## Problem 1

To enforce the necessary constraints, we introduce the Lagrangian multipliers $\lambda_1, ..., \lambda_M, \lambda_{M+1}$ and make an unconstrained maximization of

$$L = \mathbf{u}_{M+1}{}^\intercal \mathbf{S}\mathbf{u}_{M+1} + \lambda_{M+1}(1 - \mathbf{u}_{M+1}{}^\intercal \mathbf{u}_{M+1}) + \sum_{i=1}^{M} \lambda_i(\mathbf{u}_{M+1}{}^\intercal \mathbf{u}_i)$$

Now we take the derivative of L with respect to a vector $\mathbf{u}_{M+1}$ and set it to zero

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{u}_{M+1}} L = 0 &\Leftrightarrow 2\mathbf{S}\mathbf{u}_{M+1} = 2\lambda_{M+1}\mathbf{u}_{M+1} - \sum_{i=1}^{M} \lambda_i \mathbf{u}_i \\
&\Leftrightarrow 2\mathbf{u}_{M+1}{}^\intercal \mathbf{S}\mathbf{u}_{M+1} = 2\lambda_{M+1}\mathbf{u}_{M+1}{}^\intercal \mathbf{u}_{M+1} - \sum_{i=1}^{M} \lambda_i \mathbf{u}_{M+1}{}^\intercal \mathbf{u}_i && \text{(left-multiply by } \mathbf{u}_{M+1}{}^\intercal) \\
&\Leftrightarrow \mathbf{u}_{M+1}{}^\intercal \mathbf{S}\mathbf{u}_{M+1} = \lambda_{M+1} && \text{(by orthonormality of } u_i \text{ and } \mathbf{u}_{M+1})
\end{aligned}$$

By induction step, we see that the variance in direction $\mathbf{u}_{M+1}$ is maximum when we set $\mathbf{u}_{M+1}$ equal to the eigenvector having the $(m+1)^{\text{th}}$ largest eigenvalue $\lambda_{M+1}$.

## Problem 2

$$\mathbf{x_i} \sim \mathcal{N}(\mu_\mathbf{x}, \mathbf{\Phi_x}) \qquad (\mu_\mathbf{x} = \mu,\ \mathbf{\Phi_x} = \mathbf{W}\mathbf{W}^\intercal + \Phi^2\mathbf{I})$$

From lecture we know that $\mathbf{y_i} = \mathbf{A}\mathbf{x_i} \sim \mathcal{N}(\mu_\mathbf{y}, \mathbf{\Phi_y})$. Let's first derive the two moments

$$\begin{aligned}
\mu_\mathbf{y} &= \mathbf{E}[y] \\
&= \mathbf{E}[\mathbf{A}\mathbf{x}] \\
&= \mathbf{A}\mathbf{E}[\mathbf{x}] \\
&= \mathbf{A}\mu_\mathbf{x}
\end{aligned}$$

$$\begin{aligned}
\mathbf{\Phi_y} &= \mathbf{E}[(\mathbf{y} - \mu_\mathbf{y})(\mathbf{y} - \mu_\mathbf{y})^\intercal] \\
&= \mathbf{E}[(\mathbf{A}\mathbf{x} - \mathbf{A}\mu_\mathbf{x})(\mathbf{A}\mathbf{x} - \mathbf{A}\mu_\mathbf{x})^\intercal] \\
&= \mathbf{E}[(\mathbf{A}(\mathbf{x} - \mu_\mathbf{x}))(\mathbf{A}(\mathbf{x} - \mu_\mathbf{x}))^\intercal] \\
&= \mathbf{A}\mathbf{E}[(\mathbf{x} - \mu_\mathbf{x})(\mathbf{x} - \mu_\mathbf{x})^\intercal]\mathbf{A} \\
&= \mathbf{A}\mathbf{\Phi_x}\mathbf{A}
\end{aligned}$$

By pattern matching we have for the transformed Maximum Likelihood estimates

$$\begin{aligned}\mu_{y_{ML}} &= \mathbf{A}\mu_{ML} \\ \mathbf{\Phi}_{y_{ML}} &= \mathbf{A}\mathbf{\Phi}_{ML}\mathbf{A}^\intercal \\ \mathbf{W}_{y_{ML}} &= \mathbf{A}\mathbf{W}_{ML}\end{aligned}$$

