数学物理方法

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教材：
《数学物理方法讲义》杨明、石佩虎编
2018.9 (2021.3 重印)
东南大学出版社

$(d)J_\nu(x)$正交性与模值

考虑特征值问题:(*Dirichlet*边值)

$$\begin{cases} x^2y''+xy'+(\lambda x^2-\nu^2)y=0, \\ |y(0)|<\infty, \quad y(a)=0. \end{cases}$$

令$\rho=\sqrt{\lambda}x$,则方程变为$\rho^2\dfrac{d^2y}{d\rho^2}+\rho\dfrac{dy}{d\rho}+(\rho^2-\nu^2)y=0,$

所以 $y=AJ_\nu(\rho)+BY_\nu(\rho),$

原方程的通解为 $y=AJ_\nu(\sqrt{\lambda}x)+BY_\nu(\sqrt{\lambda}x),$

$|y(0)|<\infty \Rightarrow B=0, y(a)=0 \Rightarrow J_\nu(\sqrt{\lambda}a)=0 \Rightarrow \sqrt{\lambda}a=\mu_m^{(\nu)}$

结论: $e.v.\ \lambda_m=(\dfrac{\mu_m^{(\nu)}}{a})^2,\ e.f.\ J_\nu(\dfrac{\mu_m^{(\nu)}}{a}x),\ \ m=1,2,\cdots.$

将方程写成：$(xy')'-\frac{\nu^2}{x}y+\lambda xy=0$, 定义算子

$$Ly=(xy')'-\frac{\nu^2}{x}y,$$

结合边界条件 $|y(0)|<\infty, y(a)=0$, 则可以验证算子 L 是自共轭算子.

所以 L 的特征函数系 $\{J_\nu(\frac{\mu_m^{(\nu)}}{a}x),\}_1^\infty$ 构成 $L_x^2[a,b]$ 上的一组加权正交基，权函数为 x.

$J_\nu(x)$的模值

利用$Bessel$方程先证:

$$\int_0^a xJ_\nu^2(x)dx = \frac{a^2}{2}[J_\nu'(a)]^2 + \frac{1}{2}(a^2-\nu^2)J_\nu^2(a).$$

$$\Rightarrow \quad \boxed{\int_0^a xJ_\nu^2\left(\frac{\mu_m^{(\nu)}}{a}x\right)dx = \frac{a^2}{2}J_{\nu\pm1}^2(\mu_m^{(\nu)})}$$

其中$\mu_m^{(\nu)}$是$J_\nu(x)$的正零点.

换元$t=\dfrac{\mu_m^{(\nu)}}{a}x$,则

$$\int_0^a xJ_\nu^2\left(\frac{\mu_m^{(\nu)}}{a}x\right)dx = \frac{a^2}{(\mu_m^{(\nu)})^2}\int_0^{\mu_m^{(\nu)}} tJ_\nu^2(t)dt = \cdots.$$

Proof. J_ν满足方程$x^2y''+xy'+(x^2-\nu^2)y=0.$

$$xy''+y'+(x-\frac{\nu^2}{x})y=0\Rightarrow(xy')'+(x-\frac{\nu^2}{x})y=0$$

$$\Rightarrow\qquad 2xy'\cdot(xy')'+(x^2-\nu^2)2yy'=0$$

$$\Rightarrow\qquad [(xy')^2+(x^2-\nu^2)y^2]'=2xy^2$$

上式两边在$[0,a]$积分得

$$2\int_0^a xy^2dx=[(xy')^2+(x^2-\nu^2)y^2]_0^a$$

$$\Rightarrow\int_0^a xJ_\nu^2(x)dx=\frac{a^2}{2}[J_\nu'(a)]^2+\frac{1}{2}(a^2-\nu^2)J_\nu^2(a).$$

注：$J_\nu(0)=0,\quad \nu>0.$

考虑特征值问题:($Neumann$边值)

$$\begin{cases} x^2 y'' + xy' + (\lambda x^2 - \nu^2) y = 0, \\ |y(0)| < \infty, \quad y'(a) = 0. \end{cases}$$

