How to improve

Taylor expansion.

$$u(t_{n+1}) \approx u(t_n) + u'(t_n)\,h + \frac{1}{2}h^2 u''(t_n) + \cdots$$

approximate $u''(t_n)$

$$u'(t_n) = f(u(t_n), t_n) \quad . \quad u(t_n) \in \mathbb{R}^d$$

$$u''(t_n) = \underbrace{D_u f(u(t_n), t_n)}_{d \times d \text{ matrix}} \cdot f(u(t_n), t_n)$$

mat-vec. multiply.

$$+ D_t f(u(t_n), t_n)$$

2nd order Taylor scheme.

$$u_{n+1} = u_n + f(u_n, t_n) h + \frac{h^2}{2}\left[D_u f(u_n, t_n) f(u_n, t_n) + D_t f(u_n, t_n)\right]$$

Direct Taylor based scheme rarely used.

① Jacobian already expensive large d.

② higher order methods. prohibitively expensive.

Idea: Combine a few f evaluation to approximate derivative info.

$$f(u_n, t_n) + h\left[D_u f(u_n, t_n) f(u_n, t_n) + D_t f(u_n, t_n)\right]$$

$$\approx f\left(u_n + h f(u_n, t_n),\ t_{n+1}\right) + O(h^2)$$

$$\bar{u} = u_n + h f(u_n, t_n) \leftarrow \text{First order Euler.}$$

$$u_{n+1} = u_n + \frac{h}{2} f(u_n, t_n) + \frac{h}{2} f(\bar{u}, t_{n+1}) + O(h^3)$$

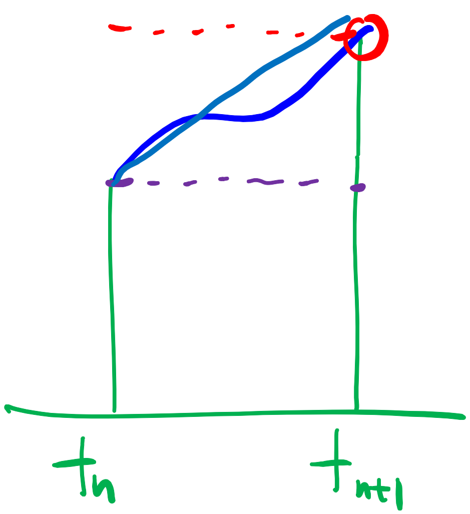
Modified Euler's method.

Integral representation.

$$\begin{cases} \dot{u}(t) = f(u(t), t) \\ u(0) = u_0 \end{cases} \Rightarrow u(t) = u_0 + \int_0^t f(u(s), s)\, ds.$$

$$u(t_{n+1}) = u(t_n) + \int_{t_n}^{t_{n+1}} f(u(s), s)\, ds$$

quadrature.



node involves t_{n+1} : implicit.

not " " : explicit.

$$\int_{t_n}^{t_{n+1}} f(u(s), s)\, ds \approx f(u(t_n), t_n) \cdot h \quad \text{forward Euler}$$

$$\approx f(u(t_{n+1}), t_{n+1})\, h \quad \text{backward Euler.}$$

$$\approx \frac{h}{2}\left[f(u(t_n), t_n) + f(u(t_{n+1}), t_{n+1})\right]$$

trapezoidal rule.

LMM.

$$\sum_{j=0}^{r} \alpha_j u_{n+j} = h \sum_{k=0}^{r} \beta_k f(u_{n+k}, t_{n+k}), \quad \alpha_r \neq 0$$

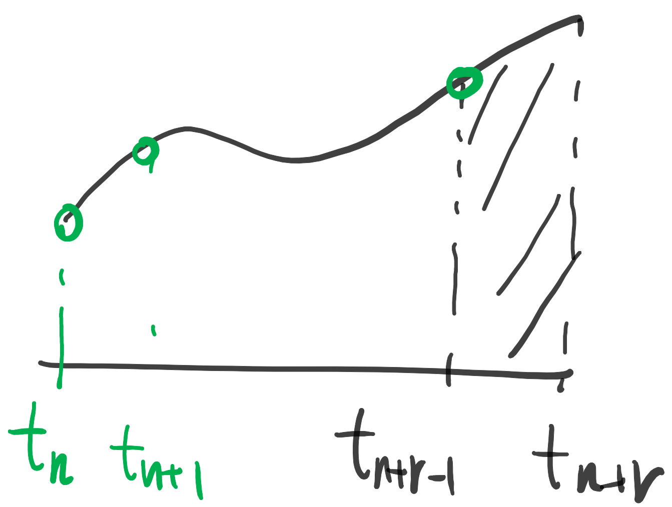
$\beta_r \neq 0$ implict. $\beta_r = 0$. explicit.

