Exploring Portfolio Choice

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1 Introduction and the basic model

Modern portfolio theory, also known as mean-variance analysis, is a framework for constructing portfolios of many risky and risk-free assets that satisfy certain constraints or properties. It was introduced by economist Harry Markowitz in 1952, who received a Nobel Prize in 1990 for his development of modern portfolio theory. This paper explores various problems in portfolio choice and their computational aspects, using real-world stock price data to construct portfolios.

Suppose that there are n stocks whose returns can be represented by a random variable R_i with mean (expected return) μ_i and variance σ_i^2. The covariance of the returns between stocks i and j is denoted by σ_{ij}. Note that $\sigma_i^2 = \sigma_{ii}$. Let $\mu = (\mu_1, \dots, \mu_n)^\top$ be the expected return vector and C be the covariance matrix with entries $C_{ij} = \sigma_{ij}$. Note that C is a symmetric $n \times n$ matrix.

A portfolio p of these n stocks is specified by a weight vector $w = (w_1, \dots, w_n)$ which gives the fraction of an investor's wealth to be invested in each stock. It is then necessary to require the sum of the weights to be 1, i.e. $w_1 + \cdots + w_n = 1$. A negative weight indicates short selling. If short selling is not allowed, then each weight must also be nonnegative, i.e. $w_i \geq 0$ for all i. We will not examine this case here.

Denote the return of a portfolio p by R_p. Then the expected return μ_p is

$$\mu_p = \mathbb{E}[R_p] = \mathbb{E}\left[\sum_{i=1}^{n} w_i R_i\right] = \sum_{i=1}^{n} \mathbb{E}[w_i R_i] = \sum_{i=1}^{n} w_i \mu_i = \mu^\top w$$

and the variance σ_p^2 is

$$\sigma_p^2 = \mathrm{Var}[R_p] = \mathrm{Var}\left[\sum_{i=1}^{n} w_i R_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Cov}[w_i R_i, w_j R_j] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j C_{ij} = w^\top C w$$

The following sections explore how μ_p and σ_p^2, among other variables, are involved in portfolio choice. In section 2, we derive minimum-variance portfolio weights and construct the efficient frontier for a market of all risky assets. In section 3, we add a risk-free asset and re-derive minimum-variance portfolio weights in this new setting. We also add investor preferences and derive the investor-optimal portfolio weights. In section 4, we apply the derivations in sections 2 and 3 to real-world stock price data, observing how portfolio choice varies depending on stock market features, such as the number of stocks n included and the country or region of the market.

2 Minimum-variance portfolios and the efficient frontier

Given μ and invertible C, the minimum-variance portfolio $p^\star(r)$ is the portfolio with expected return r that has the least possible variance. Then its weight vector $w^\star(r)$ minimizes $w^\top Cw$ subject to $e^\top w = 1$ and $\mu^\top w = r$, where e is the n-dimensional ones-vector. Note that $w^\star(r)$ solves a quadratic program in variables $w_1, \dots, w_n$:

$$\begin{aligned} \min_{w} \quad & \frac{1}{2} w^\top C w \\ \text{subject to} \quad & e^\top w = 1 \\ & \mu^\top w = r \end{aligned}$$

Since there are only equality constraints, we can use Lagrange multipliers to solve this problem. The Lagrangian is given by

$$\mathcal{L}(w, \lambda_1, \lambda_2) = \frac{1}{2} w^\top C w - \lambda_1 (e^\top w - 1) - \lambda_2 (\mu^\top w - r)$$

Since C is positive semi-definite and the constraints are linear in the w_i, the objective function is convex, so any solution $w^\star$ necessarily satisfies $\partial \mathcal{L} / \partial w_k^\star = 0$ for all k. Setting $\partial \mathcal{L} / \partial w_k = 0$ gives

$$\begin{aligned} 0 = \frac{\partial \mathcal{L}}{\partial w_k} &= \frac{\partial}{\partial w_k} \left(\frac{1}{2} w^\top C w - \lambda_1 (e^\top w - 1) - \lambda_2 (\mu^\top w - r) \right) \\ &= \frac{\partial}{\partial w_k} \left(\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j C_{ij} - \lambda_1 \left(\sum_{i=1}^{n} w_i - 1 \right) - \lambda_2 \left(\sum_{i=1}^{n} \mu_i w_i - r \right) \right) \\ &= \frac{\partial}{\partial w_k} \left(\sum_{i \neq k} w_i w_k C_{ik} + \frac{1}{2} w_k^2 C_{kk} - \lambda_1 w_k - \lambda_2 \mu_k w_k \right) = \sum_{i=1}^{n} w_i C_{ik} - \lambda_1 - \lambda_2 \mu_k \end{aligned}$$

Repeating this for $k = 1, \dots, n$ gives a system of equations

$$\sum_{i=1}^{n} C_{i1} w_i = \lambda_1 + \lambda_2 \mu_1$$
$$\vdots$$
$$\sum_{i=1}^{n} C_{in} w_i = \lambda_1 + \lambda_2 \mu_n$$

In matrix-vector notation, this system of equations is equivalent to $Cw = \lambda_1 e + \lambda_2 \mu$. Since C is invertible, then multiplying both sides by C^{-1} gives

$$w^\star = C^{-1}(\lambda_1 e + \lambda_2 \mu) \tag{2.1}$$

To solve for λ_1 and λ_2, we plug $w^\star$ into each constraints to get a system of equations

$$\begin{aligned} 1 &= e^\top w^\star = e^\top C^{-1}(\lambda_1 e + \lambda_2 \mu) = \lambda_1 e^\top C^{-1} e + \lambda_2 e^\top C^{-1} \mu \\ r &= \mu^\top w^\star = \mu^\top C^{-1}(\lambda_1 e + \lambda_2 \mu) = \lambda_1 \mu^\top C^{-1} e + \lambda_2 \mu^\top C^{-1} \mu \end{aligned}$$

