Assignment 1: Geometric Camera Models & Calibration

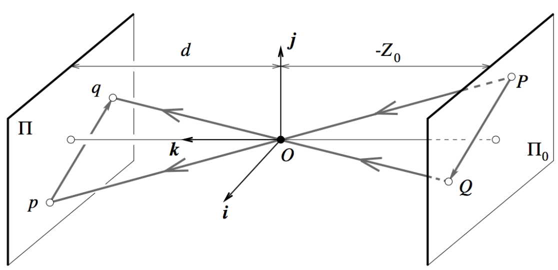
Theoretical Problems

Chuan Long

Problem 1: Pinhole Camera

A straight line in the world space is projected onto a straight line at the image plane". Prove by geometric consideration (qualitative explanation via reasoning). Assume perspective projection.

For perspective projection, all right rays emitted from a straight line pass through the pinhole, and consequently they will lie on a plane which intersects with the right rays in a straight line. Perspective projection creates inverted images, and the invert image of a straight line is definitely a straight line. Additionally, in the following figure, consider two points P and Q in Π0 and their images p and q. Obviously, the both vectors PQ and pq are parallel and are straight lines.



Show that, in the pinhole camera model, three collinear points in 3-D space are imaged into three collinear points on the image plane (show via a formal solution).

Three collinear points in world coordinates which are P1, P2, P3 form a matrix whose determinant is 0:

World Coordinates: $Det \begin{vmatrix} x1 & y1 & z1 \\ x2 & y2 & z2 \\ x3 & y3 & z3 \end{vmatrix} = 0$

(the determinant is 0 if there are two rows or cols are proportional to each other. Because these three points are collinear, the rows are proportional to each other).

For perspective projection, we assume:

$$\begin{cases} x = -mX, \\ y = -mY, \end{cases} \quad \text{where} \quad m = -\frac{d}{Z_0}.$$

and we can get the image coordinates and the corresponding determinant.

Image Coordinates: $$Det \begin{vmatrix} -m*x1 & -m*y1 & 1 \\ -m*x2 & -m*y2 & 1 \\ -m*x3 & -m*y3 & 1 \end{vmatrix} = \begin{vmatrix} x1 & y1 & 1 \\ x2 & y2 & 1 \\ x3 & y3 & 1 \end{vmatrix} = 0$$

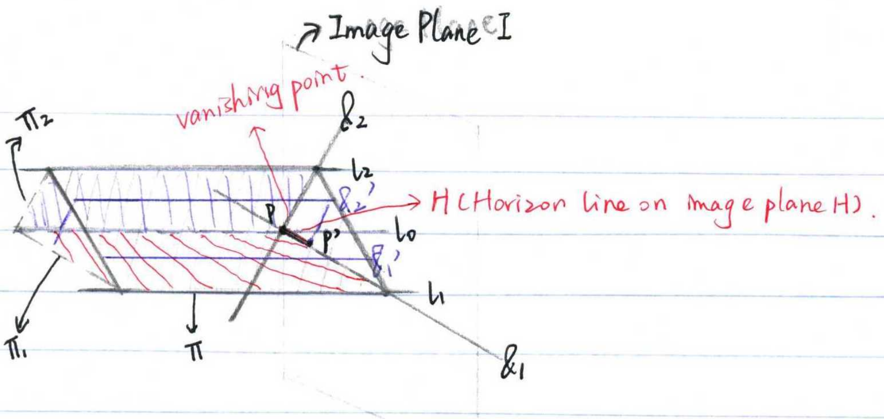
So, we can easily get that the projected points are also collinear because the determinant related to the image coordinates of projected points is 0.

Problem 2: Perspective Projection

See Fig. 1.4 from textbook on page 6 (pdf handouts) for reference.

a) Prove geometrically that the projections of two parallel lines lying in some plane appear to converge on a horizon line H formed by the intersection of the image plane with the plane parallel to and passing through the pinhole.

Problem 2: a)



Consider the above figure. Assume two parellel lines l_1 and l_2 lying in the plane π (shaded by ////). Let's say, there is a line l_0 passing through the pinhole that is parallel to l_1 and l_2. Also, lines l_0 and l_1 form a plane π_1 (shaded by \\\\ ). And lines l_0 and l_2 form plane π_2 (shaded by ||||). Now, we have plane π_1 intersects with Image Plane I as a line ℓ_1, and plane π_2 intersects with I as a line ℓ_2. These two lines ℓ_1, ℓ_2 intersect at the point p where l_0 intersects I. This point is the vanishing point associated with the lines lying on plane π and parallel to l_0, and the projection of these lines appears to converge on it.

Now let us consider two other parallel lines l_1' and l_2' lying on plane π and define as before the corresponding line l_0' and vanishing point p'. l_0 and l_0' line in a plane parallel to π that intersects the image plane along a line H passing through p and p'. This is the horizon line H, and any two parallel lines in π appears to intersect on it. They appear to converge there since any image point above the horizon is associated with a ray issued from the pinhole and pointing away from π. Horizon points correspond to rays parallel to π and points in that plane located at an infinite distance from the pinhole.

b) Prove the same result algebraically using the perspective projection equation. You can assume for simplicity that the plane is orthogonal to the image plane (as you might see in an image of railway tracks, e.g.).

Problem 2 b).

Consider the ground plane Π, for the sake of simplicity, which is orthogonal to the image plane. Consider a line L that lines in the plane Π, which is defined by the equation $ax + bz = d$.

Using perspective pinhole projections, a point on L projected on the image plane will have coordinates defined as,

$$x' = f \cdot \frac{x}{z} = f \cdot \frac{d - bz}{az}$$

$$y' = f \cdot \frac{y}{z} = f \cdot \frac{c}{z}$$

This is the parametric representation of the image L' of the line L with z as the parameter. When $z \to -\infty$, line L' stops at the point $(x', y') = (-\frac{fb}{a}, 0)$ on the x' axis of the image plane. This is the vanishing point corresponding to all parallel lines with slope $-\frac{b}{a}$ in the plane Π. All vanishing points associated with different lines with different slopes belonging to plane Π will lie on the x' axis, which is the horizon line for this case.

Problem 3: Coordinates of Optical Center

Let O denote the homogeneous coordinate vector of the optical center of a camera in some reference frame, and let M denote the corresponding perspective projection matrix. Show that MO = 0. (Hint: Think about the coordinates of the optical center in the world coordinate system, use the notion of transformations between world and camera, and plug this into the projection equation.)

Problem 3:

Defined by the extrinsic parameters of the camera, the perspective projection matrix $\mathcal{M}$ is:

$$\mathcal{M} = K(\mathcal{R} \;\; t)$$

$$\mathcal{M} = K({}^{C}_{W}\mathcal{R} \;\; {}^{C}O_{W})$$

The coordinate vector O of the camera optical center in the world coordinate system is

$$O = \begin{pmatrix} {}^{W}O_{C} \\ 1 \end{pmatrix}$$

Now,

$$\mathcal{M}O = K({}^{C}_{W}R \;\; {}^{C}O_{W})\begin{pmatrix} {}^{W}O_{C} \\ 1 \end{pmatrix}$$

$$= K[{}^{C}_{W}R \, {}^{W}O_{C} + {}^{C}O_{W}]$$

If we rotate the camera coordinate to world coordinate. ${}^{C}_{W}R$ vanishes and we are left with

$$\mathcal{M}O = K[-{}^{C}O_{W} + {}^{C}O_{W}]$$

hence, $\mathcal{M}O = 0$