Lecture 2 Orthogonal Vectors and Matrices

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1 Prerequisite

Before giving the solutions, I would like to prove some basic conclusions about this lecture

Lemma 1.1. *Given an non-singular matrix $\boldsymbol{A}$, then $\boldsymbol{A}^{-1}$ is unique*

Proof. Suppose that we have two inverse matrices $\boldsymbol{C}$ and $\boldsymbol{B}$ w.r.t $\boldsymbol{A}$. By the definition of inverse.

$$\boldsymbol{B} = \boldsymbol{B}\boldsymbol{I} = \boldsymbol{B}(\boldsymbol{A}\boldsymbol{C}) = (\boldsymbol{B}\boldsymbol{A})\boldsymbol{C} = \boldsymbol{I}\boldsymbol{C} = \boldsymbol{C},$$

and hence we can conclude that $\boldsymbol{B} = \boldsymbol{C}$. $\square$

Lemma 1.2. *Given an non-singular matrix $\boldsymbol{A} \in \mathbb{R}^{m\times m}$, its hermitian conjugate $\boldsymbol{A}^*$ is also non-singular.*

Proof.

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I},$$

We can apply the hermitian conjugate to both sides of the equation above:

$$\boldsymbol{A}^*(\boldsymbol{A}^{-1})^* = (\boldsymbol{A}^{-1})^*\boldsymbol{A}^* = \boldsymbol{I}$$

Hence we can get that $\boldsymbol{A}^*$ is non-singular. $\square$

Lemma 1.3. *Give a non-singular matrix $\boldsymbol{A}$ and its hermitian conjugate $\boldsymbol{A}^*$, we have*

$$(\boldsymbol{A}^*)^{-1} = (\boldsymbol{A}^{-1})^*$$

Proof. By lemma 1.2, $\boldsymbol{A}^*$ is non-singular and it's clear that the inverse is $(\boldsymbol{A}^{-1})^*$. However, we can get that $(\boldsymbol{A}^*)^{-1}$ is also the inverse of $\boldsymbol{A}^*$ by definition. Futher, by lemma 1.1, we have

$$(\boldsymbol{A}^*)^{-1} = (\boldsymbol{A}^{-1})^*$$

which is exactly what we need to prove. $\square$

Lemma 1.4. *Given two pure imaginary number di, bi, then*

$$(1 - di)(1 - bi) \neq 0.$$

Proof. LHS equals

$$1 - bd - (b + d)i, \tag{$\star$}$$

if $(\star) = 0$, then we have

$$\begin{aligned} 1 - bd &= 0 \\ b + d &= 0, \end{aligned}$$

which means that

$$-b^2 = 1,$$

since $b \in \mathbb{R}$, the equation above cannot be true, and hence $(1-di)(1-bi) \neq 0$. $\square$

2 Solutions

2.1 Exercise 2.1

Proof. Without loss of generality, we assume that $\boldsymbol{A}$ is upper-triangular. By the *ex. 1.3*, we can conclude that $\boldsymbol{A}^{-1}$ is also upper-triangular. It is clear that $\boldsymbol{A}^{*} = \boldsymbol{A}^{-1}$ since $\boldsymbol{A}$ is unitary. Then $\boldsymbol{A}^{\star}$ is also an upper-triangular matrix, which is

$$\boldsymbol{A}^{\star}_{i,j} = \bar{a}_{ji} = a_{ij} = 0, \quad (\forall i > j),$$

Hence, the matrix $\boldsymbol{A}$ is diagonal. The same follows if $\boldsymbol{A}$ is lower-triangular. $\square$

2.2 Exercise 2.3(a)

Let $\boldsymbol{x}$ be an eigenvector of matrix $\boldsymbol{A}$ w.r.t. the eigenvalue λ, then

$$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x},$$

multiplying both sides by $\boldsymbol{x}^{\star}$, we get that

$$\boldsymbol{x}^{\star}\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}^{\star}\boldsymbol{x} = \lambda\|\boldsymbol{x}\|^2, \tag{$\spadesuit$}$$

then

$$\lambda = \frac{\boldsymbol{x}^{\star}\boldsymbol{A}\boldsymbol{x}}{\|\boldsymbol{x}\|^2} = \frac{\boldsymbol{x}^{\star}\boldsymbol{A}^{\star}\boldsymbol{x}}{\|\boldsymbol{x}\|^2} = \frac{(\boldsymbol{x}^{\star}\boldsymbol{A}\boldsymbol{x})^{\star}}{\|\boldsymbol{x}\|^2} = \bar{\lambda},$$

which means that λ is real.

2.3 Exercise 2.3(b)

Let $\boldsymbol{x}_1, \boldsymbol{x}_2$ be two eigenvectors of the hermitian matrix $\boldsymbol{A}$. Denote λ_k the eigenvalue w.r.t $\boldsymbol{x}_k (k = 1, 2)$, where $\lambda_1 \neq \lambda_2$, then

$$\begin{aligned} \lambda_2 \boldsymbol{x}_1^\star \boldsymbol{x}_{\boldsymbol{2}} &= \boldsymbol{x}_1^\star \boldsymbol{A} \boldsymbol{x}_2, \\ \lambda_1 \boldsymbol{x}_2^\star \boldsymbol{x}_{\boldsymbol{1}} &= \boldsymbol{x}_2^\star \boldsymbol{A} \boldsymbol{x}_1. \end{aligned}$$

Note that $\boldsymbol{A}$ is hermitian, we can get that

$$\lambda_2 \boldsymbol{x}_1^\star \boldsymbol{x}_2 = \boldsymbol{x}_1^\star \boldsymbol{A} \boldsymbol{x}_2 = \boldsymbol{x}_1^\star \boldsymbol{A}^\star \boldsymbol{x}_2 = (\boldsymbol{x}_2^\star \boldsymbol{A} \boldsymbol{x}_1)^\star = \lambda_1^\star \boldsymbol{x}_1^\star \boldsymbol{x}_2,$$

then

$$(\lambda_2 - \lambda_1^\star) \boldsymbol{x}_1^\star \boldsymbol{x}_2 = 0 \Rightarrow \boldsymbol{x}_1^\star \boldsymbol{x}_2 = 0,$$

which is exactly what we need to prove.

2.4 Exercise 2.4

Let λ be an eigenvalue of $\boldsymbol{A}$, and $\boldsymbol{x}$ be the eigenvector w.r.t λ, then we have $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$ and $\|\boldsymbol{A}\boldsymbol{x}\|_2^2 = \|\lambda \boldsymbol{x}\|$, which is

$$\boldsymbol{x}^\star \boldsymbol{A}^\star \boldsymbol{A} \boldsymbol{x} = \boldsymbol{x}^\star \|\lambda^\star \lambda\| \boldsymbol{x}.$$

Since $\boldsymbol{A}$ is unitary, then

$$\boldsymbol{x}^\star \boldsymbol{A}^\star \boldsymbol{A} \boldsymbol{x} = \boldsymbol{x}^\star \boldsymbol{I} \boldsymbol{x} = \boldsymbol{x}^\star \|\lambda\|_2^2 \boldsymbol{x}.$$

Furthermore,

$$x_1^2 + x_2^2 + \cdots + x_n^2 = \|\lambda\|_2^2 (x_1^2 + x_2^2 + \cdots + x_n^2),$$

it follows that $\|\lambda\|_2^2 = 1$ since $\boldsymbol{x}$ is non-zero vector.

2.5 Exercise 2.5(a)

Let $\boldsymbol{x}$ be an eigenvector of matrix $\boldsymbol{S}$ w.r.t. the eigenvalue λ, then

$$\boldsymbol{S}\boldsymbol{x} = \lambda \boldsymbol{x}.$$

By the equation *ex.* 2.3(♠), we have

$$\lambda = \frac{\boldsymbol{x}^\star \boldsymbol{S} \boldsymbol{x}}{\|\boldsymbol{x}\|^2} = \frac{\boldsymbol{x}^\star (-\boldsymbol{S}^\star) \boldsymbol{x}}{\|\boldsymbol{x}\|^2} = \frac{-(\boldsymbol{x}^\star \boldsymbol{S} \boldsymbol{x})^\star}{\|\boldsymbol{x}\|^2} = -\bar{\lambda},$$

then we can get that $\lambda + \bar{\lambda} = 0$, which means that λ is purely imaginary.

2.6 Exercise 2.5(b)

Assume that λ is the eigenvalue of $\boldsymbol{S}$, it follows that $1-\lambda$ is the eigenvalue of $1-\boldsymbol{S}$. Since λ is purely imagnary number, then by *Lemma 2.1.4*, we have

$$\det(1-\boldsymbol{S}) = \prod_{i=1}^{n}(1-\lambda_i) \neq 0,$$

where $\lambda_i, i \in \{1, 2, \cdots, n\}$ are eigenvalues of $\boldsymbol{S}$. Hence we can conclude that $1-\boldsymbol{S}$ is non-singular.

2.7 Exercise 2.5(c)

Assume that $\boldsymbol{Q} = (\boldsymbol{I}-\boldsymbol{S})^{-1}(\boldsymbol{I}+\boldsymbol{S})$, then we have

$$\boldsymbol{Q}\boldsymbol{Q}^{\star} = (\boldsymbol{I}-\boldsymbol{S})^{-1}(\boldsymbol{I}+\boldsymbol{S})(\boldsymbol{I}+\boldsymbol{S}^{\star})\left((\boldsymbol{I}+\boldsymbol{S})^{-1}\right)^{\star},$$

by *Lemma 2.1.3*, we can get that

$$\begin{aligned}
\boldsymbol{Q}\boldsymbol{Q}^{\star} &= (\boldsymbol{I}-\boldsymbol{S})^{-1}(\boldsymbol{I}+\boldsymbol{S})(\boldsymbol{I}-\boldsymbol{S})\left((\boldsymbol{I}-\boldsymbol{S})^{-1}\right)^{\star} \\
&= (\boldsymbol{I}-\boldsymbol{S})^{-1}(\boldsymbol{I}+\boldsymbol{S})(\boldsymbol{I}-\boldsymbol{S})(\boldsymbol{I}+\boldsymbol{S})^{-1} \\
&= (\boldsymbol{I}-\boldsymbol{S})^{-1}(\boldsymbol{I}-\boldsymbol{S})(\boldsymbol{I}+\boldsymbol{S})(\boldsymbol{I}+\boldsymbol{S})^{-1} \\
&= \boldsymbol{I}.
\end{aligned}$$

Hence, we can conclude that $(1-\boldsymbol{S})^{-1}(1-\boldsymbol{S})$ is unitary.

