Principles of Communications

Chapter 5 — Presentation and Transmission of Baseband Signal

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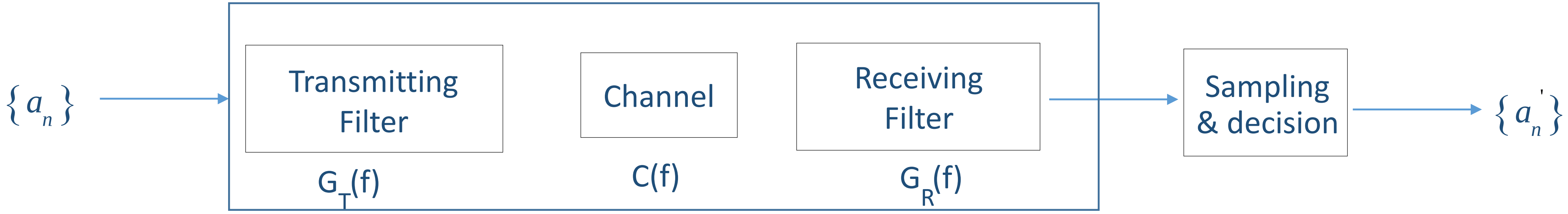
Review

Digital baseband system



$$d(t) \longrightarrow h(t) \longrightarrow y(t)$$

Receiving signal: $y(t) = h(t) * d(t) + n_R(t) = \sum_{n=-\infty}^{\infty} a_n h(t - nT_b) + n_R(t)$

$$y(kT_b + t_0) = a_k h(t_0) + \sum_{\substack{n=-\infty \\ n \neq k}}^{\infty} a_n h[(k-n)T_b + t_0] + n_R(t)$$

desired interference noise

Review

Condition of no intersymbol interference

no intersymbol interference $\sum_{\substack{n=-\infty \\ n\neq k}}^{\infty} a_n h[(k-n)T_b + t_0] = 0$

Condition in time domain

$$h(kT_b) \equiv h_k = \begin{cases} \text{constant} & k=0 \\ 0 & k \neq 0 \end{cases}$$

Condition in frequency domain

$$\frac{1}{T_b}\sum_{m=-\infty}^{\infty} H(\omega + 2\pi m/T_b) = \text{constant}, \quad |\omega| < \frac{\pi}{T_b}$$

(1) fb>2W: Must have intersymbol interference

(2) fb=2W: Only the ideal low pass system to guarantee no intersymbol interference

(3) fb<2W: If the sum of the superimposed system is constant, no intersymbol interference

Review

Ideal low pass system & Roll-off system

Ideal low pass $H_{eq}(f)=H(f)=\begin{cases} T_b, & |f|\le \frac{f_b}{2} \\ 0, & |f|>\frac{f_b}{2} \end{cases}$ Transmission limits: $\eta=\frac{R_b}{W}=2$

Hard to realize physically

Converge slowly

Roll-off system: $H(f)=\begin{cases} T_b & 0\le |f|\le \frac{1-\alpha}{2T_b} \\ \frac{T_b}{2}\left\{1+\cos\left[\frac{\pi T_b}{\alpha}\left(|f|-\frac{1-\alpha}{2T_b}\right)\right]\right\} & \frac{1-\alpha}{2T_b}\le |f|\le \frac{1+\alpha}{2T_b} \\ 0 & |f|\ge \frac{1+\alpha}{2T_b} \end{cases}$

Decreased spectrum efficiency: $\eta=\frac{R_b}{W}=\frac{2f_N}{(1+\alpha)f_N}=\frac{2}{(1+\alpha)}$

Review

Partial Response Waveforms & Partial Response System

To guarantee fast decay and maximum spectrum efficiency

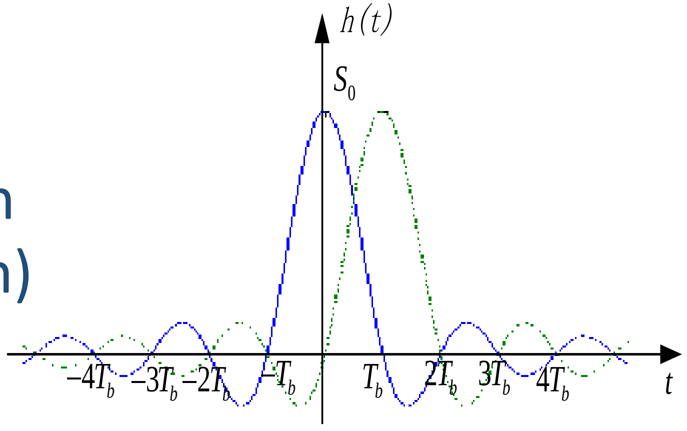
Weighted sum of several Sa function

(partial response waveform)

Decay fast

& easy to realize

& maximum spectrum efficiency



(b)

However: It introduces intersymbol interference!

Will the decision be affected by the intersymbol interference of the partial response system?

Partial Response System

How to determine the transmitted symbol at the receiver?

(Use first class partial response waveform as example)

- Assume the transmitted symbol the instant k: ak

The received signal symbol the instant k: $C_k = a_k + a_{k-1}$

The symbol at current instant & the inference from the previous symbol

The sample value Ck obtained at the k-th instant of the received waveform g(t) may have three values: -2, 0, and +2

If we know the previous transmitted symbol, then the current transmitted symbol is

$$a_k = C_k - a_{k-1}$$

Partial Response System

Problem: error will be propagate

Recovery of a_k is determined by C_k and a_{k-1}

If a_{k-1} is decided mistakenly, even if C_k is not affected by the noise, the a_k will be mis-decided due to the wrong a_{k-1}.

