数学物理方法

2025春

教材：
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7.贝塞尔函数的应用

例. 求解Helmholtz方程边值问题(球贝塞尔函数)

$$\begin{cases} \Delta u + k^2 u = 0, & x^2 + y^2 + z^2 < a^2, \\ u(x,y,z) = 1, & x^2 + y^2 + z^2 = a^2. \end{cases}$$

解 分析知u只与球坐标中的变量r有关，记为$u(r)$

$$r^2 u'' + 2ru' + k^2 r^2 u = 0,$$

令$w(r)=\sqrt{r}u(r)$，则

$$r^2w''+rw'+(k^2r^2-(\frac{1}{2})^2)w=0,$$

利用贝塞尔方程的解知

$$u(r)=\frac{A}{\sqrt{r}}J_{1/2}(kr)+\frac{B}{\sqrt{r}}Y_{1/2}(kr),$$

又因为$|u(0)|<\infty$, $u(a)=1$，解得

$$u(r)=\sqrt{\frac{a}{r}}J_{1/2}(kr)/J_{1/2}(ka).$$

例. 求解Helmholtz方程边值问题(球汉克尔函数)

$$\begin{cases} \Delta u + k^2 u = 0, & x^2+y^2+z^2 > a^2, \\ u(x,y,z) = 1, & x^2+y^2+z^2 = a^2, \\ \lim\limits_{r\to\infty} r(u_r - iku) = 0, & r = \sqrt{x^2+y^2+z^2}. \end{cases}$$

$$u(r) = \sqrt{\frac{a}{r}} H^{+}_{1/2}(kr) / H^{+}_{1/2}(ka)$$

例4.求解鼓面振动问题:

$$\begin{cases} u_{tt} = c^2(u_{xx} + u_{yy}), & x^2 + y^2 < a^2, t > 0, \\ u(x, y, t) = 0, & x^2 + y^2 = a^2, t \geq 0, \\ u(x, y, 0) = 1 - \dfrac{x^2 + y^2}{a^2}, u_t(x, y, 0) = 0, & x^2 + y^2 < a^2. \end{cases}$$

解 初值数据只与极坐标变量r有关，所以$u(x,y,t)$与极坐标变量θ无关，可以简写为$u(r,t)$.

令$u(r,t)=R(r)T(t)$，代入方程得

$$T''(t)R(r)=c^2T(t)\left[R''(r)+\frac{1}{r}R'(r)\right],$$

$$\frac{T''(t)}{c^2T(t)}=\frac{R''(r)+\frac{1}{r}R'(r)}{R(r)}=-\lambda,$$

结合边界条件得特征值问题

$$\begin{cases} r^2R''+rR'+\lambda r^2R=0, \quad 0<r<a, \\ |R(0)|<\infty, \quad R(a)=0. \end{cases}$$

该方程通解为 $R(r)=AJ_0(\sqrt{\lambda}r)+BY_0(\sqrt{\lambda}r)$,

因为$|R(0)|<\infty$，所以$B=0$，又因为$R(a)=0$，

所以$J_0(\sqrt{\lambda}a)=0$. 记μ_k 是$J_0(x)$的第k个正零点，

$$e.v.\ \lambda_k=(\frac{\mu_k}{a})^2, \qquad e.f.\ R_k(r)=J_0(\frac{\mu_k}{a}r), \qquad k=1,2,\cdots$$

$$T''(t)+c^2\lambda_k T(t)=0,\ \ T'(0)=0,$$

解得 $$T_k(t)=\cos\frac{c\mu_k t}{a}.$$

令 $$u(r,t)=\sum_1^\infty A_k\cos\frac{c\mu_k t}{a}J_0(\frac{\mu_k}{a}r),$$

代入初值得 $$1-\frac{r^2}{a^2}=\sum_1^\infty A_k J_0(\frac{\mu_k}{a}r),$$

$$A_k=\frac{\int_0^a r(1-r^2/a^2)J_0(\frac{\mu_k}{a}r)\,\mathrm{d}r}{\int_0^a rJ_0^2(\frac{\mu_k}{a}r)\,\mathrm{d}r}=\frac{2a^2J_2(\mu_k)/\mu_k^2}{a^2J_1^2(\mu_k)/2}=\frac{4J_2(\mu_k)}{\mu_k^2J_1^2(\mu_k)}.$$

