Solutions

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1 Lecture 1

1.1 (a) $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{B} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b)

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

1.2 (a)

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} -k_{12} & k_{12} & 0 & 0 \\ -k_{12} & k_{12}+k_{23} & -k_{23} & 0 \\ 0 & -k_{23} & k_{23}+k_{34} & -k_{34} \\ 0 & 0 & -k_{34} & k_{34} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} - \begin{bmatrix} -k_{12}l_{12} \\ k_{23}l_{23} - k_{12}l_{12} \\ -k_{23}l_{23} + k_{34}l_{34} \\ -k_{34}l_{34} \end{bmatrix}$$

(b) the dimension of the entry of $\mathbf{K}$ is $\dfrac{N}{m}$ (or $\dfrac{kg}{sec^2}$)

(c) $[\dfrac{kg}{sec^2}]^4$

(d) $\mathbf{K} = 1000\mathbf{K}^{'}$, $\det(\mathbf{K}) = 10^{12}\det(\mathbf{K}^{'})$

1.3 *Proof.* Let

$$\mathbf{R} = \begin{bmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_{mm} \end{bmatrix}$$

by $\mathbf{I} = \mathbf{R}^{-1}\mathbf{R} = \mathbf{R}\mathbf{R}^{-1}$, we have

$$\mathbf{I}_{m\times m} = [e_1, e_2, \cdots, e_m] = [a_1, a_2, \cdots, a_n] \begin{bmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_{mm} \end{bmatrix}$$

Since $\mathbf{R}$ is non-singular and $\det(\mathbf{R}) = \prod_{i=1}^{m} r_{ii}$, we conclude that $r_{ii} \neq 0$ $(\forall\, 1 \leq i \leq m)$. To show $\mathbf{R}^{-1}$ is upper-triangular, we work by induction. To begin with, we have $\mathbf{e}_1 = r_{11}\mathbf{a}_1$ and hence $\mathbf{a_1} = r_{11}^{-1}\mathbf{e}_1$ has zero entries except the first one. For convenience, we denote by $\mathbb{C}^m(k)$ the column space

$$\{\mathbf{v} = (v_1, \cdots, v_m)^\mathsf{T} \in \mathbb{C}^m : v_i = 0 \text{ for } i > k\}$$

Then

$$\mathbb{C}^m(1) \subset \mathbb{C}^m(2) \cdots \mathbb{C}^m(m) = \mathbb{C}^m$$

we have shown $a_1 \in \mathbb{C}^m(1)$. Assume for any $k \leq s \to \mathbf{a}_k \in \mathbb{C}^m(k)$. Then by (1.8)

$$\mathbf{e}_{s+1} = \sum_{k=1}^{m} \mathbf{a_k} r_{k,s+1}$$

Note that $r_{k,i+1} = 0 \quad (k > i+1)$, then

$$\sum_{k=1}^{m} \mathbf{a}_k r_{k,s+1} = \sum_{k=1}^{s} \mathbf{a}_k r_{k,s+1} + \mathbf{a}_{s+1} r_{s+1,s+1}$$

Therefore

$$a_{s+1} = r_{s+1,s+1}^{-1}(\mathbf{e}_{s+1} - \sum_{k=1}^{s} \mathbf{a}_k r_{k,s+1}) \in \mathbb{C}^m(s+1)$$

By induction, we have proved that $\mathbf{a}_k \in \mathbb{C}^m(k)$ for $1 \leq k \leq m$, which is equivalent to $\mathbf{R}^{-1}$ being upper-triangular. □

1.4 (a)

Proof. Denote the column vectors $(c_1, \cdots, c_n)^\mathsf{T}$, $(d_1, \cdots, d_n)^\mathsf{T}$ by notations $\mathbf{c}$ and $\mathbf{d}$, let $\mathbf{F}$ be the matrix whose $i - j$ entry is $f_j(i)$. Then, the given condition can be rephrased as:

$$\forall\ \mathbf{d} \in \mathbb{C}^8, \quad \exists\ \mathbf{c} \in \mathbb{C}^8 \xrightarrow{s.t} \mathbf{Fc} = \mathbf{d}$$

This means $range\{\mathbf{F}\} = \mathbb{C}^8$, which implies $\mathbf{F}$ has full rank by theorem 1.3. Furthermore, $\mathbf{F}$ is invertible. Therefore

$$\mathbf{c} = \mathbf{F}^{-1}\mathbf{d}$$

and hence d determines c uniquely.

□

(b)

$$\mathbf{Ad} = \mathbf{c} \to \mathbf{d} = \mathbf{A}^{-1}\mathbf{c} \Longrightarrow \mathbf{A}^{-1} = \mathbf{F} \to \mathbf{A}_{ij}^{-1} = \mathbf{F}_{ij} = f_j(i)$$

2 Lecture 2

Before giving the solutions, I would like to prove some basic conclusions about this lecture

Lemma 2.1. *The Inverse of a matrix* $\mathbf{A}$ *is unique.*

Proof. Suppose that we have two invertible matrices $\mathbf{C}$ and $\mathbf{B}$, and show that $\mathbf{C} = \mathbf{B}$

$$\mathbf{B} = \mathbf{BI} = \mathbf{B}(\mathbf{AC}) = (\mathbf{BA})\mathbf{C} = \mathbf{IC} = \mathbf{C}.$$

□

Lemma 2.2. *If a* $m \times m$ *matrix* $\mathbf{A}$ *is invertible, its hermitian conjugate* $\mathbf{A}^*$ *is also invertible*

Proof.

