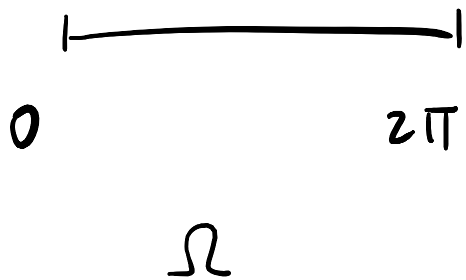
Spectral Method.

1D Poisson



$$\begin{cases} -u'' = f \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \end{cases}$$

p.b.c.

$f(x) \in C^\infty[0, 2\pi]$. periodic.

Fourier series.

$$u(x) = \sum_{k \in \mathbb{Z}} e^{ikx} \hat{u}_k, \quad f(x) = \sum_{k \in \mathbb{Z}} e^{ikx} \hat{f}_k$$

$$\hat{f}_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} f(x)\, dx$$

$$\Rightarrow \sum_{k\in\mathbb{Z}} k^2 e^{ikx} \hat{u}_k = \sum_k e^{ikx} \hat{f}_k$$

match each component.

$$k^2 \hat{u}_k = \hat{f}_k \Rightarrow \hat{u}_k = \frac{\hat{f}_k}{k^2}, \quad k\in\mathbb{Z}.$$

$k \neq 0$.
solvability
$\hat{f}_0 = 0$.

spectral method takes advantage of smoothness to obtain fast convergence.

For C^∞ problem. converges faster than any polynomial power w.r.t. N.

Define Fourier transform.

$$\hat{u}_k = (\mathcal{F}[u])_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} u(x)\,dx.$$

inverse Fourier transform.

$$\mathcal{F}^{-1}[\hat{u}](x) = \sum_{k\in\mathbb{Z}} \hat{u}_k e^{ikx}$$

1st order deriv.

If $u \in C^{(1)}_{\pi}([0,2\pi])$.

periodic

$$u(x) = \mathcal{F}^{-1}[\mathcal{F}[u]](x)$$

for every $x \in [0, 2\pi]$. $\Longleftrightarrow \mathcal{F}\circ\mathcal{F}^{-1} = \mathcal{F}^{-1}\circ\mathcal{F} = I$

Thm. $u(x) \in C_{\pi}^{(m)}([0, 2\pi])$

$$\Rightarrow |\hat{u}_k| \sim O(|k|^{-m}), \quad m \geq 1.$$

pf: integration by parts.

$$\hat{u}_k = \frac{1}{2\pi(-ik)} \int_0^{2\pi} u(x) \, d(e^{-ikx}) = \frac{1}{2\pi(ik)} \int_0^{2\pi} e^{-ikx} u'(x) \, dx$$

$$\overset{m \text{ times}}{=} \frac{1}{2\pi(ik)^m} \int_0^{2\pi} e^{-ikx} u^{(m)}(x) \, dx.$$

$$|\hat{u}_k| \leq \frac{1}{2\pi |k|^m} \int_0^{2\pi} |u^{(m)}(x)| \, dx \sim |k|^{-m}.$$

$\square$

In practice.

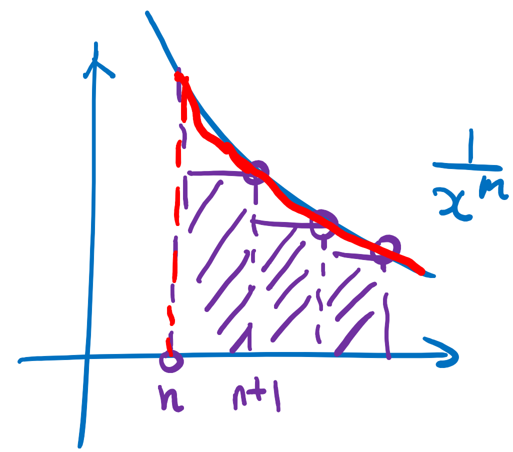
$$u_n(x) = \sum_{|k| \le n} \hat{u}_k e^{ikx}$$

$$|u(x) - u_n(x)| = \left| \sum_{|k|>n} \hat{u}_k e^{ikx} \right|$$

$$\le \sum_{|k|>n} |\hat{u}_k| \le C \sum_{|k|>n} k^{-m}$$

$$\le 2C \int_n^\infty \frac{1}{x^m} dx$$

$$= \frac{2C}{m-1} n^{-(m-1)}$$



Remark: This decay rate is NOT SHARP.