Denote $q_{ee} = e^\top C^{-1} e$, $q_{e\mu} = e^\top C^{-1} \mu$, $q_{\mu e} = \mu^\top C^{-1} e$, and $q_{\mu\mu} = \mu^\top C^{-1} \mu$. Then the system is

$$\begin{aligned} q_{ee}\lambda_1 + q_{e\mu}\lambda_2 &= 1 \\ q_{\mu e}\lambda_1 + q_{\mu\mu}\lambda_2 &= r \end{aligned}$$

This has a closed-form solution

$$\lambda_1 = \frac{q_{\mu\mu} - q_{\mu e} r}{q_{ee} q_{\mu\mu} - q_{e\mu} q_{\mu e}}, \ \lambda_2 = \frac{q_{ee} r - q_{e\mu}}{q_{ee} q_{\mu\mu} - q_{e\mu} q_{\mu e}} \tag{2.2}$$

which we can plug into equation (2.1) to get $w^\star$. The optimal variance $\sigma_p^{2\star}$ is then

$$\begin{aligned} \sigma_p^{2\star} = w^{\star\top} C w^\star &= w^{\star\top} C C^{-1}(\lambda_1 e + \lambda_2 \mu) \\ &= \lambda_1 w^{\star\top} e + \lambda_2 w^{\star\top} \mu \\ &= \lambda_1 + \lambda_2 r \\ &= \frac{q_{\mu\mu} - q_{\mu e} r + q_{ee} r^2 - q_{e\mu} r}{q_{ee} q_{\mu\mu} - q_{e\mu} q_{\mu e}} = \frac{q_{ee} r^2 - (q_{e\mu} + q_{\mu e}) r + q_{\mu\mu}}{q_{ee} q_{\mu\mu} - q_{e\mu} q_{\mu e}} \end{aligned} \tag{2.3}$$

This gives the efficient frontier: the set of all minimum variance portfolios. Note that the optimal variance $\sigma_p^{2\star}$ is a parabola in r and is thus minimized at

$$r_g = \frac{q_{e\mu} + q_{\mu e}}{2 q_{ee}} \qquad \Longrightarrow \qquad \sigma_p^{2\star}(r_g) = \frac{1}{q_{ee}} \tag{2.4}$$

so the global minimum variance portfolio weights are given by $w^\star(r_g)$.

Suppose that $r_1 > r_2$. Then

$$w^\star(r_1) - w^\star(r_2) = C^{-1}((\lambda_1(r_1) - \lambda_2(r_1))e + (\lambda_2(r_1) - \lambda_2(r_2))\mu) = C^{-1}\frac{(r_1 - r_2)(q_{ee}\mu - q_{\mu e}e)}{q_{ee}q_{\mu\mu} - q_{e\mu}q_{\mu e}}$$

When we take $r_1 - r_2 \to 0$, we then have

$$dw^\star = C^{-1}\frac{(q_{ee}\mu - q_{\mu e}e)dr}{q_{ee}q_{\mu\mu} - q_{e\mu}q_{\mu e}} \qquad \Longrightarrow \qquad \frac{dw^\star}{dr} = \frac{C^{-1}(q_{ee}\mu - q_{\mu e}e)}{q_{ee}q_{\mu\mu} - q_{e\mu}q_{\mu e}} \tag{2.5}$$

Note that $dw^\star/dr$ does not depend on r, so $w^\star$ and $w_k^\star$ for all k are linear in r.

3 Tangency portfolios and including investor preferences

Now we add a risk-free asset with expected return r_f and volatility zero. The tangency portfolio is a portfolio with expected return μ_T and volatility σ_T such that in the σ-μ plane, the line determined by $(0, r_f)$ and (σ_T, μ_T) is tangent to the parabola of the efficient frontier of the risky assets. It can be shown that in this setting, every minimum variance portfolio lies on the line determined by $(0, r_f)$ and (σ_T, μ_T), and that the tangency portfolio maximizes the Sharpe ratio $(\mu - r_f)/\sigma$. Then μ_T and σ_T solve

$$\begin{aligned} &\max_{\mu,\sigma} \ \frac{\mu - r_f}{\sigma} \\ &\text{subject to } \ \sigma = \sqrt{\frac{q_{ee}\mu^2 - (q_{e\mu} + q_{\mu e})\mu + q_{\mu\mu}}{q_{ee}q_{\mu\mu} - q_{e\mu}q_{\mu e}}} \end{aligned}$$

Substituting the constraint and factoring out terms not dependent on μ gives an unconstrained problem in one variable:

$$\max_{\mu} \ \mathcal{L}(\mu) = \frac{\mu - r_f}{\sqrt{q_{ee}\mu^2 - (q_{e\mu} + q_{\mu e})\mu + q_{\mu\mu}}}$$

Setting its derivative to zero gives

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial \mu} = \frac{(2q_{ee}r_f - q_{e\mu} - q_{\mu e})\mu - (q_{e\mu} + q_{\mu e})r_f + 2q_{\mu\mu}}{2(q_{ee}q_{\mu\mu} - q_{e\mu}q_{\mu e})^{3/2}} \\ 0 &= (2q_{ee}r_f - q_{e\mu} - q_{\mu e})\mu - (q_{e\mu} + q_{\mu e})r_f + 2q_{\mu\mu} \\ \mu_T &= \frac{(q_{e\mu} + q_{\mu e})r_f - 2q_{\mu\mu}}{2q_{ee}r_f - q_{e\mu} - q_{\mu e}} \qquad \Longrightarrow \qquad \sigma_T = \sqrt{\frac{q_{ee}\mu_T^2 - (q_{e\mu} + q_{\mu e})\mu_T + q_{\mu\mu}}{q_{ee}q_{\mu\mu} - q_{e\mu}q_{\mu e}}} \end{aligned} \tag{3.1}$$

Then the minimum variance portfolio with expected return $r = \mu_T$ is the tangency portfolio, i.e. the tangency portfolio has weights $w^T = w^\star(\mu_T)$.