The wrong a_k is then affect the decision of a_{k+1}. Wrong a_{k+1} will further influence the decision of a_{k+2}

The symbol decisions after a_{k-1} will all be wrong.

Also called: error propagation phenomenon

Partial Response System

Error propagation: example

Binary sequence	**1**	**0**	**1**	**1**	**0**	**0**	**0**	**1**	**0**	**1**	**1**
Bipolar $\{a_k\}$	**+1**	**-1**	**+1**	**+1**	**-1**	**-1**	**-1**	**+1**	**-1**	**+1**	**+1**
Received without inference $\{C_k\}$		**0**	**0**	**+ 2**	**0**	**-2**	**-2**	**0**	**0**	**0**	**+2**
Received with inference $\{C_k'\}$		**0**	**0**	**+2**	**0**	**-2**	**0***	**0**	**0**	**0**	**+2**
Recovered $\{a_k'\}$	**+1**	**-1**	**+1**	**+1**	**-1**	**-1**	**+1***	**-1***	**+1***	**-1***	**+3***

- Since {Ck'} has an error, the {ak'} recovered is all wrong after one error occurs.
- A correct initial value (+1) is a must when restoring {ak'}, otherwise, it is impossible to get the correct {ak'} sequence even if there is no transmission error

Partial Response System

The reason for error propagation:

- The partial response waveform is employed, which introduced interference of adjacent symbols
- Originally independent symbols become correlated symbols (causes error propagation)

$C_k = a_k + a_{k-1}$ is also called correlation coding

The correlation coding is necessary to obtain the expected partial response signal spectrum, but it brings the problem of error propagation.

How to solve it?

Partial Response System

Solution: Precoding

The error propagation problem is caused by correlation coding

Precoding can be performed before the correlation coding to avoid the correlations of adjacent symbols

Precoding rule: $b_k = a_k \oplus b_{k-1}$ $(a_k = b_k \oplus b_{k-1})$ $\oplus$ is modulo 2 addition

Correlation encoding: Take the precoded $\{b_k\}$ as the input symbol sequence of the system

Obtain: $C_k = b_k + b_{k-1}$

The decision rule: modulo 2 decision

$$C_k = \begin{cases} \pm 2, & \text{decision:0} \\ 0, & \text{decision:1} \end{cases}$$

$$[C_k]\text{mod}2 = [b_k + b_{k-1}]\text{mod}2 = b_k \oplus b_{k-1} = a_k \quad (a_k = [C_k]\text{mod}2)$$

Then, no need to know a_{k-1} in advance

Partial Response System

Precoding example

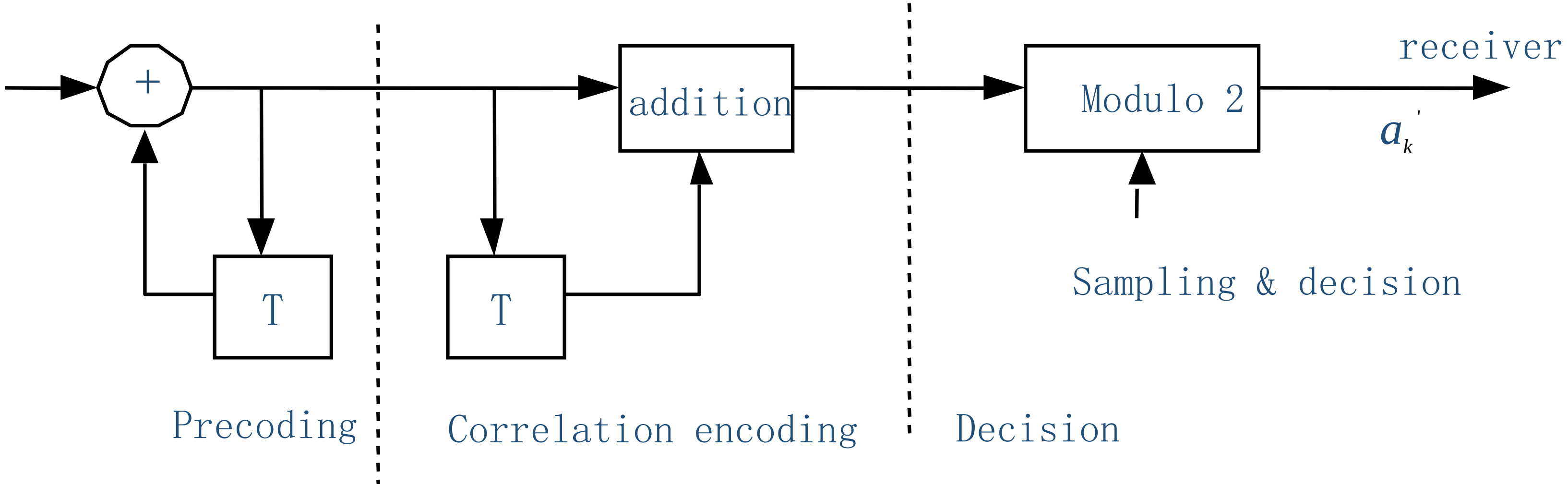
Binary sequence {ak}	**1**	**0**	**1**	**1**	**0**	**0**	**0**	**1**	**0**	**1**	**1**
{bk-1}	**0**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**0**	**0**	**1**
{bk}	**1**	**1**	**0 (-1)**	**1**	**1**	**1**	**1**	**0 (-1)**	**0 (-1)**	**1**	**0 (-1)**
Received without inference {Ck}	**0**	**+2**	**0**	**0**	**+2**	**+2**	**+2**	**0**	**-2**	**0**	**0**
Received with inference {Ck'}	**0**	**+2**	**0**	**0**	**+2**	**+2**	**+2**	**0**	**0**	**0**	**0**
Recovered {ak'}	**1**	**0**	**1**	**1**	**0**	**0**	**0**	**1**	**1**	**1**	**1**

$$C_k = \begin{cases} \pm 2, & \text{decision:0} \\ 0, & \text{decision:1} \end{cases}$$

