

w.l.o.g. scalar RK . notational convenience .

$$\vec{k} = \begin{pmatrix} k_1 \\ \vdots \\ k_r \end{pmatrix} \in \mathbb{R}^r \quad \vec{e} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$u_{n+1} = u_n + h \bar{\Phi}(u_n)$$

$$\left\{ \begin{array}{l} \vec{k} = f(u_n \vec{e} + h A \vec{k}) \\ u_{n+1} = u_n + h \underbrace{\vec{b}^T \vec{k}}_{\bar{\Phi}(u)} \end{array} \right.$$

$$e_{n+1} = e_n + (\bar{\Phi}(u(t_n)) - \bar{\Phi}(u_n)) h + \bar{e}_n$$

error propagation .

Need to show:

① (*) has a sol.

② $\Psi(\cdot)$ is Lip cont.

\Rightarrow convergence following proof of Euler's method.

① mapping $T: \mathbb{R}^r \rightarrow \mathbb{R}^r$

$$\vec{T}(\vec{k}) = f(\vec{u}_k + h A \vec{k})$$

Find a fixed pt.

$\vec{K}, \vec{K}' \in \mathbb{R}^r$

$$\|\mathbf{T}(\vec{K}) - \mathbf{T}(\vec{K}')\|_2^2 = \|f(u_n \vec{e} + h A \vec{K}) - f(u_n \vec{e} + h A \vec{K}')\|_2^2$$

$$\leq L^2 h^2 \|A(\vec{K} - \vec{K}')\|_2^2$$

$$\leq (Lh \|A\|_2)^2 \|\vec{K} - \vec{K}'\|_2^2$$

$Lh \|A\|_2 < \frac{1}{2} \Rightarrow \mathbf{T}$ contraction mapping.

$\Rightarrow \vec{K}(u_n)$ exists & unique \square .

② $\Psi(u)$ Lip cont. $\Leftrightarrow \vec{K}(u)$ Lip. cont.

$\forall u, v \in \mathbb{R}$.

$$\|\vec{K}(u) - \vec{K}(v)\|_2^2 = \|f(u\vec{e} + hA\vec{K}(u)) - f(v\vec{e} + hA\vec{K}(v))\|_2^2$$

$$\leq L^2 \|(u-v)\vec{e} + hA(\vec{K}(u) - \vec{K}(v))\|_2^2$$

$$\leq L^2 r |u-v|^2 + L^2 h^2 \|A\|_2^2 \|\vec{K}(u) - \vec{K}(v)\|_2^2$$

$$Lh \|A\|_2 < \frac{1}{2}$$

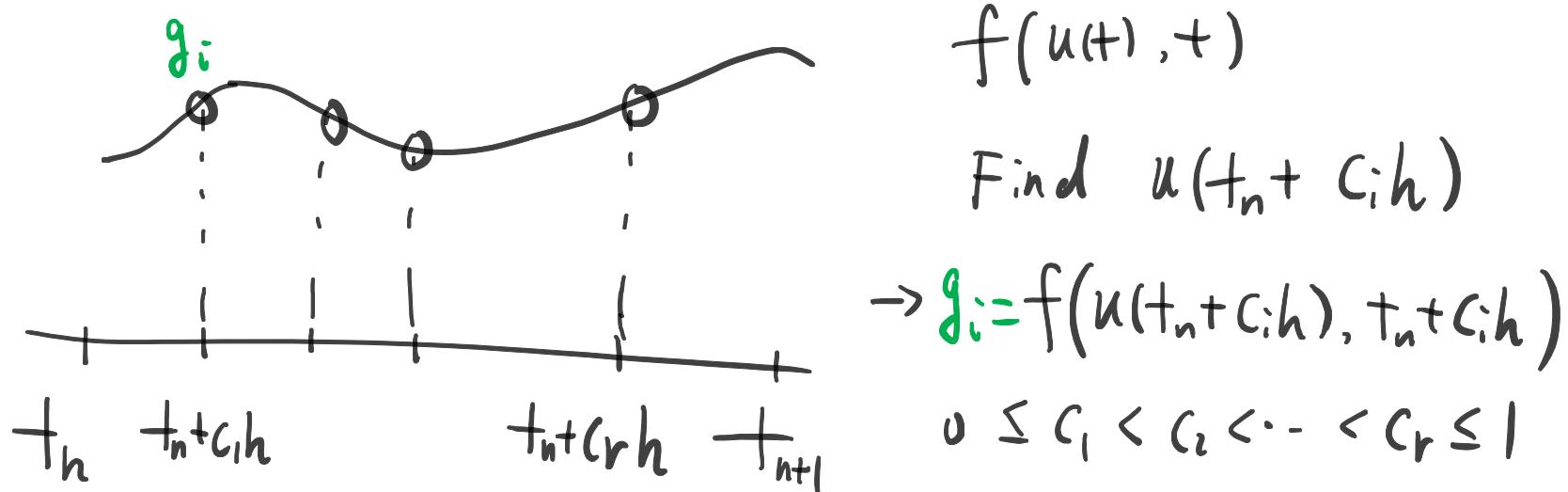
$$\|\vec{K}(u) - \vec{K}(v)\|_2^2 \leq 2 L^2 r |u-v|^2$$