$\lambda > 0$时,通解$y = AJ_\nu(\sqrt{\lambda}x) + BY_\nu(\sqrt{\lambda}x)$,

$|y(0)| < \infty \Rightarrow B = 0, y'(a) = 0 \Rightarrow \sqrt{\lambda} J_\nu'(\sqrt{\lambda}a) = 0 \Rightarrow \sqrt{\lambda}a = \mu_m^{(\nu)}$

结论: $e.v.\ \lambda_m = (\frac{\mu_m^{(\nu)}}{a})^2$, $e.f.\ J_\nu(\frac{\mu_m^{(\nu)}}{a}x)$, $\mu_m^{(\nu)}$是$J_\nu'(x)$的正零点.

$\lambda = 0$时,$x^2 y'' + xy' - \nu^2 y = 0$, $Euler$方程.

通解:$\nu = 0$时,$y = C_0 + D_0 \ln x$, $\quad \nu > 0$时,$y = Cx^\nu + Dx^{-\nu}$.

$|y(0)| < \infty \Rightarrow D_0 = D = 0. y'(a) = 0 \Rightarrow \nu > 0$时,$\lambda = 0$不是特征值,

$\nu = 0$时,$\lambda = 0$是特征值,对应的特征函数为1.

特征值问题:(Neumann边值)

$$\begin{cases} x^2 y'' + xy' + (\lambda x^2 - \nu^2) y = 0, \\ |y(0)| < \infty, \qquad y'(a) = 0. \end{cases}$$

$\nu > 0$的情况:

$$e.v.\ \lambda_m = \left(\frac{\mu_m^{(\nu)}}{a}\right)^2,\ e.f.\ \phi_m = J_\nu\left(\frac{\mu_m^{(\nu)}}{a} x\right),\ \ \mu_m^{(\nu)}\text{是}J_\nu'(x)\text{的正零点.}$$

$\nu = 0$的情况:

$$e.v.\lambda_0 = 0, \lambda_m = \left(\frac{\mu_m^{(0)}}{a}\right)^2,\ e.f.\ \phi_0 = 1,\ \phi_m = J_0\left(\frac{\mu_m^{(0)}}{a} x\right).$$

模值: $\int_0^a x\,dx = \frac{a^2}{2},\ \int_0^a x J_\nu^2\left(\frac{\mu_m^{(\nu)}}{a} x\right) dx = \frac{a^2}{2}\left(1 - \frac{\nu^2}{(\mu_m^{(\nu)})^2}\right) J_\nu^2(\mu_m^{(\nu)}).$

4.$Bessel$函数系的完备性,$Bessel$级数

自共轭算子的特征值问题的特征函数系,

$\{\phi_k(x) = J_\nu(\frac{\mu_k^{(\nu)}}{a}x), k = 1, 2, \cdots\}$构成$L_x^2(0, a)$上的一组基.

$$L_x^2(0,a) = \{f(x),\ \|f\|_{L_x^2(0,a)} = (\int_0^a xf^2(x)dx)^{1/2} < \infty\}.$$

$\forall f \in L_x^2(0,a)$,

$$f(x) = \sum c_k \phi_k(x),\quad c_k = \frac{1}{\|\phi_k\|_{L_x^2(0,a)}^2}\int_0^a xf(x)\phi_k(x)dx.$$

例2.设$\mu_k, k=1,2,\cdots$是$J_0(x)$的正零点,请将$f(x)=1$在区间$[0,1]$上展开为$\{J_0(\mu_k x)\}$的$Bessel$级数.

练习：

设$\mu_k,\ k=1,2,\cdots$是$J_0(x)$的正零点，请将

请将$f(x)=x^2$和$g(x)=a^2-x^2$在区间$[0,a]$上

展开为$\{J_0(\mu_k x/a)\}_1^\infty$ 的贝塞尔级数.