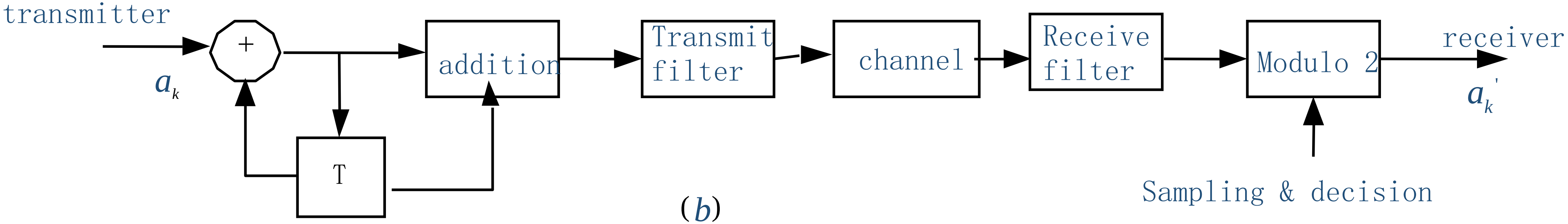
ak can be directly obtained from Ck, so the error will not propagate down and limit to the wrong symbol itself, since precoding removes the correlation between symbols.

Partial Response System

The diagram of the first class partial response system



(*a*)



(*b*)

Partial Response System

Advantage & Disadvantage

Advantage：

- Achieve the maximum spectrum efficiency of 2B/Hz (physical realizable)
- The decay of the tail is fast

Disadvantage：

- When the input data is L-ary, the number of relevant coding levels of the part response waveform must exceed L. Then, the anti-noise performance of the partial response system is worse than ideal low pass system. (decreased reliability)

Optimal receiver

How does the noise affect the transmission and how to deal with it?

- Problem: Reliability in the transmission
- Object: Noise
- Methods: How to design filters to reduce the effect of the noise

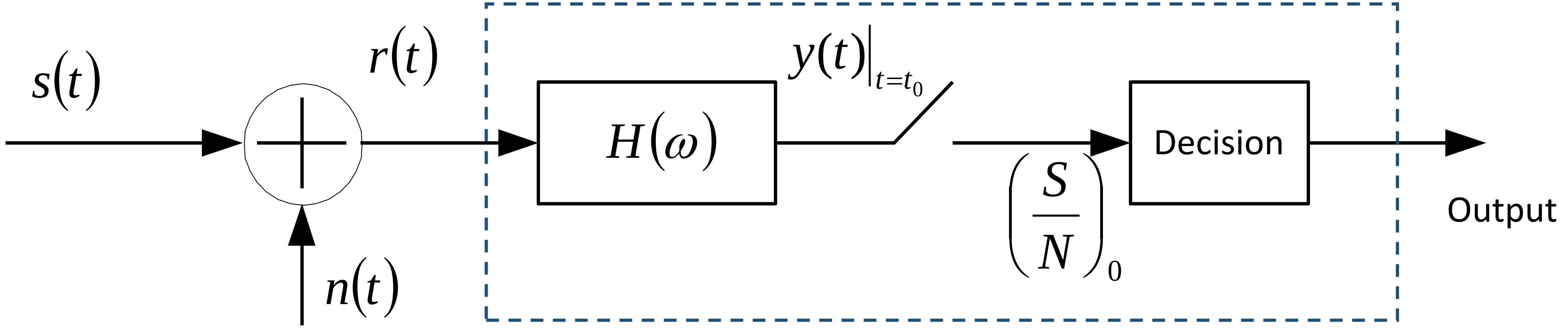
Binary deterministic signal

- Problem description
- Matched filter
- Optimal detector
- Optimal baseband transmission
- Anti-noise performance

Optimal receiver

Problem description:

Assume that no intersymbol interference but only noise, a binary symbols are transmitted



Received waveform：
$$\left.\begin{aligned} H_0 &: r(t) = s_0(t) + n(t) \\ H_1 &: r(t) = s_1(t) + n(t) \end{aligned}\right\} \quad 0 \le t \le T_b$$

The purpose of the receiver: determine the transmitted bit is 0 or 1 based on the received symbol

Intersymbol interference

Problem description:

- Optimal receive filter: For the received signal r(t), how to design a filter to output y(t) that is most conducive to the decision

When the filter can maximize the output SNR, the filter is the best linear filter (also called matched filter)

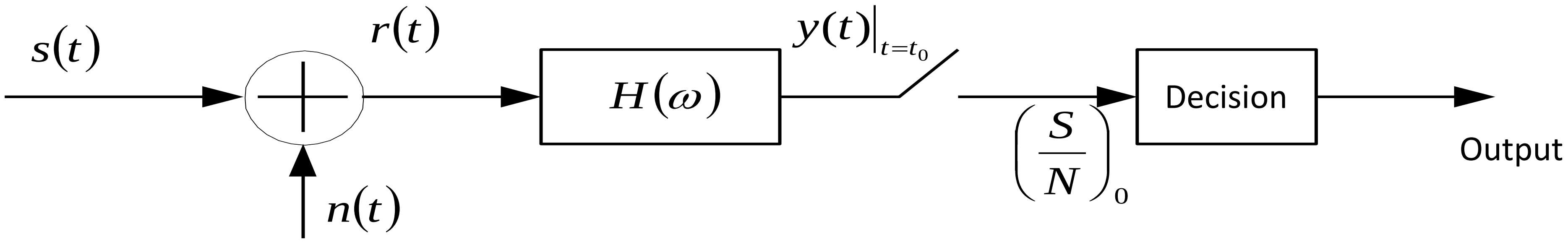
- Optimal detector: For the filter output y(t), design the optimal detector to minimize the bit error rate between the recovered sequence and the transmitted sequence

