10_homework_dim_reduction

January 15, 2018

1 Programming assignment 10: Dimensionality Reduction

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt

        %matplotlib inline
```

1.1 PCA Task

Given the data in the matrix X your tasks is to: * Calculate the covariance matrix Σ. * Calculate eigenvalues and eigenvectors of Σ. * Plot the original data X and the eigenvectors to a single diagram. What do you observe? Which eigenvector corresponds to the smallest eigenvalue? * Determine the smallest eigenvalue and remove its corresponding eigenvector. The remaining eigenvector is the basis of a new subspace. * Transform all vectors in X in this new subspace by expressing all vectors in X in this new basis.

1.1.1 The given data X

```
In [2]: X = np.array([(-3,-2),(-2,-1),(-1,0),(0,1),
                      (1,2),(2,3),(-2,-2),(-1,-1),
                      (0,0),(1,1),(2,2), (-2,-3),
                      (-1,-2),(0,-1),(1,0), (2,1),(3,2)])
```

1.1.2 Task 1: Calculate the covariance matrix Σ

```
In [3]: def get_covariance(X):
            """Calculates the covariance matrix of the input data.

            Parameters
            ----------
            X : array, shape [N, D]
                Data matrix.

            Returns
            -------
            Sigma : array, shape [D, D]
                Covariance matrix
```

```
        """
        N, D = X.shape
        mean = np.dot(X.T, np.ones((N, 1))) * 1.0/N
        cov = np.dot(X.T, X) * 1.0/N  - np.dot(mean, mean.T)
        return cov
```

Note: The covariance of the data is equal to the covariance of the centered data

```
In [4]: # covariance of data
        get_covariance(X)

Out[4]: array([[2.82352941, 2.47058824],
               [2.47058824, 2.82352941]])

In [5]: # covariance of centered data
        get_covariance(X - np.mean(X, axis=0))

Out[5]: array([[2.82352941, 2.47058824],
               [2.47058824, 2.82352941]])
```

1.1.3 Task 2: Calculate eigenvalues and eigenvectors of Σ.

```
In [6]: def get_eigen(S):
            """Calculates the eigenvalues and eigenvectors of the input matrix.

            Parameters
            ----------
            S : array, shape [D, D]
                Square symmetric positive definite matrix.

            Returns
            -------
            L : array, shape [D]
                Eigenvalues of S
            U : array, shape [D, D]
                Eigenvectors of S

            """
            steps = 10
            D = S.shape[0]
            U = np.zeros((D, D))
            L = np.zeros((D,))

            # find eigenvectors using Von Mises Power Iteration
            for d in range(D):

                # initialize arbitrary normalized vector
                w = np.random.randn(D).reshape(D,1)
                w = w / np.linalg.norm(w)
```

```
            for s in range(steps):
                w = np.dot(S, w) / np.linalg.norm(np.dot(S, w))
            U[:, d] = w[:,0]

            # find the corresponding eigenvalue
            v = np.dot(w.T, np.dot(S, w))
            L[d] = v

            # deflate the covariance matrix
            S = S - v * np.dot(w, w.T)

        return L, U
```

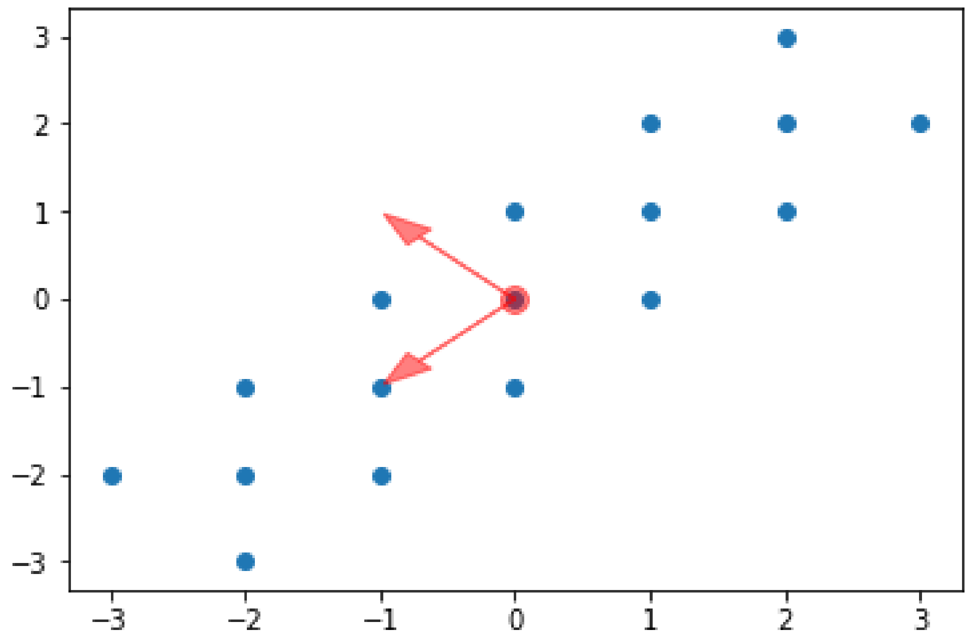
1.1.4 Task 3: Plot the original data X and the eigenvectors to a single diagram.

