LBM单组分等温两相流-2

第一代方案

格子玻尔兹曼BGK方程（LBGK方程）：

$$f_i(\mathbf{x}+\mathbf{c}_i\Delta t, t+\Delta t) = f_i(\mathbf{x},t) - \frac{\Delta t}{\tau}\left(f_i(\mathbf{x},t) - f_i^{\mathrm{eq}}(\mathbf{x},t)\right) + F_i$$

碰撞步： $f_i^{\star}(\mathbf{x},t) = f_i(\mathbf{x},t) - \frac{\Delta t}{\tau}\left(f_i(\mathbf{x},t) - f_i^{\mathrm{eq}}(\mathbf{x},t)\right)$

迁移步： $f_i(\mathbf{x}+\mathbf{c}_i\Delta t, t+\Delta t) = f_i^{\star}(\mathbf{x},t)$

Shan-Chen粒子间作用力： $\mathbf{F}_{\mathrm{int}}(\mathbf{x}) = -c_0\psi(\mathbf{x})g\nabla\psi(\mathbf{x})$

注意到： $\nabla\psi(\mathbf{x}) = \frac{\nabla\psi^2(\mathbf{x})}{2}$, $\mathbf{F}_{\mathrm{int}}(\mathbf{x}) = -c_0 g\nabla\psi^2(\mathbf{x})/2$

Gong格式粒子间作用力：

$$\mathbf{F}_{\mathrm{int}}(\mathbf{x}) = -\beta c_0\psi(\mathbf{x})g\nabla\psi(\mathbf{x}) - (1-\beta)c_0 g\nabla\psi^2(\mathbf{x})/2$$