We then consider an investor with CARA utility function

$$u(\mu, \sigma) = \mu - \frac{\alpha\sigma^2}{2} \tag{3.2}$$

where α is a parameter capturing the investor's risk aversion. Since the investor-optimal portfolio must lie on the capital allocation line, the utility-maximizing μ_{IO} and σ_{IO} solve

$$\max_{\mu,\sigma} \; \mu - \frac{\alpha\sigma^2}{2}$$
$$\text{subject to} \;\; \frac{\mu - r_f}{\sigma} = \frac{\mu_T - r_f}{\sigma_T}$$

Let $s = (\mu_T - r_f)/\sigma_T$. Then the constraint becomes $\mu = r_f + s\sigma$. Substituting the constraint gives an unconstrained problem in one variable:

$$\max_{\mu} \;\; u(\sigma) = r_f + s\sigma - \frac{\sigma^2}{2}$$

Setting its derivative to zero gives

$$0 = \frac{\partial u}{\partial \sigma} = s - \alpha\sigma \quad \implies \quad \sigma_{IO} = \frac{s}{\alpha} \quad \implies \quad \mu_{IO} = r_f + \frac{s^2}{\alpha} \tag{3.3}$$

To find the investor-optimal portfolio weights, let w_f denote the weight in the risk-free asset, and w^{IO} be the weight vector of the risky assets. Since the investor-optimal portfolio lies on the capital allocation line, w_f solves

$$w_f r_f + (1 - w_f)\mu_T = \mu_{IO} \quad \implies \quad w_f = \frac{\mu_T - r_f - s^2/\alpha}{\mu_T - r_f} \tag{3.4}$$

Then the tangency portfolio must have a total weight of $(1 - w_f)$, so $w^{IO} = (1 - w_f)w^T$.

4 Portfolio choice with real-world data

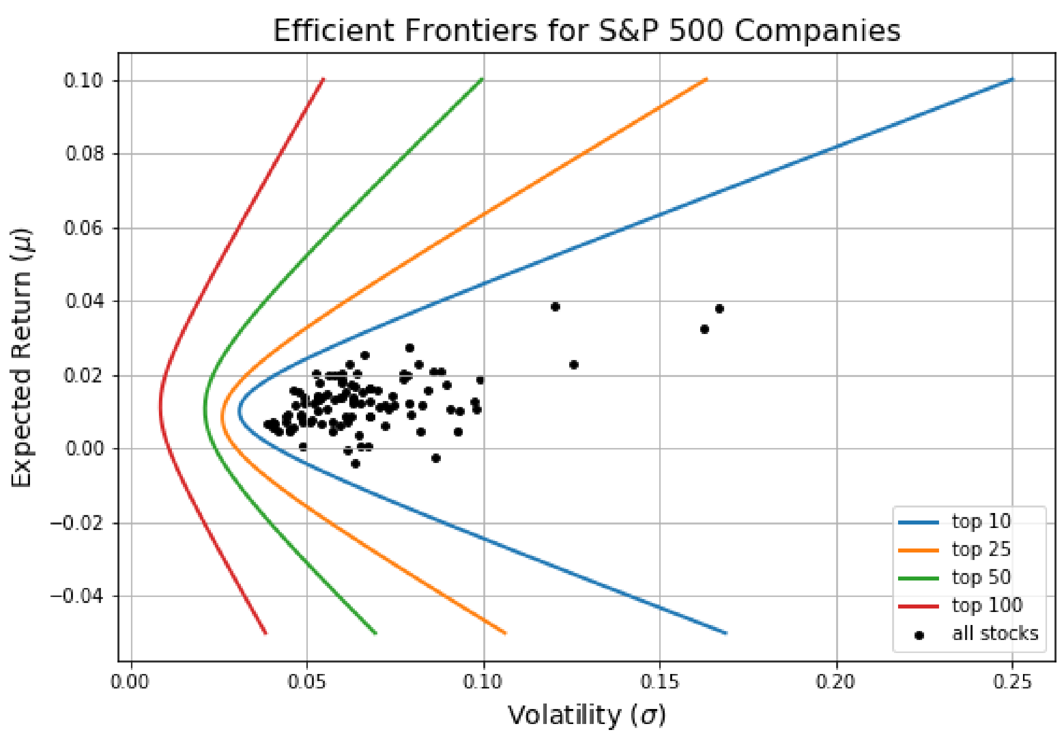
We apply our derivations in section 2 and 3 to real-world stock markets using Python. Specifically, we analyze the companies with the largest market capitalization that have complete data for the last ten years in the S&P 500 and Nikkei 225 stock market indices separately. These indices can be thought of as representatives for the American and Japanese stock markets, respectively. The market cap rankings for each index were obtained from:

- S&P 500: `https://www.slickcharts.com/sp500`
- Nikkei 225: `https://markets.businessinsider.com/index/market-capitalization/nikkei_225`

