广相复习

1.在假定空间平坦及均匀各向同性的情况下, 时空度规可写为

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

$a(t)$为宇宙尺度因子，是时间的函数。在只存在暗能量的情况下，引力场方程为$R_{uv} = -\Lambda g_{uv}$,其中，$\Lambda > 0$为常数。给定条件$a(T) = a_0$，求解$a(t)$，即尺度因子随时间的演化。
解：涉及到只求里奇张量时，可以不用求黎曼张量，具体解法如下：

$$R^{\lambda}_{\mu\lambda\nu} = -\Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\sigma\nu} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\lambda}_{\sigma\lambda}$$

$$g_{tt} = -1, g_{ii} = a^2$$

$$\Gamma^{i}_{ti} = \frac{\dot{a}}{a}, \Gamma^{t}_{ii} = a\dot{a}$$

其中不为0的分分量有：R_{tt}, R_{ii}

$$R_{tt} = R^{x}_{txt} + R^{y}_{tyt} + R^{z}_{tzt} = \frac{3\ddot{a}}{a}$$

$$R_{ii} = -(\ddot{a}a + 2\dot{a}^2)$$

对应分量相同：$\dfrac{3\ddot{a}}{a} = \Lambda, -(\ddot{a}a + 2\dot{a}^2) = -\Lambda a^2, a = a_0 e^{\sqrt{\Lambda}(t-T)/3}$

2.对于三位度规球：$ds^2 = d\theta^2 + \sin^2\theta d\varphi^2 + \cos^2\theta d\psi^2$,求联络，曲率张量，里奇张量，标量曲率。
解：首先求得度规为：$g_{\theta\theta} = 1, g_{\varphi\varphi} = \sin^2\theta, g_{\psi\psi} = \cos^2\theta$,然后计算出不为0的联络为 $\Gamma^{\varphi}_{\theta\varphi} = \frac{\cos\theta}{\sin\theta}, \Gamma^{\psi}_{\theta\psi} = -\frac{\sin\theta}{\cos\theta}, \Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta, \Gamma^{\theta}_{\psi\psi} = \sin\theta\cos\theta$

$$R^{\rho}_{\lambda\mu\nu} = -\Gamma^{\rho}_{\lambda\mu,\nu} + \Gamma^{\rho}_{\lambda\nu,\mu} - \Gamma^{\sigma}_{\lambda\mu}\Gamma^{\rho}_{\sigma\nu} + \Gamma^{\sigma}_{\lambda\nu}\Gamma^{\rho}_{\sigma\mu}$$

根据对称性和$\mu \neq \nu$不为0的黎曼张量分量为：$R_{\varphi\theta\varphi\theta}, R_{\psi\theta\psi\theta}, R_{\varphi\psi\varphi\psi}, R_{\varphi\psi\varphi\theta}, R_{\varphi\psi\psi\theta}, R_{\varphi\theta\psi\theta}$

这里的上下指标变换需要乘以度规!!!

$$R^{\psi}_{\theta\psi\theta} = 1, R^{\varphi}_{\theta\varphi\theta} = 1, R^{\varphi}_{\psi\varphi\psi} = \cos^2\theta$$

$$R^{\varphi}_{\psi\varphi\theta} = 0, R^{\varphi}_{\theta\psi\theta} = 0, R^{\varphi}_{\psi\psi\theta} = 0$$

将所有的指标写成下指标的形式：

$$R_{\varphi\theta\varphi\theta} = \sin^2\theta, R_{\psi\theta\psi\theta} = \cos^2\theta, R_{\varphi\psi\varphi\psi} = \sin^2\theta\cos^2\theta$$

$$R_{\varphi\psi\varphi\theta} = R_{\varphi\psi\psi\theta} = R_{\varphi\theta\psi\theta} = 0$$

根据里奇张量的公式：

$$
\begin{aligned}
R_{\theta\theta} &= g^{\varphi\varphi}R_{\varphi\theta\theta\varphi} + g^{\psi\psi}R_{\psi\theta\theta\psi} = -2 \\
R_{\psi\psi} &= g^{\theta\theta}R_{\theta\psi\psi\theta} + g^{\varphi\varphi}R_{\varphi\psi\psi\varphi} = -2\cos^2\theta \\
R_{\varphi\varphi} &= g^{\theta\theta}R_{\theta\varphi\varphi\theta} + g^{\psi\psi}R_{\psi\varphi\varphi\psi} = -2\sin^2\theta
\end{aligned}
$$

因此标量曲率为：注意这里还是要乘以度规!!!

$$
R = g^{\theta\theta}R_{\theta\theta} + g^{\psi\psi}R_{\psi\psi} + g^{\varphi\varphi}R_{\varphi\varphi} = -6
$$