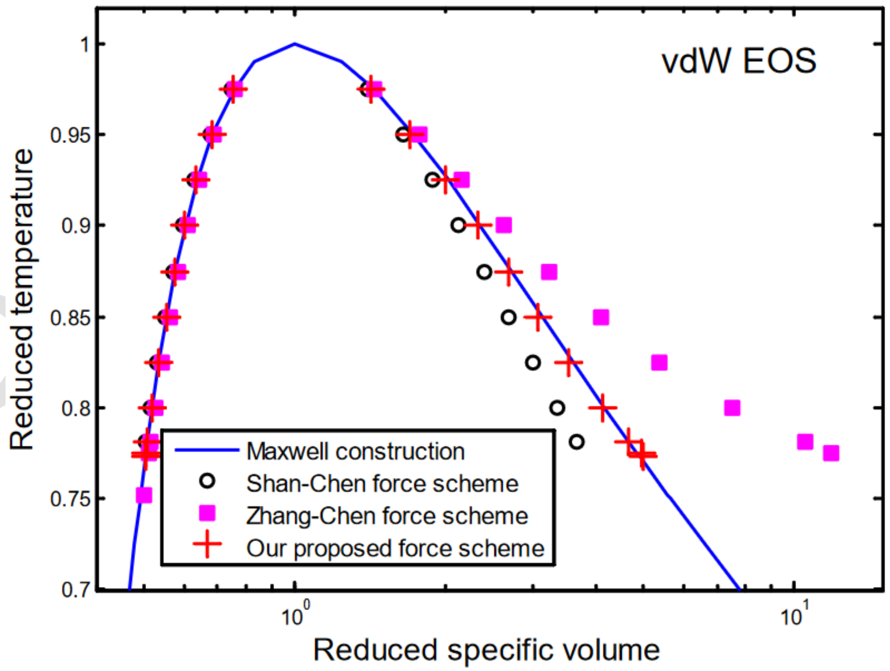
数值实施：

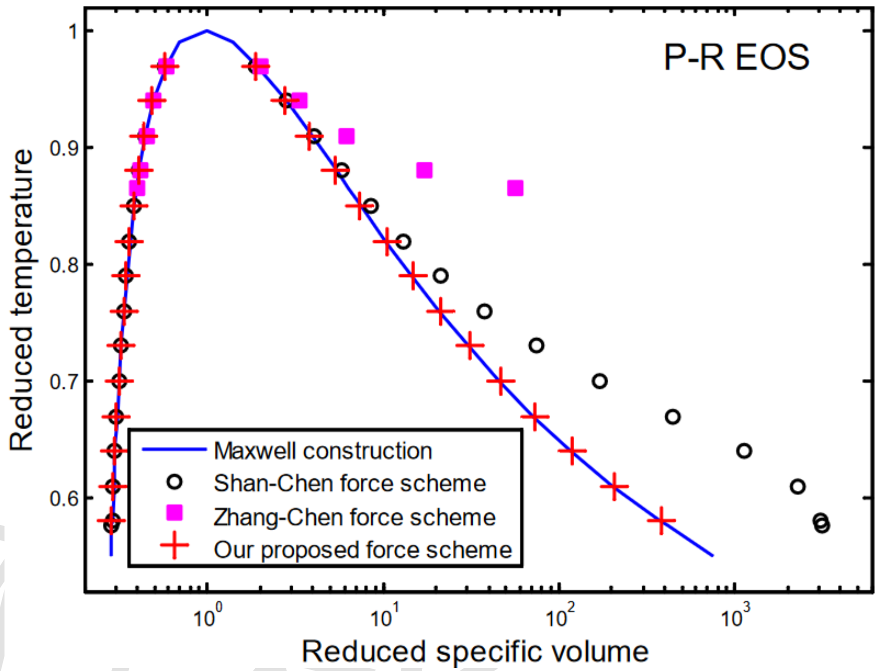
$$\mathbf{F}_{\mathrm{int}}(\mathbf{x}) = -\beta\psi(\mathbf{x})\sum_{\mathbf{x}'}G(\mathbf{x},\mathbf{x}')\psi(\mathbf{x}')(\mathbf{x}'-\mathbf{x}) - \frac{1-\beta}{2}\sum_{\mathbf{x}'}G(\mathbf{x},\mathbf{x}')\psi^2(\mathbf{x}')(\mathbf{x}'-\mathbf{x})$$

```
double tmpForce1[2] = { 0.0 }, tmpForce2[2] = { 0.0 };
for (int k = 1; k < Q; k++) {
    int ip = (i + e[k][0] + NX) % NX;
    int jp = (j + e[k][1] + NY) % NY;
    tmpForce1[0] += w_F[k] * points[ip][jp].psi * e[k][0];
    tmpForce1[1] += w_F[k] * points[ip][jp].psi * e[k][1];
    tmpForce2[0] += w_F[k] * points[ip][jp].psi * points[ip][jp].psi * e[k][0];
    tmpForce2[1] += w_F[k] * points[ip][jp].psi * points[ip][jp].psi * e[k][1];
}
points[i][j].Force[0] = (-beta * points[i][j].psi * tmpForce1[0] - 0.5 * (1 - beta) * tmpForce2[0]) * g * deltat;
points[i][j].Force[1] = (-beta * points[i][j].psi * tmpForce1[1] - 0.5 * (1 - beta) * tmpForce2[1]) * g * deltat;
```

S. Gong, P. Cheng, Numerical investigation of droplet motion and coalescence by an improved lattice Boltzmann model for phase transitions and multiphase flows, Comput Fluids 53 (2012) 93–104.

第一代方案





S. Gong, P. Cheng, Numerical investigation of droplet motion and coalescence by an improved lattice Boltzmann model for phase transitions and multiphase flows, Comput Fluids 53 (2012) 93–104.

第一代方案

1.由LBGK方程恢复得到的宏观方程中，与λ有关的项缺失
2.由LBGK方程恢复得到的宏观方程中，压力项代表理想气体状态方程
3.采用原始的粒子间作用力形式不能满足热力学一致性（√）
4.格子体系下的声速与实际体系下的声速不对应
5.模拟能够达到的最低温度不够低（气液密度比不够大），难以应用于实际问题

第二代方案

❑ MRT-Li

格子玻尔兹曼BGK方程（LBGK方程）：

$$f_i(\mathbf{x}+\mathbf{c}_i\Delta t, t+\Delta t) = f_i(\mathbf{x},t) - \frac{\Delta t}{\tau}\left(f_i(\mathbf{x},t) - f_i^{\mathrm{eq}}(\mathbf{x},t)\right)$$

碰撞步： $$f_i^{\star}(\mathbf{x},t) = f_i(\mathbf{x},t) - \frac{\Delta t}{\tau}\left(f_i(\mathbf{x},t) - f_i^{\mathrm{eq}}(\mathbf{x},t)\right)$$

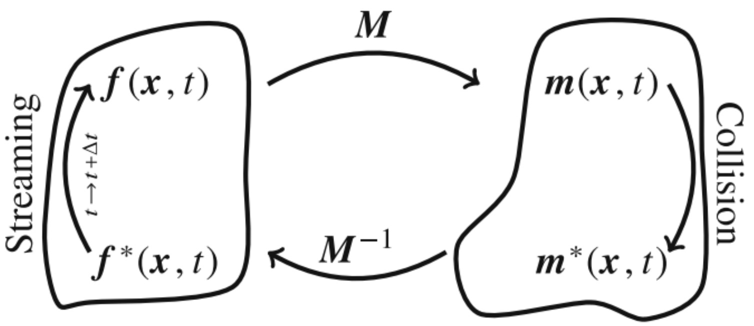
迁移步： $$f_i(\mathbf{x}+\mathbf{c}_i\Delta t, t+\Delta t) = f_i^{\star}(\mathbf{x},t)$$

