Problem 1

We define the random variables S ("Scanner") and T ("Terrorist") as follows

$$S(\text{'person is predicted as terrorist'}) = 1$$
$$S(\text{'person is predicted as not terrorist'}) = 0$$

$$T(\text{'person is an actual terrorist'}) = 1$$
$$T(\text{'person is not an actual terrorist'}) = 0$$

By problem statement we have

$$\begin{aligned} P(T=1) &= 0.01 \Rightarrow P(T=0) = 0.99 \\ P(S=1|T=1) &= 0.95 \Rightarrow P(S=0|T=1) = 0.05 \\ P(S=0|T=0) &= 0.95 \Rightarrow P(S=1|T=0) = 0.05 \end{aligned}$$

Goal: Find $P(T=1|S=1)$.
Solution.

$$\begin{aligned} P(T=1|S=1) &= \frac{P(S=1|T=1)\cdot P(T=1)}{P(S=1)} && \text{(by Bayes' Theorem)} \\ &= \frac{P(S=1|T=1)\cdot P(T=1)}{P(S=1|T=1)\cdot P(T=1) + P(S=1|T=0)\cdot P(T=0)} \\ & && \text{(by Law of Total Probability)} \\ &= \frac{0.95\cdot 0.01}{0.95\cdot 0.01 + 0.05\cdot 0.99} \\ &= 0.16 \end{aligned}$$

Problem 2

Let $\Omega = \{w, r\}$. Let's define all possible events of putting two balls in the basket.

$$P(\text{"rr"}) = P(\text{"wr"}) = P(\text{"rw"}) = P(\text{"ww"}) = \frac{1}{4}$$

Let's define also $\Omega' = \{W, R\}$ and let "RRR" be the event of drawing three red balls from the basket.
Goal: Find $P(\text{"}rr\text{"}|\text{"}RRR\text{"})$.
Solution.

$$\begin{aligned} P(\text{"}rr\text{"}|\text{"}RRR\text{"}) &= \frac{P(\text{"}RRR\text{"}|\text{"}rr\text{"})\cdot P(\text{"}rr\text{"})}{P(\text{"}RRR\text{"})} && \text{(by Bayes' Theorem)} \\ &= \frac{1\cdot 1/4}{1\cdot 1/4 + 1/8\cdot 1/4 + 1/8\cdot 1/4 + 0\cdot 1/4} && \text{(by Law of Total Probability)} \\ &= 0.8 \end{aligned}$$

Problem 3

$$X(\text{X=x}) = \begin{cases} 1, \text{ if x = "H"} \\ 0, \text{ if x = "T"} \end{cases}$$

$$Y(Y = y_i) = i, \text{ if } \quad y_i\text{="Coin X is tossed i times, where the first i-1 are T and the i-th is H".}$$

Solution.

$$\mathbb{E}_Y[X = 1] = 1 \qquad \text{(by problem statement)}$$

$$\begin{aligned} \mathbb{E}_Y[X = 0] &= \sum_{i=0}^{\infty} \cdot (\frac{1}{2})^i \cdot \frac{1}{2} && \text{(by expectation definition)} \\ &= \frac{1}{2} \cdot \sum_{i=0}^{\infty} \cdot (\frac{1}{2})^i \\ &= \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} && \text{(by geometric series definition)} \\ &= 1 \end{aligned}$$

Problem 4

Goal: Compute $\mathbb{E}[X], Var[X]$.
Solution.

$$\begin{aligned} \mathbb{E}[X] &= \int_{-\infty}^{\infty} x \cdot p(x)\,\mathrm{d}x && \text{(by definition expectation of continous random variable)} \\ &= \int_a^b x \cdot p(x)\,\mathrm{d}x && \text{(by definition p(x) is zero outside these boundaries)} \\ &= \frac{1}{b-a} \int_a^b x\,\mathrm{d}x \\ &= \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b \\ &= \frac{1}{b-a} \cdot \frac{b^2 - a^2}{2} \\ &= \frac{a+b}{2} \end{aligned}$$

