

Sparse Direct Method .

$$Ax=b, \quad A \text{ sparse}$$

Sparse direct method = Gauss elimination  
for sparse matrices.

$$A = LU. \quad \text{Then}$$

$$Ax = b \Rightarrow LUx = b$$

$$\underline{y = L^{-1}b} \quad . \quad \underline{x = U^{-1}y}$$

forward  
substitution

backward  
substitution.

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{R}^{n \times n}. \quad A_{11} \in \mathbb{R}, \quad A_{11} \neq 0$$

$$\begin{pmatrix} 1 & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix} \begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \underline{A_{22} - A_{21}A_{11}^{-1}A_{12}} \end{pmatrix} \rightarrow \text{block-diagonal.}$$

Schur complement =  $S_{22}$

equivalently.

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ +A_{21}A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} A_{11} + A_{12} \\ 0 & I \end{pmatrix}$$

$$A = L_1 \begin{pmatrix} 1 & 0 \\ 0 & S_{22} \end{pmatrix} U_1$$

$S_{22}$  is denser than  $A_{22}$ , due to

fill-in  $(-A_{21}A_{11}^{-1}A_{12})$

But in general still sparse.

$$S_{22} = \tilde{L}_2 \begin{pmatrix} 1 & 0 \\ 0 & S_{33} \end{pmatrix} \tilde{U}_2$$

$$L_2 = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{L}_2 \end{pmatrix} \quad U_2 = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{U}_2 \end{pmatrix}$$

$$A = L_1 L_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & S_{33} \end{pmatrix} U_2 U_1$$

recursively

$$= \underbrace{L_1 L_2 \cdots L_n}_L \underbrace{U_n \cdots U_1}_U$$

Alg. (LU factorization. column-wise)

for  $k=1, n$ .

$$L_{kk} = 1, \quad U_{kk} = A_{kk}.$$

$$L_{ik} = A_{ik} A_{kk}^{-1}, \quad i > k$$

$$U_{kj} = A_{kj}, \quad j > k$$

$$A_{ij} \leftarrow A_{ij} - L_{ik} U_{kj}, \quad i, j > k.$$

} only need to  
work with  
 $A_{ik} \neq 0, A_{kj} \neq 0$ .

end.

$A$  sym. pos. def.  $\in \mathbb{R}^{n \times n}$

$A = LL^T$  Cholesky factorization.

$L$  lower triangular. non-unitary diagonal.

$$\begin{pmatrix} A_{11} & A_{21}^T \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11}^{\frac{1}{2}} & 0 \\ A_{21} A_{11}^{-\frac{1}{2}} & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A_{22} - A_{21} A_{11}^{-1} A_{21}^T \end{pmatrix} \cdot \begin{pmatrix} A_{11}^{\frac{1}{2}} & A_{11}^{-\frac{1}{2}} A_{21}^T \\ 0 & I \end{pmatrix}$$

Note:  $A_{11}$  is matrix.  $A_{11} = L_1 L_1^T$   
 $\downarrow$   
"  $A_{11}^{\frac{1}{2}}$  "

Alg. Cholesky. column-wise.

for  $k=1, n$ .