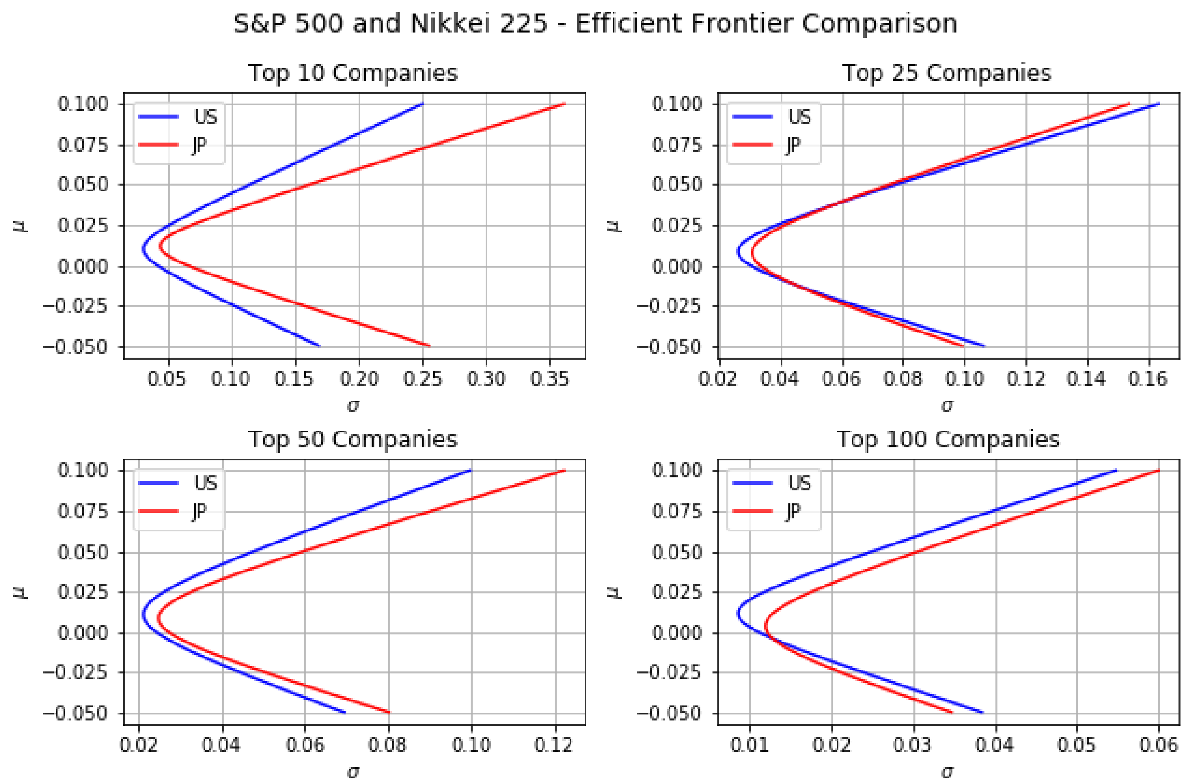
Assuming no dividends, we compute monthly returns on stocks using historical stock price data over the last ten years, specifically using the opening price at the start of each month given by *Yahoo Finance*. Forming a data matrix with the returns at each month, we can then compute the expected return vector, variance-covariance matrix, and other important quantities required for the derivations in sections 2 and 3.

Besides analyzing how portfolio choice varies across countries, we are also interested in the relationship between the number of stocks included in a market and portfolio choice. To this end, we analyze portfolio choice for the top 10, 25, 50, and 100 companies by market cap in each of the indices.

