Assignment 3: Photometric Stereo

Theoretical Problems

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1.1 Reflectance map

1.1.1 Special case

(Book problem 2.1). We see a diffuse sphere centered at the origin, with radius one and albedo ρ, in an orthographic camera, looking down the z-axis. The sphere is illuminated by a distant point light source whose source direction is (0,0,1). There is no other illumination. Show that the shading field (reflectance map) in the camera is:

$$\boldsymbol{\rho\sqrt{(1 - x^2 \text{-} y^2)}}$$

Solution:

As we can see, the equation for the sphere is $x^2 + y^2 + z^2 = 1$.

So, the depth $z = \sqrt{1 - x^2 - y^2}$. In this case, we just consider the visible part, which z >= 0. Then, we get the partial derivative for each equation:

$$\frac{\partial \mathrm{z}}{\partial \mathrm{x}} = \frac{-x}{\sqrt{1 - x^2 - y^2}} = p$$

$$\frac{\partial \mathrm{z}}{\partial \mathrm{y}} = \frac{-y}{\sqrt{1 - x^2 - y^2}} = q$$

We knew that the source direction is (0, 0, 1). So, the equation for reflectance map is:

$$R(p, q) = \rho < \widehat{N}, [0, 0, 1] > = \rho \frac{1}{\sqrt{1 + p^2 + q^2}}$$

Substitute the p and q, we can easily get the reflectance map equation very straightforwardly:

$$R(p, q) = \rho\sqrt{1 - x^2 - y^2}$$

1.1.2 General light source direction

Starting from the reflectance map equation for a general light source direction (see slides and book), show that reflectance map isophotes (curves of constant brightness in $\mathbf{R(p, q)}$) represent conic sections, i.e. curves with squared and linear terms in p and q. You may

choose an example for a light source direction and plot the result in $(\mathbf{p}, \mathbf{q})$ - space for several values of constant brightness.

Solution:

As we can see, the equation for the Lambertian surface from general source position is:

$$R(p,q) = \frac{p \cdot p_L + q \cdot q_L + 1}{\sqrt{p^2 + q^2 + 1}\sqrt{p_L{}^2 + q_L{}^2 + 1}}$$

Now, I define a sphere which is the same as the problem 1, centered at the origin with radius one. But now I choose a different light source direction, which is [0, 1, 1].

And, we can get the reflectance map:

$$R(p,q) \; = \frac{0 * p + 1 * q + 1}{\sqrt{1 \; + \; p^2 \; + \; q^2}\sqrt{0 \; + \; 1 + 1}}$$

And we already know:

$$\frac{\partial \mathrm{z}}{\partial \mathrm{x}} \; = \; \frac{-\,x}{\sqrt{1 - x^2 - y^2}} = p$$

$$\frac{\partial \mathrm{z}}{\partial \mathrm{y}} \; = \; \frac{-\,y}{\sqrt{1 - x^2 - y^2}} = q$$

So,

$$R(p,q) \; = \frac{-x - y + \sqrt{1 - x^2 - y^2}}{\sqrt{3}}$$

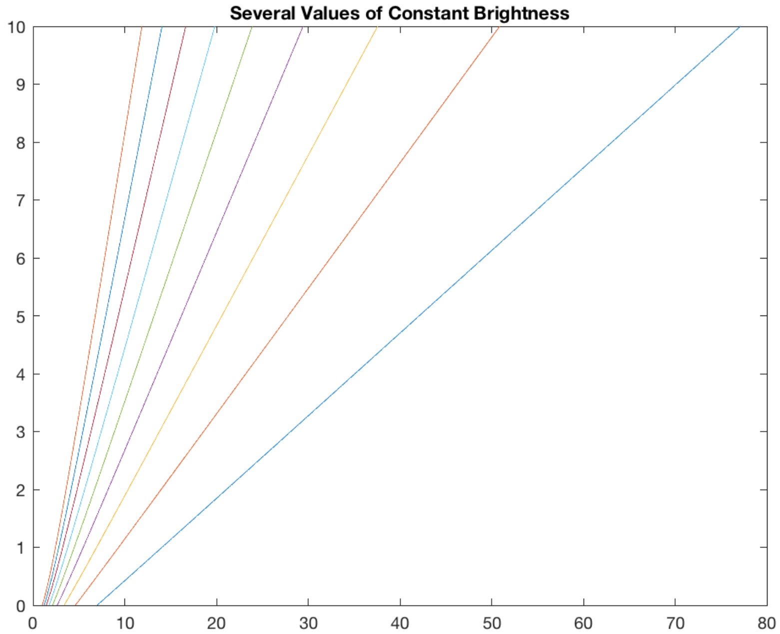
Now, I want to plot the figure $R(p,q)$ = c.

Then, we can get the equation in terms of $p \; and \; q$, which is:

$$p = \sqrt{\frac{q^2+1+2q}{2c^2} - 1 - q^2}$$

for example, if c = 0.5, $p = \sqrt{q^2 + 4q + 1}$

Now, we have constant brightness c = (0.1:0.1:1.0), and we can get the plot results with several values of constant brightness.



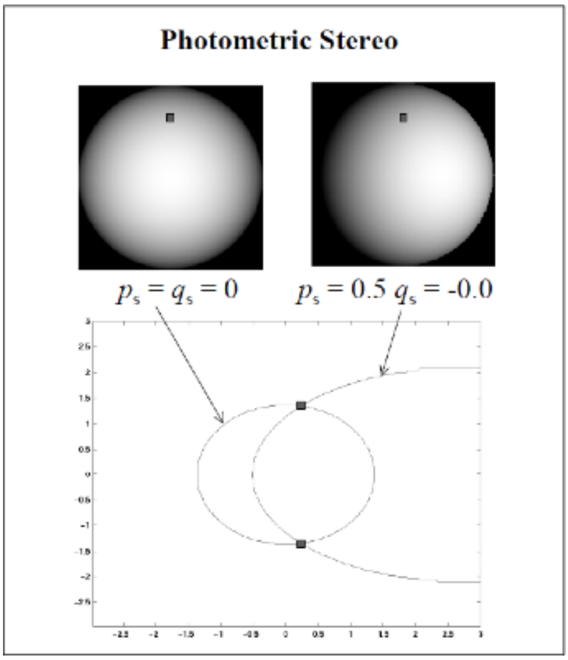
1.1.3 Number of images for reconstruction of normal?

Following the discussions in the book chapter 2 and slides, explain:

Why is one image with on light source direction not enough?

Getting images with more than one light source directions, justify how many images we would need as a minimum. You can use drawings or equations if you like.

Solution:
1. The reason why is one image with one light source direction not enough is that we will get a contour map, which is like problem 2 and what I have plotted, and any point (a pair of p, q) at the same line could be the exact point in the image. So, it is obviously not enough.
2. We have to use at least three images to determine the reconstruction of normal.



As we can see, if we have only two images, although we can narrow down our range to only two points, they are still arbitrary. And if we have the third image, we can determine the only one point of these two that could represent the normal, which is like the following figure: