Selected Solutions of Numerical Linear Algebra

Fang Zhu

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Part I

Fundamentals

Lecture 1. Matrix-Vector Multiplication

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todo...

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1.3 *Proof.* We denote a non-singular matrix $\boldsymbol{R}$ as

$$\boldsymbol{R} = \begin{pmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_{mm} \end{pmatrix},$$

it is clear that $r_{ii} \neq 0$, otherwise $\boldsymbol{R}$ is singular. Since $\boldsymbol{R}$ is non-singular, we assume that

$$\boldsymbol{I} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_m) = (\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n) \begin{pmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_{mm} \end{pmatrix}$$

where $(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n) = \boldsymbol{R}^{-1}$. To show $\boldsymbol{R}^{-1}$ is upper-triangular, we work by induction. To begin with, we have $\boldsymbol{e}_1 = r_{11}\boldsymbol{a}_1$ and hence $\boldsymbol{a_1} = r_{11}^{-1}\mathbf{e}_1$ has *zero entries* except the first one. For convenience, we denote by $\mathbb{C}_k^m$ the column space

$$\mathbb{C}_k^m = \{\boldsymbol{v} = (v_1, \cdots, v_k, 0, \cdots, 0)^T, v_i \neq 0 \ (1 \leq i \leq k)\},$$

Then

$$\mathbb{C}_1^m \subset \mathbb{C}_2^m \cdots \mathbb{C}_m^m = \mathbb{C}^m.$$

We have shown that $a_1 \in \mathbb{C}^m(1)$, assume that for any $k \leq s$, we have that $\mathbf{a}_k \in \mathbb{C}_k^m$. Then by equation *Page 8, (1.8)*, we have

$$\boldsymbol{e}_{s+1} = \sum_{k=1}^{m} \boldsymbol{a}_k r_{k,s+1}.$$

Note that $r_{k,s+1} = 0, \ \forall k > s+1$, then

$$\sum_{k=1}^{m} \boldsymbol{a}_k r_{k,s+1} = \sum_{k=1}^{s} \boldsymbol{a}_k r_{k,s+1} + \boldsymbol{a}_{s+1} r_{s+1,s+1} = \boldsymbol{e}_{s+1},$$

Therefore

$$\boldsymbol{a}_{s+1} = r_{s+1,s+1}^{-1}(\mathbf{e}_{s+1} - \sum_{k=1}^{s} \boldsymbol{a}_k r_{k,s+1}) \in \mathbb{C}_{s+1}^{m}$$

By induction, we have proved that $\boldsymbol{a}_k \in \mathbb{C}_k^m$ for $1 \leq k \leq m$, which is equivalent to the fact that $\boldsymbol{R}^{-1}$ is upper-triangular. □

1.4(a) *Proof.* Denote the column vectors $(c_1, \cdots, c_n)^T$, $(d_1, \cdots, d_n)^T$ by notations $\boldsymbol{c}$ and $\boldsymbol{d}$, let $\boldsymbol{F}$ be the matrix whose (i, j) entry is $f_j(i)$. Then, the given condition can be rephrased as: ForAll $\boldsymbol{d} \in \mathbb{C}^8$, there must exist a vector $\boldsymbol{c}$ such that $\boldsymbol{Fc} = \boldsymbol{d}$. This means that

$$\text{range}\{\boldsymbol{F}\} = \mathbb{C}^8,$$

which implies that $\boldsymbol{F}$ has full rank by *theorem 1.3*. Furthermore, $\mathbf{F}$ is non-singular. Therefore

$$\boldsymbol{c} = \boldsymbol{F}^{-1}\boldsymbol{d}$$

and hence $\boldsymbol{d}$ determines $\boldsymbol{c}$ uniquely.

□

1.4(b) The given condition can be reformatted as

$$\boldsymbol{Ad} = \boldsymbol{c}.$$

Note that $\boldsymbol{c} = \boldsymbol{F}^{-1}\boldsymbol{d}$, then

$$\boldsymbol{Ad} = \boldsymbol{c} = \boldsymbol{F}^{-1}\boldsymbol{d},$$

then we have

$$(\boldsymbol{FA} - \boldsymbol{I})\boldsymbol{d} = \mathbf{0},$$

note that this equation above is true for any $d \in \mathbb{C}^8$, then $\boldsymbol{FA} - \boldsymbol{I}$ must be *zero matrix*, which is $\boldsymbol{FA} = \boldsymbol{I}$. Hence the i, j entry of $\boldsymbol{A}^{-1}$ is the i, j entry of $\boldsymbol{F}$ we defined in (a).