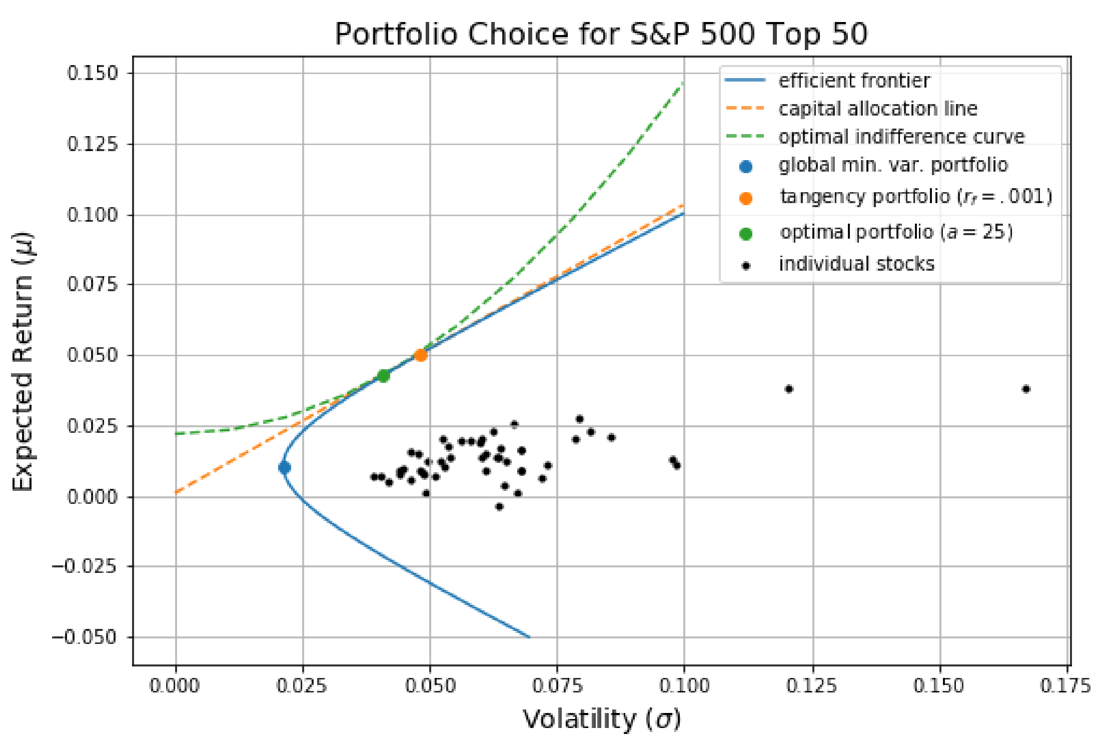
4.1 Efficient frontiers



The above figure shows the efficient frontiers for portfolios with different numbers of stocks from the S&P 500, and the individual stock statistics in the σ-μ plane. Note that the number of stocks in the portfolio increases, the efficient frontier shifts to the left, meaning that portfolios with the same expected return can be achieved with lower probability. We then compare the efficient frontiers for the S&P 500 with those for the Nikkei 225:

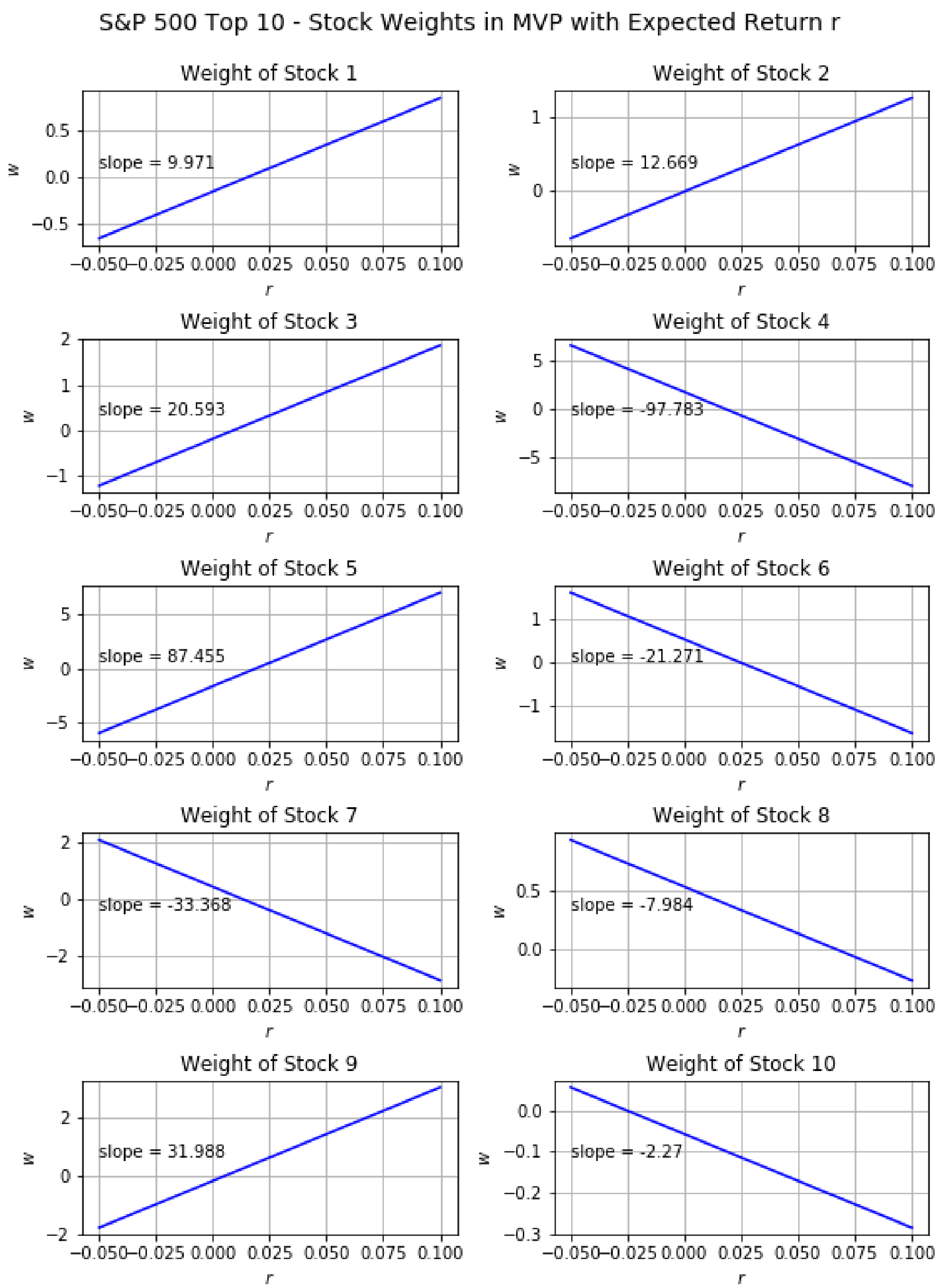


4.2 Portfolio choices



The above figure summarizes the results in sections 2 and 3 for the top fifty companies of the S&P 500 by market cap. We used the one-month T-bill yield at the time of writing for a risk-free rate of $r_f = 0.001$ to compute the capital allocation line and the tangency portfolio. An investor with utility function of the form in (3.2) with $\alpha = 25$ was used to compute the optimal portfolio.

