Lecture 4 The Singular Value Decomposition

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1 Prerequisite

Lemma 1.1. *Given summetrix matrix $\boldsymbol{A}$, then the eigenvalues of $\boldsymbol{A}$ are real.*

Theorem 1.1. *Given symmetric matrix $\boldsymbol{A}$, then $\boldsymbol{A}$ can be factored as*

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^{\star},$$

where

- $\boldsymbol{Q}$ *is unitary;*
- Λ *is diagonal, with the eigenvalues of* $\boldsymbol{A}$ *on its dagonal.*

Proof. By induction of the dimension of $\boldsymbol{A}$. □

2 Solutions

2.1 Exercise 4.1(e)

First we compute the singular values σ_i by finding the eigenvalues of $\boldsymbol{A}^{\star}\boldsymbol{A}$:

$$\boldsymbol{A}^{\star}\boldsymbol{A} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

the characteristic polynomial of $\boldsymbol{A}^{\star}\boldsymbol{A}$ is

$$\det(\boldsymbol{A}^{\star}\boldsymbol{A} - \lambda\boldsymbol{I}) = \lambda(\lambda - 4) = 0,$$

so the singular values are $\sigma_1 = 0, \sigma_2 = 2$. For $\lambda = 4$, we have

$$\boldsymbol{A}^{\star}\boldsymbol{A} - 4\boldsymbol{I} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix},$$

a unit vector in the kernel of the matrix is $\boldsymbol{v}_2 = \left(1/\sqrt{2}, 1/\sqrt{2}\right)^T$. For $\lambda = 0$, we have

$$\boldsymbol{A}^{\star}\boldsymbol{A} - 0\boldsymbol{I} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

a unit vector in the kernel of the matrix is $\boldsymbol{v}_2 = \left(-1/\sqrt{2}, 1/\sqrt{2}\right)^T$. So at this point we know that

$$\boldsymbol{A} = \boldsymbol{U\Sigma V}^\star = (\boldsymbol{u}_1, \boldsymbol{u}_2)\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Finally we can compute $\boldsymbol{u}_1$ by the formula $\sigma_i \boldsymbol{u}_i = \boldsymbol{A}\boldsymbol{v}_i$, this gives $\boldsymbol{u}_i = \left(\sqrt{2}/2, \sqrt{2}/2\right)$, then by $\boldsymbol{u}_2^\star \boldsymbol{u}_1 = 0$ and$\|\boldsymbol{u}_2\|_2 = 1$ we can get a $\boldsymbol{u}_2 = \left(-\sqrt{2}/2, \sqrt{2}/2\right)$. So in this full glory the SVD is

$$\boldsymbol{A} = \boldsymbol{U\Sigma V}^\star = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \boldsymbol{U\Sigma V}^\star.$$

2.2 Exercise 4.2

Assume that

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots a_{2n} \\ \vdots & \vdots & \ddots \vdots \\ a_{m1} & a_{m2} & \cdots a_{mn} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}_1^T \\ \boldsymbol{\alpha}_1^T \\ \vdots \\ \boldsymbol{\alpha}_m^T \end{pmatrix}$$

then we can get that matrix $\boldsymbol{B}$

$$\boldsymbol{B} = \begin{pmatrix} a_{m1} & \cdots & a_{21} & a_{1n} \\ a_{m2} & \cdots & a_{22} & a_{12} \\ \vdots & \vdots & \ddots \vdots & \\ a_{mn} & \cdots & a_{2n} & a_{1n} \end{pmatrix} = (\boldsymbol{\alpha}_m, \cdots, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1)$$

that is

$$\boldsymbol{B} = \boldsymbol{A}^T \begin{pmatrix} 0 & 0 & \cdots 1 \\ \vdots & \vdots & \ddots \vdots \\ 0 & 1 & \cdots 0 \\ 1 & 0 & \cdots 0 \end{pmatrix} = \boldsymbol{A}^T \boldsymbol{P}$$

it is clear that $\boldsymbol{P}$ is a othogonomal matrix since $\boldsymbol{P}\boldsymbol{P}^T = \boldsymbol{P}^T\boldsymbol{P} = \boldsymbol{I}_m$, then

$$\boldsymbol{B}\boldsymbol{B}^T = \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{P}^T\boldsymbol{A} = \boldsymbol{A}^T\boldsymbol{A},$$

which means that $\boldsymbol{B}$ and $\boldsymbol{A}$ have that same sigular values.

2.3 Exercise 4.3

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clc; clear;

A = [1 2; 0 2];

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[U,S,V] = svd(A);

[m, n] = size(A); % U:m*m, V:n*n

ss = diag(S);

v_end_points_xsub = V(1, :);
v_end_points_ysub = V(2, :);

u_end_points_xsub = U(1, :);
u_end_points_ysub = U(2, :);

% Plot a circle with radius = 1
ths = linspace(0, 2*pi, 100);
x = cos(ths);
y = sin(ths);
plot(x, y);
axis equal;
hold on;

% Plot the ellipse transformed by A
xy = [x;y];
transformed_xy = A*xy;
transformed_x = transformed_xy(1,:);
transformed_y = transformed_xy(2,:);
plot(transformed_x, transformed_y);
hold on;



% Plot the right singular vector v1, v2
quiver(0,0,v_end_points_xsub(1),v_end_points_ysub(1), "AutoScale","off");
hold on;
quiver(0,0,v_end_points_xsub(2),v_end_points_ysub(2), "AutoScale","off");

% Plot the left singular vector u1,u2 scaled by the singular vales
quiver(0,0,ss(1)*u_end_points_xsub(1),ss(1)*u_end_points_ysub(1), "AutoScale","off");
hold on;
quiver(0,0,ss(2)*u_end_points_xsub(2),ss(2)*u_end_points_ysub(2), "AutoScale","off");
```

2.4 Exercise 4.4

If $\boldsymbol{A}, \boldsymbol{B}$ are unitary quivalent, we can use the same argument as in Exercise 4.2 to show that they have the same singular values. It is evident that the matrices $\boldsymbol{I}$ and $-\boldsymbol{I}$ have the same singular values. However, $\boldsymbol{I}$ and $-\boldsymbol{I}$ can not be unitary equivalent because $\boldsymbol{I}$ is not equal to $\boldsymbol{Q}\,(-\boldsymbol{I})\,\boldsymbol{Q}^\star$, where $\boldsymbol{Q}$ is unitary matrix, and thus, the latter is equal to $-\boldsymbol{I}$.

2.5 Exercise 4.5

By Theorem 1.1, if $\boldsymbol{A}$ is a real matrix, then $\boldsymbol{A}^\star\boldsymbol{A}$ is a real symmetric matrix with a real eigen decomposition:

$$\boldsymbol{A}^\star\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^\star,$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix whose entries are the eigenvalues of $\boldsymbol{A}^\star\boldsymbol{A}$. We can obtain $\boldsymbol{U}$ by solving $\boldsymbol{A}\boldsymbol{v}_i = \sigma_i\boldsymbol{u}_i$, which gives us the real singular value decomposition(SVD) of $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\star,$$

where $\boldsymbol{\Sigma}$ is a diagonal matrix whose entries are the singular values of $\boldsymbol{A}$.