$$L_{kk} = A_{kk}^{\frac{1}{2}}$$

$$L_{ik} = A_{ik} A_{kk}^{-\frac{1}{2}}, \quad i > k$$

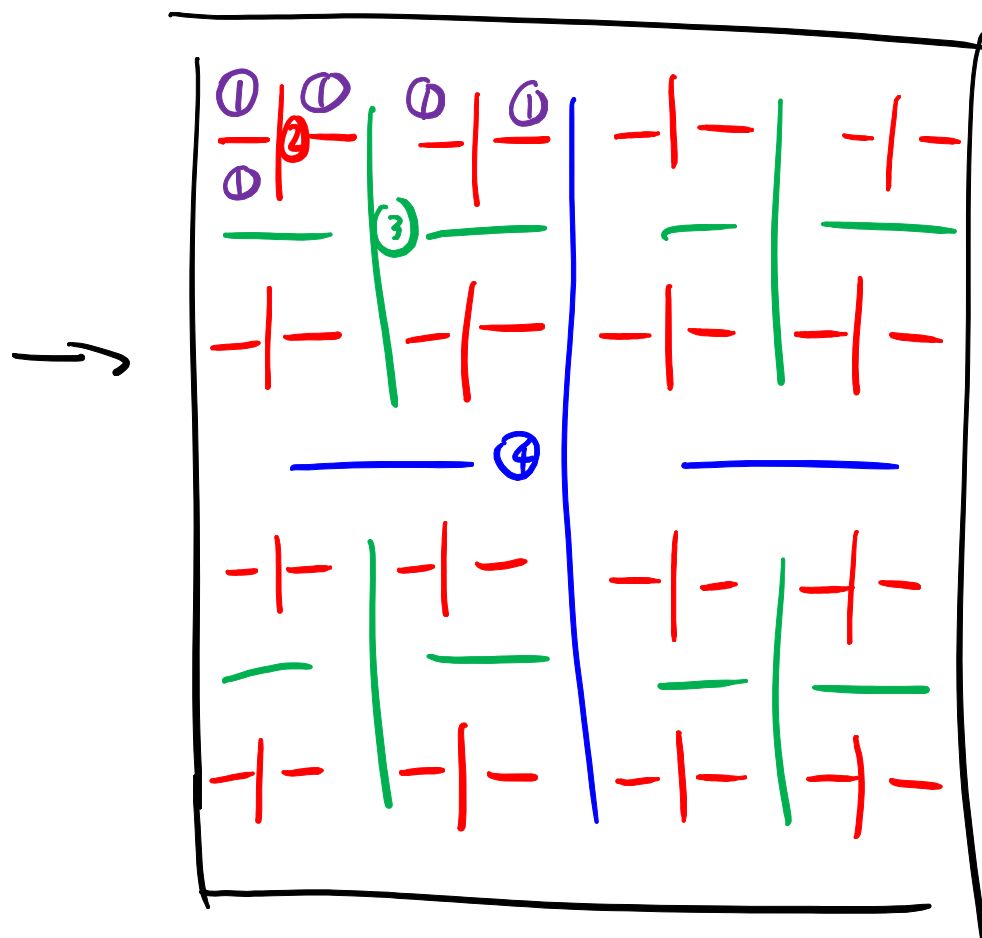
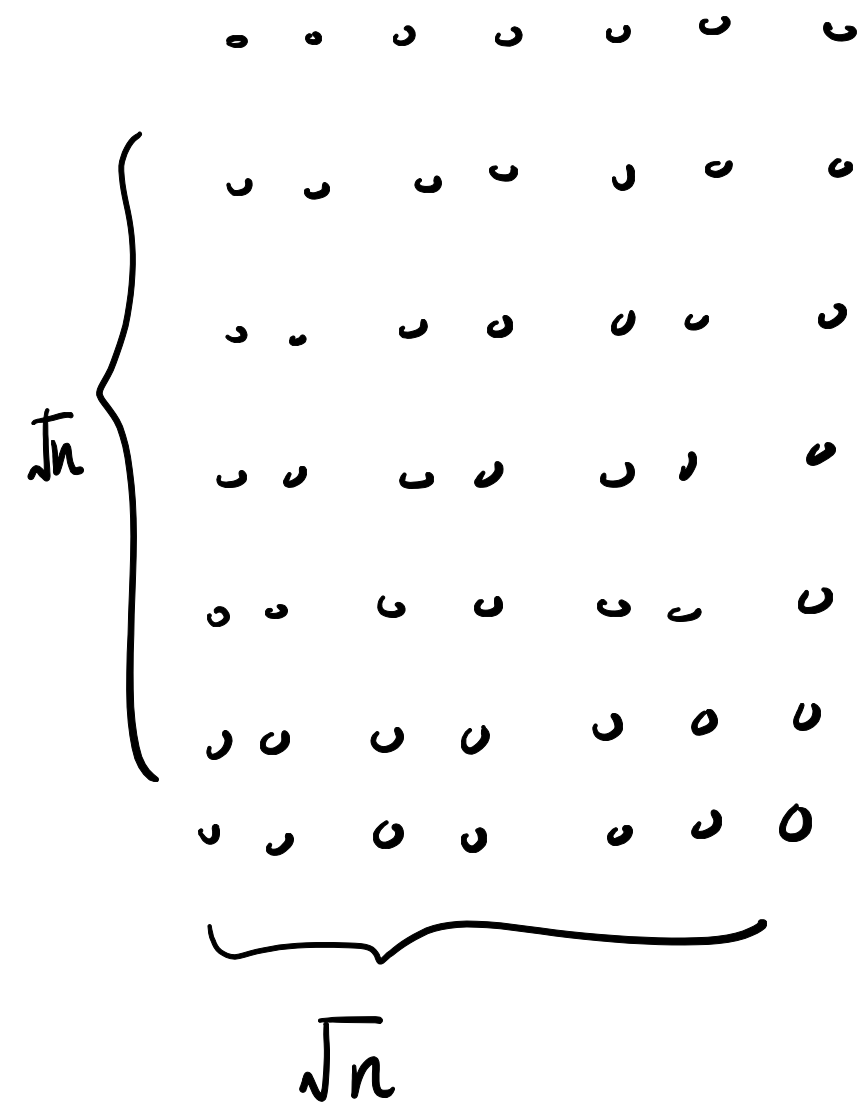
$$A_{ij} \leftarrow A_{ij} - L_{ik} L_{jk}, \quad i, j > k$$

} allows  
sparsity.

end.



