Lecture 6 Projectors

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1 Prerequisite

2 Solutions

2.1 Exercise 6.1

Proof. Since $\boldsymbol{P}$ is an orthogonal projector, then we have $\boldsymbol{P}^2 = \boldsymbol{P}$ and $\boldsymbol{P}^\star = \boldsymbol{P}$, hence

$$(\boldsymbol{I} - 2\boldsymbol{P})^\star(\boldsymbol{I} - 2\boldsymbol{P}) = \boldsymbol{I} - 2\boldsymbol{P}^\star - 2\boldsymbol{P} + 4\boldsymbol{P}^\star\boldsymbol{P} = 0, \tag{1}$$

which means that $\boldsymbol{I} - 2\boldsymbol{P}$ is unitary. A geometric interpretation is given by Figure 1. ☐

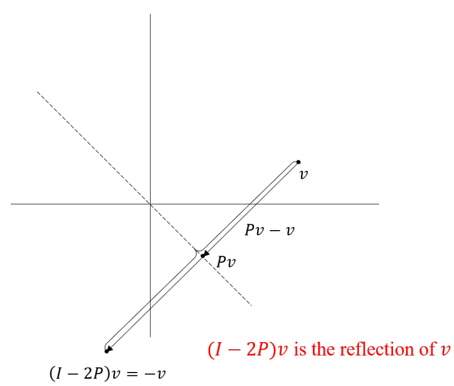

Figure 1: Geometric interpretation of $\boldsymbol{I} - 2\boldsymbol{P}$.

2.2 Exercise 6.2

We begin by considering the matrix $\boldsymbol{F}$ defined by

$$\boldsymbol{F} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix},$$

where $\boldsymbol{F}$ is a permutation matrix that reverses the order of the basis vectors. Consequently, we define

$$\boldsymbol{E} = \frac{\boldsymbol{I} + \boldsymbol{F}}{2} = \begin{pmatrix} \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdots & \frac{1}{2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \cdots & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

where $\boldsymbol{I}$ is the identity matrix. Observing that $\boldsymbol{F}$ is its own inverse, i.e., $\boldsymbol{F}^2 = \boldsymbol{I}$, and that the adjoint of a matrix is equal to its conjugate transpose, we deduce that the adjoint of $\boldsymbol{E}$, denoted $\boldsymbol{E}^\star$, satisfies

$$\boldsymbol{E}^\star = \left(\frac{\boldsymbol{I} + \boldsymbol{F}}{2}\right)^\star = \boldsymbol{E},$$

since both $\boldsymbol{I}$ and $\boldsymbol{F}$ are real and symmetric, and thus equal to their own adjoints.

Next, we compute $\boldsymbol{E}^2$ as follows:

$$\boldsymbol{E}^2 = \left(\frac{\boldsymbol{I} + \boldsymbol{F}}{2}\right)^2 = \frac{\boldsymbol{I} + 2\boldsymbol{F} + \boldsymbol{F}^2}{4} = \frac{2\boldsymbol{I} + 2\boldsymbol{F}}{4} = \frac{\boldsymbol{I} + \boldsymbol{F}}{2} = \boldsymbol{E}.$$

Hence, $\boldsymbol{E}^2 = \boldsymbol{E}$, which implies that $\boldsymbol{E}$ is idempotent. Additionally, since for any vector $\boldsymbol{v}$, the equality $\langle \boldsymbol{E}\boldsymbol{v}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{E}\boldsymbol{v} \rangle$ holds, where $\langle \cdot, \cdot \rangle$ denotes the inner product, it follows that $\boldsymbol{E}$ is self-adjoint. Combining the idempotence and self-adjointness of $\boldsymbol{E}$, we conclude that $\boldsymbol{E}$ is an orthogonal projector.

2.3 Exercise 5.3

Suppose that $\boldsymbol{A}$ is full rank. This implies that $\boldsymbol{A}$ has n non-zero singular values. Consequently, the matrix $\boldsymbol{A}^\star \boldsymbol{A}$, where $\boldsymbol{A}^\star$ denotes the conjugate transpose of $\boldsymbol{A}$, has n non-zero eigenvalues $\lambda_1, \ldots, \lambda_n$. The determinant of $\boldsymbol{A}^\star \boldsymbol{A}$ is then given by the product of its eigenvalues:

$$\det(\boldsymbol{A}^\star \boldsymbol{A}) = \prod_{i=1}^{n} \lambda_i \neq 0,$$

which indicates that $\boldsymbol{A}^\star\boldsymbol{A}$ is non-singular.

For the "only if" part of the proof, we invoke the singular value decomposition (SVD). Since $\boldsymbol{A}^\star\boldsymbol{A}$ is non-singular, it follows from Theorem 5.2 that

$$\mathrm{range}(\boldsymbol{A}) = \langle \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle,$$

where the $\boldsymbol{u}_i$ are left singular vectors of $\boldsymbol{A}^\star\boldsymbol{A}$. This space is n-dimensional, which confirms that the matrix $\boldsymbol{A}$ is indeed full rank.

2.4 Exercise 6.4 (a)

We first compute $\boldsymbol{P}$ as

$$\boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A}^\star\boldsymbol{A})^{-1}\boldsymbol{A}^\star = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

then

$$\boldsymbol{P}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

2.5 Exercise 6.4(b)

We first compute $\boldsymbol{P}$ as

$$\boldsymbol{P} = \boldsymbol{B}(\boldsymbol{B}^\star\boldsymbol{B})^{-1}\boldsymbol{B}^\star = \begin{pmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{pmatrix}$$

then

$$\boldsymbol{P}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}.$$

2.6 Exercise 6.5

We first show that $\|\boldsymbol{P}\|_2 \geq 1$. Suppose that there exists a matrix $\boldsymbol{P}$ such that $\|\boldsymbol{P}\|_2 < 1$, then we know that the largest singular value $\sigma_{\max}$ of $\boldsymbol{P}$ is less than 1 then

$$|\det(\boldsymbol{P}^2)| = |\det(\boldsymbol{P})^2| = |\prod_i \sigma_i^2| < |\prod_i \sigma_i| = |\det(\boldsymbol{P})| \tag{2}$$

then $\boldsymbol{P}^2 \neq \boldsymbol{P}$, which is contradicted with that the matrix $\boldsymbol{P}$ is a projector, and hence we conclude that $\|\boldsymbol{P}\|_2 \geq 1$. Suppose that the SVD of $\boldsymbol{P}$ are $\boldsymbol{P} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\star$, then if $\|\boldsymbol{P}\| = 1$, then by (2), we can get that all the singular values of $\boldsymbol{P}$ are 1, i.e., $\boldsymbol{\Sigma}$ is the identity matrix and hence $\boldsymbol{P}^\star =$