2.8 Exercise 2.6

Proof. If $\boldsymbol{A}$ is singular, there exists a vector $\boldsymbol{x} \in \mathbb{C} \setminus \{0\}$ such that

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x} + \boldsymbol{u}\boldsymbol{v}^{\star}\boldsymbol{x} = 0,$$

then $\boldsymbol{x} = -\boldsymbol{u}(\boldsymbol{v}^{\star}\boldsymbol{x})$ where $\boldsymbol{v}^{\star}\boldsymbol{u}$ is scalar. Let $\boldsymbol{x} = t\boldsymbol{u} (t \in \mathbb{R})$, then we can get that

$$t\boldsymbol{u} + \boldsymbol{u}(\boldsymbol{v}^{\star}t\boldsymbol{u}) = t\boldsymbol{u}(1 + \boldsymbol{v}^{\star}\boldsymbol{u}) = 0,$$

It follows that $\boldsymbol{v}^{\star}\boldsymbol{u} = -1$ since $\boldsymbol{x} = t\boldsymbol{u} \neq 0$. Assume that $\alpha = -1/(1+\boldsymbol{v}^{\star}\boldsymbol{u})$, then

$$(\boldsymbol{I} + \boldsymbol{u}\boldsymbol{v}^{\star})(\boldsymbol{I} + \alpha\boldsymbol{u}\boldsymbol{v}^{\star}) = \boldsymbol{I}.$$

Note that we have shown that $\boldsymbol{v}^{\star}\boldsymbol{u} = -1$ is a necessary condition of $\boldsymbol{A}$ is singular. For suffciency, we assume that $\boldsymbol{v}^{\star}\boldsymbol{u} = -1$. Then for any $t \in \mathbb{C} \setminus \{0\}$, we have

$$\boldsymbol{A}\boldsymbol{u} = t\boldsymbol{u} + \boldsymbol{u}\boldsymbol{v}^{\star}t\boldsymbol{u} = t\boldsymbol{u} + t\boldsymbol{u}(\boldsymbol{v}^{\star}\boldsymbol{u}) = 0,$$

which implies that $\boldsymbol{A}$ is singular. Combined, we conclude that $\boldsymbol{A}$ is singular iff. $\boldsymbol{v}^{\star}\boldsymbol{u} = -1$. In this case,

$$\text{null}(\boldsymbol{A}) = \{t\boldsymbol{u}, t \in \mathbb{R}\},$$

the linear subspace spanned by $\boldsymbol{u}$. □

2.9 Exercise 2.7

Proof. We can verify that $\boldsymbol{H}_{k+1}$ is Hadamard matrix directly,

$$\begin{aligned}
\boldsymbol{H}_{k+1}^T \boldsymbol{H}_{k+1} &= \begin{pmatrix} \boldsymbol{H}_k^T & \boldsymbol{H}_k^T \\ \boldsymbol{H}_k^T & -\boldsymbol{H}_k^T \end{pmatrix} \begin{pmatrix} \boldsymbol{H}_k & \boldsymbol{H}_k \\ \boldsymbol{H}_k & -\boldsymbol{H}_k \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{H}_k^T \boldsymbol{H}_k + \boldsymbol{H}_k^T \boldsymbol{H}_k & \boldsymbol{H}_k^T \boldsymbol{H}_k - \boldsymbol{H}_k^T \boldsymbol{H}_k \\ \boldsymbol{H}_k^T \boldsymbol{H}_k - \boldsymbol{H}_k^T \boldsymbol{H}_k & \boldsymbol{H}_k^T \boldsymbol{H}_k + \boldsymbol{H}_k^T \boldsymbol{H}_k \end{pmatrix} \\
&= \begin{pmatrix} 2\boldsymbol{H}_k^T \boldsymbol{H}_k & \boldsymbol{0} \\ \boldsymbol{0} & 2\boldsymbol{H}_k^T \boldsymbol{H}_k \end{pmatrix} \\
&= \begin{pmatrix} 2c\boldsymbol{I}_k & \boldsymbol{0} \\ \boldsymbol{0} & 2c\boldsymbol{I}_k \end{pmatrix} \\
&= 2c \cdot \boldsymbol{I}_{2k}.
\end{aligned}$$

Then we can get that $\boldsymbol{H}_{k+1}^T = 2c\boldsymbol{H}_{k+1}^{-1}$. Note that the entries of $\boldsymbol{H}_{k+1}$ are also all ± 1 by the recursion formula. Hence $\boldsymbol{H}_{k+1}$ is also a Hadamard matrix. □