Lecture 6 Projectors

Fang Zhu

November 16, 2023

1 Prerequisite

2 Solutions

2.1 Exercise 6.1

Proof. Since $\boldsymbol{P}$ is an orthogonal projector, then we have $\boldsymbol{P}^2 = \boldsymbol{P}$ and $\boldsymbol{P}^\star = \boldsymbol{P}$, hence

$$(\boldsymbol{I} - 2\boldsymbol{P})^\star(\boldsymbol{I} - 2\boldsymbol{P}) = \boldsymbol{I} - 2\boldsymbol{P}^\star - 2\boldsymbol{P} + 4\boldsymbol{P}^\star\boldsymbol{P} = 0, \tag{1}$$

which means that $\boldsymbol{I} - 2\boldsymbol{P}$ is unitary. A geometric interpretation is given by Figure 1. □

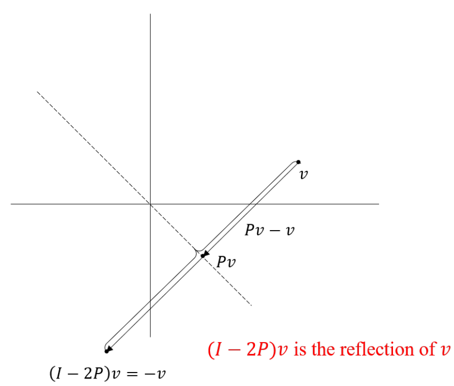

Figure 1: Geometric interpretation of $\boldsymbol{I} - 2\boldsymbol{P}$.

2.2 Exercise 6.2

We begin by considering the matrix $\boldsymbol{F}$ defined by

$$\boldsymbol{F} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix},$$

where $\boldsymbol{F}$ is a permutation matrix that reverses the order of the basis vectors. Consequently, we define

$$\boldsymbol{E} = \frac{\boldsymbol{I}+\boldsymbol{F}}{2} = \begin{pmatrix} \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdots & \frac{1}{2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \cdots & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

where $\boldsymbol{I}$ is the identity matrix. Observing that $\boldsymbol{F}$ is its own inverse, i.e., $\boldsymbol{F}^2 = \boldsymbol{I}$, and that the adjoint of a matrix is equal to its conjugate transpose, we deduce that the adjoint of $\boldsymbol{E}$, denoted $\boldsymbol{E}^\star$, satisfies

$$\boldsymbol{E}^\star = \left(\frac{\boldsymbol{I}+\boldsymbol{F}}{2}\right)^\star = \boldsymbol{E},$$

since both $\boldsymbol{I}$ and $\boldsymbol{F}$ are real and symmetric, and thus equal to their own adjoints.

Next, we compute $\boldsymbol{E}^2$ as follows:

$$\boldsymbol{E}^2 = \left(\frac{\boldsymbol{I}+\boldsymbol{F}}{2}\right)^2 = \frac{\boldsymbol{I}+2\boldsymbol{F}+\boldsymbol{F}^2}{4} = \frac{2\boldsymbol{I}+2\boldsymbol{F}}{4} = \frac{\boldsymbol{I}+\boldsymbol{F}}{2} = \boldsymbol{E}.$$

Hence, $\boldsymbol{E}^2 = \boldsymbol{E}$, which implies that $\boldsymbol{E}$ is idempotent. Additionally, since for any vector $\boldsymbol{v}$, the equality $\langle \boldsymbol{E}\boldsymbol{v}, \boldsymbol{v}\rangle = \langle \boldsymbol{v}, \boldsymbol{E}\boldsymbol{v}\rangle$ holds, where $\langle\cdot,\cdot\rangle$ denotes the inner product, it follows that $\boldsymbol{E}$ is self-adjoint. Combining the idempotence and self-adjointness of $\boldsymbol{E}$, we conclude that $\boldsymbol{E}$ is an orthogonal projector.

2.3 Exercise 5.3

Suppose that $\boldsymbol{A}$ is full rank. This implies that $\boldsymbol{A}$ has n non-zero singular values. Consequently, the matrix $\boldsymbol{A}^\star\boldsymbol{A}$, where $\boldsymbol{A}^\star$ denotes the conjugate transpose of $\boldsymbol{A}$, has n non-zero eigenvalues $\lambda_1, \ldots, \lambda_n$. The determinant of $\boldsymbol{A}^\star\boldsymbol{A}$ is then given by the product of its eigenvalues:

$$\det(\boldsymbol{A}^\star\boldsymbol{A}) = \prod_{i=1}^{n} \lambda_i \neq 0,$$

which indicates that $\boldsymbol{A}^\star \boldsymbol{A}$ is non-singular.