Adams - Bashforth (ABn) explicit.

$$u_{n+r} - u_{n+r-1} = h \sum_{k=0}^{r-1} \beta_k f(u_{n+k}, t_{n+k})$$

$\alpha_r = 1$, $\alpha_{r-1} = -1$, $\alpha_j = 0$, $0 \le j \le r-2$

Determine $\beta \rightarrow$ quadrature.



$$u(t_{n+r}) = u(t_{n+r-1}) + \int_{t_{n+r-1}}^{t_{n+r}} f(u(s), s)\, ds$$

Lagrange interpolation.

$P_k(t)$ polynomials w.r.t. t, $0 \le k \le r-1$

$$P_k(t_{n+l}) = \delta_{kl} = \begin{cases} 1, & k=l \\ 0, & k \ne l. \end{cases}$$

Kronecker δ symbol

$$\tilde{f}(t) = \sum_{k=0}^{r-1} f_{n+k} P_k(t) \approx f(t)$$

interpolatory: $\tilde{f}(t_{n+l}) = \sum_{k=0}^{r-1} f_{n+k} P_k(t_{n+l}) = f_{n+l}.$

$$= f(t_{n+l})$$

Uniquely determine $P_k(t) \in \mathbb{P}_{r-1}$

explicit formula.

$$P_k(t) = \prod_{\substack{j=0 \\ j \neq k}}^{r-1} \left(\frac{t - t_{n+j}}{t_{n+k} - t_{n+j}} \right) \qquad 0 \le k \le r-1.$$

$P_k(t_{n+k}) = 1$, $P_k(t_{n+l}) = 0$. $l \neq k$.

Lagrange interpolation polynomial.

$$U_{n+r} - U_{n+r-1} = \int_{t_{n+r-1}}^{t_{n+r}} \sum_{k=0}^{r-1} f_{n+k} P_k(s)\, ds$$

$$= h \sum_k \beta_k f_{n+k}, \quad \beta_k = \frac{1}{h} \int_{t_{n+r-1}}^{t_{n+r}} P_k(s)\, ds$$

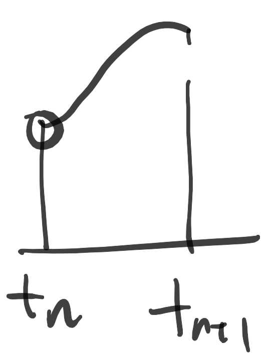
Assume uniform time discretization.

$$s = t_{n+r-1} + \theta h \quad , \quad \theta \in [0,1]$$

$$\beta_k = \int_0^1 P_k (t_{n+r-1} + \theta h) \, d\theta$$

$$= \int_0^1 \prod_{\substack{j=0 \\ j \neq k}}^{r-1} \left(\frac{r-1-j+\theta}{k-j} \right) d\theta =: \int_0^1 \tilde{P}_k(\theta) \, d\theta .$$

<u>Ex</u>. $r=1$



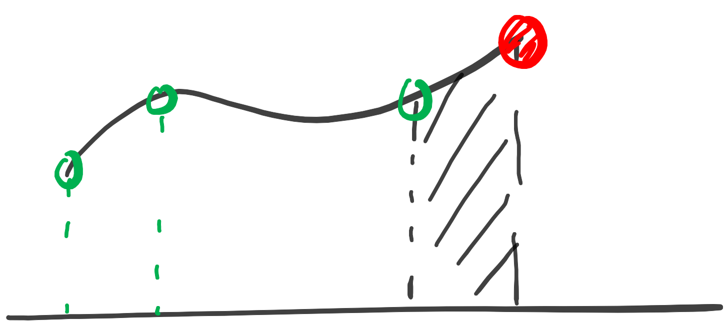
$\beta_0 = 1$. forward Euler.

$r=2$. $\tilde{P}_0(\theta) = \dfrac{2-1-1+\theta}{0-1} = -\theta$. $\beta_0 = -\frac{1}{2}$

$\tilde{P}_1(\theta) = \dfrac{2-1-0+\theta}{1-0} = 1+\theta$. $\beta_1 = \frac{3}{2}$

AB2 : $u_{n+2} - u_{n+1} = h\left(-\frac{1}{2} f_n + \frac{3}{2} f_{n+1}\right)$.