例3.设$\mu_k, k=1,2,\cdots$是$J_0'(x)$的正零点,请将$f(x)=1-x^2$在区间[0,1]上展开为$\{1, J_0(\mu_k x), k=1,2,\cdots\}$的$Bessel$级数.

$$f(x)=A_0+\sum_{k=1}^{\infty} A_k J_0(\widehat{\mu}_k x)$$

$$A_0=\frac{\int_0^1 x\cdot(1-x^2)\,\mathrm{d}x}{\int_0^1 x\,\mathrm{d}x}=\frac{1}{2},\ A_k=\frac{\int_0^1 x\cdot(1-x^2)\cdot J_0(\widehat{\mu}_k x)\,\mathrm{d}x}{\int_0^1 xJ_0^2(\widehat{\mu}_k x)\,\mathrm{d}x}=\frac{4J_2(\widehat{\mu}_k)}{\widehat{\mu}_k{}^2 J_0^2(\widehat{\mu}_k)}.$$

5.半整数阶的$Bessel$函数* $J_\nu(x), \nu = n + \frac{1}{2}.$

$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$, 令$u = \sqrt{x}y$, 则

$$u'' + (1 + \frac{\nu^2 - 1/4}{x^2})u = 0.$$

当 $\nu = \frac{1}{2}$时, $u'' + u = 0 \Rightarrow u = A\cos x + B\sin x.$

$\therefore y = \frac{A}{\sqrt{x}}\cos x + \frac{B}{\sqrt{x}}\sin x, \because J_{1/2}(0)$有界 $\therefore A = 0.$

取$B = \sqrt{\frac{2}{\pi}}$得$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$, 类似得$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x.$

由递推公式得:

$$J_{n+\frac{1}{2}}(x) = (-1)^n\sqrt{\frac{2}{\pi}}x^{n+\frac{1}{2}}(\frac{1}{x}\frac{d}{dx})^n\frac{\sin x}{x}, J_{-n-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}x^{n+\frac{1}{2}}(\frac{1}{x}\frac{d}{dx})^n\frac{\cos x}{x}.$$

6. $J_n(x)$的生成函数

考虑函数 $e^{\frac{x}{2}(z-\frac{1}{z})} = e^{\frac{xz}{2}} e^{-\frac{x}{2z}}$,

利用指数函数的级数表达式得

$$e^{\frac{xz}{2}} e^{-\frac{x}{2z}} = \Big(\sum_{j=0}^{\infty} \frac{1}{j!}\Big(\frac{xz}{2}\Big)^j\Big) \cdot \Big(\sum_{k=0}^{\infty} \frac{1}{k!}\Big(-\frac{x}{2z}\Big)^k\Big)$$

令 $j-k=n$，则

$$e^{\frac{x}{2}(z-\frac{1}{z})} = \sum_{n=-\infty}^{\infty} \Big(\sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!}\Big(\frac{x}{2}\Big)^{n+2k}\Big) z^n,$$

$$e^{\frac{1}{2}x\left(z-\frac{1}{z}\right)}=\sum_{-\infty}^{\infty}J_n(x)z^n.$$

称等式左端的函数为$J_n(x)$的生成函数

令$z=e^{i\theta}$，代入生成函数得

$$e^{ix\sin\theta}=\sum_{-\infty}^{\infty}J_n(x)e^{in\theta},$$

即将函数$e^{ix\sin\theta}$做变量θ的复傅立叶级数展开，

$$J_n(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ix\sin\theta}\cdot e^{-in\theta}\,\mathrm{d}\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(x\sin\theta-n\theta)}\,\mathrm{d}\theta$$

特别地 $$J_0(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ix\sin\theta}\,\mathrm{d}\theta.$$

在生成函数中取 $x=kr,\ z=ie^{i\theta}$ 得

$$e^{ikr\cos\theta}=\sum_{-\infty}^{\infty}J_n(kr)i^n e^{in\theta}=J_0(kr)+2\sum_{1}^{\infty}i^n J_n(kr)\cos n\theta.$$

左端是沿x轴正向传播的平面波，而右端各项中的 $J_n(kr)$表示的是柱面波. 因此，上式的物理意义就是平面波按柱面波展开.