Lecture 2. Orthogonal Vectors and Matrices

2.1 Prerequisite

Before giving the solutions, I would like to prove some basic conclusions about this lecture

Lemma 2.1. *Given an non-singular matrix $\boldsymbol{A}$, then $\boldsymbol{A}^{-1}$ is unique*

Proof. Suppose that we have two inverse matrices $\boldsymbol{C}$ and $\boldsymbol{B}$ w.r.t $\boldsymbol{A}$. By the definition of inverse.

$$\boldsymbol{B} = \boldsymbol{B}\boldsymbol{I} = \boldsymbol{B}(\boldsymbol{A}\boldsymbol{C}) = (\boldsymbol{B}\boldsymbol{A})\boldsymbol{C} = \boldsymbol{I}\boldsymbol{C} = \boldsymbol{C},$$

and hence we can conclude that $\boldsymbol{B} = \boldsymbol{C}$. □

Lemma 2.2. *Given an non-singular matrix $\boldsymbol{A} \in \mathbb{R}^{m\times m}$, its hermitian conjugate $\boldsymbol{A}^*$ is also non-singular.*

Proof.

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I},$$

We can apply the hermitian conjugate to both sides of the equation above:

$$\boldsymbol{A}^*(\boldsymbol{A}^{-1})^* = (\boldsymbol{A}^{-1})^*\boldsymbol{A}^* = \boldsymbol{I}$$

Hence we can get that $\boldsymbol{A}^*$ is non-singular. □

Lemma 2.3. *Give a non-singular matrix $\boldsymbol{A}$ and its hermitian conjugate $\boldsymbol{A}^*$, we have*

$$(\boldsymbol{A}^*)^{-1} = (\boldsymbol{A}^{-1})^*$$

Proof. By lemma 2.2, $\boldsymbol{A}^*$ is non-singular and it's clear that the inverse is $(\boldsymbol{A}^{-1})^*$. However, we can get that $(\boldsymbol{A}^*)^{-1}$ is also the inverse of $\boldsymbol{A}^*$ by definition. Futher, by lemma 2.1, we have

$$(\boldsymbol{A}^*)^{-1} = (\boldsymbol{A}^{-1})^*$$

which is exactly what we need to prove. □

Lemma 2.4. *Given two pure imaginary number di, bi, then*

$$(1-di)(1-bi) \neq 0.$$

Proof. LHS equals

$$1 - bd - (b+d)i, \tag{$\star$}$$

if $(\star) = 0$, then we have

$$\begin{aligned} 1 - bd &= 0 \\ b + d &= 0, \end{aligned}$$

which means that

$$-b^2 = 1,$$

since $b \in \mathbb{R}$, the equation above cannot be true, and hence $(1-di)(1-bi) \neq 0$. □

2.2 Solutions

2.1 *Proof.* Without loss of generality, we assume that $\boldsymbol{A}$ is upper-triangular. By the *ex. 1.3*, we can conclude that $\boldsymbol{A}^{-1}$ is also upper-triangular. It is clear that $\boldsymbol{A}^* = \boldsymbol{A}^{-1}$ since $\boldsymbol{A}$ is unitary. Then $\boldsymbol{A}^\star$ is also an upper-triangular matrix, which is

$$\boldsymbol{A}^\star_{i,j} = \bar{a}_{ji} = a_{ij} = 0, \quad (\forall i > j),$$

Hence, the matrix $\boldsymbol{A}$ is diagonal. The same follows if $\boldsymbol{A}$ is lower-triangular. □

2.3 (a) Let $\boldsymbol{x}$ be an eigenvector of matrix $\boldsymbol{A}$ w.r.t. the eigenvalue λ, then

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x},$$

multiplying both sides by $\boldsymbol{x}^\star$, we get that

$$\boldsymbol{x}^\star \boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}^\star \boldsymbol{x} = \lambda \|\boldsymbol{x}\|^2, \tag{$\spadesuit$}$$

then

$$\lambda = \frac{\boldsymbol{x}^\star \boldsymbol{A}\boldsymbol{x}}{\|\boldsymbol{x}\|^2} = \frac{\boldsymbol{x}^\star \boldsymbol{A}^\star \boldsymbol{x}}{\|\boldsymbol{x}\|^2} = \frac{(\boldsymbol{x}^\star \boldsymbol{A}\boldsymbol{x})^\star}{\|\boldsymbol{x}\|^2} = \bar{\lambda},$$

which means that λ is real.