If $m=1$, above predicts no decay.

But we care about here if when m is large (e.g. ∞)

Question: can $\hat{u}_k$ be evaluated sufficiently accurately?

$$P_n = \left\{ u(x) \,\middle|\, u(x) = \sum_{k \in K_n} \hat{u}_k e^{ikx}, \ \hat{u}_k \in \mathbb{C} \right\}.$$

$$K_n = \left\{ -\frac{n}{2}+1, \cdots, -1, 0, \cdots, \frac{n}{2} \right\}. \quad n \text{ is even}$$

$$= \left\{ -\frac{(n-1)}{2}, \cdots, \frac{n-1}{2} \right\} \quad n \text{ is odd.}$$

$\# K_n = n$.

Galerkin condition. $-u'' = f$

weak form. $(v', u_n') = (v, f) = \int_0^{2\pi} v^*(x) f(x) dx$.

$u_n, v \in P_n$. $f \notin P_n$.

Take $v = e^{ilx}$, $l \in K_n$

$$\Rightarrow \quad \hat{u}_l \cdot |l|^2 = \hat{f}_l$$

$$\Rightarrow \quad \hat{u}_l = \frac{\hat{f}_l}{|l|^2}, \quad l \neq 0.$$

Final piece. evaluate

$$\hat{f}_l = \frac{1}{2\pi} \int_0^{2\pi} e^{-ilx} f(x)\, dx.$$

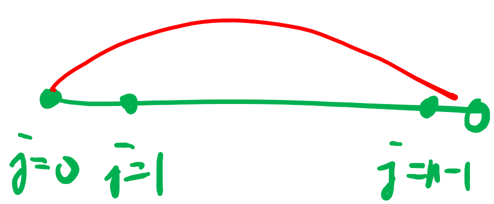
a quadrature problem.

Accuracy of trapezoidal rule for periodic functions.

$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} f(x)\,dx$$

$$= \frac{1}{n}\sum_{j=0}^{n-1} e^{-ikx_j} f(x_j) + \varepsilon_{n,k}$$

$x_j = \frac{2\pi j}{n}$, $j = 0, \cdots, n-1$



Thm. $f \in C_\pi^{(m)}([0,2\pi])$, $\varepsilon_{n,k} \sim O(n^{-(m-1)})$, $k \in K_n$

Pf: Consider.

$$f_n(x) = \sum_{k \in K_n} e^{ikx} \hat{f}_k$$

$\forall\, k, l \in K_n$. $|k-l| < n$.

$$\frac{1}{n} \sum_{j=0}^{n-1} e^{-ikx_j} e^{ilx_j} = \frac{1}{n} \sum_{j=0}^{n-1} e^{-i(k-l)x_j}$$

$\longrightarrow$ quad.

$$= \frac{1}{n} \sum_{j=0}^{n-1} e^{-i\frac{(k-l)2\pi j}{n}} = \delta_{k,l}$$

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} e^{ilx}\, dx = \delta_{k,l}.$$

$\longrightarrow$ analytic.

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} f_n(x)\,dx = \frac{1}{n}\sum_{j=0}^{n-1} e^{-ikx_j} f_n(x_j) \longrightarrow \text{exact!}$$

$$\varepsilon_{n,k} = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx}\left(f(x) - f_n(x)\right)dx$$

$$|\varepsilon_{n,k}| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(x) - f_n(x)|\,dx \sim O(n^{-(m-1)}). \quad \square$$

Super-algebraic decay!

L. Trefethen. SIAM Rev.

Discrete Fourier transform.

$$u \in \mathbb{C}^n, \quad (\mathcal{F}_n u)_k = \sum_{j=0}^{n-1} e^{-i\frac{2\pi jk}{n}} u_j \equiv \hat{u}$$

$k = 0, \cdots, n-1$

Inverse

$$\hat{u} \in \mathbb{C}^n \quad (\mathcal{F}_n^{-1} \hat{u})_j = \frac{1}{n} \sum_{k=0}^{n-1} e^{i\frac{2\pi jk}{n}} \hat{u}_k$$

MATLAB/Julia/etc. convention.