3.引入场强张量 $F^{\mu v}, F^{\mu v} = -F^{v\mu}, \mu, v = 0,1,2,3$ ，其中 $F^{0i} = E^i, F^{ij} = \varepsilon^{ijk}B_k, i,j,k = 1,2,3,$ 证明麦克斯韦方程组

$$
\begin{aligned}
&\varepsilon^{ijk}\partial_j B_k - \partial_0 E^i = 4\pi J^i \quad && \partial_i E^i = 4\pi J^0 \\
&\varepsilon^{ijk}\partial_j E_k + \partial_0 B^i = 0 && \partial_i B^i = 0
\end{aligned}
$$

可写为

$$
\partial_\mu F^{\mu\nu} = -4\pi J^\nu \qquad \partial_\mu F_{v\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu v} = 0
$$

并得出电荷守恒方程 $\quad \partial_\mu J^\mu = 0.$ (注意: $F_{\mu v} = \eta_{\mu\nu}\eta_{v\sigma}F^{\lambda\sigma}$, 所以 $F_{0i} = \eta_{00}F^{0i} = -F^{0i}, F_{ij} = F^{ij}$)

解：ijk都是表示的分量，$i,j,k = 1,2,3$

$$
\begin{aligned}
\text{由前两个式子可以得出：} &\partial_j F^{ij} - \partial_0 F^{oi} = 4\pi J_i \\
&\partial_0 F^{i0} = 0, \partial_0 F^{00} = 0 \\
&\partial_i F^{0i} = 4\pi J^0
\end{aligned}
$$

由上述式子可以推出要证明的第一个式子

$$
\begin{aligned}
\text{乘以度规进行上下指标的置换：} & -F_{0i} = E_i, B^i = \frac{1}{2}\varepsilon^{ijk}F_{jk} \\
\text{代入第三个式子} & \frac{1}{2}\varepsilon^{ijk}(2\partial_j F_{k0} + \partial_0 F_{jk}) = 0 \\
& \frac{1}{2}\varepsilon^{ijk}(\partial_k F_{0j} + \partial_j F_{k0} + \partial_0 F_{jk}) = 0 \\
\text{上式对jk求和可以得出：} & \partial_k F_{0j} + \partial_j F_{k0} + \partial_0 F_{jk} = 0
\end{aligned}
$$

$$
\text{对第四个式子：} \partial_i F_{jk} + \partial_k F_{ij} + \partial_j F_{ki} = 0 \tag{1}
$$

上面已经得出来对于一个指标一个0和没有0的情况，由于三个指标都需要变成$\mu\nu\lambda$，所以还有两个0和三个0的情况，上面是式子是对i 指标的情况，其他的指标和i指标是类似的，因此只需要对一个指标求出来了，其他指标也就求出来了。

对最后一个式子的证明,利用对称性：

$$
\begin{aligned}
\partial_\nu F^{\nu\mu} &= -4\pi J^\mu \\
\partial_\mu\partial_\nu F^{\nu\mu} &= -4\pi\partial_\mu J^\mu \\
&= -\partial_\mu\partial_\nu F^{\mu\nu} \\
&= -\partial_\nu\partial_\mu F^{\nu\mu}
\end{aligned}
$$

4.电磁场和电流耦合的作用量为

$$S\left[A_{\mu}\right]=\int d^{4}x\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+A_{\mu}J^{\mu}\right),$$

其中 $F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}$, $J^{\mu}=(\rho,\vec{J})$ 为给定的四维电流密度, 对 A_{μ} 作变分, 得出电磁场运动方程 $\partial_{\mu}F^{\mu v}=-J^{v}$

解：注意这里的电磁场张量是反对称的,并且这里有$\nu\mu$代表求和

$$\begin{aligned}
&\delta(F^{\nu\mu}F_{\nu\mu})=2F^{\mu\nu}\delta F_{\nu\mu}\\
&F^{\mu\nu}\delta(\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu})=F^{\mu\nu}\partial_{\mu}\delta A_{\nu}+F^{\nu\mu}\partial_{\nu}\delta A_{\mu}=2F^{\mu\nu}\partial_{\mu}\delta A_{\nu}
\end{aligned}$$

$$\begin{aligned}
\delta S[A_{\nu}]&=\int d^{4}x(-F^{\mu\nu}\partial_{\mu}\delta A_{\nu}+J^{\nu}\delta A_{\nu})\\
&=\int d^{4}x(-\partial_{-}\mu(F^{\mu\nu}\delta A_{\nu})+\delta A_{n}u\partial_{\mu}F^{\mu\nu}+J^{\nu}\delta A_{n}u)\\
&=\int d^{4}x(\partial_{\mu}F^{\mu\nu}+J^{\nu})\delta A_{\nu}=0
\end{aligned}$$