例5.求解热扩散问题:

$$\begin{cases} u_t = k(u_{xx} + u_{yy}), & x^2 + y^2 < a^2, t > 0, \\ u(x, y, t) = B \ (\text{常数}), & x^2 + y^2 = a^2, t \geq 0, \\ u(x, y, 0) = 0, & x^2 + y^2 < a^2. \end{cases}$$

解 令 $v = u - B$，则 v 满足条件

$$\begin{cases} v_t = k(v_{xx} + u_{yy}), & x^2 + y^2 < a^2,\ t > 0, \\ v(x,y,t) = 0, & x^2 + y^2 = a^2,\ t \geq 0, \\ v(x,y,0) = -B, & x^2 + y^2 < a^2. \end{cases}$$

在极坐标下用分离变量法来求解v，v的初值与θ无关，所以v与θ无关，于是令

$$v(r,t) = T(t)\phi(r),$$

$$T'(t)\phi(r) = kT(t)\Delta\phi(r)$$

$$\Rightarrow \quad \frac{T'(t)}{kT(t)} = \frac{\phi''(r) + \frac{1}{r}\phi(r)}{\phi(r)} = -\lambda,$$

其中 λ 是常数. 结合边界条件考虑特征值问题

$$\begin{cases} r^2\phi''(r) + r\phi(r) + \lambda r^2\phi(r) = 0, \quad 0 < r < a, \\ |\phi(0)| < \infty, \qquad \phi(a) = 0. \end{cases}$$

解得通解为 $\quad \phi(r) = AJ_0(\sqrt{\lambda}r) + BY_0(\sqrt{\lambda}r),$

因为$|\phi(0)|<\infty$，所以$B=0$，又因为$\phi(a)=0$，

所以 $J_0(\sqrt{\lambda}a)=0$，于是记μ_n是J_0的第n个正零点，

则$\sqrt{\lambda}a=\mu_n$，从而

$$e.v.\ \lambda_n=(\frac{\mu_n}{a})^2,\qquad e.f.\ \phi_n(r)=J_0(\frac{\mu_n}{a}r),\qquad n=1,2,\cdots.$$

$$T'(t)+k\lambda_n T(t)=0\quad\Rightarrow\quad T_n(t)=e^{-k\lambda_n t},$$

设
$$v(r,t)=\sum_{n=1}^{\infty}A_n e^{-k\lambda_n t}J_0(\frac{\mu_n}{a}r),$$

代入初始条件得 $-B=\sum_{n=1}^{\infty}A_nJ_0(\frac{\mu_n}{a}r),$

解得 $$A_n=\frac{-B\int_0^a rJ_0(\frac{\mu_n}{a}r)\,\mathrm{d}r}{\int_0^a rJ_0^2(\frac{\mu_n}{a}r)\,\mathrm{d}r}=-\frac{2B}{\mu_nJ_1(\mu_n)}$$

所以 $$u(r)=B-\sum_1^{\infty}\frac{2B}{\mu_nJ_1(\mu_n)}e^{-k(\frac{\mu_n}{a})^2t}J_0(\frac{\mu_n}{a}r).$$

例6.求解圆柱区域上的$Laplace$方程.

设圆柱区域$D=\{(r,\theta,z):0\le r<a,0<z<l\}$,

$$\begin{cases} u_{rr}+\dfrac{1}{r}u_r+\dfrac{1}{r^2}u_{\theta\theta}+u_{zz}=0, \qquad\qquad in\ \ D, \\ u(r,\theta,0)=0,\ \ u(r,\theta,l)=g(r)\sin\theta,\ \ u(a,\theta,z)=0, \end{cases}$$

给出解的表达式.

