

Lecture 4 The Singular Value Decomposition

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April 20, 2023

1 Prerequisite

Lemma 1.1. *Given symmetric matrix \mathbf{A} , then the eigenvalues of \mathbf{A} are real.*

Theorem 1.1. *Given symmetric matrix \mathbf{A} , then \mathbf{A} can be factored as*

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^*,$$

where

- \mathbf{Q} is unitary;
- $\mathbf{\Lambda}$ is diagonal, with the eigenvalues of \mathbf{A} on its diagonal.

Proof. By induction of the dimension of \mathbf{A} . □

2 Solutions

2.1 Exercise 4.1(e)

First we compute the singular values σ_i by finding the eigenvalues of $\mathbf{A}^*\mathbf{A}$:

$$\mathbf{A}^*\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

the characteristic polynomial of $\mathbf{A}^*\mathbf{A}$ is

$$\det(\mathbf{A}^*\mathbf{A} - \lambda\mathbf{I}) = \lambda(\lambda - 4) = 0,$$

so the singular values are $\sigma_1 = 0, \sigma_2 = 2$. For $\lambda = 4$, we have

$$\mathbf{A}^*\mathbf{A} - 4\mathbf{I} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix},$$

a unit vector in the kernel of the matrix is $\mathbf{v}_2 = (1/\sqrt{2}, 1/\sqrt{2})^T$. For $\lambda = 0$, we have

$$\mathbf{A}^*\mathbf{A} - 0\mathbf{I} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

a unit vector in the kernel of the matrix is $\mathbf{v}_2 = (-1/\sqrt{2}, 1/\sqrt{2})^T$. So at this point we know that

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^* = (\mathbf{u}_1, \mathbf{u}_2) \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Finally we can compute \mathbf{u}_1 by the formula $\sigma_i \mathbf{u}_i = \mathbf{A} \mathbf{v}_i$, this gives $\mathbf{u}_i = (\sqrt{2}/2, \sqrt{2}/2)$, then by $\mathbf{u}_2^* \mathbf{u}_1 = 0$ and $\|\mathbf{u}_2\|_2 = 1$ we can get a $\mathbf{u}_2 = (-\sqrt{2}/2, \sqrt{2}/2)$. So in this full glory the SVD is

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^* = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*.$$

2.2 Exercise 4.2

Assume that

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \alpha_1^T \\ \alpha_1^T \\ \vdots \\ \alpha_m^T \end{pmatrix}$$

then we can get that matrix \mathbf{B}

$$\mathbf{B} = \begin{pmatrix} a_{m1} & \cdots & a_{21} & a_{1n} \\ a_{m2} & \cdots & a_{22} & a_{12} \\ \vdots & \vdots & \ddots & \vdots \\ a_{mn} & \cdots & a_{2n} & a_{1n} \end{pmatrix} = (\alpha_m, \cdots, \alpha_2, \alpha_1)$$

that is

$$\mathbf{B} = \mathbf{A}^T \begin{pmatrix} 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} = \mathbf{A}^T \mathbf{P}$$

it is clear that \mathbf{P} is a orthogonal matrix since $\mathbf{P}\mathbf{P}^T = \mathbf{P}^T\mathbf{P} = \mathbf{I}_m$, then

$$\mathbf{B}\mathbf{B}^T = \mathbf{A}^T \mathbf{P}\mathbf{P}^T \mathbf{A} = \mathbf{A}^T \mathbf{A},$$

which means that \mathbf{B} and \mathbf{A} have that same singular values.

2.3 Exercise 4.3

```
1 clc; clear;
2
3 A = [1 2; 0 2];
4
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5  [U,S,V] = svd(A);
6
7  [m, n] = size(A); % U:m*m, V:n*n
8
9  ss = diag(S);
10
11 v_end_points_xsub = V(1, :);
12 v_end_points_ysub = V(2, :);
13
14 u_end_points_xsub = U(1, :);
15 u_end_points_ysub = U(2, :);
16
17 % Plot a circle with radius = 1
18 ths = linspace(0, 2*pi, 100);
19 x = cos(ths);
20 y = sin(ths);
21 plot(x, y);
22 axis equal;
23 hold on;
24
25 % Plot the ellipse transformed by A
26 xy = [x;y];
27 transformed_xy = A*xy;
28 transformed_x = transformed_xy(1,:);
29 transformed_y = transformed_xy(2,:);
30 plot(transformed_x, transformed_y);
31 hold on;
32
33
34
35 % Plot the right singular vector v1, v2
36 quiver(0,0,v_end_points_xsub(1),v_end_points_ysub(1),
37        "AutoScale","off");
38 hold on;
39 quiver(0,0,v_end_points_xsub(2),v_end_points_ysub(2),
40        "AutoScale","off");
41
42 % Plot the left singular vector u1,u2 scaled by the
43 % singular vales
44 quiver(0,0,ss(1)*u_end_points_xsub(1),ss(1)*
45        u_end_points_ysub(1), "AutoScale","off");
46 hold on;
47 quiver(0,0,ss(2)*u_end_points_xsub(2),ss(2)*
48        u_end_points_ysub(2), "AutoScale","off");

```

2.4 Exercise 4.4

If \mathbf{A}, \mathbf{B} are unitary equivalent, we can use the same argument as in Exercise 4.2 to show that they have the same singular values. It is evident that the matrices \mathbf{I} and $-\mathbf{I}$ have the same singular values. However, \mathbf{I} and $-\mathbf{I}$ can not be unitary equivalent because \mathbf{I} is not equal to $\mathbf{Q}(-\mathbf{I})\mathbf{Q}^*$, where \mathbf{Q} is unitary matrix, and thus, the latter is equal to $-\mathbf{I}$.

2.5 Exercise 4.5

By [Theorem 1.1](#), if \mathbf{A} is a real matrix, then $\mathbf{A}^*\mathbf{A}$ is a real symmetric matrix with a real eigen decomposition:

$$\mathbf{A}^*\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^*,$$

where $\mathbf{\Lambda}$ is a diagonal matrix whose entries are the eigenvalues of $\mathbf{A}^*\mathbf{A}$. We can obtain \mathbf{U} by solving $\mathbf{A}\mathbf{v}_i = \sigma_i\mathbf{u}_i$, which gives us the real singular value decomposition(SVD) of \mathbf{A} :

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*,$$

where $\mathbf{\Sigma}$ is a diagonal matrix whose entries are the singular values of \mathbf{A} .