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_{m\times m}$$

We can apply hermitian conjugate to both sides of the equation

$$\mathbf{A}^*(\mathbf{A}^{-1})^* = (\mathbf{A}^{-1})^*\mathbf{A}^* = \mathbf{I}_{m\times m}$$

Hence we can get that $\mathbf{A}^*$ is invertible. □

Property 1. *Give a invertible matrix* $\mathbf{A}$ *and its hermitian conjugate* $\mathbf{A}^*$*, we have*

$$(\mathbf{A}^*)^{-1} = (\mathbf{A}^{-1})^*$$

Proof. By lemma 2.2, $\mathbf{A}^*$ is invertible and its inverse has the form of $(\mathbf{A}^{-1})^*$. However, we can get that $(\mathbf{A}^*)^{-1}$ is also the inverse of $\mathbf{A}^*$ by definition. Futher, by lemma 2.1, we have

$$(\mathbf{A}^*)^{-1} = (\mathbf{A}^{-1})^*$$

This is exactly what we need to prove.

□

2.1 *Proof.* Assume that matrix $\mathbf{A}$ is upper-triangular. By the definition of unitary, $\mathbf{A}^* = \mathbf{A}^{-1}$. Since $\mathbf{A}$ is triangular, from ex 1.3 that $\mathbf{A}^{-1}$ is also upper-triangular. Moreover, since $\mathbf{A}$ is unitary, we know that $\mathbf{A}^*$ is upper-triangular as well. In order for $\mathbf{A}$ and its transpose to be upper-triangular. $\mathbf{A}$ must be diagonal. The same follows if $\mathbf{A}$ is lower-triangular □

2.2 (a)

$$||\mathbf{x}_1 + \mathbf{x}_2||^2 = \mathbf{x}_1^\mathsf{T}\mathbf{x}_1 + \mathbf{x}_2^\mathsf{T}\mathbf{x}_2 + 2\mathbf{x}_1^\mathsf{T}\mathbf{x}_2 = ||\mathbf{x}_1||^2 + ||\mathbf{x}_2||^2$$

(b)

Assume for any $k \leq m$, $||\sum_{i=1}^{k} \mathbf{x}_i||^2 = \sum_{i=1}^{k} ||\mathbf{x}_i||^2$. Then,

$$\begin{aligned}
||\sum_{i=1}^{m+1} \mathbf{x}_i||^2 &= ||\sum_{i=1}^{m} \mathbf{x}_i||^2 + ||\mathbf{x}_{m+1}||^2 \\
&= \sum_{i=1}^{m} ||\mathbf{x}_i||^2 + ||\mathbf{x}_{m+1}|| \\
&= \sum_{i=1}^{m} ||\mathbf{x}_i^2|| + ||\mathbf{x}_{m+1}||^2 \\
&= \sum_{i=1}^{m+1} ||\mathbf{x}_i||
\end{aligned}$$

By induction,

$$||\sum_{k=1}^{n} \mathbf{x}_i||^2 = \sum_{k=1}^{n} ||\mathbf{x}_i||^2$$

2.3 (a) By the definition of eigenvalue, we have

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

Multiplying both sides by $\mathbf{x}^*$, we get

$$\mathbf{x}^*\mathbf{A}\mathbf{x} = \lambda \mathbf{x}^*\mathbf{x} = \lambda ||\mathbf{x}||^2$$

In other words

$$\frac{\mathbf{x}^*\mathbf{A}\mathbf{x}}{||\mathbf{x}||^2} = \lambda$$

First, we need to know that $\mathbf{x}^*\mathbf{A}\mathbf{x}$ is real. We will start by showing $\mathbf{x}^*\mathbf{A}\mathbf{x}$ is Hermitian

$$\begin{aligned}
(\mathbf{x}^*\mathbf{A}\mathbf{x})^* &= (\mathbf{A}\mathbf{x})^*\mathbf{x} \\
&= \mathbf{x}^*\mathbf{A}^*\mathbf{x} \\
&= \mathbf{x}^*\mathbf{A}\mathbf{x}
\end{aligned}$$

Therefore it must have reals on the main diagonal. We know $||\mathbf{x}||^2$ is real by the definition of norm and hence the eigenvalue λ must be real value.

(b) If $\boldsymbol{x}, \boldsymbol{y}$ are eigenvectors corresponding to the different eigenvalues λ_1, λ_2 of hermitian matrix $\boldsymbol{A}$, we have

$$\begin{aligned}
&\boldsymbol{y}^\star \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}^\star \lambda_1 \boldsymbol{x} = \lambda_1 \boldsymbol{y}^\star \boldsymbol{x}. \\
&(\boldsymbol{A}\boldsymbol{y})^\star \boldsymbol{x} = (\lambda_2 \boldsymbol{y})^\star \boldsymbol{x} \Rightarrow \boldsymbol{y}^\star \boldsymbol{A}^\star \mathbf{x} = \lambda_2 \boldsymbol{y}^\star \boldsymbol{x}
\end{aligned}$$

Since $\boldsymbol{A}$ is hermitian,

$$\lambda_1 \boldsymbol{y}^\star \boldsymbol{x} = \boldsymbol{y}^\star \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}^\star \boldsymbol{A}^\star \boldsymbol{x} = \lambda_2 \boldsymbol{y}^\star \boldsymbol{x},$$

then, $(\lambda_1 - \lambda_2)\boldsymbol{y}^\star\boldsymbol{x} = 0$, where $(\lambda_1 - \lambda_2) \neq 0$, thereby $\boldsymbol{x}^\star\boldsymbol{y}$ could only be 0, which means that $\boldsymbol{x}, \boldsymbol{y}$ are orthogonal.

2.4 Conclusion: All eigenvalues of $\boldsymbol{A}$ have length 1.

Proof. Since $\boldsymbol{A}$ is unitary,

$$\boldsymbol{A}^\star\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^\star = \boldsymbol{I}.$$

□

2.5

2.6

2.7

Remark. *Here is another solution of ex 2.3(a)*

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$