# Nested dissection reordering



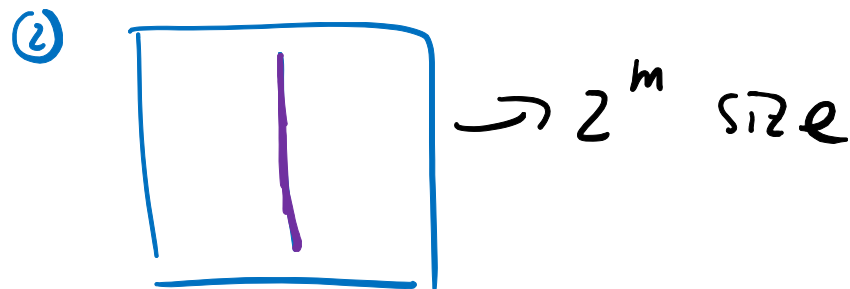
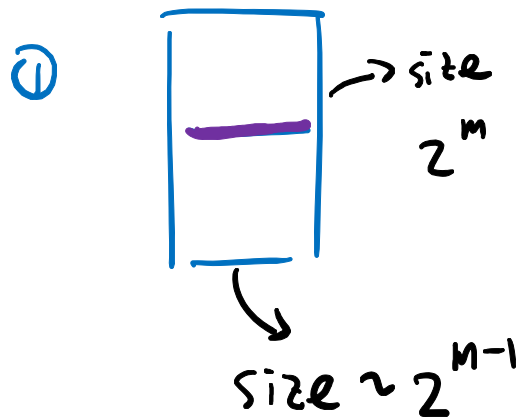
total # levels.  $\log_2 \sqrt{n}$

level 1: size of block  $\sim O(1)$

$n$  blocks.

tot non-zeros in  $L$ :  $O(n)$

level  $m$ .



dense matrix

(dense) matrix

$$2^m \times (5 \cdot 2^m) = 5 \cdot 2^{2m}$$

$$2^{m-1} \times (3 \cdot 2^{m-1} + 2 \cdot 2^m) = \frac{7}{4} \cdot 2^{2m}$$

tot nz in each block.

$$2 \times \frac{7}{4} \times 2^{2m} + 5 \cdot 2^{2m} = \frac{17}{2} \cdot 2^{2m} \sim O(2^{2m})$$

# blocks at level  $m$ .  $O\left(\frac{n}{2^{2m}}\right)$

tot # nz of  $L$  at level  $m$ .

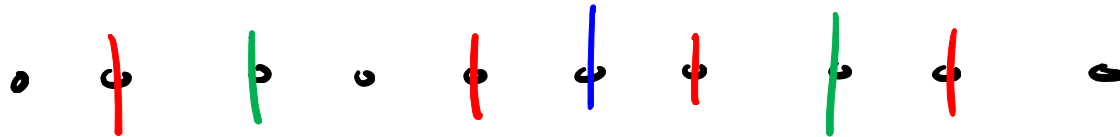
$$O\left(2^{2m} \cdot \frac{n}{2^{2m}}\right) = O(n)$$

tot # nz in  $L$  in nested dissection is

$$O(n \log_2 n)$$

1D / 2D / 3D.

1D.



size of separator is  $O(1)$ .

$\Rightarrow$  each level  $m$ , # nz in  $L$  for each block  $O(1)$

$\frac{n}{2^m}$  blocks.

$\Rightarrow$  tot # nz in  $L$   $\sum_{l=1}^{\log_2 n} \frac{n}{2^m} \sim O(n)$

computational complexity  $O(n)$ .