2.3 (b) Let $\boldsymbol{x}_1, \boldsymbol{x}_2$ be two eigenvectors of the hermitian matrix $\boldsymbol{A}$. Denote λ_k the eigenvalue w.r.t $\boldsymbol{x}_k (k = 1, 2)$, where $\lambda_1 \neq \lambda_2$, then

$$\begin{aligned} \lambda_2 \boldsymbol{x}_1^\star \boldsymbol{x}_{\boldsymbol{2}} &= \boldsymbol{x}_1^\star \boldsymbol{A}\boldsymbol{x}_2, \\ \lambda_1 \boldsymbol{x}_2^\star \boldsymbol{x}_{\boldsymbol{1}} &= \boldsymbol{x}_2^\star \boldsymbol{A}\boldsymbol{x}_1. \end{aligned}$$

Note that $\boldsymbol{A}$ is hermitian, we can get that

$$\lambda_2 \boldsymbol{x}_1^\star \boldsymbol{x}_2 = \boldsymbol{x}_1^\star \boldsymbol{A}\boldsymbol{x}_2 = \boldsymbol{x}_1^\star \boldsymbol{A}^\star \boldsymbol{x}_2 = (\boldsymbol{x}_2^\star \boldsymbol{A}\boldsymbol{x}_1)^\star = \lambda_1^\star \boldsymbol{x}_1^\star \boldsymbol{x}_2,$$

then

$$(\lambda_2 - \lambda_1^\star)\boldsymbol{x}_1^\star \boldsymbol{x}_2 = 0 \Rightarrow \boldsymbol{x}_1^\star \boldsymbol{x}_2 = 0,$$

which is exactly what we need to prove.

2.4 Let λ be an eigenvalue of $\boldsymbol{A}$, and $\boldsymbol{x}$ be the eigenvector w.r.t λ, then we have $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ and $\|\boldsymbol{A}\boldsymbol{x}\|_2^2 = \|\lambda\boldsymbol{x}\|$, which is

$$\boldsymbol{x}^\star \boldsymbol{A}^\star \boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}^\star \|\lambda^\star \lambda\| \boldsymbol{x}.$$

Since $\boldsymbol{A}$ is unitary, then

$$\boldsymbol{x}^\star \boldsymbol{A}^\star \boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}^\star \boldsymbol{I}\boldsymbol{x} = \boldsymbol{x}^\star \|\lambda\|_2^2 \boldsymbol{x}.$$

Furthermore,

$$x_1^2 + x_2^2 + \cdots + x_n^2 = \|\lambda\|_2^2 (x_1^2 + x_2^2 + \cdots + x_n^2),$$

it follows that $\|\lambda\|_2^2 = 1$ since $\boldsymbol{x}$ is non-zero vector.

2.5(a) Let $\boldsymbol{x}$ be an eigenvector of matrix $\boldsymbol{S}$ w.r.t. the eigenvalue λ, then

$$\boldsymbol{S}\boldsymbol{x} = \lambda\boldsymbol{x}.$$

By the equation *ex.* 2.3(♠), we have

$$\lambda = \frac{\boldsymbol{x}^\star \boldsymbol{S}\boldsymbol{x}}{\|\boldsymbol{x}\|^2} = \frac{\boldsymbol{x}^\star(-\boldsymbol{S}^\star)\boldsymbol{x}}{\|\boldsymbol{x}\|^2} = \frac{-(\boldsymbol{x}^\star \boldsymbol{S}\boldsymbol{x})^\star}{\|\boldsymbol{x}\|^2} = -\bar{\lambda},$$

then we can get that $\lambda + \bar{\lambda} = 0$, which means that λ is purely imaginary.

2.5(b) Assume that λ is the eigenvalue of $\boldsymbol{S}$, it follows that $1 - \lambda$ is the eigenvalue of $1 - \boldsymbol{S}$. Since λ is purely imagnary number, then by *Lemma 2.1.4*, we have

$$\det(1 - \boldsymbol{S}) = \prod_{i=1}^{n} (1 - \lambda_i) \neq 0,$$

where $\lambda_i, i \in \{1, 2, \cdots, n\}$ are eigenvalues of $\boldsymbol{S}$. Hence we can conclude that $1 - \boldsymbol{S}$ is non-singular.