By orthogonality $\mathbf{A}\mathbf{A}^\intercal = \mathbf{A}^\intercal\mathbf{A} = \mathbf{I}$ and $\mathbf{\Phi} = \sigma^2\mathbf{I}$ we have

$$\begin{aligned}\mathbf{A}\mathbf{\Phi}\mathbf{A}^\intercal &= \mathbf{A}\sigma^2\mathbf{I}\mathbf{A}^\intercal \\ &= \sigma^2\mathbf{A}\mathbf{A}^\intercal \\ &= \sigma^2\mathbf{I}\end{aligned}$$

**Problem 3**

Let $\mathbf{x} \in \mathbb{R}^5$ hold the movie ratings given by Leslie.
By the SVD projection in concept space we have

$$\mathbf{V} \cdot \mathbf{x} = [1.74, 2.84]^\intercal$$

By SVD decomposition and reconstruction of the input using the projected space we have

$$[1.74, 2.84]^\intercal \cdot \mathbf{V}^\intercal = [1.0092, 1.0092, 1.0092, 2.0164, 2.0164]^\intercal$$

E.g. we can predict that Leslie will rate Titanic movie with 2.0164.

**Problem 4**

See below.

# 10_homework_dim_reduction

January 13, 2018

## 1 Programming assignment 10: Dimensionality Reduction

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt

        %matplotlib inline
```

### 1.1 PCA Task

Given the data in the matrix X your tasks is to: * Calculate the covariance matrix $\Sigma$. * Calculate eigenvalues and eigenvectors of $\Sigma$. * Plot the original data $X$ and the eigenvectors to a single diagram. What do you observe? Which eigenvector corresponds to the smallest eigenvalue? * Determine the smallest eigenvalue and remove its corresponding eigenvector. The remaining eigenvector is the basis of a new subspace. * Transform all vectors in X in this new subspace by expressing all vectors in X in this new basis.

#### 1.1.1 The given data X

```
In [7]: X = np.array([(-3,-2),(-2,-1),(-1,0),(0,1),
                      (1,2),(2,3),(-2,-2),(-1,-1),
                      (0,0),(1,1),(2,2), (-2,-3),
                      (-1,-2),(0,-1),(1,0), (2,1),(3,2)])
```

#### 1.1.2 Task 1: Calculate the covariance matrix $\Sigma$

```
In [8]: def get_covariance(X):
            """Calculates the covariance matrix of the input data.

            Parameters
            ----------
            X : array, shape [N, D]
                Data matrix.

            Returns
            -------
            Sigma : array, shape [D, D]
                Covariance matrix
```

```
    """
    N, D = X.shape
    mean = np.dot(X.T, np.ones((N, 1))) * 1.0/N
    cov = np.dot(X.T, X) * 1.0/N  - np.dot(mean, mean.T)
    return cov
```

Note: The covariance of the data is equal to the covariance of the centered data

```
In [10]: # covariance of data
         get_covariance(X)
```

```
Out[10]: array([[2.82352941, 2.47058824],
                [2.47058824, 2.82352941]])
```

```
In [11]: # covariance of centered data
         get_covariance(X - np.mean(X, axis=0))
```

```
Out[11]: array([[2.82352941, 2.47058824],
                [2.47058824, 2.82352941]])
```

### 1.1.3 Task 2: Calculate eigenvalues and eigenvectors of $\Sigma$.

```
In [4]: def get_eigen(S):
            """Calculates the eigenvalues and eigenvectors of the input matrix.

            Parameters
            ----------
            S : array, shape [D, D]
                Square symmetric positive definite matrix.

            Returns
            -------
            L : array, shape [D]
                Eigenvalues of S
            U : array, shape [D, D]
                Eigenvectors of S

            """
            steps = 10
            D = S.shape[0]
            U = np.zeros((D, D))
            L = np.zeros((D,))

            # find eigenvectors using Von Mises Power Iteration
            for d in range(D):

                # initialize arbitrary normalized vector
                w = np.random.randn(D).reshape(D,1)
                w = w / np.linalg.norm(w)
```

```
        for s in range(steps):
            w = np.dot(S, w) / np.linalg.norm(np.dot(S, w))
        U[d, :] = w[:,0]