Adams-Moulton (AMn)



t_n t_{n+1} t_{n+r-1} t_{n+r}

r-th order Lag. poly.

$$P_k(t) = \prod_{\substack{j=0 \\ j\neq k}}^{r} \left(\frac{t - t_{n+j}}{t_{n+k} - t_{n+j}}\right)$$

$r = 1, \quad P_0(t) = \frac{t - t_{n+1}}{t_n - t_{n+1}}, \quad P_1(t) = \frac{t - t_n}{t_{n+1} - t_n}.$

$\beta_0 = \frac{1}{2} = \beta_1$

$u_{n+1} = u_n + \frac{h}{2}(f_n + f_{n+1})$. AM1 : trapezoidal.

How to start a LMM?

AB 2

$$u_{n+2} = u_{n+1} + h\left(-\frac{1}{2} f_n + \frac{3}{2} f_{n+1}\right)$$

Initial data. u_0 , u_1 ?

e.g. AB1.

$$u_1 = u_0 + h f_0 .$$

"error analysis".

$$\|e_N\| \le C_1 \|e_0\| + C_2 \|e_1\| + \frac{C_3}{h} \tau_{AB2} \sim O(h^3)$$

no error

LTE Euler $O(h^2)$

$\lesssim h^2$

Trapezoidal rule.

Ex. Linear eq.

$$\begin{cases} u'(t) = A(t)\,u(t) \\ u(0) = u_0 \end{cases} \qquad u(t) \in \mathbb{R}^d,\quad A(t) \in \mathbb{R}^{d\times d}.$$

$$u_{n+1} = u_n + \frac{h}{2}\left(A(t_n)\,u_n + A(t_{n+1})\,u_{n+1}\right)$$

$$\Rightarrow u_{n+1} = \left(I - \frac{h}{2}A(t_{n+1})\right)^{-1}\left[I + \frac{h}{2}A(t_n)\right]u_n$$

Generally.

$$u_{n+1} = u_n + \frac{h}{2}\left[f(u_n, t_n) + f(u_{n+1}, t_{n+1}) \right]$$

Fixed point problem. w.r.t. u_{n+1}.

$x \in \mathbb{R}^d$. $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$

$$x^* = T(x^*)$$

Here $T(x) = u_n + \frac{h}{2} f(u_n, t_n) + \frac{h}{2} f(x, t_{n+1})$

Simplest idea: fixed point iteration.

Alg. $k=0$, $x^{(0)}$, $e^{(0)} = x^{(0)} - T(x^{(0)})$

while ($\|e^{(k)}\| > \tau$)

$x^{(k+1)} = T(x^{(k)})$.

$k \leftarrow k+1$

Convergence $\Rightarrow$ error propagation.

Assume x^* exists. T Lip. cont. α

$$x^{(k+1)} = T(x^{(k)})$$

$$x^* = T(x^*)$$

$$\|x^{(k+1)} - x^*\| = \|T(x^{(k)}) - T(x^*)\|$$

$$\leq \alpha \|x^{(k)} - x^*\|$$

$$\|e^{(k+1)}\| \leq \alpha^{k+1} \|e^{(0)}\|$$

$\alpha < 1$. global convergence

Def (Contraction map) $T: \mathbb{R}^d \to \mathbb{R}^d$

$$\|T(u) - T(v)\| \le \alpha \|u - v\|, \quad \forall u, v \in \mathbb{R}^d. \qquad \alpha < 1.$$

Thm. T contraction map. Fixed pt problem

$x = T(x)$ sol. exists. & unique.

Trapezoidal rule.

$$T(x) = u_n + \frac{h}{2}\left[f(u_n, t_n) + f(x, t_{n+1})\right]$$

$$\|T(x) - T(y)\| = \frac{h}{2}\|f(x, t_{n+1}) - f(y, t_{n+1})\|$$

$$\le \frac{hL}{2}$$

$h < \frac{2}{L}$ Fixed pt iteration converge. globally.

Thm. Trapezoidal rule is convergent of order 2.

Pf: ① Consistency (LTE)

$$\tau_n = \underline{u(t_{n+1})} - u(t_n) - \frac{h}{2}\left[f(u(t_n), t_n) + \underline{f(u(t_{n+1}), t_{n+1})}\right]$$

$$u(t_{n+1}) = u(t_n) + h\,u'(t_n) + \frac{h^2}{2}u''(t_n) + \int_{t_n}^{t_{n+1}} u'''(s)\,\frac{(t_{n+1}-s)^2}{2}\,ds$$

$$u'(t_{n+1}) = u'(t_n) + h\,u''(t_n) + \int_{t_n}^{t_{n+1}} u'''(s)\,(t_{n+1}-s)\,ds$$

$$\tau_n = \int_{t_n}^{t_{n+1}} u'''(s)\left[\frac{(t_{n+1}-s)^2}{2} - \frac{h}{2}(t_{n+1}-s)\right]ds$$

$$M = \sup_{0 \le t \le T} \| u^{(4)}(t) \|$$

$$\| \tau_n \| \le \frac{M h^3}{12}$$

② Stability (error propagation).