2D.

each level  $m$ . #nz in each block  $O(2^{2m})$ .

comp. complexity each block  $O(2^{3m})$

$$\text{tot \#nz in } L \sim \sum_{l=1}^{\log_2 \sqrt{n}} 2^{2l} \cdot \frac{n}{2^{2l}} \sim n \log_2 n$$

$$\begin{aligned} \text{tot comp. complexity} &\sim \sum_{l=1}^{\log_2 \sqrt{n}} 2^{3l} \cdot \frac{n}{2^{2l}} \sim \sum_{l=1}^{\log_2 \sqrt{n}} 2^l \cdot n \\ &\quad \uparrow \text{\# blocks} \\ &\sim O(n^{1.5}) \end{aligned}$$

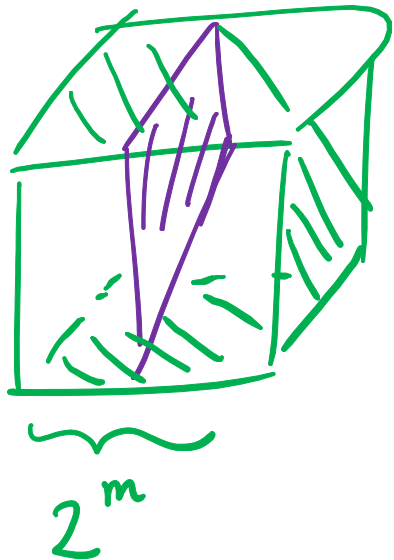
exer: check #nz. & complexity in natural ordering.

3D

at level  $m$ . # nz in each block of  $L$   $(2^{2m})^2 \sim 2^{4m}$

comp. complexity each block  $(2^{2m})^3 \sim 2^{6m}$

tot #level.  $\log_2 n^{\frac{1}{3}}$



# in plane  $\cdot 2^{2m}$

# in plane  $\sim 2^{2m} \cdot 6$

tot # nz in  $L$ .

$$\sum_{l=1}^{\log_2 n^{\frac{1}{3}}} 2^{4l} \cdot \frac{n}{2^{3l}} \sim n^{\frac{4}{3}}$$

tot complexity

$$\sum_{l=1}^{\log_2 n^{\frac{1}{3}}} 2^{6l} \frac{n}{2^{3l}} \sim n^2.$$

Lap, nearest neighbor like stencil.

	1D	2D	3D
nz in L	$n$	$n \log_2 n$	$n^{\frac{4}{3}}$
comp. complexity	$n$	$n^{\frac{3}{2}}$	$n^2$

sparse direct method  
most successful.

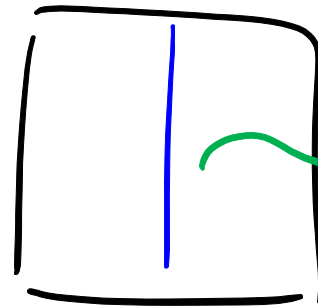
less competitive.  
consider iterative  
methods first

Qualitative understanding.

1D root  $\sim$  size 1.

# blocks  $n$ . tot cost  $\sim n$ .

2D in the end skeleton



size  $\sqrt{n}$   
dense mat.

complexity.  $(\sqrt{n})^3 = n^{1.5}$

root (last separator) dominates.



3D. last skeleton

comp. complexity.

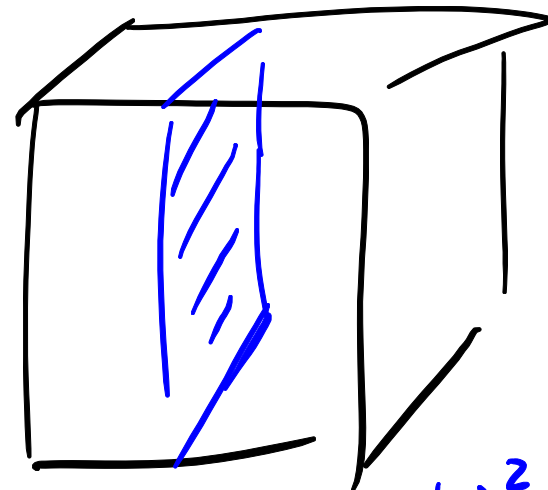
$$\left(n^{\frac{2}{3}}\right)^3 \sim O(n^2)$$

root dominates.

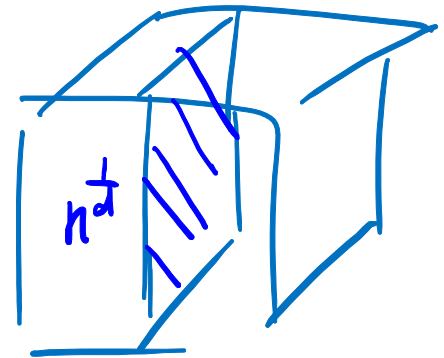
d-dimensional Laplacian.

size of separator.  $n^{\frac{d-1}{d}}$

$$\text{cost } n^{3\left(\frac{d-1}{d}\right)} \xrightarrow{d \rightarrow \infty} n^3$$



size  $\left(n^{\frac{1}{3}}\right)^2$



hypercube

infinite dimensional space. sparse direct  
solve NO BENEFIT.