When the symbol error is minimized, the detector is the optimal detector.

$$P_e = P(\text{decide on } 1 | \text{ transmit } 0)P(\text{transmit } 0) + P(\text{decide on } 0 | \text{ transmit } 1)P(\text{transmit } 1)$$

Matched filter

Matched filter: A linear filter that output the maximum SNR (the ratio of the instantaneous signal power and the noise power)



The waveform in the receiver end (sum of signal and noise):

$$r(t) = s(t) + n(t)$$

AWGN whose power spectrum density is $n_0/2$

The input digital signal (Frequency spectrum denoted as P(ω))

Matched filter

Assume the output of the filter:

$$y(t) = s_0(t) + n_0(t)$$

where

Signal: $s_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_0(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega) H(\omega) e^{j\omega t} d\omega$

Spectrum of the filter

The average power of the noise：

$$N_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_{n_0}(\omega) d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_{n_i}(\omega) |H(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{n_0}{2} |H(\omega)|^2 d\omega = \frac{n_0}{4\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega$$

Matched filter

The SNR of the linear filter output signal at time t0 is

$$r_0 = \frac{|s_0(t_0)|^2}{N_0} = \frac{\left| \frac{1}{2\pi} \int H(\omega) S(\omega) e^{j\omega t_0} d\omega \right|^2}{\frac{n_0}{4\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega}$$

r0 is related to S(ω) and H(ω)

When given the input signal s(t), then, r0 is only related to H(ω)

Given s(t), H(ω) should be designed to find the maximum SNR

Matched filter

How to find the matched filter?

$$r_0 = \frac{\left|\frac{1}{2\pi}\int H(\omega)S(\omega)e^{j\omega t_0}d\omega\right|^2}{\frac{n_0}{4\pi}\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega}$$

Cauchy Schwartz Inequality:

$$\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)Y(\omega)d\omega\right|^2 \le \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega \frac{1}{2\pi}\int_{-\infty}^{\infty}|Y(\omega)|^2 d\omega$$

Equation holds when: $X(\omega) = KY^*(\omega)$

let $\begin{pmatrix} X(\omega) = H(\omega) \\ Y(\omega) = S(\omega)e^{j\omega t_0} \end{pmatrix}$ $\Rightarrow$ $r_0 \le \dfrac{\frac{1}{4\pi^2}\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega \int_{-\infty}^{\infty}\left|S(\omega)e^{j\omega t_0}\right|^2 d\omega}{\frac{n_0}{4\pi}\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega} = \dfrac{\frac{1}{2\pi}\int_{-\infty}^{\infty}|S(\omega)|^2 d\omega}{\frac{n_0}{2}}$

Since $\frac{1}{2\pi}\int_{-\infty}^{\infty}|S(\omega)|^2 d\omega = \int_{-\infty}^{\infty}s^2(t)dt = E$ $\Rightarrow$ $r_0 \le \dfrac{2E}{n_0}$

Matched filter

The maximum output SNR of the filter: $$r_{0\max} = \frac{2E}{n_0}$$

When the output SNR is maximal, the corresponding filter:

$$H(\omega) = KS^*(\omega)e^{-j\omega t_0}$$

Transfer function of the optimal linear filter

1、With AWGN, the linear filter based on the above formula will be able to obtain the maximum SNR 2E/n0 at time t0. (optimal linear filter in the sense of maximum SNR)

2、Transmission characteristics are consistent with the complex conjugate of the signal spectrum, so it is also called a matched filter.

Matched filter

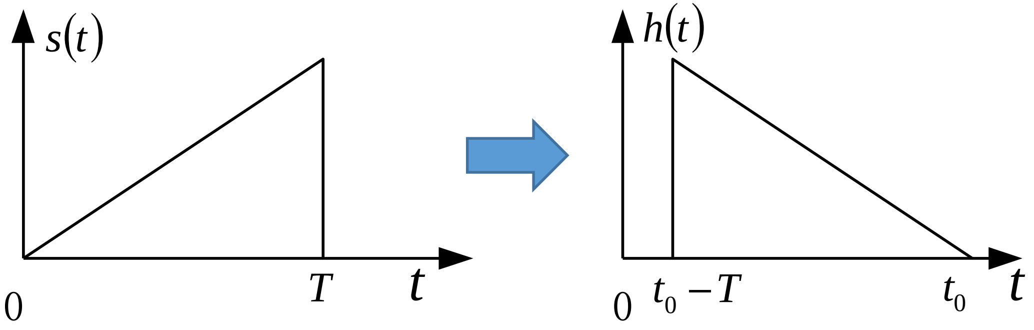
The expression of the matched filter in time domain:

$$h(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}H(\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}KS^{*}(\omega)e^{-j\omega t_0}e^{j\omega t}d\omega$$

$$=\frac{K}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}s(\tau)e^{-j\omega\tau}d\tau\right]^{*}e^{-j\omega(t_0-t)}d\omega=K\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{j\omega(\tau-t_{0t})}d\omega\right]s(\tau)d\tau$$

$$=K\int_{-\infty}^{\infty}s(\tau)\delta(\tau-t_0+t)d\tau=Ks(t_0-t)$$

$$h(t)=Ks(t_0-t)$$



1、The unit impulse response h (t) of the matched filter is the mirror function of the input signal s (t)

2、t0 is the instant that output the maximum SNR.