MRT：碰撞步在矩空间执行

$$\mathbf{m} = \mathbf{M}\mathbf{f}$$

$$\mathbf{m}^{*} = \mathbf{m} - \Lambda(\mathbf{m} - \mathbf{m}^{\mathbf{eq}}) + \delta_t\left(\mathbf{I} - \frac{\Lambda}{2}\right)\mathbf{S}$$

$$\mathbf{f} = \mathbf{M}^{-1}\mathbf{m}$$




```
void collide() {
    for (int i = 0; i < NX; i++) {
        for (int j = 0; j < NY; j++) {
            //计算MRT分布函数
            for (int k = 0; k < Q; k++){ ... }
            //执行碰撞步
            for (int k = 0; k < Q; k++) {
                points[i][j].mout[k] = points[i][j].min[k] - A[k] * (points[i][j].min[k] -
                    meq(k, points[i][j].rho, points[i][j].u)) + (1.0 - 0.5 * A[k]) * points[i][j].S[k] * deltat;
            }
            //从矩空间返回分布函数空间以执行后续迁移步
            for (int k = 0; k < Q; k++){ ... }
        }
    }
}
```

Q. Li, K.H. Luo, X.J. Li, Lattice Boltzmann modeling of multiphase flows at large density ratio with an improved pseudopotential model, Phys Rev E 87 (2013) 053301.

第二代方案

❑ 多松弛时间LB模型（multi-relaxation time，MRT）

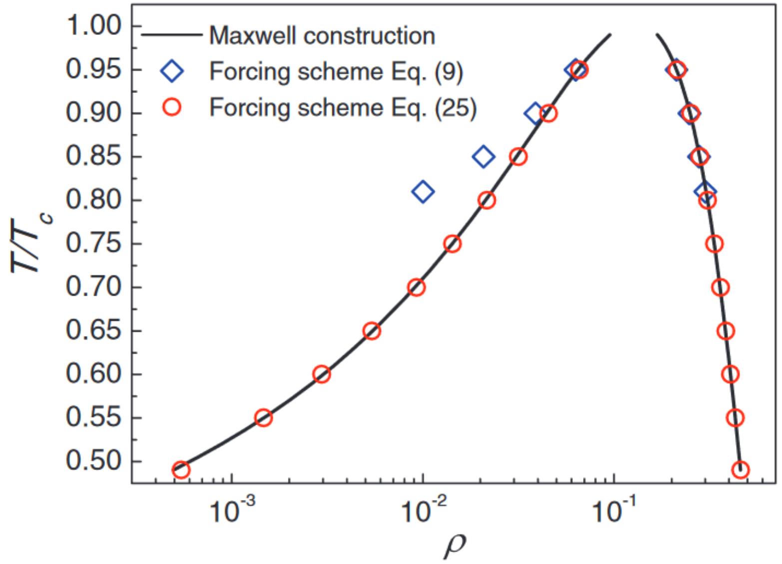
$$\mathbf{m}^{*}=\mathbf{m}-\Lambda(\mathbf{m}-\mathbf{m}^{\mathrm{eq}})+\delta_t\left(\mathbf{I}-\frac{\Lambda}{2}\right)\mathbf{F}_m$$

Eq(9) 原始的矩空间作用力项

$$\mathbf{F}_m=\begin{bmatrix}0\\6(v_xF_x+v_yF_y)\\-6(v_xF_x+v_yF_y)\\F_x\\-F_x\\F_y\\-F_y\\2(v_xF_x-v_yF_y)\\(v_xF_y+v_yF_x)\end{bmatrix}$$

Eq(25) 改进后的矩空间作用力项

$$\mathbf{F}_m=\begin{bmatrix}0\\6(v_xF_x+v_yF_y)+\dfrac{12\sigma|\mathbf{F}|^2}{\psi^2\delta_t(\tau_e-0.5)}\\-6(v_xF_x+v_yF_y)-\dfrac{12\sigma|\mathbf{F}|^2}{\psi^2\delta_t(\tau_\xi-0.5)}\\F_x\\-F_x\\F_y\\-F_y\\2(v_xF_x-v_yF_y)\\(v_xF_y+v_yF_x)\end{bmatrix}$$



满足热力学一致性的同时能够达到更低的模拟温度

但是无法独立调节气液密度比与表面张力

Q. Li, K.H. Luo, X.J. Li, Lattice Boltzmann modeling of multiphase flows at large density ratio with an improved pseudopotential model, Phys Rev E 87 (2013) 053301.

第二代方案

MRT-Huang

$$\overline{\mathbf{m}}(\mathbf{x},t)=\mathbf{m}(\mathbf{x},t)-\mathbf{S}\left[\mathbf{m}(\mathbf{x},t)-\mathbf{m}^{eq}(\mathbf{x},t)\right]+\delta_t\left(\mathbf{I}-\frac{\mathbf{S}}{2}\right)\mathbf{F}_m(\mathbf{x},t)+\mathbf{S}\mathbf{Q}_m(\mathbf{x},t)$$

$$\mathbf{m}^{eq}=\left(\rho,-2\rho+3\rho\frac{|\mathbf{u}|^2}{c^2},\rho-3\rho\frac{|\mathbf{u}|^2}{c^2},\rho\frac{u_x}{c},-\rho\frac{u_x}{c},\rho\frac{u_y}{c},-\rho\frac{u_y}{c},\rho\frac{u_x^2-u_y^2}{c^2},\rho\frac{u_xu_y}{c^2}\right)^{\mathrm{T}}$$