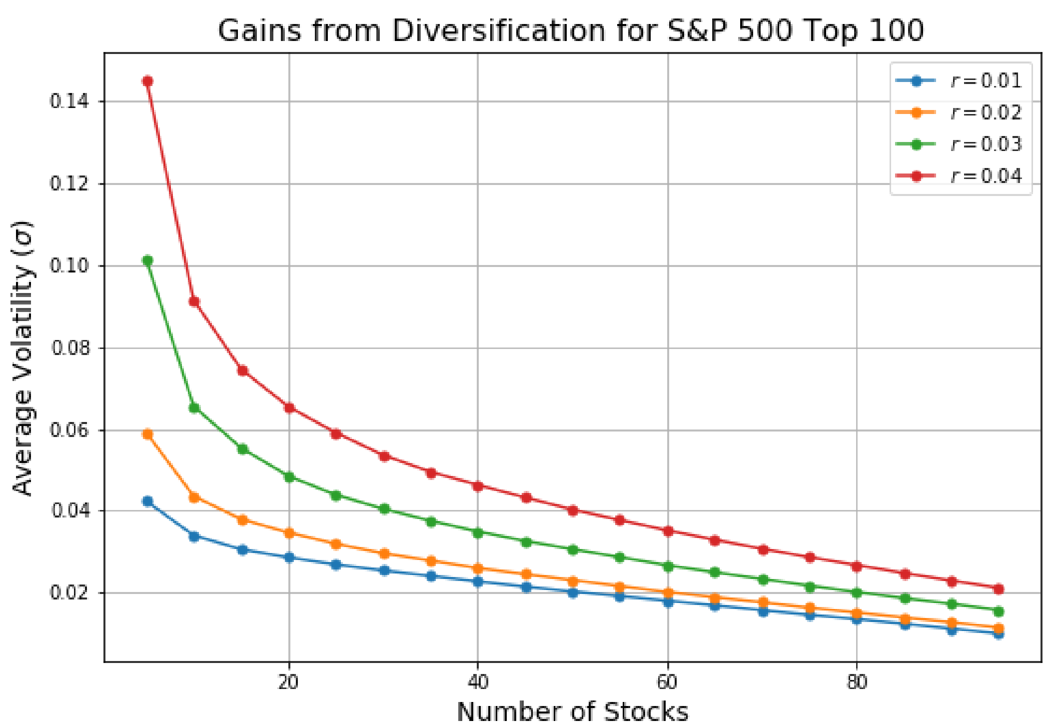
4.3 Individual stock weights



The above figure shows the weight of each stock in portfolios of the top ten S&P 500 companies by market cap as r varies. Note that each weight is linear in r as equation (2.5) suggests, with the slopes of each line also consistent with equation (2.5), as can be verified in the companion notebooks.

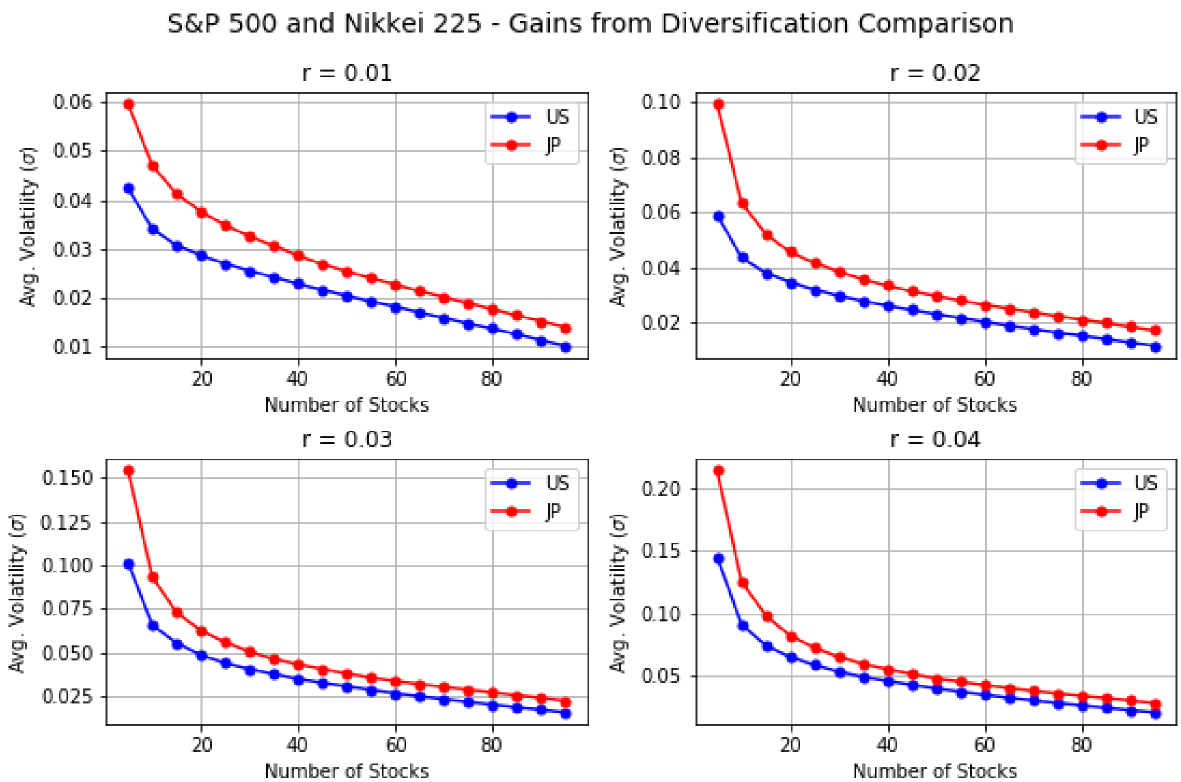
4.4 Gains from diversification

To illustrate the gains from diversification for these markets, we considered values of $k = 5, 10, \ldots, 95$. For each k, we randomly sampled 1000 subsets of k stocks from the top 100 companies of each index by market cap. For each subset sampled, we computed the volatility of the minimum variance portfolio for different values of r, and averaged them for each k to obtain an average volatility for a a k-stock portfolio of an index. Our results are shown below:



This figure clearly illustrates the gains from diversification, especially for higher levels of desired expected return. Note that the average volatility is decreasing in the number of stocks, with higher levels of desired return exhibiting larger decreases in average volatility.