2.5(c) Assume that $\boldsymbol{Q} = (\boldsymbol{I} - \boldsymbol{S})^{-1}(\boldsymbol{I} + \boldsymbol{S})$, then we have

$$\boldsymbol{Q}\boldsymbol{Q}^\star = (\boldsymbol{I} - \boldsymbol{S})^{-1}(\boldsymbol{I} + \boldsymbol{S})(\boldsymbol{I} + \boldsymbol{S}^\star)\left((\boldsymbol{I} + \boldsymbol{S})^{-1}\right)^\star,$$

by *Lemma 2.1.3*, we can get that

$$\begin{aligned}
\boldsymbol{Q}\boldsymbol{Q}^\star &= (\boldsymbol{I} - \boldsymbol{S})^{-1}(\boldsymbol{I} + \boldsymbol{S})(\boldsymbol{I} - \boldsymbol{S})\left((\boldsymbol{I} - \boldsymbol{S})^{-1}\right)^\star \\
&= (\boldsymbol{I} - \boldsymbol{S})^{-1}(\boldsymbol{I} + \boldsymbol{S})(\boldsymbol{I} - \boldsymbol{S})(\boldsymbol{I} + \boldsymbol{S})^{-1} \\
&= (\boldsymbol{I} - \boldsymbol{S})^{-1}(\boldsymbol{I} - \boldsymbol{S})(\boldsymbol{I} + \boldsymbol{S})(\boldsymbol{I} + \boldsymbol{S})^{-1} \\
&= \boldsymbol{I}.
\end{aligned}$$

Hence, we can conclude that $(1 - \boldsymbol{S})^{-1}(1 - \boldsymbol{S})$ is unitary.

2.6 *Proof.* If $\boldsymbol{A}$ is singular, there exists a vector $\boldsymbol{x} \in \mathbb{C} \setminus \{0\}$ such that

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x} + \boldsymbol{u}\boldsymbol{v}^\star \boldsymbol{x} = 0,$$

then $\boldsymbol{x} = -\boldsymbol{u}(\boldsymbol{v}^\star \boldsymbol{x})$ where $\boldsymbol{v}^\star \boldsymbol{u}$ is scalar. Let $\boldsymbol{x} = t\boldsymbol{u}(t \in \mathbb{R})$, then we can get that

$$t\boldsymbol{u} + \boldsymbol{u}(\boldsymbol{v}^\star t\boldsymbol{u}) = t\boldsymbol{u}(1 + \boldsymbol{v}^\star \boldsymbol{u}) = 0,$$

It follows that $\boldsymbol{v}^\star \boldsymbol{u} = -1$ since $\boldsymbol{x} = t\boldsymbol{u} \neq 0$. Assume that $\alpha = -1/(1 + \boldsymbol{v}^\star \boldsymbol{u})$, then

$$(\boldsymbol{I} + \boldsymbol{u}\boldsymbol{v}^\star)(\boldsymbol{I} + \alpha \boldsymbol{u}\boldsymbol{v}^\star) = \boldsymbol{I}.$$

Note that we have shown that $\boldsymbol{v}^\star \boldsymbol{u} = -1$ is a necessary condition of $\boldsymbol{A}$ is singular. For suffciency, we assume that $\boldsymbol{v}^\star \boldsymbol{u} = -1$. Then for any $t \in \mathbb{C} \setminus \{0\}$, we have

$$\boldsymbol{A}\boldsymbol{u} = t\boldsymbol{u} + \boldsymbol{u}\boldsymbol{v}^\star t\boldsymbol{u} = t\boldsymbol{u} + t\boldsymbol{u}(\boldsymbol{v}^\star \boldsymbol{u}) = 0,$$

which implies that $\boldsymbol{A}$ is singular. Combined, we conclude that $\boldsymbol{A}$ is singular iff. $\boldsymbol{v}^\star \boldsymbol{u} = -1$. In this case,

$$\mathrm{null}(\boldsymbol{A}) = \{t\boldsymbol{u}, t \in \mathbb{R}\},$$

the linear subspace spanned by $\boldsymbol{u}$. $\square$

2.7 *Proof.* We can verify that $\boldsymbol{H}_{k+1}$ is Hadamard matrix directly,

