Lecture 4 The Singular Value Decomposition

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1 Prerequisite

Lemma 1.1. *Given summetrix matrix $\boldsymbol{A}$, then the eigenvalues of $\boldsymbol{A}$ are real.*

Theorem 1.1. *Given symmetric matrix $\boldsymbol{A}$, then $\boldsymbol{A}$ can be factored as*

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^\star,$$

where

- $\boldsymbol{Q}$ *is unitary;*
- Λ *is diagonal, with the eigenvalues of $\boldsymbol{A}$ on its dagonal.*

Proof. By induction of the dimension of $\boldsymbol{A}$. □

2 Solutions

2.1 Exercise 4.1(e)

First we compute the singular values σ_i by finding the eigenvalues of $\boldsymbol{A}^\star\boldsymbol{A}$:

$$\boldsymbol{A}^\star\boldsymbol{A} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

the characteristic polynomial of $\boldsymbol{A}^\star\boldsymbol{A}$ is

$$\det(\boldsymbol{A}^\star\boldsymbol{A} - \lambda\boldsymbol{I}) = \lambda(\lambda - 4) = 0,$$

so the singular values are $\sigma_1 = 0, \sigma_2 = 2$. For $\lambda = 4$, we have

$$\boldsymbol{A}^\star\boldsymbol{A} - 4\boldsymbol{I} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix},$$

a unit vector in the kernel of the matrix is $\boldsymbol{v}_2 = \left(1/\sqrt{2}, 1/\sqrt{2}\right)^T$. For $\lambda = 0$, we have

$$\boldsymbol{A}^\star\boldsymbol{A} - 0\boldsymbol{I} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

a unit vector in the kernel of the matrix is $\boldsymbol{v}_2 = \left(-1/\sqrt{2}, 1/\sqrt{2}\right)^T$. So at this point we know that

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\star = (\boldsymbol{u}_1, \boldsymbol{u}_2) \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Finally we can compute $\boldsymbol{u}_1$ by the formula $\sigma_i \boldsymbol{u}_i = \boldsymbol{A}\boldsymbol{v}_i$, this gives $\boldsymbol{u}_i = \left(\sqrt{2}/2, \sqrt{2}/2\right)$, then by $\boldsymbol{u}_2^\star \boldsymbol{u}_1 = 0$ and $\|\boldsymbol{u}_2\|_2 = 1$ we can get a $\boldsymbol{u}_2 = \left(-\sqrt{2}/2, \sqrt{2}/2\right)$. So in this full glory the SVD is

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\star = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\star.$$

2.2 Exercise 4.2

Assume that

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots a_{2n} \\ \vdots & \vdots & \ddots \vdots \\ a_{m1} & a_{m2} & \cdots a_{mn} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}_1^T \\ \boldsymbol{\alpha}_1^T \\ \vdots \\ \boldsymbol{\alpha}_m^T \end{pmatrix}$$

then we can get that matrix $\boldsymbol{B}$

$$\boldsymbol{B} = \begin{pmatrix} a_{m1} & \cdots & a_{21} & a_{1n} \\ a_{m2} & \cdots & a_{22} & a_{12} \\ \vdots & \vdots & \ddots \vdots & \\ a_{mn} & \cdots & a_{2n} & a_{1n} \end{pmatrix} = (\boldsymbol{\alpha}_m, \cdots, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1)$$

that is

$$\boldsymbol{B} = \boldsymbol{A}^T \begin{pmatrix} 0 & 0 & \cdots 1 \\ \vdots & \vdots & \ddots \vdots \\ 0 & 1 & \cdots 0 \\ 1 & 0 & \cdots 0 \end{pmatrix} = \boldsymbol{A}^T \boldsymbol{P}$$

it is clear that $\boldsymbol{P}$ is a othogonomal matrix since $\boldsymbol{P}\boldsymbol{P}^T = \boldsymbol{P}^T\boldsymbol{P} = \boldsymbol{I}_m$, then

$$\boldsymbol{B}\boldsymbol{B}^T = \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{P}^T\boldsymbol{A} = \boldsymbol{A}^T\boldsymbol{A},$$

which means that $\boldsymbol{B}$ and $\boldsymbol{A}$ have that same sigular values.

2.3 Exercise 4.3

See matlab code.

2.4 Exercise 4.4

If $\boldsymbol{A}, \boldsymbol{B}$ are unitary quivalent, we can get that $\boldsymbol{A}, \boldsymbol{B}$ have the same singular values by the same argument in 4.2. It is clear that $\boldsymbol{I}, -\boldsymbol{I}$ have the same singular values, but $\boldsymbol{I}$ and $-\boldsymbol{I}$ can't be unitary equivalent since $\boldsymbol{I} \neq \boldsymbol{Q}\left(-\boldsymbol{I}\right)\boldsymbol{Q}^\star = -\boldsymbol{I}$.

2.5 Exercise 4.5

If $\boldsymbol{A}$ is a real matrix, then by Theorem 1.1, $\boldsymbol{A}^\star \boldsymbol{A}$ is a real symmetric matrix and $\boldsymbol{A}^\star \boldsymbol{A}$ has a real eigen decomposition

$$\boldsymbol{A}^\star \boldsymbol{A} = \boldsymbol{V} \boldsymbol{\Lambda} \boldsymbol{V}^\star$$

where Λ is a diagonal matrix with its entries are eigenvalues of $\boldsymbol{A}^\star \boldsymbol{A}$, then we can get $\boldsymbol{U}$ by $\boldsymbol{A}\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$, we get the real SVD of $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{U} \boldsymbol{\Sigma} \boldsymbol{V}^\star.$$