

Ihm. Gauss-Legendre quadrature w. r points

is exact for all polynomials of order
 $\leq 2r-1$.

Pf: For any $f(x) \in P_{2r-1}$. $P_r(x) \leftarrow GL$

$$f(x) = g(x)P_r(x) + h(x), \quad g, h \in P_{r-1}$$

$$\int_{-1}^1 f(x) dx = \int_{-1}^1 g(x)P_r(x) dx + \int_{-1}^1 h(x) dx$$

apply quad.

$$= \sum_{j=1}^r w_j g(x_j) P_r(x_j) + \sum_{j=1}^r w_j h(x_j)$$

\parallel
0

exact
(orthogonality &
choice of nodes)

exact .
(Lag. interpolation)

□ .

algebraic accuracy.

More general $f \in C^\infty([-1, 1])$

$$a = -1, b = 1$$

$$\left| \int_a^b f(x) dx - \sum_{j=1}^r f(x_j) w_j \right|$$

$$= \left| \int_a^b \sum_{n=0}^{2r-1} \frac{f^{(n)}(a)}{n!} (x-a)^n dx + \sum_{n=2r}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n dx \right|$$

$$- \sum_{l=1}^r f(x_l) w_l \right|$$

↑
Taylor expand as well.

$$\leq C \sup_{a \leq x \leq b} |f^{(2r)}(x)| (b-a)^{2r+1}$$

G-L quadrature from $[-1, 1] \rightarrow [a, a+h]$

$$\rightarrow \left| \int_a^{a+h} f(x) dx - \sum_{l=1}^r f(x_l) w_l \right| \leq C h^{2r+1} \sup_{a \leq x \leq a+h} |f^{(2r)}(x)|$$

G-L w. r points is of order. $2r$.

Why 3-term recurrence?

$$\beta_l P_l(x) = (\cancel{x} - \alpha_l) P_{l-1}(x) - \gamma_l P_{l-2}(x)$$

$$+ \sum_{j=0}^{l-3} c_j P_j(x).$$

Project w. P_j ($j=0, \dots, l-3$)

$$\int_{-1}^1 P_l \cancel{P_j} P_{l-2} dx = \int_{-1}^1 \cancel{x P_j} P_{l-1} dx - \alpha_l \int_{-1}^1 \cancel{P_j} P_{l-1} dx \\ - \gamma_l \int_{-1}^1 \cancel{P_j} P_{l-2} dx + \sum_{j=0}^{l-3} c_j' \int_{-1}^1 P_j \cancel{P_j} dx$$

$$\left(\int_{-1}^1 P_j^2 dx \right) c_j = 0 \Rightarrow c_j = 0, \quad j=0, \dots, l-3.$$

Three - term recurrence .

$$x P_{l-1}(x) = \gamma_l P_{l-2}(x) + \alpha_l P_{l-1}(x) + \beta_l P_l(x).$$

"matrix A"

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$\{x_i\}_{i=1}^M$

-

|

$$A = \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_M \end{bmatrix} \quad x_1 = -1, x_M = 1$$

$$P_l = \begin{bmatrix} P_l(x_1) \\ \vdots \\ P_l(x_M) \end{bmatrix}$$

$$V_l = [P_0, P_1, \dots, P_{l-1}] \in \mathbb{R}^{n \times l}$$

$$P_l^T P_{l'} = 0, \quad l \neq l'$$

$$AV_l = V_{l+1} \begin{bmatrix} \alpha_0 & \gamma_1 & & 0 \\ \beta_0 & \alpha_1 & & \\ & \ddots & \ddots & \\ 0 & & \ddots & \gamma_{l-1} \\ & & & \ddots & \alpha_{l-1} \\ & & & & \beta_{l-1} \end{bmatrix} \xrightarrow{\text{green arrow}} \begin{bmatrix} T \\ \vdots \\ 0 & \beta_{l-1} \end{bmatrix}$$

$$\mathbb{R}^{(l+1) \times l}$$

$$AV_l = V_l T + \beta_{l-1} P_l \cdot e_l^T, \quad e_l = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \}_{l+1}$$

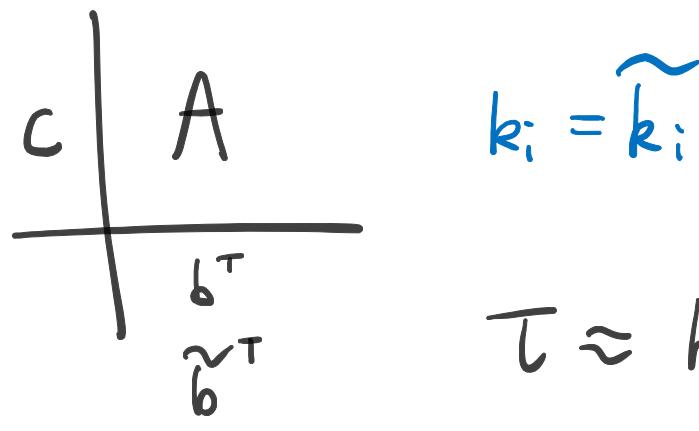
Lanczos procedure.