        # find the corresponding eigenvalue
        v = np.dot(w.T, np.dot(S, w))
        L[d] = v

        # deflate the covariance matrix
        S = S - v * np.dot(w, w.T)

    return L, U
```

### 1.1.4 Task 3: Plot the original data X and the eigenvectors to a single diagram.

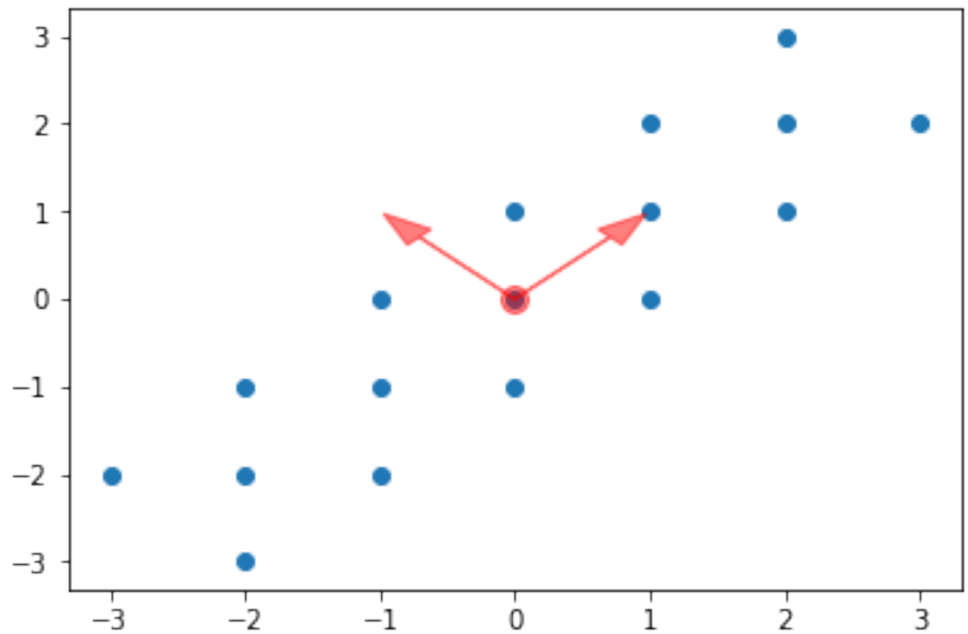
```
In [12]: # plot the original data
         plt.scatter(X[:, 0], X[:, 1])

         # plot the mean of the data
         mean_d1, mean_d2 = X.mean(0)
         plt.plot(mean_d1, mean_d2, 'o', markersize=10, color='red', alpha=0.5)

         # calculate the covariance matrix
         Sigma = get_covariance(X)

         # calculate the eigenvector and eigenvalues of Sigma
         L, U = get_eigen(Sigma)

         plt.arrow(mean_d1, mean_d2, U[0, 0], U[0, 1], width=0.01, color='red', alpha=0.5, head
         plt.arrow(mean_d1, mean_d2, U[1, 0], U[1, 1], width=0.01, color='red', alpha=0.5, head
```




What do you observe in the above plot? Which eigenvector corresponds to the smallest eigenvalue?

Write your answer here:

**[ANSWER]**

By repeatedly using *Von Mises Power Iteration* we compute the eigenvector of `Sigma` with the k-th greatest absolute value, thus the second eigenvector corresponds to the smallest eigenvalues, which is depicted in the plot by the coordinates `U[1, 0]`, `U[1, 1]`.

#### 1.1.5 Task 4: Transform the data

Determine the smallest eigenvalue and remove its corresponding eigenvector. The remaining eigenvector is the basis of a new subspace. Transform all vectors in X in this new subspace by expressing all vectors in X in this new basis.

```
In [13]: def transform(X, U, L):
             """Transforms the data in the new subspace spanned by the eigenvector correspondi

             Parameters
             ----------
             X : array, shape [N, D]
                 Data matrix.
             L : array, shape [D]
                 Eigenvalues of Sigma_X
             U : array, shape [D, D]
                 Eigenvectors of Sigma_X

             Returns
```

```
    -------
    X_t : array, shape [N, 1]
        Transformed data

    """
    # get smallest eigenvalue
    wmin_idx = np.argmin(L)
    # define new basis by removing smallest eigenvector
    basis = U[0:wmin_idx, :]
    # project data on new basis
    X_t = np.dot(X, basis.T)
    return X_t
```

```
In [85]: X_t = transform(X, U, L)
```

## 1.2 Task SVD

### 1.2.1 Task 5: Given the matrix $M$ find its SVD decomposition $M = U \cdot \Sigma \cdot V$ and reduce it to one dimension using the approach described in the lecture.

```
In [86]: M = np.array([[1, 2], [6, 3],[0, 2]])
```

```
In [89]: def reduce_to_one_dimension(M):
             """Reduces the input matrix to one dimension using its SVD decomposition.

             Parameters
             ----------
             M : array, shape [N, D]
                 Input matrix.

             Returns
             -------
             M_t: array, shape [N, 1]
                 Reduce matrix.

             """
             U, S, V = np.linalg.svd(M, full_matrices=False)
             M_t = np.dot(M, V.T)
             return M_t
```

```
In [90]: M_t = reduce_to_one_dimension(M)
```