解 令$u(r,\theta,z)=\phi(r,\theta)Z(z)$，代入方程得

$$\Delta\phi(r,\theta)Z(z)+\phi(r,\theta)Z''(z)=0,$$

分离变量得 $$-\frac{Z''(z)}{Z(z)}=\frac{\Delta\phi(r,\theta)}{\phi(r,\theta)}=-\lambda,$$

其中λ为常数. 结合边界条件得特征值问题

$$\begin{cases}\Delta\phi(r,\theta)+\lambda\phi(r,\theta)=0, \quad 0\le r<a,\\ \phi(a,\theta)=0.\end{cases}$$

继续分离变量，令$\phi(r,\theta)=R(r)\Theta(\theta)$，代入得

$$R''(r)\Theta(\theta)+\frac{1}{r}R'(r)\Theta(\theta)+\frac{1}{r^2}R(r)\Theta''(\theta)+\lambda R(r)\Theta(\theta)=0,$$

变量分离得 $$\frac{r^2R''+rR'}{R}+\lambda r^2=-\frac{\Theta''}{\Theta}=\gamma,$$

其中γ为常数，结合周期条件得到子特征值问题(I)

$$\begin{cases}\Theta''(\theta)+\gamma\Theta(\theta)=0,\\ \Theta(\theta)=\Theta(\theta+2\pi).\end{cases}$$

解得特征值为 $\gamma_n = n^2,\ n = 0, 1, 2, \cdots,$

对应的特征函数是

$$\Theta_0 = 1,\ \Theta_n = A_n \cos n\theta + B_n \sin n\theta.$$

将此结果代入得子特征值问题(II)

$$\begin{cases} r^2 R''(r) + rR'(r) + (\lambda r^2 - n^2)R(r) = 0, \quad 0 < r < a, \\ |R(0)| < \infty, \quad R(a) = 0. \end{cases}$$

$$R(r) = AJ_n(\sqrt{\lambda} r) + BY_n(\sqrt{\lambda} r),$$

因为$|R(0)| < \infty$，所以$B = 0$，又因为$R(a) = 0$，

所以$J_n(\sqrt{\lambda}a) = 0$. 记$\mu_k^{(n)}$是$J_n(x)$的第k个正零点，

则$\sqrt{\lambda}a = \mu_k^{(n)}$，所以

$$e.v.\ \ \lambda_{kn} = (\frac{\mu_k^{(n)}}{a})^2, \quad e.f.\ \ R_{kn}(r) = J_n(\frac{\mu_k^{(n)}}{a}r), \quad k = 1, 2, \cdots.$$

将特征值代入z的方程得

$$Z''(z) - \lambda_{kn}Z(z) = 0, \quad Z(0) = 0 \quad \Rightarrow \quad Z_{kn}(z) = \sinh\frac{\mu_k^{(n)}}{a}z.$$

令

$$u(r,\theta,z)=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(A_{kn}\cos n\theta+B_{kn}\sin n\theta)J_n(\frac{\mu_k^{(n)}}{a}r)\sinh\frac{\mu_k^{(n)}}{a}z,$$

则

$$g(r)\sin\theta=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(A_{kn}\cos n\theta+B_{kn}\sin n\theta)J_n(\frac{\mu_k^{(n)}}{a}r)\sinh\frac{\mu_k^{(n)}}{a}l,$$

所以 $A_{kn}=0,\ \ n=0,1,2,\cdots,\quad B_{kn}=0,\ \ n\neq 1,$

$$g(r)=\sum_{k=1}^{\infty}B_{k1}J_1(\frac{\mu_k^{(1)}}{a}r)\sinh\frac{\mu_k^{(1)}}{a}l,$$

$$B_{k1} = \frac{1}{\sinh \frac{\mu_k^{(1)}}{a} l} \cdot \frac{\int_0^a r g(r) J_1(\frac{\mu_k^{(1)}}{a} r) \,\mathrm{d}r}{\int_0^a r J_1^2(\frac{\mu_k^{(1)}}{a} r) \,\mathrm{d}r} = \frac{2 \int_0^a r g(r) J_1(\frac{\mu_k^{(1)}}{a} r) \,\mathrm{d}r}{a^2 \sinh \frac{\mu_k^{(1)}}{a} l \cdot J_2^2(\mu_k^{(1)})},$$

综上

$$u(r,\theta,z) = \sum_{k=1}^{\infty} \frac{2 \int_0^a r g(r) J_1(\frac{\mu_k^{(1)}}{a} r) \,\mathrm{d}r}{a^2 \sinh \frac{\mu_k^{(1)}}{a} l \cdot J_2^2(\mu_k^{(1)})} \sin\theta \, J_1(\frac{\mu_k^{(1)}}{a} r) \sinh \frac{\mu_k^{(1)}}{a} z.$$

课堂练习

1.有一个半径为1的圆板，表面绝热，边缘处保持温度为零，若初始温度为$f(r)$，板的热传导系数为k，求圆板在任一时刻的温度.