课堂练习

1. 讨论Robin边界条件下的特征值问题

$$\begin{cases} r^2R''(r)+rR'(r)+\lambda r^2R(r)=0, & 0<r<a, \\ |R(0)|<\infty, \quad R'(a)+\sigma R(a)=0, & \text{其中}\sigma\text{为正常数.} \end{cases}$$

2. 设$\mu_k, k=1,2,\cdots$是$J_0(x)$的正零点，令

$\phi_k(x)=J_0(\mu_k\sqrt{x/l})$. 证明：$\{\phi_k\}$是$L^2[0,l]$的

一组正交基并计算$\|\phi_k\|^2$.

证明　对于任意的$f\in L^2[0,l]$，令$r=\sqrt{l\,x}$，$g(r)=f(x)$，则

$$\int_0^l rg^2(r)\,\mathrm{d}r=\frac{l}{2}\int_0^l f^2(x)\,\mathrm{d}x<\infty \ \Rightarrow\ g\in L_r^2[0,l].$$

又因为 $\{J_0(\mu_k\frac{r}{l})\}_1^\infty$ 是$L_r^2[0,l]$的一组正交基，则

$$f(x)=g(r)=\sum_1^\infty A_kJ_0(\mu_k\frac{r}{l})=\sum_1^\infty A_kJ_0(\mu_k\sqrt{\frac{x}{l}}).$$

另外利用 $\{J_0(\mu_k\frac{r}{l})\}_1^\infty$ 在$[0,l]$上的正交性和模值，计算可得

$$\int_0^l \phi_j(x)\phi_k(x)\,\mathrm{d}x=\int_0^l J_0(\mu_j\sqrt{x/l})J_0(\mu_k\sqrt{x/l})\,\mathrm{d}x$$

$$=\frac{2}{l}\int_0^l rJ_0(\mu_j\frac{r}{l})J_0(\mu_k\frac{r}{l})\,\mathrm{d}r=\delta_{jk}\cdot l\,J_1^2(\mu_k).$$

3. 讨论特征值问题

$$\begin{cases} (x\phi'(x))' + \lambda\phi(x) = 0, & 0 < x < a, \\ |\phi(0)| < \infty, \quad \phi(a) = 0, \end{cases}$$

提示：做变换 $t = 2\sqrt{\lambda x}$

解　讨论可知特征值$\lambda > 0$，于是可做变换$t = 2\sqrt{\lambda x}$，则

$$x = \frac{t^2}{4\lambda}, \quad \partial_x = \sqrt{\frac{\lambda}{x}}\partial_t = \frac{2\lambda}{t}\partial_t,$$

所以方程变为

$$\lambda\phi_{tt} + \frac{\lambda}{t}\phi_t + \lambda\phi(x) = 0 \quad \Rightarrow \quad t^2\phi_{tt} + t\phi_t + t^2\phi = 0,$$

解得 $\phi(t) = AJ_0(t) + BY_0(t)$，即

$$\phi(x) = AJ_0(2\sqrt{\lambda x}) + BY_0(2\sqrt{\lambda x}).$$

因为 $|\phi(0)| < \infty$，所以$B = 0$，又因为$\phi(a) = 0$，所以$J_0(2\sqrt{\lambda a}) = 0.$

记μ_k是$J_0(x)$的第k个正零点，则 $2\sqrt{\lambda a} = \mu_k$，

$$e.v.\quad \lambda_k = \frac{\mu_k^2}{4a}, \qquad e.f.\quad \phi_k(x) = J_0(\mu_k\sqrt{\frac{x}{a}}), \quad k = 1, 2, \cdots.$$