

Lecture 6 Projectors

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November 15, 2023

1 Prerequisite

2 Solutions

2.1 Exercise 6.1

Proof. Since \mathbf{P} is an orthogonal projector, then we have $\mathbf{P}^2 = \mathbf{P}$ and $\mathbf{P}^* = \mathbf{P}$, hence

$$(\mathbf{I} - 2\mathbf{P})^*(\mathbf{I} - 2\mathbf{P}) = \mathbf{I} - 2\mathbf{P}^* - 2\mathbf{P} + 4\mathbf{P}^*\mathbf{P} = 0, \quad (1)$$

which means that $\mathbf{I} - 2\mathbf{P}$ is unitary. A geometric interpretation is given by Figure 1. \square

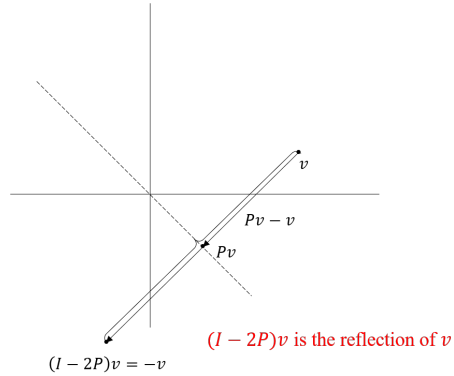


Figure 1: Geometric interpretation of $\mathbf{I} - 2\mathbf{P}$.

2.2 Exercise 6.2

We begin by considering the matrix \mathbf{F} defined by

$$\mathbf{F} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix},$$

where \mathbf{F} is a permutation matrix that reverses the order of the basis vectors. Consequently, we define

$$\mathbf{E} = \frac{\mathbf{I} + \mathbf{F}}{2} = \begin{pmatrix} \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdots & \frac{1}{2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \cdots & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

where \mathbf{I} is the identity matrix. Observing that \mathbf{F} is its own inverse, i.e., $\mathbf{F}^2 = \mathbf{I}$, and that the adjoint of a matrix is equal to its conjugate transpose, we deduce that the adjoint of \mathbf{E} , denoted \mathbf{E}^* , satisfies

$$\mathbf{E}^* = \left(\frac{\mathbf{I} + \mathbf{F}}{2} \right)^* = \mathbf{E},$$

since both \mathbf{I} and \mathbf{F} are real and symmetric, and thus equal to their own adjoints.

Next, we compute \mathbf{E}^2 as follows:

$$\mathbf{E}^2 = \left(\frac{\mathbf{I} + \mathbf{F}}{2} \right)^2 = \frac{\mathbf{I} + 2\mathbf{F} + \mathbf{F}^2}{4} = \frac{2\mathbf{I} + 2\mathbf{F}}{4} = \frac{\mathbf{I} + \mathbf{F}}{2} = \mathbf{E}.$$

Hence, $\mathbf{E}^2 = \mathbf{E}$, which implies that \mathbf{E} is idempotent. Additionally, since for any vector \mathbf{v} , the equality $\langle \mathbf{E}\mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{E}\mathbf{v} \rangle$ holds, where $\langle \cdot, \cdot \rangle$ denotes the inner product, it follows that \mathbf{E} is self-adjoint. Combining the idempotence and self-adjointness of \mathbf{E} , we conclude that \mathbf{E} is an orthogonal projector.

2.3 Exercise 5.3

Suppose that \mathbf{A} is full rank. This implies that \mathbf{A} has n non-zero singular values. Consequently, the matrix $\mathbf{A}^*\mathbf{A}$, where \mathbf{A}^* denotes the conjugate transpose of \mathbf{A} , has n non-zero eigenvalues $\lambda_1, \dots, \lambda_n$. The determinant of $\mathbf{A}^*\mathbf{A}$ is then given by the product of its eigenvalues:

$$\det(\mathbf{A}^*\mathbf{A}) = \prod_{i=1}^n \lambda_i \neq 0,$$

which indicates that $\mathbf{A}^* \mathbf{A}$ is non-singular.

For the “only if” part of the proof, we invoke the singular value decomposition (SVD). Since $\mathbf{A}^* \mathbf{A}$ is non-singular, it follows from Theorem 5.2 that

$$\text{range}(\mathbf{A}) = \langle \mathbf{u}_1, \dots, \mathbf{u}_n \rangle,$$

where the \mathbf{u}_i are left singular vectors of $\mathbf{A}^* \mathbf{A}$. This space is n -dimensional, which confirms that the matrix \mathbf{A} is indeed full rank.

2.4 Exercise 6.4 (a)

We first compute \mathbf{P} as

$$\mathbf{P} = \mathbf{A}(\mathbf{A}^* \mathbf{A})^{-1} \mathbf{A}^* = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

then

$$\mathbf{P} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

2.5 Exercise 6.4(b)

We first compute \mathbf{P} as

$$\mathbf{P} = \mathbf{B}(\mathbf{B}^* \mathbf{B})^{-1} \mathbf{B}^* = \begin{pmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{pmatrix}$$

then

$$\mathbf{P} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}.$$

2.6 Exercise 6.5

We first show that $\|\mathbf{P}\|_2 \geq 1$. Choose \mathbf{x} such that $\mathbf{P}\mathbf{x} = \mathbf{y} \neq \mathbf{0}$, then $\mathbf{P}\mathbf{y} = \mathbf{P}^2\mathbf{x} = \mathbf{P}\mathbf{x} = \mathbf{y}$, then we have

$$\|\mathbf{P}\|_2 = \max_{\mathbf{x}} \frac{\|\mathbf{P}\mathbf{x}\|_2}{\|\mathbf{x}\|_2} \geq \frac{\|\mathbf{P}\mathbf{y}\|_2}{\|\mathbf{y}\|_2} = 1.$$

Assume that \mathbf{P} is orthogonal projector, then we have that

$$\mathbf{P} = \hat{\mathbf{Q}}\hat{\mathbf{Q}}^*$$

where the columns of $\hat{\mathbf{Q}}$ are orthonormal. Hence we can get that

$$1 \leq \|\mathbf{P}\|_2 \leq \|\hat{\mathbf{Q}}\|_2 \|\hat{\mathbf{Q}}^*\|_2 \leq 1$$

which means that $\|\mathbf{P}\|_2 = 1$. If $\|\mathbf{P}\|_2 = 1$, then we have that

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