$$\mathbf{S}=\mathrm{diag}(s_0,s_e,s_\varepsilon,s_j,s_q,s_j,s_q,s_p,s_p)$$

$$\mathbf{F}_{\mathrm{m}}=\left(0,6\frac{\mathbf{F}\cdot\mathbf{u}}{c^2},-6\frac{\mathbf{F}\cdot\mathbf{u}}{c^2},\frac{F_x}{c},-\frac{F_x}{c},\frac{F_y}{c},-\frac{F_y}{c},2\frac{F_xu_x-F_yu_y}{c^2},\frac{F_xu_y+F_yu_x}{c^2}\right)^{\mathrm{T}}$$

$$\mathbf{Q}_{\mathrm{m}}=\left(0,3(k_1+2k_2)\frac{|\mathbf{F}|^2}{G\psi^2c^2},-3(k_1+2k_2)\frac{|\mathbf{F}|^2}{G\psi^2c^2},0,0,0,0,k_1\frac{F_x^2-F_y^2}{G\psi^2c^2},k_1\frac{F_xF_y}{G\psi^2c^2}\right)^{\mathrm{T}}$$

$$\epsilon=-8(k_1+k_2)$$

```
for (int k = 0; k < Q; k++) {
    points[i][j].mout[k] = points[i][j].min[k] - S[k] * (points[i][j].min[k] - meq(k, points[i][j].rho, points[i][j].u))
        + (1.0 - 0.5 * S[k]) * points[i][j].Fm[k] * deltat + S[k] * points[i][j].Qm[k];
}
```

```
double meq(const int k, const double rho, const double u[2]) {
    double u_squ = u[0] * u[0] + u[1] * u[1];
    if (k == 0) {
        return rho;
    }
    else if (k == 1) {
        return (-2.0 + 3.0 * u_squ) * rho;
    }
    else if (k == 2) {
        return (1.0 - 3.0 * u_squ) * rho;
    }
    else if (k == 3) {
        return u[0] * rho;
    }
    else if (k == 4) {
        return -u[0] * rho;
    }
    else if (k == 5) {
        return u[1] * rho;
    }
    else if (k == 6) {
        return -u[1] * rho;
    }
    else if (k == 7) {
        return (u[0] * u[0] - u[1] * u[1]) * rho;
    }
    else if (k == 8) {
        return (u[0] * u[1]) * rho;
    }
}
```

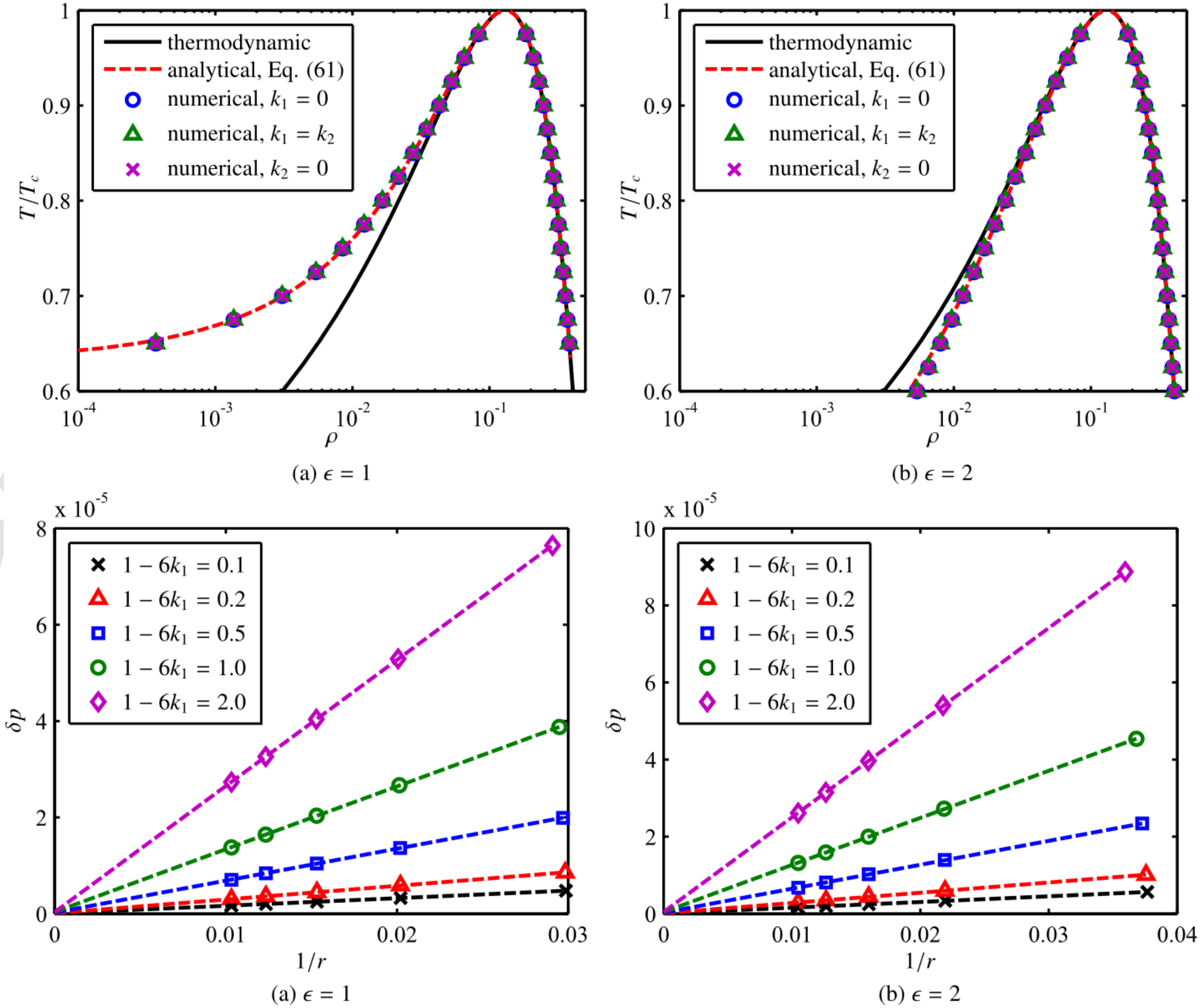
R. Huang, H. Wu, Third-order analysis of pseudopotential lattice Boltzmann model for multiphase flow, J Comput Phys 327 (2016) 121–139.