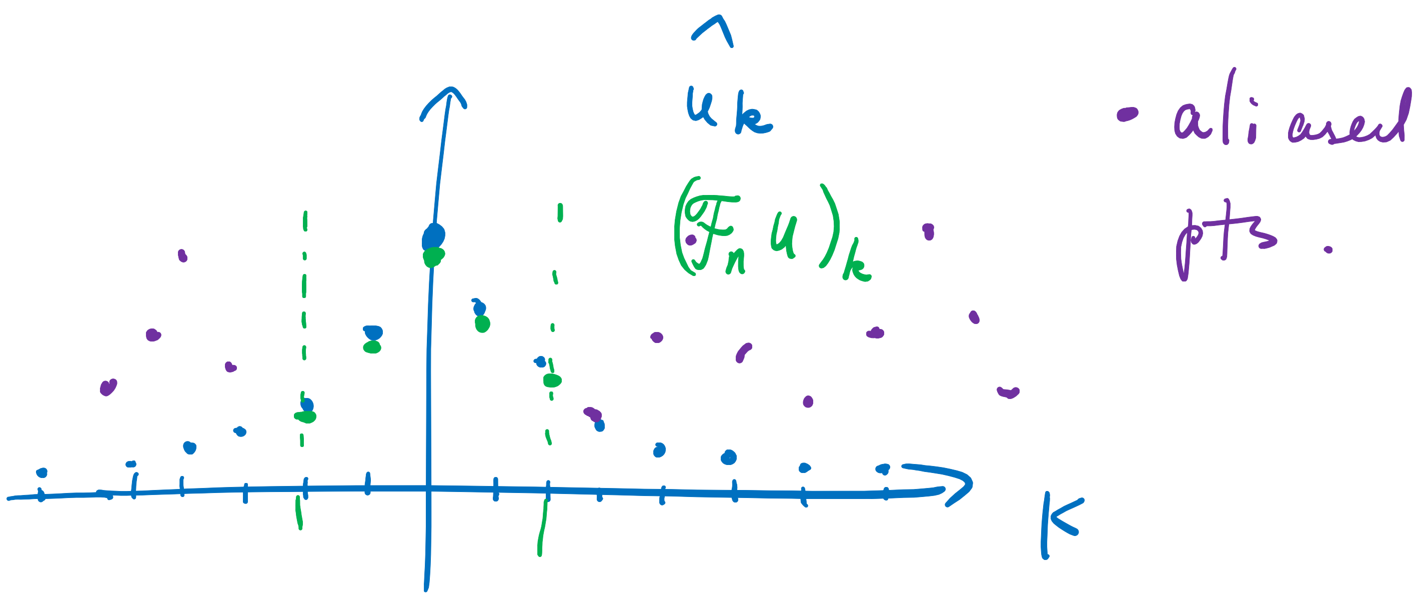
$$\mathcal{F}_n \circ \mathcal{F}_n^{-1} = \underline{I}.$$

Aliasing.

$\forall c \in \mathbb{Z}. \quad u \in \mathbb{C}^n$

$$(\mathcal{F}_n u)_{k+cn} = \sum_{j=0}^{n-1} e^{-\frac{2\pi i (k+cn) j}{n}} u_j.$$

$$= \sum_{j=0}^{n-1} e^{-\frac{2\pi i kj}{n}} u_j = (\mathcal{F} u)_k.$$



$f(x) \quad , \; g(x)$

$$f(x)g(x) \approx \left(\sum_{k \in K_n} e^{ikx} \hat{f}_k \right) \left(\sum_{l \in K_n} e^{ilx} \hat{g}_l \right)$$

$$= \sum_{k,l \in K_n} e^{i(k+l)x} \hat{f}_k \hat{g}_l$$

$k+l \notin K_n$

Fast Fourier transform.

Compute $\displaystyle\sum_{j=0}^{n-1} e^{-i\frac{kj2\pi}{n}} f_j \quad O(n \log_2 n)$ cost.

Alg. Solve Poisson.

Input: $f_j = f(x_j)$, $x_j = \frac{2\pi j}{n}$, $j = 0, \cdots, n-1$.

Output: $u_j \approx u(x_j)$

1. $\hat{f} = \text{fft}(f)$

2. $\hat{g} = [0, \cdots, \frac{n}{2}, \underbrace{-\frac{n}{2}+1, \cdots, -1}_{\text{aliasing.}}]$

3. $\hat{u}_k = \begin{cases} \hat{g}_k^{-2} \hat{f}_k, & k \neq 0 \\ 0, & k = 0. \end{cases}$

4. $u = \text{ifft}(\hat{u})$