Matched filter

Physical realizability

Requirement on the signal

$$h(t)=\begin{cases} Ks(t_0-t), & t\ge 0 \\ 0, & t<0 \end{cases}$$

$$\Rightarrow \quad s(t_0-t)=0, t<0$$

$$\Rightarrow \quad s(t)=0, t>t_0$$

For a physically realizable matched filter, its input signal s(t) must end before it outputs the maximum SNR (t0)

- If the input signal ends at time T , for a physically realizable matched filter, the time t0 when output maximum SNR must be after the end of its input signal, (t0 ≥ T)
- For the receiver, t0 is the time delay, and it is usually hoped that the time delay may be small, (in general, t0=T)

Matched filter

The output of the matched filter:

Input signal: s(t)

Output signal: $s_o(t)$

$$s_0(t) = s(t) * h(t)$$
$$= \int_{-\infty}^{\infty} s(t-\tau) h(\tau) d\tau$$
$$= \int_{-\infty}^{\infty} s(t-\tau) K s(t_0-\tau) d\tau$$

Let $t_0-\tau=x$

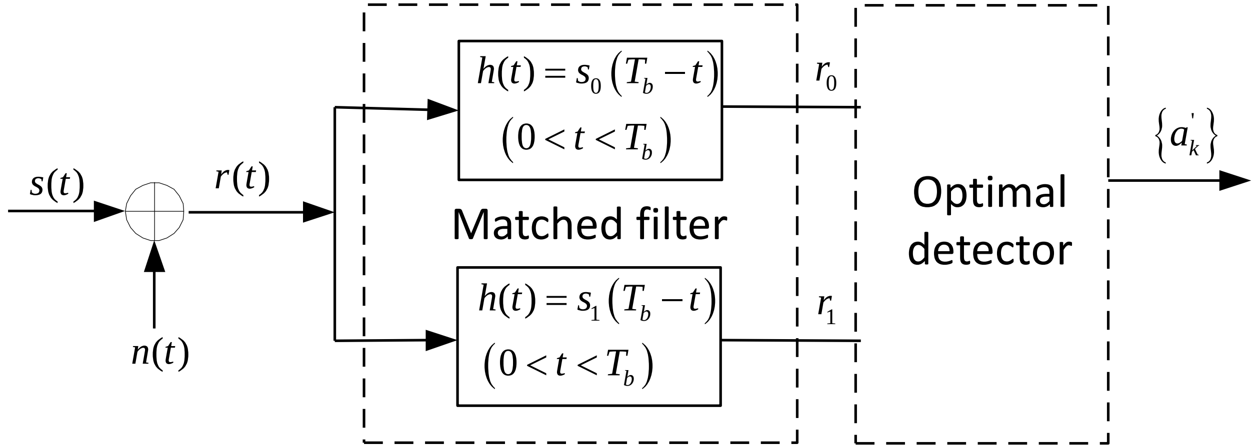
$$s_0(t) = K\int_{-\infty}^{\infty} s(x)s(x+t-t_0)dx = KR(t-t_0)$$

(R(t) is the autocorrelation function of the input signal s(t))

- The matched filter can be seen as a correlator that finds the autocorrelation function of the input signal, and it obtains the maximum SNR r_{0max}=2E/n0 at time t0.
- Since the output SNR has nothing to do with the constant K, usually K=1.

Matched filter

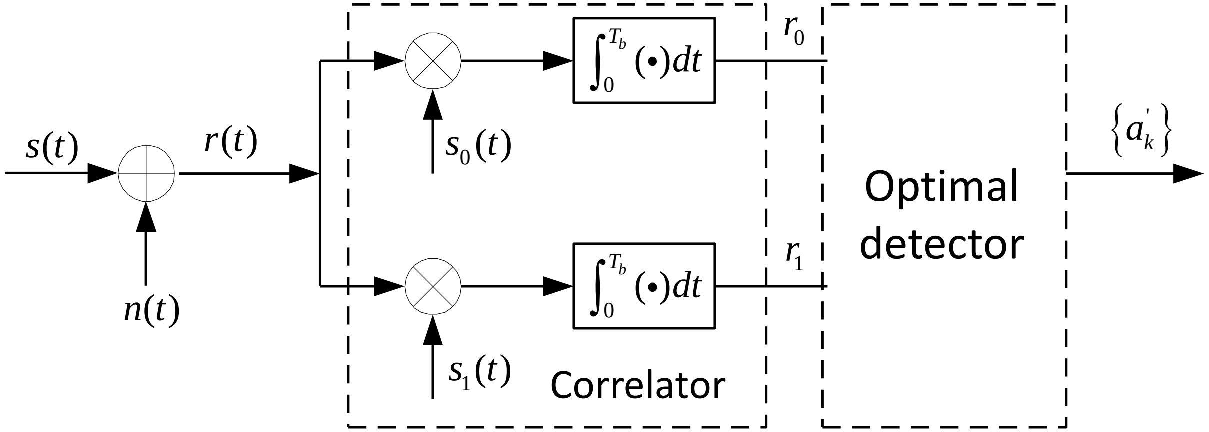
Optimal receiving (binary signal):



When the transmitted signal is S0(t)

$$r_0 = K\int_0^{T_b} s_0(t)s_0(t)dt$$

$$r_1 = K\int_0^{T_b} s_0(t)s_1(t)dt$$



The matched filter output at t=Tb is exactly same as the correlator output, so the matched filter can be replaced by the correlator

Optimal Detector

Question: How to design a detection criterion according to decision variable y to minimize the total error probability ?