$$u_{n+1} = u_n + \frac{h}{2} \left[f(u_n, t_n) + f(u_{n+1}, t_{n+1}) \right]$$

$$u(t_{n+1}) = u(t_n) + \frac{h}{2} \left[f(u(t_n), t_n) + f(u(t_{n+1}), t_{n+1}) \right] + \tau_n$$

$$\|e_{n+1}\| \le \|e_n\| + \frac{h}{2} L \left(\|e_n\| + \|e_{n+1}\| \right) + \|\tau_n\|$$

Assume $\frac{hL}{2} < 1$

$\tau = \frac{Mh^3}{12}$

$$\left(1 - \frac{hL}{2}\right) \|e_{n+1}\| \le \left(1 + \frac{hL}{2}\right) \|e_n\| + \tau$$

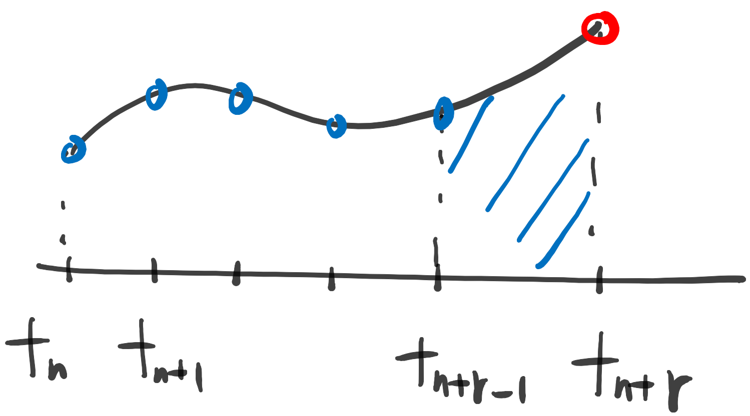
$$\Rightarrow \quad \|e_N\| \le \left(\frac{1 + \frac{hL}{2}}{1 - \frac{hL}{2}} \right)^N \|e_0\| + \left[\left(\frac{1 + \frac{hL}{2}}{1 - \frac{hL}{2}} \right)^N - 1 \right] \frac{\tau}{hL}$$

$$\left(\frac{1 + \frac{hL}{2}}{1 - \frac{hL}{2}} \right)^N = \left(1 + \frac{hL}{1 - \frac{1}{2}hL} \right)^N \le e^{\frac{LT}{1 - \frac{1}{2}hL}}$$

$$\|e_N\| \le e^{\frac{LT}{1-\frac{1}{2}hL}} \|e_0\| + \frac{e^{\frac{LT}{1-\frac{hL}{2}}} - 1}{L} \frac{M}{12} h^2$$

If $\|e_0\| \sim h^2$ $\square$.

Consistency of LMM



AB $\circ$

AM $\circ + \circ$

LTE.

$$\tau_n = \sum_{j=0}^{r} \alpha_j u(t_{n+j}) - h \sum_{k=0}^{r} \beta_k f(u(t_{n+k}), t_{n+k}).$$

$$u(t_{n+j}) = u(t_n) + (jh) u'(t_n) + \cdots + \frac{1}{p!}(jh)^p u^{(p)}(t_n) + O(h^{p+1})$$

$$u'(t_{n+j}) = u'(t_n) + (jh) u''(t_n) \qquad + \frac{1}{p!}(jh)^p u^{(p+1)}(t_n) + O(h^{p+1})$$

$$\tau_n = \sum_{k=0}^{p} C_k h^k u^{(k)}(t_n) + O(h^{p+1}).$$

$$C_0 = \sum_{j=0}^{r} \alpha_j$$

$$C_1 = \sum_{j=0}^{r} (j\alpha_j - \beta_j)$$

$$\vdots$$

$$C_k = \sum_{j=0}^{r} \left(\frac{j^k}{k!}\alpha_j - \frac{j^{k-1}}{(k-1)!}\beta_j \right) = \frac{1}{k!}\left(\sum_{j=0}^{r} j^k \alpha_j - k j^{k-1} \beta_j \right)$$

Thm. LTE is $O(h^{p+1}) \iff C_0 = \cdots = C_p = 0$.