第二代方案

❑ MRT-Huang

通过ϵ调整力学稳定性（气液共存密度）

通过k_1调整表面张力



R. Huang, H. Wu, Third-order analysis of pseudopotential lattice Boltzmann model for multiphase flow, J Comput Phys 327 (2016) 121–139.

第二代方案

❑ MRT-Huang

$$\overline{\mathbf{m}}(\mathbf{x},t)=\mathbf{m}(\mathbf{x},t)-\mathbf{S}\left[\mathbf{m}(\mathbf{x},t)-\mathbf{m}^{\mathrm{eq}}(\mathbf{x},t)\right]+\delta_t\left(\mathbf{I}-\frac{\mathbf{S}}{2}\right)\mathbf{F}_m(\mathbf{x},t)+\mathbf{S}\mathbf{Q}_m(\mathbf{x},t)$$

$$\mathbf{S}=\mathrm{diag}(s_0,s_e,s_\varepsilon,s_j,s_q,s_j,s_q,s_p,s_p)$$

3. Second-order analysis

To establish a starting point for the third-order Chapman–Enskog analysis, we first perform the standard second-order Chapman–Enskog analysis of the MRT pseudopotential LB model in this section. Through a second-order Taylor series expansion of $f_i(\mathbf{x}+\mathbf{e}_i\delta_t,t+\delta_t)$ centered at $(\mathbf{x},t)$, the streaming step (i.e., Eq. (3)) can be written as

$$f_i+\delta_t(\partial_t+\mathbf{e}_i\cdot\nabla)f_i+\frac{\delta_t^2}{2}(\partial_t+\mathbf{e}_i\cdot\nabla)^2 f_i+O(\delta_t^3)=\bar{f}_i. \quad (11)$$

Transforming Eq. (11) into the moment space, and then combining it with the collision step (i.e., Eq. (2)), we obtain

$$(\mathbf{I}\partial_t+\mathbf{D})\mathbf{m}+\frac{\delta_t}{2}(\mathbf{I}\partial_t+\mathbf{D})^2\mathbf{m}+O(\delta_t^2)=-\frac{\mathbf{S}}{\delta_t}(\mathbf{m}-\mathbf{m}^{\mathrm{eq}})+\left(\mathbf{I}-\frac{\mathbf{S}}{2}\right)\mathbf{F}_m, \quad (12)$$

where $\mathbf{D}=\mathbf{M}\left[\mathrm{diag}(\mathbf{e}_0\cdot\nabla,\cdots,\mathbf{e}_8\cdot\nabla)\right]\mathbf{M}^{-1}$. Eq. (12) is called the Taylor series expansion of the MRT LBE in the moment space. Introducing the following Chapman–Enskog expansions [48]

$$\partial_t=\sum_{n=1}^{+\infty}\varepsilon^n\partial_{tn},\qquad \nabla=\varepsilon\nabla_1,\qquad f_i=\sum_{n=0}^{+\infty}\varepsilon^n f_i^{(n)},\qquad \mathbf{F}=\varepsilon\mathbf{F}^{(1)}, \quad (13)$$