We then compare the gains from diversification between the American and Japanese stock markets, here represented by the S&P 500 and Nikkei 225, respectively, using the same method. The figure below suggests investing in the largest companies in the Nikkei 225 is riskier on average than investing in those in the S&P 500.



The historical stock price data and the Python notebooks used in this section to produce the figures can be found at `https://github.com/zhubrian/portfolio-choice`.

5 Conclusion and further exploration

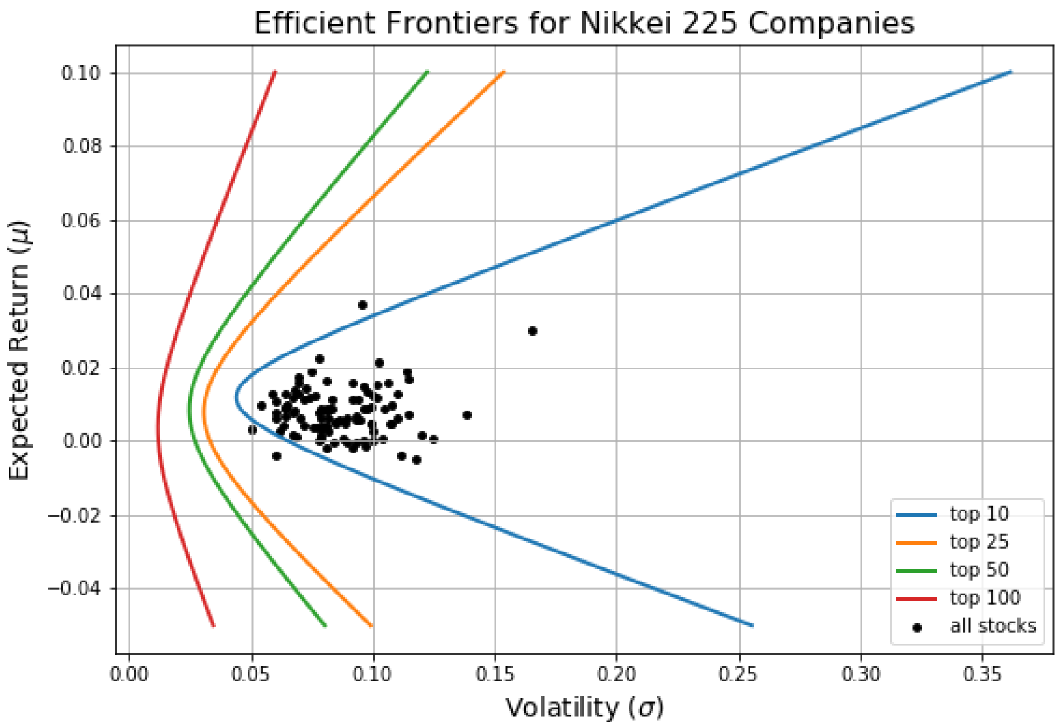
In summary, modern portfolio theory is a powerful and versatile tool that can be used to construct optimal portfolios in a variety of settings. Sections 2 and 3 demonstrate that in all of the settings described (all risky assets, adding a risk-free asset, and adding investor preferences), the corresponding optimal portfolio has a concise solution in terms of the parameters μ and C. Section 4 illustrates the results in sections 2 and 3 with real-world data, and also highlights the ability of modern computers to produce those results very quickly. The idea of gains from diversification is captured in sections 4.1 and 4.4. Finally, computation allows us to compare aspects of stock markets by sector, country, or size with speed and efficiency.

One area of further exploration in portfolio choice is the case where no short selling is allowed. This restriction would impose inequality constraints in the quadratic program, preventing Lagrange multipliers from being used to find a closed-form solution. In this case, portfolio choice will have to rely on optimization algorithms.