$$
\begin{aligned}
Var[X] &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 && \text{(by definition variance of continous random variable)} \\
&= \mathbb{E}[X^2] - (\frac{a+b}{2})^2 \\
&= \int_{-\infty}^{\infty} x^2 \cdot p(x)\,\mathrm{d}x - (\frac{a+b}{2})^2 \\
&= \int_{a}^{b} x^2 \cdot p(x)\,\mathrm{d}x - (\frac{a+b}{2})^2 && \text{(by definition p(x) is zero outside these boundaries)} \\
&= \frac{1}{b-a} \int_{a}^{b} x^2\,\mathrm{d}x - (\frac{a+b}{2})^2 \\
&= \frac{1}{b-a} \left[\frac{x^3}{3}\right]_a^b - (\frac{a+b}{2})^2 \\
&= \frac{1}{b-a} \cdot \frac{b^3-a^3}{3} - (\frac{a+b}{2})^2 \\
&= \frac{b^2+ab+a^2}{3} - (\frac{a+b}{2})^2 && \text{(by difference of cubes)} \\
&= \frac{b^2-2ab+a^2}{12} \\
&= \frac{(a-b)^2}{12}
\end{aligned}
$$

Problem 5

Goal: Prove

$$
\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}_Y[\mathbb{E}_{X|Y}[X]] \\
Var[X] &= \mathbb{E}_Y[Var_{X|Y}[X]] + Var_Y[E_{X|Y}[X]]
\end{aligned}
$$

Solution. For simplicity, we prove for the discrete case however going to continous case requires only the use of integrals instead of sums.

$$\begin{aligned}
\mathbb{E}_Y[\mathbb{E}_{X|Y}[X]] &= \sum_y p(y)\cdot \mathbb{E}_{X|Y}[X] \\
&= \sum_y p(y)\cdot (\sum_x x \cdot p(x|y)) \\
&= \sum_y \sum_x x\cdot p(x|y)\cdot p(y) \\
&= \sum_y \sum_x x\cdot p(x,y) \\
&= \sum_x \sum_y x\cdot p(x,y) \\
&= \sum_x x \sum_y p(x,y) \\
&= \sum_x x\cdot p(x) \\
&= \mathbb{E}[X]
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_Y[Var_{X|Y}[X]] + Var_Y[E_{X|Y}[X]] &= \mathbb{E}_Y[\mathbb{E}_{X|Y}[X^2] - (E_{X|Y}[X])^2] + \mathbb{E}_Y[(E_{X|Y}[X])^2] - (\mathbb{E}_Y[E_{X|Y}[X]])^2 \\
&= \mathbb{E}_Y[\mathbb{E}_{X|Y}[X^2]] - E_Y[(E_{X|Y}[X])^2] + \mathbb{E}_Y[(E_{X|Y}[X])^2] - (\mathbb{E}_Y[E_{X|Y}[X]])^2 \\
&= \mathbb{E}_Y[\mathbb{E}_{X|Y}[X^2]] - (\mathbb{E}_Y[E_{X|Y}[X]])^2 \\
&= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 && \text{(by first proof)} \\
&= Var[X]
\end{aligned}$$

Problem 6

Goal: Prove $p(|\frac{1}{n}\sum_{i=1}^n X_i - \mathbb{E}[X_i]| > \epsilon) \to 0$.
Solution.
Let $X = \frac{1}{n}\sum_{i=1}^n X_i$.

$$\begin{aligned}
p(|\frac{1}{n}\sum_{i=1}^n X_i - \mathbb{E}[X_i]| > \epsilon) &= p(|X - \frac{1}{n}\sum_{i=1}^n \mathbb{E}[X_i]| > \epsilon) && \text{(linearity of expectation)} \\
&= p(|X - \mathbb{E}[\frac{1}{n}\sum_{i=1}^n X_i]| > \epsilon) && \text{(by substitution)} \\
&= p(|X - \mathbb{E}[X| > \epsilon) && \text{(by Chebyshev inequality)} \\
&\le \frac{Var(X)}{\epsilon^2} && \text{(by linearity of variance under i.i.d assumption)} \\
&= \frac{\frac{1}{n^2}\sum_{i=1}^n Var(X_i)}{\epsilon^2} && \text{(X_i have same variance σ^2)} \\
&= \frac{\sigma^2}{\epsilon^2 n} \to 0 && (n \to \infty)
\end{aligned}$$