Lecture 1 Matrix-Vector Multiplication

Fang Zhu

February 17, 2023

1 Prerequisite

todo...

2 Solutions

2.1 Exercise 1.3

Proof. We denote a non-singular matrix $\boldsymbol{R}$ as

$$\boldsymbol{R} = \begin{pmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_{mm} \end{pmatrix},$$

it is clear that $r_{ii} \neq 0$, otherwise $\boldsymbol{R}$ is singular. Since $\boldsymbol{R}$ is non-singular, we assume that

$$\boldsymbol{I} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_m) = (\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n) \begin{pmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_{mm} \end{pmatrix}$$

where $(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n) = \boldsymbol{R}^{-1}$. To show $\boldsymbol{R}^{-1}$ is upper-triangular, we work by induction. To begin with, we have $\boldsymbol{e}_1 = r_{11}\boldsymbol{a}_1$ and hence $\boldsymbol{a_1} = r_{11}^{-1}\mathbf{e}_1$ has *zero entries* except the first one. For convenience, we denote by $\mathbb{C}_k^m$ the column space

$$\mathbb{C}_k^m = \{\boldsymbol{v} = (v_1, \cdots, v_k, 0, \cdots, 0)^T, v_i \neq 0 \ (1 \leq i \leq k)\},$$

Then

$$\mathbb{C}_1^m \subset \mathbb{C}_2^m \cdots \mathbb{C}_m^m = \mathbb{C}^m.$$

We have shown that $a_1 \in \mathbb{C}^m(1)$, assume that for any $k \leq s$, we have that $\mathbf{a}_k \in \mathbb{C}_k^m$. Then by equation *Page 8, (1.8)*, we have

$$\boldsymbol{e}_{s+1} = \sum_{k=1}^{m} \boldsymbol{a}_k r_{k,s+1}.$$

Note that $r_{k,s+1} = 0,\ \forall k > s+1$, then

$$\sum_{k=1}^{m} \boldsymbol{a}_k r_{k,s+1} = \sum_{k=1}^{s} \boldsymbol{a}_k r_{k,s+1} + \boldsymbol{a}_{s+1} r_{s+1,s+1} = \boldsymbol{e}_{s+1},$$

Therefore

$$\boldsymbol{a}_{s+1} = r_{s+1,s+1}^{-1}(\mathbf{e}_{s+1} - \sum_{k=1}^{s} \boldsymbol{a}_k r_{k,s+1}) \in \mathbb{C}_{s+1}^{m}$$

By induction, we have proved that $\boldsymbol{a}_k \in \mathbb{C}_k^m$ for $1 \leq k \leq m$, which is equivalent to the fact that $\boldsymbol{R}^{-1}$ is upper-triangular. $\square$

2.2 Exercise 1.4(a)

Proof. Denote the column vectors $(c_1, \cdots, c_n)^T$, $(d_1, \cdots, d_n)^T$ by notations $\boldsymbol{c}$ and $\boldsymbol{d}$, let $\boldsymbol{F}$ be the matrix whose (i,j) entry is $f_j(i)$. Then, the given condition can be rephrased as: ForAll $\boldsymbol{d} \in \mathbb{C}^8$, there must exist a vector $\boldsymbol{c}$ such that $\boldsymbol{F}\boldsymbol{c} = \boldsymbol{d}$. This means that

$$\text{range}\{\boldsymbol{F}\} = \mathbb{C}^8,$$

which implies that $\boldsymbol{F}$ has full rank by *theorem 1.3.* Furthermore, $\mathbf{F}$ is non-singular. Therefore

$$\boldsymbol{c} = \boldsymbol{F}^{-1}\boldsymbol{d}$$

and hence $\boldsymbol{d}$ determines $\boldsymbol{c}$ uniquely.

$\square$

2.3 Exerciese 1.4(b)

The given condition can be reformatted as

$$\boldsymbol{A}\boldsymbol{d} = \boldsymbol{c}.$$

Note that $\boldsymbol{c} = \boldsymbol{F}^{-1}\boldsymbol{d}$, then

$$\boldsymbol{A}\boldsymbol{d} = \boldsymbol{c} = \boldsymbol{F}^{-1}\boldsymbol{d},$$

then we have

$$(\boldsymbol{F}\boldsymbol{A} - \boldsymbol{I})\boldsymbol{d} = \boldsymbol{0},$$

note that this equation above is true for any $d \in \mathbb{C}^8$, then $\boldsymbol{F}\boldsymbol{A} - \boldsymbol{I}$ must be *zero matrix*, which is $\boldsymbol{F}\boldsymbol{A} = \boldsymbol{I}$. Hence the i,j entry of $\boldsymbol{A}^{-1}$ is the i,j entry of $\boldsymbol{F}$ we defined in 1.4(a).