Galerkin Method for the Inviscid Burgers' Equation

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Abstract

This document provides a detailed derivation of the numerical solution of the inviscid Burgers' equation using the Galerkin approach with sinusoidal basis functions. This transforms the original partial differential equation into a system of ordinary differential equations, which can be solved using standard numerical techniques.

1 Introduction

We start with the inviscid Burgers' equation given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{1}$$

where $u(x,t)$ is the dependent variable and $x \in [-L, L]$ is the spatial coordinate on a periodic domain.

2 Galerkin Projection

To approximate the solution to the equation, we use the Galerkin method. We represent $u(x,t)$ as a finite series expansion of sinusoidal basis functions:

$$u(x,t) \approx \sum_{i=1}^{N} a_i(t)\phi_i(x), \tag{2}$$

where the coefficients $a_i(t)$ are functions of time and the basis functions $\phi_i(x)$ are given by

$$\phi_i(x) = \sqrt{\frac{1}{L}}\sin(i\pi x). \tag{3}$$

We substitute the approximation (2) into the original Burgers' equation (1) to obtain the residual $R(x,t)$:

$$R(x,t) = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}, \tag{4}$$

which simplifies to

$$R(x,t) = \frac{\partial}{\partial t}\left(\sum_{i=1}^{N} a_i(t)\phi_i(x)\right) + \left(\sum_{i=1}^{N} a_i(t)\phi_i(x)\right)\frac{\partial}{\partial x}\left(\sum_{j=1}^{N} a_j(t)\phi_j(x)\right). \quad (5)$$

In the Galerkin method, we set the projection of the residual $R(x,t)$ onto each of the basis functions in the series expansion to be zero, leading to:

$$\int_{-L}^{L} R(x,t)\phi_j(x)dx = 0, \quad \text{for } j = 1, ..., N. \quad (6)$$

Substituting the expression for $R(x,t)$ into the above integral and separating the terms inside the integral, we obtain:

$$0 = \int_{-L}^{L} \frac{\partial}{\partial t}\left(\sum_{i=1}^{N} a_i(t)\phi_i(x)\right)\phi_j(x)dx + \int_{-L}^{L}\left(\sum_{i=1}^{N} a_i(t)\phi_i(x)\right)\frac{\partial}{\partial x}\left(\sum_{k=1}^{N} a_k(t)\phi_k(x)\right)\phi_j(x)dx. \quad (7)$$

We can bring the time derivative and the coefficients $a_i(t)$ outside the first integral as they do not act on the x variable:

$$0 = \sum_{i=1}^{N}\frac{da_i}{dt}\int_{-L}^{L}\phi_i(x)\phi_j(x)dx + \sum_{i=1}^{N}\sum_{k=1}^{N} a_i a_k \int_{-L}^{L}\phi_i(x)\frac{\partial\phi_k(x)}{\partial x}\phi_j(x)dx. \quad (8)$$

By simplifying further, we get a system of N ordinary differential equations for the coefficients $a_i(t)$:

$$\frac{da_j}{dt} = -\sum_{i=1}^{N}\sum_{k=1}^{N} a_i a_k \int_{-L}^{L}\phi_i(x)\frac{\partial\phi_k(x)}{\partial x}\phi_j(x)dx. \quad (9)$$

This gives us a system of N ODEs for $a_j(t)$, which can be written in a more compact form as:

$$\frac{da_j}{dt} = -\sum_{i=1}^{N}\sum_{k=1}^{N} C_j^{(ik)} a_k a_i, \quad (10)$$

where

$$C_j^{(ik)} = \int_{-L}^{L}\phi_i(x)\phi_j(x)\frac{d\phi_k(x)}{dx}dx. \quad (11)$$

3 Numerical Solution

We can solve the system of ordinary differential equations numerically. Once we have the coefficients $a_i(t)$, we can obtain the approximate solution $u(x,t)$ using equation (2).

4 Conclusion

We have presented a Galerkin method for solving the inviscid Burgers' equation using sinusoidal basis functions. This transforms the partial differential equation into a system of ordinary differential equations which can be solved numerically.