1.分析知u与θ无关，设为$u(r,t)$,

$$\begin{cases} u_t = k(u_{rr} + \dfrac{1}{r}u_r), & 0 < r < 1, \\ u(1,t) = 0, & t \geq 0, \\ u(r,0) = f(r), & 0 \leq r < 1. \end{cases}$$

$$u(r,t) = \sum_{k=1}^{\infty} \frac{2\int_0^1 rf(r)J_0(\mu_k r)dr}{J_1^2(\mu_k)} e^{-k\mu_k{}^2 t} J_0(\mu_k r).$$

2.半径为a的圆形薄膜，边缘固定，受到周期外力$f(r,t)=A\sin\omega t$的作用，薄膜的初始位移和初始速度均为零，求此薄膜的振动规律.

2.

$$\begin{cases} u_{tt} - c^2(u_{xx} + u_{yy}) = A\sin\omega t, & x^2 + y^2 < a^2,\ t > 0, \\ u(x,y,t) = 0, & x^2 + y^2 = a^2,\ t \geq 0, \\ u(x,y,0) = 0, \quad u_t(x,y,0) = 0, & x^2 + y^2 < a^2. \end{cases}$$

其中 $\omega \neq \omega_k$，$\omega_k = \mu_k c/a$，μ_k是J_0的第k个零点.

解 解u与极坐标中的变量θ无关，故可设解为$u(r,t)$.

考虑特征值问题

$$\begin{cases} r^2R''(r) + rR'(r) + \lambda r^2 R = 0, & 0 < r < a, \\ |R(0)| < \infty, \quad R(a) = 0. \end{cases}$$

$$e.v.\ \ \lambda_k = (\frac{\mu_k}{a})^2, \quad e.f.\ \ R_k(r) = J_0(\frac{\mu_k}{a}r), \quad k = 1, 2, \cdots,$$

令 $$u(r,t) = \sum_{k=1}^{\infty} T_k(t) J_0(\frac{\mu_k}{a}r),$$

$$A\sin\omega t = \sum_{k=1}^{\infty} \frac{2A}{\mu_k J_1(\mu_k)} \sin\omega t \cdot J_0(\frac{\mu_k}{a}r),$$

代入方程得

$$\begin{cases} T_k''(t) + c^2\lambda_k T_k(t) = \frac{2A}{\mu_k J_1(\mu_k)} \sin\omega t, \quad t > 0, \\ T_k(0) = 0, \quad T_k'(0) = 0. \end{cases}$$

采用拉普拉斯变换求解得

$$p^2\widetilde{T}_k(p)+c^2\lambda_k\widetilde{T}_k(p)=\frac{2A}{\mu_k J_1(\mu_k)}\cdot\frac{\omega}{p^2+\omega^2},$$

$$\widetilde{T}_k(p)=\frac{2A}{\mu_k J_1(\mu_k)}\cdot\frac{\omega}{(p^2+\omega^2)(p^2+c^2\lambda_k)}$$

$$=\frac{2A}{\mu_k J_1(\mu_k)}\cdot\frac{1}{\omega^2-\omega_k^2}\left(\frac{\omega}{p^2+\omega^2}-\frac{\omega}{\omega_k}\frac{\omega_k}{p^2+\omega_k^2}\right)$$

求拉普拉斯逆变换得

$$T_k(t)=\frac{2A}{\mu_k J_1(\mu_k)}\cdot\frac{1}{\omega^2-\omega_k^2}\left(\sin\omega t-\frac{\omega}{\omega_k}\sin\omega_k t\right).$$