$$\begin{cases} H_0 & \text{transmitted ``0''} \\ H_1 & \text{transmitted ``1''} \end{cases}$$

$P(\text{decide on } 1 \mid \text{transmit } 0) \longrightarrow P\,(D_1/H_0) \rightarrow$ The probability of error when the assumption H0 is true but the detection result is H1

False Alarm Probability $\longleftarrow$ When transmitted "0", the probability of deciding 1 mistakenly

$P(\text{decide on } 0 \mid \text{transmit } 1) \longrightarrow P\,(D_0/H_1) \rightarrow$ The probability of error when the assumption H1 is true but the detection result is H0

Missing alarm Probability $\longleftarrow$ When transmitted "1", the probability of deciding 0 mistakenly

$$P_e = P(\text{decide on } 1 \mid \text{transmit } 0)P(\text{transmit } 0) + P(\text{decide on } 0 \mid \text{transmit } 1)P(\text{transmit } 1)$$

$$p_e = p\left(D_1 \middle| H_0\right) p\left(H_0\right) + p\left(D_0 \middle| H_1\right) p\left(H_1\right)$$

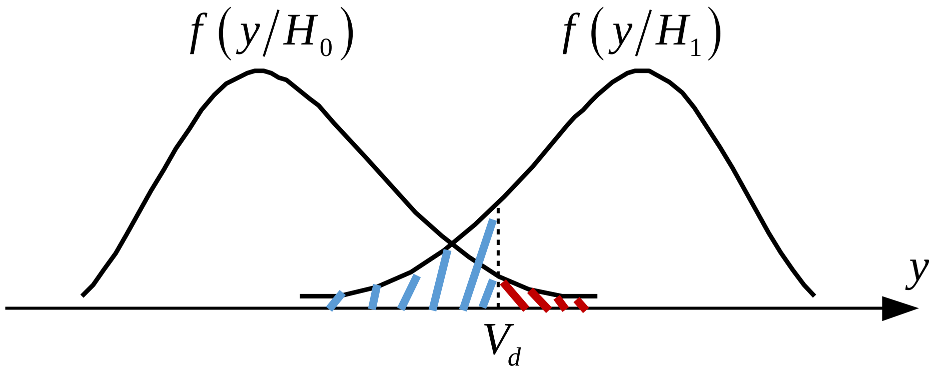
Need to find the probability distribution of decision variable y and decision threshold

Optimal Detector

Mathematical description: How to design a detection criterion according to decision variable y to minimize the total error probability ?

When given the characteristic of the noise is given, the probability density function f(y/H0) & f(y/H1) of the decision variable y can be written.

When the probability density function f(y/H0) & f(y/H1) of the decision variable y, how to find the threshold to minimize the error rate?



Optimal Detector

A General solution：

When given a threshold Vd, the decision criterion is

$$\begin{cases} y > V_d & \text{decide on } 1 \\ y < V_d & \text{decide on } 0 \end{cases}$$

Then the error rate Pe:

$$p_e = P(H_0)\int_{V_d}^{\infty} f(y/H_0)dy + P(H_1)\int_{-\infty}^{V_d} f(y/H_1)dy$$

Let dPe/dVd=0 to find the optimal threshold Vd* that minimize the error rate

$dp_e/dV_d = P(H_0) f(V_d/H_0) - P(H_1) f(V_d/H_1) = 0$

$$\frac{f(V_d/H_1)}{f(V_d/H_0)} = \frac{P(H_0)}{P(H_1)} = \lambda_o$$

Solve it to get Vd*

Substitute the threshold Vd* into the Pe and get the minimum error rate

$$p_e = P(H_0\int_{V_d^*}^{\infty} f(y/H_0)dy + P(H_1)\int_{-\infty}^{V_d^*} f(y/H_1)dy$$

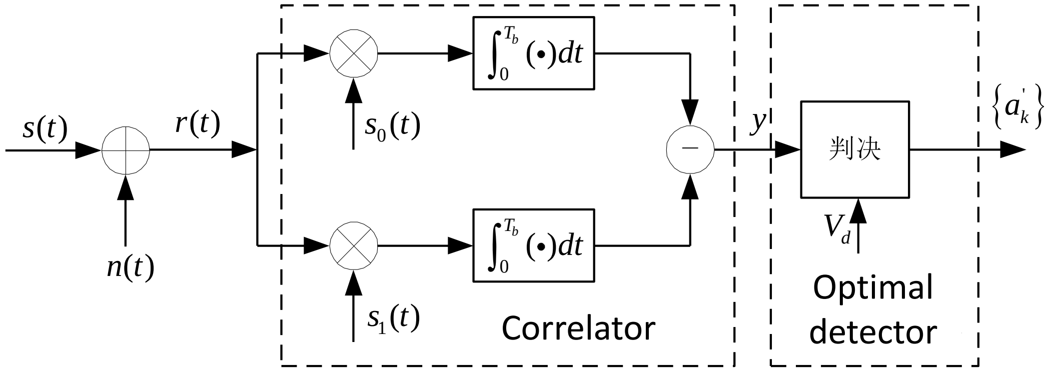
Optimal Detector

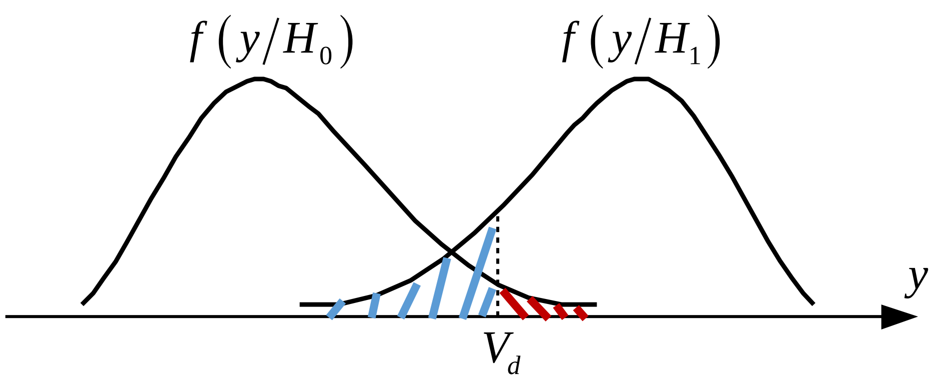
The optimal detector of binary deterministic signal