For the "only if" part of the proof, we invoke the singular value decomposition (SVD). Since $\boldsymbol{A}^\star \boldsymbol{A}$ is non-singular, it follows from Theorem 5.2 that

$$\text{range}(\boldsymbol{A}) = \langle \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle,$$

where the $\boldsymbol{u}_i$ are left singular vectors of $\boldsymbol{A}^\star \boldsymbol{A}$. This space is n-dimensional, which confirms that the matrix $\boldsymbol{A}$ is indeed full rank.

2.4 Exercise 6.4 (a)

We first compute $\boldsymbol{P}$ as

$$\boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A}^\star \boldsymbol{A})^{-1}\boldsymbol{A}^\star = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

then

$$\boldsymbol{P} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

2.5 Exercise 6.4(b)

We first compute $\boldsymbol{P}$ as

$$\boldsymbol{P} = \boldsymbol{B}(\boldsymbol{B}^\star \boldsymbol{B})^{-1}\boldsymbol{B}^\star = \begin{pmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{pmatrix}$$

then

$$\boldsymbol{P} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}.$$

2.6 Exercise 6.5

The solution is based on the answer on Mathematics Stack Exchange. We first show that $\|\boldsymbol{P}\|_2 \geq 1$. Choose $\boldsymbol{x}$ such that $\boldsymbol{P}\boldsymbol{x} = \boldsymbol{y} \neq \boldsymbol{0}$, then $\boldsymbol{P}\boldsymbol{y} = \boldsymbol{P}^2\boldsymbol{x} = \boldsymbol{P}\boldsymbol{x} = \boldsymbol{y}$, then we have

$$\|\boldsymbol{P}\|_2 = \max_{\boldsymbol{x}} \frac{\|\boldsymbol{P}\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2} \geq \frac{\|\boldsymbol{P}\boldsymbol{y}\|_2}{\|\boldsymbol{y}\|_2} = 1.$$

Assume that $\boldsymbol{P}$ is orthogonal projector, then we have that

$$\boldsymbol{P} = \hat{\boldsymbol{Q}}\hat{\boldsymbol{Q}}^\star$$

where the columns of $\hat{\boldsymbol{Q}}$ are orthonormal. Hence we can get that

$$1 \leq \|\boldsymbol{P}\|_2 \leq \|\hat{\boldsymbol{Q}}\|_2 \|\hat{\boldsymbol{Q}}^\star\|_2 \leq 1$$

which means that $\|\boldsymbol{P}\|_2 = 1$. If $\|\boldsymbol{P}\| = 1$, we only to show that $\text{range}(\boldsymbol{P}) \perp \text{null}(\boldsymbol{P})$. Let $\boldsymbol{x}$ be a vector such that $\boldsymbol{x} \in \text{null}(\boldsymbol{P})^\perp$ and consider $\boldsymbol{y} := \boldsymbol{P}\boldsymbol{x} - \boldsymbol{x}$. It is clear that $\boldsymbol{y} \in \text{null}(\boldsymbol{P})$, then $\boldsymbol{x} \perp \boldsymbol{y}$. Since $\|\boldsymbol{P}\|_2 = 1$, we have $\|\boldsymbol{P}\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_2$. It follows that

$$\|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_2 + \|\boldsymbol{y}\|_2 = \|\boldsymbol{x} + \boldsymbol{y}\|_2 = \|\boldsymbol{P}\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_2$$

and hence $\boldsymbol{y} = \boldsymbol{0}$, which implies that $\|\boldsymbol{P}\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$ and thus $\boldsymbol{x} \in \text{range}(\boldsymbol{P})$. Therefore $\text{null}(\boldsymbol{P})^\perp \subset \text{range}(\boldsymbol{P})$. On the other hand, if $\boldsymbol{z} \in \text{range}(\boldsymbol{P}), \boldsymbol{P}\boldsymbol{z} = \boldsymbol{z}$, Let $\boldsymbol{x} \in \text{null}(\boldsymbol{P})$ and $\boldsymbol{y} \in \text{null}(\boldsymbol{P})^\perp$ such that $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$, so $\boldsymbol{P}\boldsymbol{z} = \boldsymbol{P}\boldsymbol{x} + \boldsymbol{P}\boldsymbol{y} = \boldsymbol{P}\boldsymbol{y}$. Since $\text{null}(\boldsymbol{P})^\perp \subset \text{range}(\boldsymbol{P})$, we have $\boldsymbol{P}\boldsymbol{y} = \boldsymbol{y}$ and $\boldsymbol{y} = \boldsymbol{z}$ and $\boldsymbol{z} \in \text{null}(\boldsymbol{P})^\perp$. Therefore $\text{range}(\boldsymbol{P}) \subset \text{null}(\boldsymbol{P})^\perp$. From $\text{range}(\boldsymbol{P}) \subset \text{null}(\boldsymbol{P})^\perp$ and $\text{range}(\boldsymbol{P}) \supset \text{null}(\boldsymbol{P})^\perp$, it follows that $\text{range}(\boldsymbol{P}) = \text{null}(\boldsymbol{P})^\perp$.