Problem 1

To enforce the necessary constraints, we introduce the Lagrangian multipliers $\lambda_1, ..., \lambda_M, \lambda_{M+1}$ and make an unconstrained maximization of

$$L = \mathbf{u}_{M+1}{}^\intercal \mathbf{S} \mathbf{u}_{M+1} + \lambda_{M+1}(1 - \mathbf{u}_{M+1}{}^\intercal \mathbf{u}_{M+1}) + \sum_{i=1}^{M} \lambda_i (\mathbf{u}_{M+1}{}^\intercal \mathbf{u}_i)$$

Now we take the derivative of L with respect to a vector $\mathbf{u}_{M+1}$ and set it to zero

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{u}_{M+1}} L = 0 &\Leftrightarrow 2\mathbf{S}\mathbf{u}_{M+1} = 2\lambda_{M+1}\mathbf{u}_{M+1} - \sum_{i=1}^{M} \lambda_i \mathbf{u}_i \\
&\Leftrightarrow 2\mathbf{u}_{M+1}{}^\intercal \mathbf{S}\mathbf{u}_{M+1} = 2\lambda_{M+1}\mathbf{u}_{M+1}{}^\intercal \mathbf{u}_{M+1} - \sum_{i=1}^{M} \lambda_i \mathbf{u}_{M+1}{}^\intercal \mathbf{u}_i \qquad \text{(left-multiply by } \mathbf{u}_{M+1}{}^\intercal\text{)} \\
&\Leftrightarrow \mathbf{u}_{M+1}{}^\intercal \mathbf{S}\mathbf{u}_{M+1} = \lambda_{M+1} \qquad \text{(by orthonormality of } u_i \text{ and } \mathbf{u}_{M+1}\text{)}
\end{aligned}$$

By induction step, we see that the variance in direction $\mathbf{u}_{M+1}$ is maximum when we set $\mathbf{u}_{M+1}$ equal to the eigenvector having the $(m+1)^{\text{th}}$ largest eigenvalue λ_{M+1}.

Problem 2

Problem 3

Let $\mathbf{x} \in \mathbb{R}^5$ hold the movie ratings given by Leslie.
By the SVD projection in concept space we have

$$\mathbf{v} \cdot \mathbf{x} = [1.74, 2.84]^\intercal$$

By SVD decomposition and reconstruction of the input using the projected space we have

$$[1.74, 2.84]^\intercal \cdot \mathbf{v}^\intercal = [1.0092, 1.0092, 1.0092, 2.0164, 2.0164]^\intercal$$

E.g. we can predict Leslie to rate Titanic movie with 2.0164.

Problem 4

See below.