$$\begin{aligned}
\boldsymbol{H}_{k+1}^T \boldsymbol{H}_{k+1} &= \begin{pmatrix} \boldsymbol{H}_k^T & \boldsymbol{H}_k^T \\ \boldsymbol{H}_k^T & -\boldsymbol{H}_k^T \end{pmatrix} \begin{pmatrix} \boldsymbol{H}_k & \boldsymbol{H}_k \\ \boldsymbol{H}_k & -\boldsymbol{H}_k \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{H}_k^T \boldsymbol{H}_k + \boldsymbol{H}_k^T \boldsymbol{H}_k & \boldsymbol{H}_k^T \boldsymbol{H}_k - \boldsymbol{H}_k^T \boldsymbol{H}_k \\ \boldsymbol{H}_k^T \boldsymbol{H}_k - \boldsymbol{H}_k^T \boldsymbol{H}_k & \boldsymbol{H}_k^T \boldsymbol{H}_k + \boldsymbol{H}_k^T \boldsymbol{H}_k \end{pmatrix} \\
&= \begin{pmatrix} 2\boldsymbol{H}_k^T \boldsymbol{H}_k & \boldsymbol{0} \\ \boldsymbol{0} & 2\boldsymbol{H}_k^T \boldsymbol{H}_k \end{pmatrix} \\
&= \begin{pmatrix} 2\boldsymbol{I}_k & \boldsymbol{0} \\ \boldsymbol{0} & 2\boldsymbol{I}_k \end{pmatrix} \\
&= 2c \cdot \boldsymbol{I}_{2k}.
\end{aligned}$$

Then we can get that $\boldsymbol{H}_{k+1}^T = 2c\boldsymbol{H}_{k+1}^{-1}$. Note that the entries of $\boldsymbol{H}_{k+1}$ are also all ± 1 by the recursion formula. Hence $\boldsymbol{H}_{k+1}$ is also a Hadamard matrix. $\square$

Lecture 3. Norms

3.1 Prerequisite

Lemma 3.1. *Given a permutation matrix* $\mathbf{P} \in \mathcal{M}_{\mathbf{m}\times\mathbf{n}}$*, and a vector* $\boldsymbol{x} \in \mathbb{C}^n$*, then*

$$\|\boldsymbol{P}\boldsymbol{x}\|_p = \|\boldsymbol{x}\|_p.$$

Proof. By the definition of vector norm, then

$$\|\boldsymbol{x}\|_p = \begin{cases} \left(\sum_{i=1}^m |x_i|^p\right)^{\frac{1}{p}}, & 1 \le p < \infty; \\ \max_i\{x_i\}, & p = \infty. \end{cases}$$

It is clear that $\|\boldsymbol{x}\|_p$ won't be changed after permutation of entries. Therefore, for any permutation matrix $\boldsymbol{P}$,

$$\|\boldsymbol{P}\boldsymbol{x}\|_p = \|\boldsymbol{x}\|_p,$$

which is what we need to prove. □

Corollary 3.1. *Given matrix* $\boldsymbol{A} \in \mathcal{M}_{m\times n}$ *and two permutation matrix* $\boldsymbol{P} \in \mathcal{M}_{m\times m}, \boldsymbol{Q} \in \mathcal{M}_{n\times n}$*. Then,*

$$\|\boldsymbol{PAQ}\|_p = \|\boldsymbol{A}\|_p.$$

3.2 Solutions

3.1 By equation (3.3), we can get that

$$\|\boldsymbol{x}\|_{\boldsymbol{W}} = \|\boldsymbol{W}\boldsymbol{x}\|,$$

where $\|\cdot\|$ is a vector norm. It is clear that $\|\cdot\|_{\boldsymbol{W}}$ meets (2), (3) of the vector norm's definition. Furthermore, we assume that

$$\boldsymbol{W}\boldsymbol{x} = \mathbf{0}. \tag{$\star$}$$

Since $\boldsymbol{W}$ is non-singular, ($\star$) is true iif. $\boldsymbol{x} = \mathbf{0}$. Then $\|x\|_{\boldsymbol{W}} = \|\boldsymbol{W}\boldsymbol{x}\| \ge 0$, and $\|\boldsymbol{x}\| = 0$ iif. $\|\boldsymbol{x}\| = 0$, which meets condition (1) of vector norm's definition. Hence, we can conclude that $\|\cdot\|_{\boldsymbol{W}}$ is a vector norm.