embedded RK . Adaptive time stepping .

estimate τ .

$$u(t_{n+1}) = u(t_n) + h \sum_{i=1}^r b_i k_i + \bar{\tau} + O(h^{p+2})$$

$$u(t_{n+1}) = u(t_n) + h \sum_{i=1}^r \tilde{b}_i \tilde{k}_i + O(h^{p+2})$$



$$\bar{\tau} \approx h \sum_{i=1}^r (\tilde{b}_i - b_i) k_i$$

Ode 45 . 4-th & 5-th order , 7 - stage scheme
for better (absolute) stability properties .

Absolute stability.

recall zero stability.

$$u' = 0 \quad \text{special RHS.}$$

abs. stability

$$u' = \lambda u, \quad \lambda \in \mathbb{C}$$

$\operatorname{Re} \lambda < 0$. $u(t) = e^{\lambda t} u(0)$ $|u(t)|$ decays exponentially

numerical scheme to produce exp. decay.

$\operatorname{Re}\lambda = 0$, $\lambda = i\omega$, $\omega \in \mathbb{R}$.

often require special treatment. $\omega \gg 1$.

$\operatorname{Re}\lambda > 0$. should just use small time step
to resolve exp. growth.

Forward Euler

$$u_{n+1} = u_n + \lambda h u_n = (H \lambda h) u_n$$

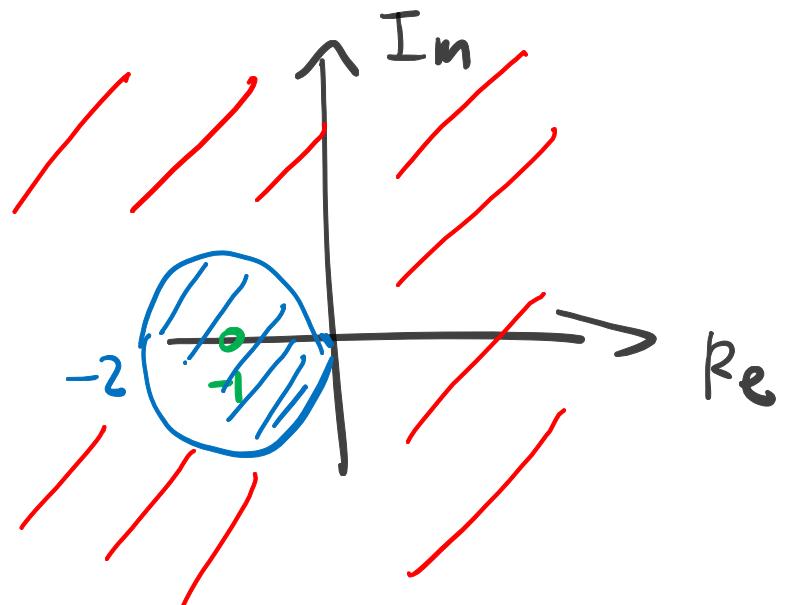
$$\gamma = \lambda h$$

$$u_n = (1+\gamma)^n u_0$$

$$|1+\gamma| \leq 1$$

region of absolute
stability (RAS)

$$RAS := \{ \gamma \mid |1+\gamma| \leq 1 \}.$$

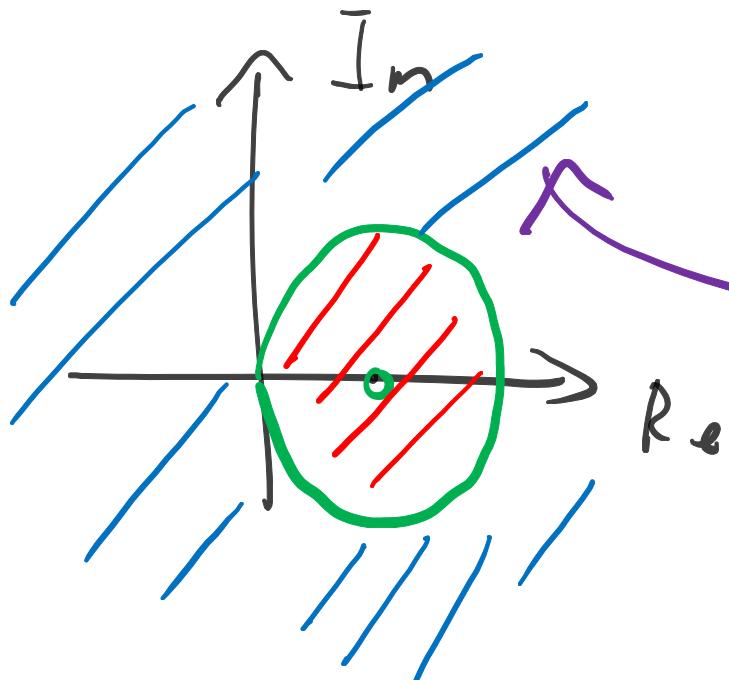


Backward Euler.