- Binary deterministic signal : A signal whose variables (such as frequency, phase, etc) of the transmitted binary signal are known at the receiving end
- The decision variable is a two-dimensional signal $\vec{r} = [r_0, r_1]$

(The decision area can be determined by one dimension threshold Vd)

- If the noise n(t) is AWGN, it is easy to obtain the optimal detector





Optimal Detector

How to find $f(y/H_1)$

If H1 is true, then the received signal should be r(t)=s1(t)+n(t)

The output of the correlator:

$$y=\int_0^{T_b} r(t)[s_1(t)-s_0(t)]dt=\int_0^{T_b}[s_1(t)+n(t)][s_1(t)-s_0(t)]dt$$

$$=\int_0^{T_b} s_1(t)[s_1(t)-s_0(t)]dt+\int_0^{T_b} n(t)[s_1(t)-s_0(t)]dt$$

$$y=E_b(1-\rho)+\int_0^{T_b} n(t)[s_1(t)-s_0(t)]dt$$

$$E_b=\int_0^{T_b} S_0^2(t)dt=\int_0^{T_b} S_1^2(t)dt$$

$$\rho=\frac{\int_0^{T_b} s_1(t)s_0(t)dt}{E_b}$$

$\Longrightarrow$ Correlation coefficient between two signals

the signal after passes through the receiving filter

Noise after passing through the receive filter

Optimal Detector

Noise $\xi = \int_0^{T_b} n(t)\left[s_1(t) - s_0(t)\right]dt$

- n(t): Gaussian stationary random process
- After passing through a linear system: Gaussian stationary random process.
- Its probability density function is determined only by its mean and variance

Mean $E[\xi] = E\left\{\int_0^{T_b} n(t)\left[s_1(t) - s_0(t)\right]dt\right\} = \int_0^{T_b} E\{n(t)\}\left[s_1(t) - s_0(t)\right]dt = 0$

Variance $\sigma_\xi^2 = D[\xi] = E\left[\xi^2\right] = E\left\{\int_0^{T_b}\int_0^{T_b} n(t)\left[s_1(t) - s_0(t)\right]n(\tau)\left[s_1(\tau) - s_0(\tau)\right]d\tau dt\right\}$

$$= \int_0^{T_b}\int_0^{T_b} E\left[n(t)n(\tau)\right]\left[s_1(t) - s_0(t)\right]\left[s_1(\tau) - s_0(\tau)\right]d\tau dt$$

$$E\left[n(t)n(\tau)\right] = \frac{n_0}{2}\delta(t-\tau) = \begin{cases} \dfrac{n_0}{2} & t = \tau \\ 0 & t \neq \tau \end{cases}$$

$$\sigma_\xi^2 = \frac{n_0}{2}\int_0^{T_b}\left[s_1(t) - s_0(t)\right]^2 dt = \frac{n_0}{2}\cdot 2E_b(1-\rho) = n_0 E_b(1-\rho)$$

Optimal Detector

The PDF of ξ $\quad f(\xi)=\frac{1}{\sqrt{2\pi}\sigma_\xi}\exp\left\{-\frac{\xi^2}{2\sigma_\xi^2}\right\} \quad \sigma_\xi^2=n_0E_b(1-\rho)$

The PDF of the decision variable y

$$f(y|H_1)=\frac{1}{\sqrt{2\pi}\sigma_\xi}\exp\left\{-\frac{(y-a)^2}{2\sigma_\xi^2}\right\}$$

Let a=$E_b(1-\rho)$

If H_0 is true, then the received signal is $r(t)=s_0(t)+n(t)$. After the correlator:

$$\begin{aligned} y=\int_0^{T_b} r(t)[s_1(t)-s_0(t)]dt &= \int_0^{T_b}[s_0(t)+n(t)][s_1(t)-s_0(t)]dt \\ &= \int_0^{T_b} s_0(t)[s_1(t)-s_0(t)]dt+\int_0^{T_b} n(t)[s_1(t)-s_0(t)]dt \\ &= E_b(\rho-1)+\int_0^{T_b} n(t)[s_1(t)-s_0(t)]dt \end{aligned}$$

Optimal Detector

If "1" is transmitted $y = E_b(1-\rho) + \int_0^{T_b} n(t)\left[s_1(t) - s_0(t)\right]dt$ $f(y|H_1) = \frac{1}{\sqrt{2\pi}\sigma_\xi}\exp\left\{-\frac{(y-a)^2}{2\sigma_\xi^2}\right\}$

If "0" is transmitted $y = E_b(\rho-1) + \int_0^{T_b} n(t)\left[s_1(t) - s_0(t)\right]dt$ $f(y|H_0) = \frac{1}{\sqrt{2\pi}\sigma_\xi}\exp\left\{-\frac{(y+a)^2}{2\sigma_\xi^2}\right\}$