there have $\mathbf{D}=\varepsilon\mathbf{D}_1$, $\mathbf{m}=\sum_{n=0}^{+\infty}\varepsilon^n\mathbf{m}^{(n)}$, and $\mathbf{F}_m=\varepsilon\mathbf{F}_m^{(1)}$, where ε is the small expansion parameter. Substituting these Chapman–Enskog expansions into Eq. (12), we can rewrite Eq. (12) in the consecutive orders of ε as

$$\varepsilon^0:\ \mathbf{m}^{(0)}=\mathbf{m}^{\mathrm{eq}}, \quad (14a)$$

$$\varepsilon^1:\ (\mathbf{I}\partial_{t1}+\mathbf{D}_1)\mathbf{m}^{(0)}-\mathbf{F}_m^{(1)}=-\frac{\mathbf{S}}{\delta_t}\left(\mathbf{m}^{(1)}+\frac{\delta_t}{2}\mathbf{F}_m^{(1)}\right), \quad (14b)$$

$$\varepsilon^2:\ \partial_{t2}\mathbf{m}^{(0)}+(\mathbf{I}\partial_{t1}+\mathbf{D}_1)\left(\mathbf{I}-\frac{\mathbf{S}}{2}\right)\left(\mathbf{m}^{(1)}+\frac{\delta_t}{2}\mathbf{F}_m^{(1)}\right)=-\frac{\mathbf{S}}{\delta_t}\mathbf{m}^{(2)}, \quad (14c)$$

where the first-order (ε^1) equation has been used to simplify the second-order (ε^2) equation.

To deduce the macroscopic equation, we extract the equations for the conserved moments (m_0, m_3, and m_5) from Eq. (14) as

$$\varepsilon^0:\ \begin{cases} m_0^{(0)}=m_0^{\mathrm{eq}},\\ m_3^{(0)}=m_3^{\mathrm{eq}},\\ m_5^{(0)}=m_5^{\mathrm{eq}}, \end{cases} \quad (15a)$$

$$\varepsilon^1:\ \begin{cases} \partial_{t1}m_0^{(0)}+c\partial_{x1}m_3^{(0)}+c\partial_{y1}m_5^{(0)}-F_{m0}^{(1)}=-\frac{s_0}{\delta_t}\left(m_0^{(1)}+\frac{\delta_t}{2}F_{m0}^{(1)}\right),\\ \partial_{t1}m_3^{(0)}+c\partial_{x1}\left(\frac{2}{3}m_0^{(0)}+\frac{1}{6}m_1^{(0)}+\frac{1}{2}m_7^{(0)}\right)+c\partial_{y1}m_8^{(0)}-F_{m3}^{(1)}=-\frac{s_j}{\delta_t}\left(m_3^{(1)}+\frac{\delta_t}{2}F_{m3}^{(1)}\right),\\ \partial_{t1}m_5^{(0)}+c\partial_{x1}m_8^{(0)}+c\partial_{y1}\left(\frac{2}{3}m_0^{(0)}+\frac{1}{6}m_1^{(0)}-\frac{1}{2}m_7^{(0)}\right)-F_{m5}^{(1)}=-\frac{s_j}{\delta_t}\left(m_5^{(1)}+\frac{\delta_t}{2}F_{m5}^{(1)}\right), \end{cases} \quad (15b)$$

$$\varepsilon^2:\ \begin{cases} \left(\begin{array}{l}\partial_{t2}m_0^{(0)}+\partial_{t1}\left(1-\frac{s_0}{2}\right)\left(m_0^{(1)}+\frac{\delta_t}{2}F_{m0}^{(1)}\right)+\\ c\partial_{x1}\left(1-\frac{s_j}{2}\right)\left(m_3^{(1)}+\frac{\delta_t}{2}F_{m3}^{(1)}\right)+c\partial_{y1}\left(1-\frac{s_j}{2}\right)\left(m_5^{(1)}+\frac{\delta_t}{2}F_{m5}^{(1)}\right)\end{array}\right)=-\frac{s_0}{\delta_t}m_0^{(2)},\\ \left(\begin{array}{l}\partial_{t2}m_3^{(0)}+\partial_{t1}\left(1-\frac{s_j}{2}\right)\left(m_3^{(1)}+\frac{\delta_t}{2}F_{m3}^{(1)}\right)+c\partial_{y1}\left(1-\frac{s_p}{2}\right)\left(m_8^{(1)}+\frac{\delta_t}{2}F_{m8}^{(1)}\right)+\\ c\partial_{x1}\left[\frac{2}{3}\left(1-\frac{s_0}{2}\right)\left(m_0^{(1)}+\frac{\delta_t}{2}F_{m0}^{(1)}\right)+\frac{1}{6}\left(1-\frac{s_e}{2}\right)\left(m_1^{(1)}+\frac{\delta_t}{2}F_{m1}^{(1)}\right)+\frac{1}{2}\left(1-\frac{s_p}{2}\right)\left(m_7^{(1)}+\frac{\delta_t}{2}F_{m7}^{(1)}\right)\right]\end{array}\right)\\ =-\frac{s_j}{\delta_t}m_3^{(2)},\\ \left(\begin{array}{l}\partial_{t2}m_5^{(0)}+\partial_{t1}\left(1-\frac{s_j}{2}\right)\left(m_5^{(1)}+\frac{\delta_t}{2}F_{m5}^{(1)}\right)+c\partial_{x1}\left(1-\frac{s_p}{2}\right)\left(m_8^{(1)}+\frac{\delta_t}{2}F_{m8}^{(1)}\right)+\\ c\partial_{y1}\left[\frac{2}{3}\left(1-\frac{s_0}{2}\right)\left(m_0^{(1)}+\frac{\delta_t}{2}F_{m0}^{(1)}\right)+\frac{1}{6}\left(1-\frac{s_e}{2}\right)\left(m_1^{(1)}+\frac{\delta_t}{2}F_{m1}^{(1)}\right)-\frac{1}{2}\left(1-\frac{s_p}{2}\right)\left(m_7^{(1)}+\frac{\delta_t}{2}F_{m7}^{(1)}\right)\right]\end{array}\right)\\ =-\frac{s_j}{\delta_t}m_5^{(2)}. \end{cases} \quad (15c)$$