$$u_{n+1} = u_n + \gamma u_{n+1}$$

$$u_{n+1} = \frac{1}{1-\gamma} u_n$$

$$RAS = \{ \gamma \mid |(-\gamma)^{-1}| \leq 1 \} \supseteq \{ \gamma \mid \operatorname{Re} \lambda < 0 \}$$



A-stable

discrepancy
between stability γ
accuracy (consistency).

General LMM.

$$\sum_{j=0}^r \alpha_j u_{n+j} = h \sum_{k=0}^r \beta_k f_{n+k}. \quad f_{n+k} = \lambda u_{n+k}.$$

$$\Rightarrow \sum_{j=0}^r (\alpha_j - \lambda \beta_j) u_{n+j} = 0.$$

which value(s) of λ \Rightarrow difference eq.

are stable.

root condition (for a fixed z)

$$\begin{aligned}\tilde{P}(\omega; z) &= P(\omega) - z \sigma(\omega) \\ &= \sum_{j=0}^r (\alpha_j - z \beta_j) \omega^j \in \mathbb{P}_r \text{ w.r.t. } \omega.\end{aligned}$$

- 1) all roots of $\tilde{P}(\omega; z)$ have modulus ≤ 1 .
- 2) if a root $|\omega| = 1$, then ω must be simple.

$\text{RAS} = \{ z \mid \tilde{\rho}(\cdot; z) \text{ satisfies root condition} \}.$

“brute force” computing RAS.

mesh on \mathbb{C} . check root cond. on each point.

Ex. F - Euler.

$$\tilde{\rho}(\omega; z) = \omega - 1 - z \cdot \underline{1} = 0.$$

root $\omega = 1 + z$. $\Rightarrow |1 + z| \leq 1$

Ex. Trapezoidal.

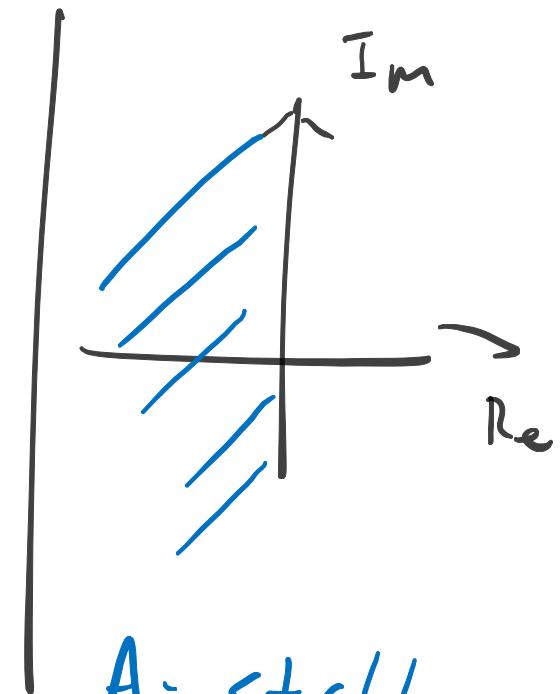
$$\tilde{\rho}(\omega; z) = (\omega - 1) - z \left(\frac{1}{2}\omega + \frac{1}{2} \right) = 0.$$

$$\Rightarrow \omega = \frac{1 + \frac{1}{2}z}{1 - \frac{1}{2}z}$$

$$\left| 1 + \frac{1}{2}z \right| \leq \left| 1 - \frac{1}{2}z \right|$$

$$\Rightarrow 1 + \frac{1}{4}|z|^2 + \operatorname{Re} z \leq 1 + \frac{1}{4}|z|^2 - \operatorname{Re} z$$

$$\Rightarrow \operatorname{Re} z \leq 0.$$



A-stable

Ex. Explicit mid point rule (a.k.a. leap frog)

$$u_{n+2} - u_n = 2h f_{n+1}$$

zero stable. 2nd order accurate.

