7.17 a) The maximum flow is given by the following sequence of updates: route 4 units of flow along $S - A - C - T$, route 2 units along $S - A - D - T$. Route 2 units along $S - B - C - T$, route 3 units along $S - B - D - T$. The resulting flow is feasible and has value 11. It produces the mincut $(\{S, A, B\}, \{C, D, T\})$ of capacity 11, certifying the optimality of the flow. Notice that $(\{S, A, B, D\}, \{C, T\})$ is also a mincut.

b) The residual graph is shown in the figure. Vertices S,A and B are reachable from S. T can be reached from vertices C and T.

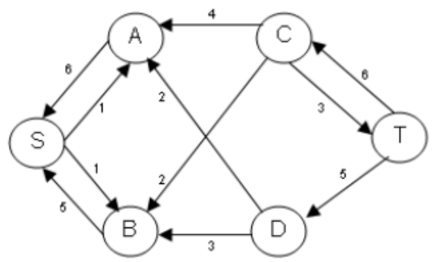

Figure 2: The residual graph

c) (A, C) and (B, C) are bottleneck edges. The other edges belonging to a mincut are not bottleneck, as increasing their capacity does not increase the capacity of the minimum (s, t)-cut.

d) The following figure shows an example with no bottleneck edges. The optimal flow saturates all edges, but augmenting the capacity of any of them does not increase the capacity of the minimum cut, i.e. does not open up any new path for flow to run from source to sink.

e) Run the usual network flow algorithm and consider the final residual graph. Let S be the set of vertices reachable from s and T the set of vertices from which t is reachable in this graph. By the optimality of the flow S and T must be disjoint, as they are separated by a saturated cut. Suppose now that $e = (u, v)$ is a bottleneck edge. As we increase e's capacity we are able to route flow from s to u, through e and from v to t in the residual graph. This implies that $u \in S$ and

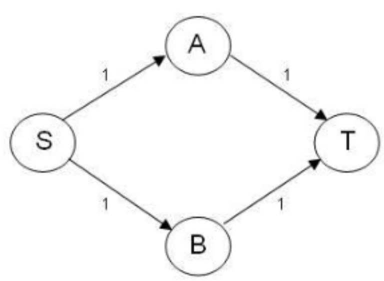

Figure 3: An example with no bottleneck edges

$v \in T$. Moreover, if an edge $e = (u, v)$ has $u \in S$ and $v \in T$ (notice that e must then be in a minimum cut), increasing its capacity allows us to route flow from s to u (as $u \in S$), through the new capacity of e and to t (as $v \in T$). Hence, the set of bottleneck edges is the set $E(S, T)$, i.e. the set of edges which originate in S and end in T.

7.21 Note that each critical edge must be saturated in any max-flow solution otherwise the same solution is still feasible even when its capacity is decreased by one unit. For each such edge, we can check in time $O(|V|+|E|)$ if decreasing its capacity decreases the max flow (see next problem). Hence, we can find a max-flow and check each saturated edge for criticality in $O(|E| \cdot (|V|+|E|))$ time.

8.4 (a) Given a clique in the graph, it is easy to verify in polynomial time that there is an edge between every pair of vertices. Hence a solution to CLIQUE-3 can be checked in polynomial time.

(b) The reduction is in the wrong direction. We must reduce CLIQUE to CLIQUE-3, if we intend to show that CLIQUE-3 is at least as hard as CLIQUE.

(c) The statement "a subset $C \subseteq V$ is a vertex cover in G is and only if the complimentary set $V - C$ is a clique in G" used in the reduction is false. C is a vertex cover if and only if $V - C$ is an *independent set* in G.

(d) The largest clique in the graph can be of size at most 4, since every vertex in a clique of size k must have degree at least $k-1$. Thus, there is no solution for $k > 4$, and for $k \leq 4$ we can check every k-tuple of vertices, which takes $O(|V|^k) = O(|V|^4)$ time.

8.15 This is a generalization of CLIQUE. Given (G, k) as an instance of CLIQUE with n vertices, take $G_1 = G$, $b = k$ and G_2 as a clique of size n. Then (G_1, G_2, b) has a solution if and only if G has a clique of size at least k.