Lip const. $\vec{K}(\cdot)$ is bounded by $\sqrt{2r} L$. \square .

How to construct (implicit) RK methods
of arbitrary order.

One possibility: quadrature. Collocation method.

"Co-locates" all functions of interest
at the same set of points.



Lag range interpolation . w.r.t . $\{g_i\}_{i=1}^r$

$$P_i(t) = \prod_{j \neq i} \frac{t - (t_n + c_j h)}{h(c_i - c_j)}$$

$$P_i(t_n + c_k h) = \delta_{ik}$$

$$g(t) = \sum_{i=1}^r P_i(t) g_i \approx f(u(t), t)$$

$$u(t_{n+1}) = u(t_n) + \int_{t_n}^{t_{n+1}} \underbrace{f(u(s), s)}_{g(s)} ds$$

As a scheme .

$$u_{n+1} = u_n + \sum_{i=1}^r \left(\int_{t_n}^{t_{n+1}} P_i(s) ds \right) g_i$$

Define $b_i = \frac{1}{h} \int_{t_n}^{t_{n+1}} p_i(s) ds$

$$\rightarrow u_{n+1} = u_n + h \sum_{i=1}^r b_i g_i$$

need to show

g_i 's are k_i 's.

To compute g_i

$$u(t_n + c_i h) = u(t_n) + \int_{t_n}^{t_n + c_i h} f(u(s), s) ds$$

$f(u(s), s)$
replace by $g(s)$

$$\tilde{u}(t_n + c_i h) = u_n + \sum_j \int_{t_n}^{t_n + c_i h} p_j(s) ds g_j$$

$$\text{Define } a_{ij} = \frac{1}{h} \int_{t_n}^{t_n + c_i h} p_j(s) ds$$

$$\tilde{u}(t_n + c_i h) = u_n + h \sum_j a_{ij} g_j$$

$$g_i := f(\tilde{u}(t_n + c_i h), t_n + c_i h)$$

$$= f(u_n + h \sum_j a_{ij} g_j, t_n + c_i h)$$

$$g_i \equiv k_i$$

Def. Collocation RK method. (fully implicit)

Given $0 \leq c_1 < \dots < c_r \leq 1$ only input

$$k_i = f(u_n + \sum_j a_{ij} k_j, t_n + c_i h)$$

$$u_{n+1} = u_n + h \sum_{i=1}^r b_i k_i$$

where.

$$a_{ij} = \int_0^{c_i} \tilde{P}_j(\theta) d\theta$$

$$\tilde{P}_i(\theta) = \prod_{j \neq i} \frac{\theta - c_j}{c_i - c_j}$$

$$b_i = \int_0^1 \tilde{P}_i(\theta) d\theta.$$

Whether Kutta's condition is satisfied?

$$\sum_j a_{ij} = \int_0^{c_i} \sum_j \tilde{P}_j(\theta) d\theta$$

$$f(\theta) := \sum_j \tilde{P}_j(\theta), \quad f(c_i) = 1, \quad i=1, \dots, r$$

$$f(\theta) \in P_{r-1} \Rightarrow f(\theta) \equiv 1.$$

$$\boxed{\sum_j a_{ij} = c_i}$$

Kutta's condition ✓.

Ex. $c_1 = 0$.

$$\begin{array}{c|c} 0 & a \\ \hline & b \end{array}$$

$$a=0. \quad b=\int_0^1 d\theta = 1.$$

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

forward Enter.

$$c_1 = 1.$$

$$a=1, b=1$$

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

backward Enter.

Ex. $c_1 = \frac{1}{2}$. $a = \frac{1}{2}$ $b = 1$.