$$\tilde{P}(\omega; z) = \omega^2 - 1 - z \cdot 2 \cdot \omega = 0$$

$$\omega = \frac{z \mp \sqrt{z^2 + 4}}{2} = z \pm \sqrt{z^2 + 1}$$

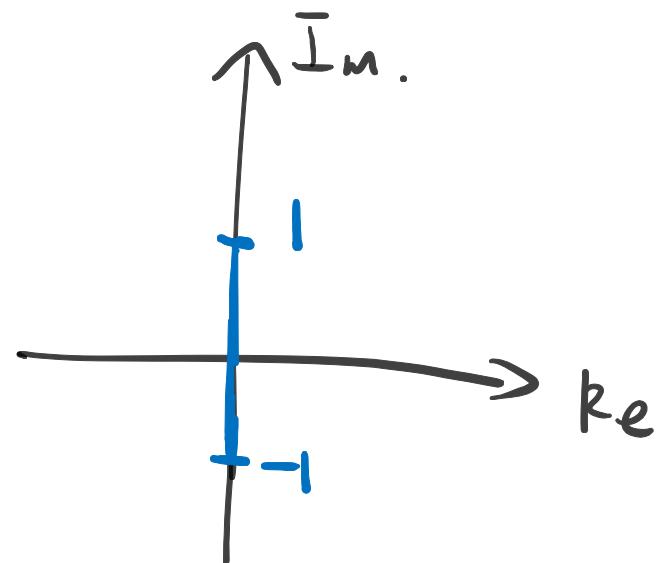
$$|\omega| \leq 1. \quad \omega_1 \omega_2 = -1 \Rightarrow |\omega_1| \cdot |\omega_2| = 1$$

$$\Rightarrow |\omega_1| = |\omega_2| = 1 .$$

$$\omega = e^{i\theta}, \quad \theta \in [0, 2\pi).$$

$$e^{2i\theta} - 1 = 2z e^{i\theta}$$

$$\Rightarrow z = \frac{1}{2}(e^{i\theta} - e^{-i\theta}) = i \sin \theta$$



only works

- 1) eq. purely oscillatory
- 2) small time step.

General trend.

For LMM. RAS general **shrinkage**

w.r.t increase of order.

⇒ high order LMM. more difficult to
use

Boundary locus method for drawing RAS.

RAS is a "proper" region.

∂ RAS is a line

characterize $|\omega| = 1$. $\omega = e^{i\theta}$, $\theta \in [0, 2\pi]$.

$$\widehat{P}(\omega; z) = p(e^{i\theta}) - z \sigma(e^{i\theta}) = 0.$$

$$\Rightarrow z = \frac{p(e^{i\theta})}{\sigma(e^{i\theta})}$$

Pick a point z inside both intervals
exterior to see whether region is stable.

RAS for RK methods

$$k_i = f(u_n + h \sum_{j=1}^r a_{ij} k_j) = \lambda (u_n + h \sum_j a_{ij} k_j)$$

$$u_{n+1} = u_n + h \sum_i b_i k_i$$

$$\vec{k} = \lambda u_n \vec{e} + \lambda h A \vec{k}$$

$$\Rightarrow \vec{k} = (\bar{I} - zA)^{-1} \vec{e} \cdot \lambda u_n$$

$$u_{n+1} = u_n + \vec{b}^\top (\bar{I} - zA)^{-1} \vec{e} \cdot z u_n$$

$$= R(z) u_n. \quad R(z) = 1 + z \vec{b}^\top (\bar{I} - zA)^{-1} \vec{e}$$

Def. RAS for RK method

$$\{z \in \mathbb{C} \mid |R(z)| \leq 1\}.$$

Ex. GL 1.

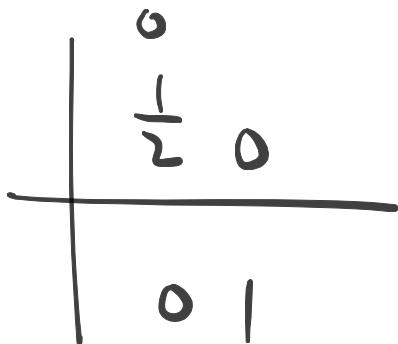
$$k = f(u_n + \frac{1}{2}hk)$$
$$u_{n+1} = u_n + hk$$

$\overset{\circ}{u_n} \overset{\Delta}{\rightarrow} \overset{\circ}{u_{n+1}}$ implicit midpoint rule.

$$R(z) = 1 + z \cdot \left(1 - \frac{1}{2}z\right)^{-1} = \frac{1 + \frac{1}{2}z}{1 - \frac{1}{2}z}$$

$$|R(z)| \leq 1 \Rightarrow \operatorname{Re} z \leq 0, \text{ A-stable}$$

Ex.



$$R(z) = | + z \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{1}{2}z \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= | + z \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{2}z & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= | + z \left(| + \frac{1}{2}z \right) = | + z + \frac{1}{2}z^2$$

Boundary locus.

$$R(z) = 1 + z + \frac{1}{2}z^2 = e^{i\theta}, \quad \theta \in [0, 2\pi].$$

$$z = -1 \pm \sqrt{-2(-e^{i\theta})}$$