Considering $m_0=\rho$, $m_3=\rho u_x/c-\frac{\delta_t}{2}F_x/c$, and $m_5=\rho u_y/c-\frac{\delta_t}{2}F_y/c$ (see Eq. (7)), Eq. (15a) indicates that

$$\begin{cases} m_0^{(1)}+\frac{\delta_t}{2}F_{m0}^{(1)}=0, & m_0^{(n)}=0\ (\forall n\ge 2),\\ m_3^{(1)}+\frac{\delta_t}{2}F_{m3}^{(1)}=0, & m_3^{(n)}=0\ (\forall n\ge 2),\\ m_5^{(1)}+\frac{\delta_t}{2}F_{m5}^{(1)}=0, & m_5^{(n)}=0\ (\forall n\ge 2). \end{cases} \quad (16)$$

Therefore, the first-order equation (i.e., Eq. (15b)) can be simplified as

$$\varepsilon^1:\ \begin{cases} \partial_{t1}\rho+\partial_{x1}(\rho u_x)+\partial_{y1}(\rho u_y)=0,\\ \partial_{t1}(\rho u_x)+\partial_{x1}(\rho u_x^2)+\partial_{y1}(\rho u_x u_y)=-\partial_{x1}(\frac{1}{3}\rho c^2)+F_x^{(1)},\\ \partial_{t1}(\rho u_y)+\partial_{x1}(\rho u_x u_y)+\partial_{y1}(\rho u_y^2)=-\partial_{y1}(\frac{1}{3}\rho c^2)+F_y^{(1)}. \end{cases} \quad (17)$$

Based on Eq. (17), the following relation can be obtained

$$\begin{aligned}\partial_{t1}(\rho\mathbf{u}\mathbf{u})&=[\partial_{t1}(\rho\mathbf{u})]\mathbf{u}+\mathbf{u}[\partial_{t1}(\rho\mathbf{u})]-\mathbf{u}\mathbf{u}(\partial_{t1}\rho)\\&=-\frac{1}{3}c^2[(\nabla_1\rho)\mathbf{u}+\mathbf{u}(\nabla_1\rho)]+\mathbf{F}^{(1)}\mathbf{u}+\mathbf{u}\mathbf{F}^{(1)}+O(|\mathbf{u}|^3),\end{aligned} \quad (18)$$

where the cubic term of velocity will be neglected with the low Mach number condition. In order to simplify the second-order equation (i.e., Eq. (15c)), the involved first-order terms on the non-conserved moments, i.e., $m_1^{(1)}+\frac{\delta_t}{2}F_{m1}^{(1)}$, $m_7^{(1)}+\frac{\delta_t}{2}F_{m7}^{(1)}$, and $m_8^{(1)}+\frac{\delta_t}{2}F_{m8}^{(1)}$, should be calculated firstly. These first-order terms are obtained from Eq. (14b) and then simplified with the aid of Eqs. (14a) and (18) as:

$$\begin{aligned}-\frac{s_e}{\delta_t}\left(m_1^{(1)}+\frac{\delta_t}{2}F_{m1}^{(1)}\right)&=\partial_{t1}m_1^{(0)}+c\partial_{x1}\left(m_3^{(0)}+m_4^{(0)}\right)+c\partial_{y1}\left(m_5^{(0)}+m_6^{(0)}\right)-F_{m1}^{(1)}\\&\approx 2\rho(\partial_{x1}u_x+\partial_{y1}u_y),\end{aligned} \quad (19a)$$