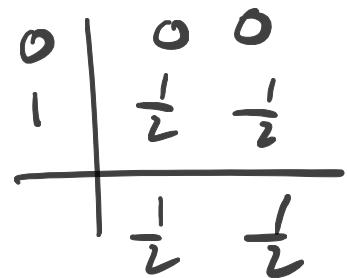
$$\begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}$$

$$\left\{ \begin{array}{l} k = f(u_n + \frac{1}{2}hk, t_n + \frac{1}{2}h) \\ u_{n+1} = u_n + hk \end{array} \right.$$

2nd order.
(exer)

Gauss-Legendre 1 (implicit midpoint rule)

$\Sigma_x. \quad c_1 = 0, \quad c_2 = 1.$

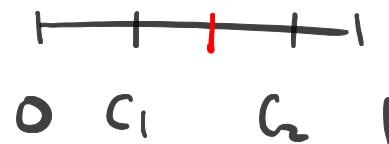


trapezoidal. (exer).

Ex. GL 2.

4th order

$$\begin{array}{|cc|} \hline * & \frac{1}{4} \quad \frac{1}{4} - \frac{\sqrt{3}}{6} \\ * & \frac{1}{4} + \frac{\sqrt{3}}{6} \quad \frac{1}{4} \\ \hline & \frac{1}{2} \quad \frac{1}{2} \\ \hline \end{array}$$



$$c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6} \quad c_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}$$

accuracy of quadrature.

$$\int_a^b f(x) dx \approx \sum_{l=1}^r f(x_l) \omega_l$$

\uparrow \uparrow
nodes weights

weights can be computed via Lag poly.

$$P_K(x) = \prod_{l \neq K} \frac{x - x_l}{x_k - x_l}, \quad P_K(x_l) = \delta_{kl}.$$

$$\omega_k = \int_a^b P_k(x) dx.$$

Def. $\left| \int_a^b f(x) dx - \sum_{l=1}^r f(x_l) \omega_l \right| \leq C(b-a) \sup_{\substack{a \leq x \leq b \\ P+1}} |f^{(P)}(x)|$

holds for all $f \in C^{(P)}([a,b])$, $c > 0$

indep. of a, b, f . Then quadrature is of
order \underline{P} .

Consistency of collocation RK methods.

Thm. Quadrature order p . r-stages (points)

\Rightarrow LTE $O(h^{p+1})$. \leftarrow only need to prove.

\Rightarrow Scheme order p .

Duhamel's principle.

Thy. $\dot{u}(t) = Au(t) + g(t)$, $u(0)=u_0$

$$\Rightarrow u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}g(s)ds.$$

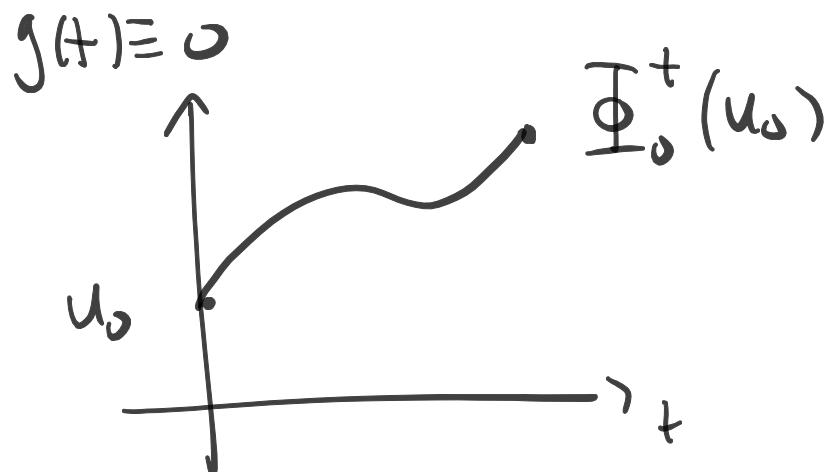
Pf: $\dot{u}(t) = A e^{At} u(0) + g(t) + \int_0^t A e^{A(t-s)} g(s)ds$
 $= A \left(e^{At} u(0) + \int_0^t e^{A(t-s)} g(s)ds \right) + g(t)$
 $= A u(t) + g(t)$