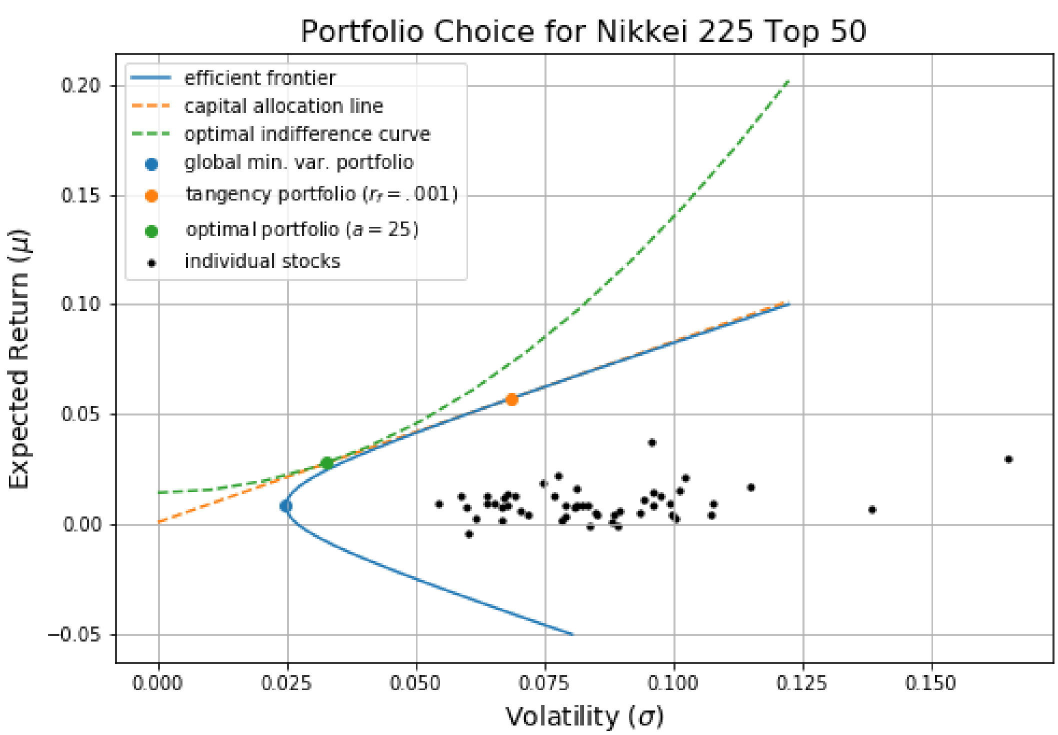
References

- Markowitz, Harry. (1952). "Portfolio Selection." *Journal of Finance 7*: 77-91.
- Campbell, John Y. (2018). "Static Portfolio Choice." *Financial Decisions and Markets*: 23-46. Princeton University Press.
- Data on historical stock prices provided by *Yahoo Finance*. `https://finance.yahoo.com/`
- Data on S&P 500 companies by market capitalization provided by *Slickcharts*. `https://www.slickcharts.com`
- Data on Nikkei 225 companies by market capitalization provided by *Business Insider*. `https://markets.businessinsider.com/index/market-capitalization`
- Notebooks used to generate figures available on the author's Github. `https://github.com/zhubrian/portfolio-choice`.

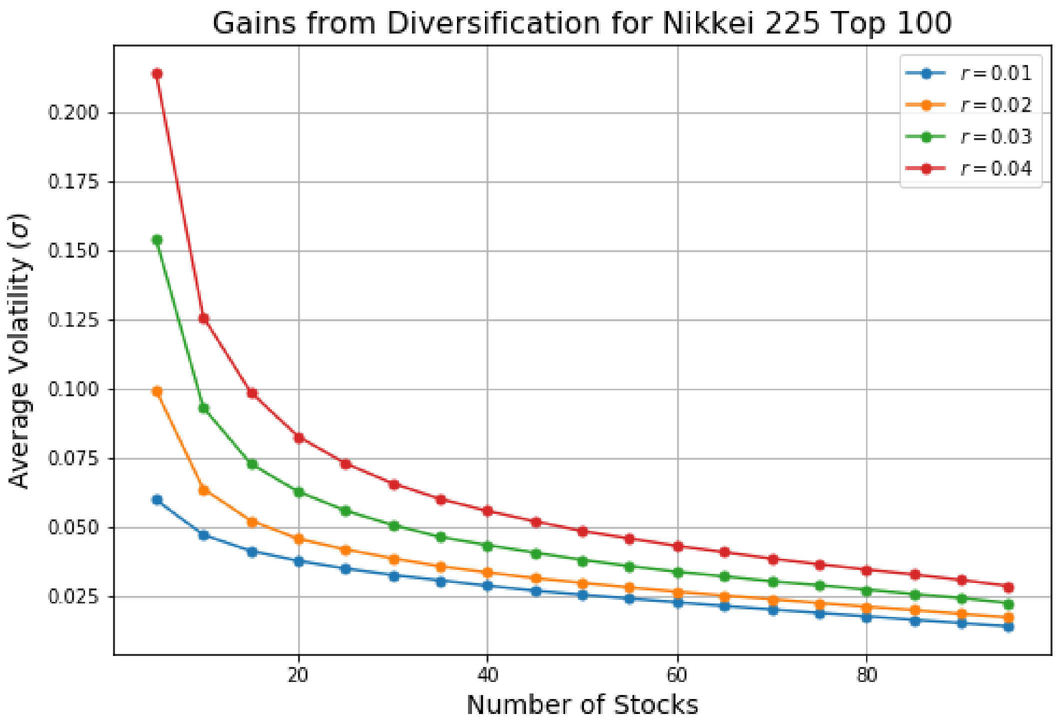
Appendix



Efficient Frontiers for Nikkei 225 Companies
0.10
0.08
0.06
0.04
0.02
0.00
−0.02
−0.04
Expected Return (μ)
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
Volatility (σ)
top 10
top 25
top 50
top 100
all stocks



Portfolio Choice for Nikkei 225 Top 50
efficient frontier
capital allocation line
optimal indifference curve
global min. var. portfolio
tangency portfolio ($r_f = .001$)
optimal portfolio ($a = 25$)
individual stocks
0.20
0.15
0.10
0.05
0.00
−0.05
Expected Return (μ)
0.000
0.025
0.050
0.075
0.100
0.125
0.150
Volatility (σ)


Gains from Diversification for Nikkei 225 Top 100
Average Volatility (σ)
Number of Stocks
$r = 0.01$
$r = 0.02$
$r = 0.03$
$r = 0.04$
0.200
0.175
0.150
0.125
0.100
0.075
0.050
0.025
20
40
60
80