$$\begin{aligned}-\frac{s_p}{\delta_t}\left(m_7^{(1)}+\frac{\delta_t}{2}F_{m7}^{(1)}\right)&=\partial_{t1}m_7^{(0)}+c\partial_{x1}\left(\frac{1}{3}m_3^{(0)}-\frac{1}{3}m_4^{(0)}\right)-c\partial_{y1}\left(\frac{1}{3}m_5^{(0)}-\frac{1}{3}m_6^{(0)}\right)-F_{m7}^{(1)}\\&\approx\frac{2}{3}\rho(\partial_{x1}u_x-\partial_{y1}u_y),\end{aligned} \quad (19b)$$

$$\begin{aligned}-\frac{s_p}{\delta_t}\left(m_8^{(1)}+\frac{\delta_t}{2}F_{m8}^{(1)}\right)&=\partial_{t1}m_8^{(0)}+c\partial_{x1}\left(\frac{2}{3}m_5^{(0)}+\frac{1}{3}m_6^{(0)}\right)+c\partial_{y1}\left(\frac{2}{3}m_3^{(0)}+\frac{1}{3}m_4^{(0)}\right)-F_{m8}^{(1)}\\&\approx\frac{1}{3}\rho(\partial_{x1}u_y+\partial_{y1}u_x),\end{aligned} \quad (19c)$$

where the sign "≈" means the cubic term of velocity is neglected. With the aid of Eqs. (14a), (16), and (19), the second-order equation (i.e., Eq. (15c)) can be finally simplified as

$$\varepsilon^2:\ \begin{cases} \partial_{t2}\rho=0,\\ \partial_{t2}(\rho u_x)=\partial_{x1}[\rho\nu(\partial_{x1}u_x-\partial_{y1}u_y)]+\partial_{y1}[\rho\nu(\partial_{y1}u_x+\partial_{x1}u_y)]+\partial_{x1}[\rho\varsigma(\partial_{x1}u_x+\partial_{y1}u_y)],\\ \partial_{t2}(\rho u_y)=\partial_{x1}[\rho\nu(\partial_{x1}u_y+\partial_{y1}u_x)]+\partial_{y1}[\rho\nu(\partial_{y1}u_y-\partial_{x1}u_x)]+\partial_{y1}[\rho\varsigma(\partial_{x1}u_x+\partial_{y1}u_y)], \end{cases} \quad (20)$$

where $\nu=c^2\delta_t(s_p^{-1}-0.5)/3$ is the kinematic viscosity, $\varsigma=c^2\delta_t(s_e^{-1}-0.5)/3$ is the bulk viscosity. Combining the first- and second-order equations (i.e., Eqs. (17) and (20)), the following macroscopic equation at the Navier–Stokes level (second-order) can be recovered

$$\begin{cases} \partial_t\rho+\nabla\cdot(\rho\mathbf{u})=0,\\ \partial_t(\rho\mathbf{u})+\nabla\cdot(\rho\mathbf{u}\mathbf{u})=-\nabla(\frac{1}{3}\rho c^2)+\mathbf{F}+\nabla\cdot\{\rho\nu[\nabla\mathbf{u}+\mathbf{u}\nabla-(\nabla\cdot\mathbf{u})\mathbf{I}]\}+\nabla(\rho\varsigma\nabla\cdot\mathbf{u}). \end{cases} \quad (21)$$

MRT模型CE展开后得到的宏观方程：

$$\partial_t\rho+\nabla\cdot(\rho\mathbf{u})=0$$

$$\partial_t(\rho\mathbf{u})+\nabla\cdot(\rho\mathbf{u}\mathbf{u})=-\nabla(\frac{1}{3}\rho c^2)+\mathbf{F}+\nabla\cdot\{\rho\nu[\nabla\mathbf{u}+\mathbf{u}\nabla-(\nabla\cdot\mathbf{u})\mathbf{I}]+\rho\zeta\nabla\cdot\mathbf{u}\}$$

$$\nu=c_s^2(s_p^{-1}-1/2)\delta_t,\quad \varsigma=\varpi c_s^2\left(s_e^{-1}-1/2\right)\delta_t$$

动量方程

$$\frac{\partial(\rho\mathbf{u})}{\partial t}+\nabla\cdot(\rho\mathbf{u}\mathbf{u})=-\nabla p+\rho\mathbf{F}+\nabla\cdot\left(\lambda(\nabla\cdot\mathbf{u})\mathbf{I}+\mu\left[\nabla\mathbf{u}+(\nabla\mathbf{u})^{\mathrm{T}}\right]\right)$$

能够恢复粘性耗散项与体积粘度项
但压力梯度项仍然是理想气体状态方程

R. Huang, H. Wu, Third-order analysis of pseudopotential lattice Boltzmann model for multiphase flow, J Comput Phys 327 (2016) 121–139.

第二代方案

❑ MRT模型总结

1.由LBGK方程恢复得到的宏观方程中，与λ有关的项缺失（√）

2.由LBGK方程恢复得到的宏观方程中，压力项代表理想气体状态方程

3.采用原始的粒子间作用力形式不能满足热力学一致性（√）

4.格子体系下的声速与实际体系下的声速不对应

5.模拟能够达到的最低温度不够低（气液密度比不够大），难以应用于实际问题（√）