$$u(0) = u_0 \quad \square .$$

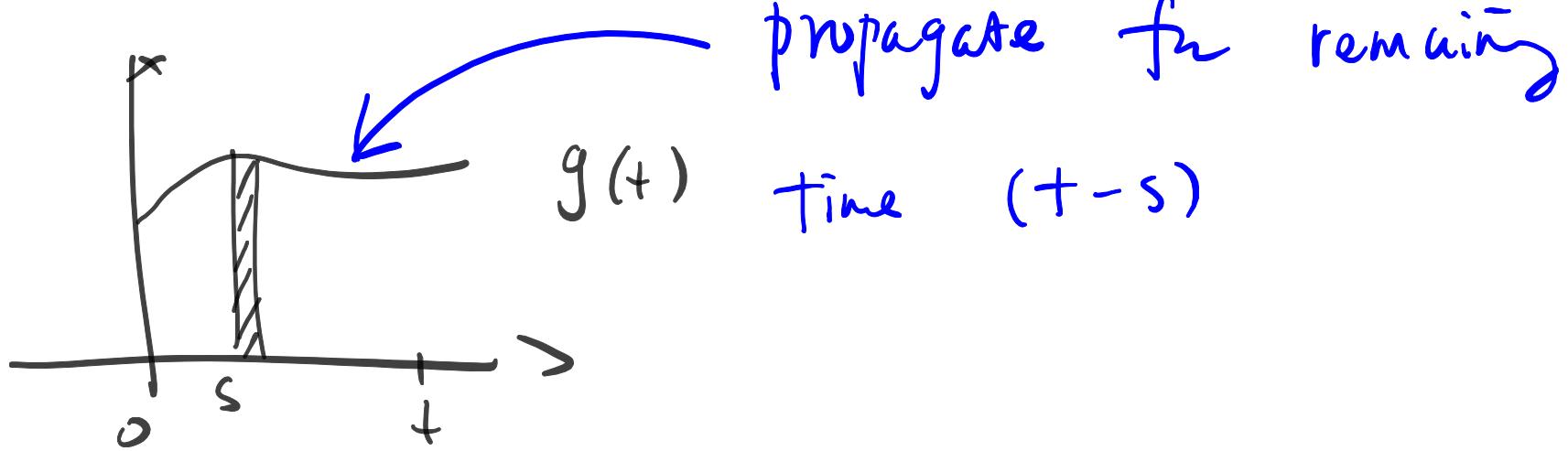
$$\begin{cases} \frac{d}{dt} \underline{\Phi}_s^t(w) = A \underline{\Phi}_s^t(w) \\ \underline{\Phi}_s^s(w) = w \end{cases} \rightarrow \underline{\Phi}_s^t(w) = e^{A(t-s)}w$$

Duhamel's principle.

$$u(t) = \underline{\Phi}_0^t(u_0) + \int_0^t \underline{\Phi}_s^t(g(s)) ds.$$



$$\int_0^t \underline{\Phi}_s^t(g(s)) ds \approx \sum_{i=0}^N \underline{\Phi}_{s_i}^t(g(s_i)) (s_{i+1} - s_i)$$



Thm (Alekseev - Gröbner)

Non linear version of Duhamel's principle.

[Hai] I.14.

$$\begin{cases} \dot{u}(t) = f(u(t), t) \\ u(0) = u_0 \end{cases} \rightarrow \begin{cases} \frac{d}{dt} \underline{\Phi}_s^+(w) = f(\underline{\Phi}_s^+(w), t) \\ \underline{\Phi}_s'(w) = w \end{cases}$$

$$\begin{cases} \dot{v}(t) = f(v(t), t) + \underline{g(v(t), t)} \\ v(0) = v_0 \end{cases}$$

$$\Rightarrow v(t) = u(t) + \left[\int_0^t \left[\frac{\partial}{\partial w} \underline{\Phi}_s^+(w) \right] g(v(s), s) ds \right]_{w=v(s)}$$

(exer) use A-G to recover Duhamel's principle

$$\text{i.e. } f(u(t), t) = A u(t)$$

Pf. (Convergence of Wllution RK).

LTE.

$$\begin{cases} \dot{u}(t) = f(u(t), t) \\ u(t_n) = u(t_n) \end{cases}$$

numerical scheme

$$\begin{cases} \dot{v}(t) = \tilde{f}(v(t), t) & \leftarrow \text{Lag. interp.} \\ v(t_n) = u(t_n) & \text{at } t_n + c_i h \end{cases}$$

$$\dot{v}(t) = \tilde{f}(v(t), t) = f(v(t), t) + \frac{(\tilde{f}(v(t), t) - f(v(t), t))}{g(v(t), t)}$$

interpolating: $g(v(t_n + c_i h), t_n + c_i h) = 0$.

$$v(t_{n+1}) = u(t_{n+1}) + \int_{t_n}^{t_{n+1}} \left[\frac{\partial \Phi_s^{t_{n+1}}(w)}{\partial w} \right] \Big|_{w=v(s)} g(v(s), s) ds$$

$$= u(t_{n+1}) + \underset{\substack{\uparrow \\ \text{quadrature}}}{O} + O(h^{p+1}) \quad \text{D .}$$

\uparrow

$O(h^{p+1})$

\uparrow

$\text{err of quadrature .}$

