Handbook on geometry

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§1. Notations and conventions

Transformation (active transformation) of manifold $M \subset \mathbb{R}^n$ is one-to-one map $x \to x' = f(x)$. Each transformation f can be considered as *coordinate transformation* (passive transformation, transformation of coordinate system) in such a way that the old x and new ξ coordinates are related by $x = f(\xi)$. If not specified we distinguish active and passive transformations by using the decoration of the same symbol (x and x') for active and different symbol (x and ξ) for passive transformations. In most cases we treat transformations as active.

§2. Linear transformations

2.1. General formulas

Consider a linear space $\mathbb{R}^n$. We denote the coordinates of vector $x \in \mathbb{R}^n$ and matrix (operator) $A : \mathbb{R}^n \to \mathbb{R}^n$ by indexes below: x_i and A_{ij}.

Linear transformation is a linear map $x \to x' = Tx$, where T is a nondegenarate matrix. They compose a group $GL(n, \mathbb{R})$. *Translation* is a map $x \to x' = x + a$, where a is a vector. The corresponding group is denoted by $T(n, \mathbb{R})$. Combinations of linear transformations and translations form *affine transformations*, $x \to x' = Tx + a$. The affine group $Aff(n, \mathbb{R}) = GL(n, \mathbb{R}) \ltimes T(n, \mathbb{R})$ since $(T, a)(1, b)(T, a)^{-1} = (1, Tb)$. Any linear transformation of vectors (or linear part of affine transformation) induces the following transformation of matrices: $A \to A' = TAT^{-1}$.

Any basis $\{e^i\}$ transforms to basis $\varepsilon^i = Te^i$ so that for any $x \in \mathbb{R}^n$ we have $x = \sum_i x_i e^i = \sum_i \xi_i \varepsilon^i$ and thus $x_i = \sum_j T_{ij} \xi_j$ and $\varepsilon^i = \sum_j T_{ji} e^j$.

In Euclidean space the *orthogonal transformations* are defined as those preserving the scalar product: $(Ox, Oy) = (x, y)$. The corresponding group is denoted by $O(n)$ and it is the group of matrices O with $\det O = \pm 1$. *Rotation* R (proper) is the orthogonal transformation with $\det R = 1$, the corresponding group is denoted by $SO(n)$. Any orthogonal transformations is either rotation or composition of a rotation and *inversion* ($x \to -x$) yielding $O(n) = SO(n) \times I$, where $I = \{1, -1\}$. Any