Problem 1

Proofs make use of the properties/rules from Lecture 06 - slides (7, 15, 16, 17).

- $f(x, y, z) = 3x + e^{y+z} - \min\{-x^2, \quad \log(y)\}$, on $D = (\text{-}100, 100) \times (1, 50) \times (10, 20)$ is convex.
  - $3x$ is convex by Rule 1.
  - e^{y+z} is convex by Rule 1 and 2.
  - x^2 is convex (i.e. its Hessian, a 3 $\times$ 3 matrix, which is zero everywhere except $\frac{\partial^2 x^2}{\partial^2 x} = 2$ is positive-semi-definite on D, thus making $-x^2$ concave. $\log(y)$ is concave. $\min\{-x^2, \quad \log(y)\}$ returns $-x^2$ by construction of D, making $-\min\{-x^2, \quad \log(y)\}$ convex.

  By Rule 1, the sum of these convex functions is also convex. It follows, $f(x, y, z)$ is convex.
- $f(x, y) = yx^3 - 2yx^2 + y + 4$, $D = (\text{-}10, 10) \times (\text{-}10, 10)$ is not convex because the Hessian of x^3, a 2 $\times$ 2 matrix, which is zero everywhere except $\frac{\partial^2 x^3}{\partial^2 x} = 6x$ is clearly not positive-semi-definite for all $z \in D$.
- $f(x) = \log(x) + x^3$ and $D = (1, \infty)$ is not convex because $\log(x)$ is concave by definition.
- $f(x) = \text{-}\min(2\log(2x), \text{-}x^2 + 4x \quad \text{-}32)$, $D = \mathbb{R}^+$ is convex, because $\text{-}x^2 + 4x \quad \text{-}32$ is a concave quadratic function which reaches its global maximum of 28 at $x = 2$ (e.g. use high-school math to determine maximum). On the other hand $\log(2x)$ is non-negative, meaning that $\min(\cdot, \cdot)$ always returns the quadratic function. Finally, the negative sign makes $f(x)$ convex.

Problem 2

Let $x, y \in \mathbb{R}^d, \alpha \in [0, 1]$.

$$\begin{aligned}
h(\alpha x + (1-\alpha)y) &= f_1(\alpha x + (1-\alpha)y) + f_2(\alpha x + (1-\alpha)y) && \text{(by def.)} \\
&\leq \alpha f_1(x) + (1-\alpha)f_1(y) + \alpha f_2(x) + (1-\alpha)f_2(y) && \text{(by convexity of } f_1, f_2\text{)} \\
&= \alpha f_1(x) + \alpha f_2(x) + (1-\alpha)f_1(y) + (1-\alpha)f_2(y) \\
&= \alpha h(x) + (1-\alpha)h(y)
\end{aligned}$$

Problem 3

Proof by counter example. Let $f_1(x) = x^2$ and $f_2(x) = x$. Both functions are convex (by Exercise 1). But, $g(x) = f_1(x) \cdot f_2(x) = x^3$ is not convex on $\mathbb{R}$ because its second order derivative $6x$ is not non-negative over this interval.

Problem 4

Proof by contradiction. Assume $\nabla_\theta f(\theta^*) = 0$ and θ^* is not a global minimum. By first order convexity of f we have $\forall x, y : f(y) \geq f(x) + (y-x)^\intercal \nabla_x f(x)$. For $x = \theta^*$ we get $f(y) \geq f(\theta^*)$. But θ^* is not global minimum by assumption thus we get a contradiction. Thus θ^* must be the global minimum.

Problem 5

See notebook below.