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Characterization of the Negative Binomial and Gamma Distributions

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Summary

Characterization of the negative binomial and gamma distributions by a conditional distribution and a linear regression, and the gamma distribution by the negative binomial distribution are given. An application to a random shock model is discussed.

Key words: Conditional distribution; Linear regression; Negative Binomial; Random shock model; Gamma.

1. Introduction

Characterizations of distributions, particularly the gamma distribution, have attracted the attention of many people and several results are found in the literature. Surveys may be found for instance in Kagan, Linnik and Rao (1973) and Galambos and Kotz (1978). Recently, Osaki and Li (1988) characterized the gamma and negative binomial distributions in terms of failure rates (or conditional expectations). Motivated by an inventory problem, Engel and Zijlstra (1980) have also given a characterization of the gamma distribution by the negative binomial distribution (see also Cacoullos and Papageorgiou, 1982).

The purpose of this note is to give yet another characterization of the negative binomial and gamma distributions by a conditional distribution and linear regression and point out an application to a simple random shock model (Downton, 1970). The characterization is a particular case of a general result considered by Cacoullos and Papageorgiou (1984, Proposition 3.2) for mixtures of continuous distributions. Analogous results for some discrete distribution mixtures are to be found in Cacoullos and Papageorgiou (1983) and Papageorgiou (1984). Cacoullos and Papageorgiou (1983) have indicated the application of such characterizations to goodness-of-fit tests.

2. Characterization of the Negative Binomial and Gamma Distributions

CACOULLOS and PAPAGEORGIOU (1984) considered mixtures of normal or gamma densities $f(x|\theta)$ with respect to the parameter θ (discrete or continuous) which are characterized by the posterior mean of θ given the continuous random variable (rv) X. The mixing distribution of θ is also characterized since mixtures are identifiable. Of particular interest is the following result on gamma mixtures with respect to the shape parameter α.

Theorem 2.1. (CACOULLOS and PAPAGEORGIOU, 1984, p. 28)
If the gamma density $f(x|\alpha, \lambda)$ is given by

$$f(x|\alpha, \lambda) = \lambda^\alpha x^{\alpha-1} e^{-\lambda x}/\Gamma(\alpha) \tag{1}$$

with λ fixed, and

$$f(x) = \int_0^\infty f(x|\alpha, \lambda)\, g(\alpha)\, d\alpha, \tag{2}$$

then $E[\alpha|X]$ determines $f(x)$ and $g(\alpha)$ uniquely.

Since linear regression is rather simple in form, it is interesting to know the specific densities $f(x)$ and $g(\alpha)$ corresponding to linear $E[\alpha|X]$. As a special case of Theorem 2.1 we have the following characterization of the negative binomial and gamma distributions.

Theorem 2.2.

(a) Let $X|\alpha$ be a gamma *rv* with density given by (1) where α is a positive integer. Let A be a positive *rv* with pmf $g(\alpha)$. Then

$$E[A|X] = \gamma + \beta X, \tag{3}$$

where γ is a positive integer and $\beta \neq 0$, if, and only if, A has the negative binomial distribution.

(b) A has the negative binomial distribution with pmf

$$g(\alpha) = \binom{\alpha-1}{r-1} p^r q^{\alpha-r}, \quad \alpha \geqq r, \quad 0 < q = 1 - p < 1, \tag{4}$$

$r = 1, 2, 3, \ldots$, if, and only if, the unconditional X has the gamma (Erlang) distribution with density

$$f(x) = (\lambda p)^r x^{r-1} e^{-\lambda p x}/\Gamma(r), \quad \lambda p > 0, \quad r > 0. \tag{5}$$

Proof: It is easy to show that

$$\begin{aligned} f(x) &= \sum_{\alpha=r}^{\infty} f(x)|\alpha, \lambda)\, g(\alpha) \\ &= (\lambda p)^r\, x^{r-1}\, e^{-\lambda p x}/\Gamma(r), \quad r>0 \end{aligned}$$

by substituting (1) and (4) in (2) and replacing integration by summation.

The conditional pmf $g(\alpha|x)$ is given by

$$\begin{aligned} g(\alpha|x) &= f(x|\alpha)\, g(\alpha)/f(x) \\ &= e^{-\lambda q x}(\lambda q x)^{\alpha-r}/(\alpha-r)!, \quad \alpha \geqq r, \end{aligned}$$

a Poisson pmf. Thus

$$\begin{aligned} E[A|x] &= \sum_{n=r}^{\infty} r\, \frac{(\lambda q x)^{n-r}}{(n-r)!}\, e^{-\lambda q x} \\ &= \sum_{i=0}^{\infty} (i+r)\, \frac{(\lambda q x)^i}{i!}\, e^{-\lambda q x}, \quad i=n-r, \\ &= (\lambda q)\, x + r \end{aligned}$$

which is of the form (3) with $\gamma = r$ and $\beta = \lambda q$.

The result follows from Theorem 2.1. □

3. Connection with a Random Shock Model

Although the characterization given by Theorem 2.2 is restricted to integer α, it is of interest because of its relationship with a simple random shock model (Downton, 1970, p. 410). A generalization of this shock model is considered here which might be useful in the study of survivorship of an organism.

Let an organism (component) be subjected to random shocks generated by a Poisson process such that the intervals between shocks have an exponential distribution with density

$$h(t) = \lambda e^{-\lambda t}$$

and the number of shocks n needed to cause failure (death) of the organism (component) is a negative binomial rv N with pmf given by (4) (Downton assumed N to be a geometric rv). The time-to-failure of the organism, T, conditional on n shocks has a gamma distribution with density

$$f(t|n) = \lambda^n\, t^{n-1}\, e^{-\lambda t}/\Gamma(n).$$

Then the unconditional T has gamma density

$$f(t) = (\lambda p)^r\, t^{r-1}\, e^{-\lambda p t}/\Gamma(r).$$

As a consequence of Theorem 2.2, the assumptions on the distributions of N and T of this simple model may be checked by considering the conditional expected values $E[N|t]$, if they are known. Since $E[N|t]$ determines the distributions of N and T uniquely (Theorem 2.1), the linearity of $E[N|t]$ would indicate that the assumptions on N and T are acceptable. Alternatively, if a sample of failure times are given and found acceptable as a gamma sample, then N is a negative binomial rv.

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