```
In [7]: # plot the original data
        plt.scatter(X[:, 0], X[:, 1])

        # plot the mean of the data
        mean_d1, mean_d2 = X.mean(0)
        plt.plot(mean_d1, mean_d2, 'o', markersize=10, color='red', alpha=0.5)

        # calculate the covariance matrix
        Sigma = get_covariance(X)

        # calculate the eigenvector and eigenvalues of Sigma
        L, U = get_eigen(Sigma)
        U = U.T
        plt.arrow(mean_d1, mean_d2, U[0, 0], U[0, 1], width=0.01, color='red', alpha=0.5, head_
        plt.arrow(mean_d1, mean_d2, U[1, 0], U[1, 1], width=0.01, color='red', alpha=0.5, head_
```



What do you observe in the above plot? Which eigenvector corresponds to the smallest eigenvalue?

Write your answer here:

[ANSWER]

By repeatedly using *Von Mises Power Iteration* we compute the eigenvector of `Sigma` with the k-th greatest absolute value, thus the second eigenvector corresponds to the smallest eigenvalues, which is depicted in the plot by the coordinates `U[1, 0]`, `U[1, 1]`.

1.1.5 Task 4: Transform the data

Determine the smallest eigenvalue and remove its corresponding eigenvector. The remaining eigenvector is the basis of a new subspace. Transform all vectors in X in this new subspace by expressing all vectors in X in this new basis.

```
In [19]: def transform(X, U, L):
             """Transforms the data in the new subspace spanned by the eigenvector correspondi

             Parameters
             ----------
             X : array, shape [N, D]
                 Data matrix.
             L : array, shape [D]
                 Eigenvalues of Sigma_X
             U : array, shape [D, D]
                 Eigenvectors of Sigma_X

             Returns
```

```
    -------
    X_t : array, shape [N, 1]
        Transformed data

    """
    # get smallest eigenvalue
    wmin_idx = np.argmin(L)
    # define new basis by removing smallest eigenvector
    basis = U[:, wmin_idx]
    # project data on new basis
    X_t = np.dot(X, basis.T)
    return X_t
```

```
In [20]: X_t = transform(X, U, L)
         X_t, X_t.shape
```

```
Out[20]: (array([ 7.07106782e-01,  7.07106781e-01,  7.07106781e-01,  7.07106781e-01,
                  7.07106781e-01,  7.07106780e-01,  5.71604986e-10,  2.85802493e-10,
                  0.00000000e+00, -2.85802493e-10, -5.71604986e-10, -7.07106780e-01,
                 -7.07106781e-01, -7.07106781e-01, -7.07106781e-01, -7.07106781e-01,
                 -7.07106782e-01]), (17,))
```

1.2 Task SVD

1.2.1 Task 5: Given the matrix M find its SVD decomposition $M = U \cdot \Sigma \cdot V$ and reduce it to one dimension using the approach described in the lecture.

```
In [16]: M = np.array([[1, 2], [6, 3],[0, 2]])
```

```
In [17]: def reduce_to_one_dimension(M):
             """Reduces the input matrix to one dimension using its SVD decomposition.

             Parameters
             ----------
             M : array, shape [N, D]
                 Input matrix.

             Returns
             -------
             M_t: array, shape [N, 1]
                 Reduce matrix.

             """
             U, S, V = np.linalg.svd(M, full_matrices=False)
             M_t = np.dot(M, V.T)
             return M_t
```

```
In [18]: M_t = reduce_to_one_dimension(M)
         M_t, M_t.shape
```

```
Out[18]: (array([[-1.90211303,  1.1755705 ],
                 [-6.68109819, -0.60243425],
                 [-1.05146222,  1.70130162]]), (3, 2))
```