki. 2019. "Dynamic Mechanism Design: An Introduction." *Journal of Economic Literature*, 57 (2): 235-74.

Myerson, Roger. 1981. "Optimal Auction Design." *Mathematics of Operations Research*, 6 (1): 58-73.

Appendix

Code used to implement the methods and generate the figures can be found at

- `https://github.com/zhubrian/dynamic-auction`