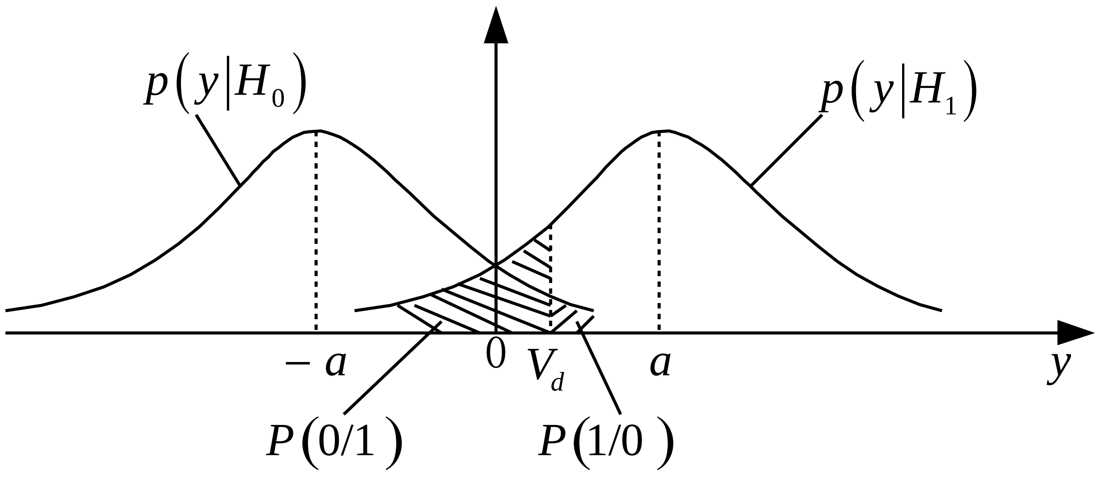
$$\frac{f(V_d/H_1)}{f(V_d/H_0)} = \frac{P(H_0)}{P(H_1)} = \lambda_o$$

$$\Downarrow$$

$$V_d^* = \frac{\sigma_\xi^2}{2a}\ln\frac{p(H_0)}{p(H_1)} = \frac{n_0}{2}\ln\frac{p(H_0)}{p(H_1)}$$

$$\Downarrow$$

0



(1、0 have the same probabilities)

Optimal Detector

$$p_{e\min} = P(H_0)\int_{-\infty}^{V_d^*} f(y/H_0)dy + P(H_1)\int_{V_d^*}^{\infty} f(y/H_1)dy \ = \int_0^{\infty} f(y/H_1)dy$$

$$= \frac{1}{\sqrt{2\pi\sigma_\xi^2}}\int_0^{\infty}\frac{1}{\sqrt{2\pi\sigma_\xi^2}}\exp\left(-\frac{(y-a)^2}{2\sigma_\xi^2}\right)dy = \frac{1}{2}erfc\left[\frac{a}{\sqrt{2}\sigma_\xi}\right]$$

$a=E_b(1-\rho)$ $\quad$ $\sigma_\xi^2 = n_0 E_b(1-\rho)$

When “1”、“0” have the same probabilities

$$p_e = \frac{1}{2}erfc\left[\sqrt{\frac{E_b(1-\rho)}{2n_0}}\right]$$

The general form of the error rate of the optimal receiver of the binary deterministic signal

Assume “1”、“0” have the same power

Optimal Detector

The complementary error function erfc(x) is a strictly monotonically decreasing function. Therefore, as the independent variable x increases, the value of the function decreases

$$p_e = \frac{1}{2} erfc\left[\sqrt{\frac{E_b(1-\rho)}{2n_0}}\right] \quad \text{min} \quad \xrightarrow{\rho = \int_0^{T_b} s_1(t)s_0(t)dt / E_b} \quad E_b(1-\rho)/2n_0 \quad \text{max}$$

When the signal energy Eb and the noise power n0 are fixed, the error rate is a function of the cross-correlation coefficient ρ

Design the signals at the transmitter so that the cross-correlation coefficient ρ between the signals is as small as possible

$-1 \le \rho \le 1$

$\rho = -1$ $\quad p_e = \frac{1}{2} erfc\left[\sqrt{\frac{E_b}{n_0}}\right]$ When the transmitted signal is the optimal waveform (such as bipolar signal)

$\rho = 0$ $\quad p_e = \frac{1}{2} erfc\left[\sqrt{\frac{E_b}{2n_0}}\right]$ When the transmitted signal is not the optimal waveform (such as orthogonal frequency signal)

$\rho = 1$ $\quad p_e = \frac{1}{2}$ When the transmitted signal is totally the same

Optimal Detector

Assume：

“1”“0” have the same probability

“1”“0” have different power

→ For example, unipolar baseband signal, 2ASK signal

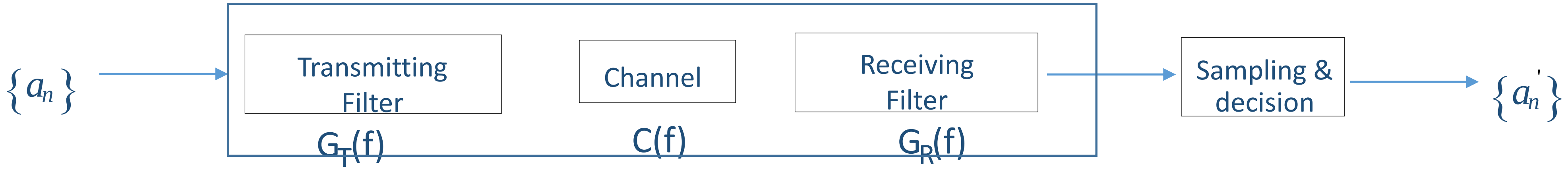
Symbol error rate $$p_e = \frac{1}{2} erfc\left[\sqrt{\frac{E_b}{4n_0}}\right]$$

Optimal decision threshold

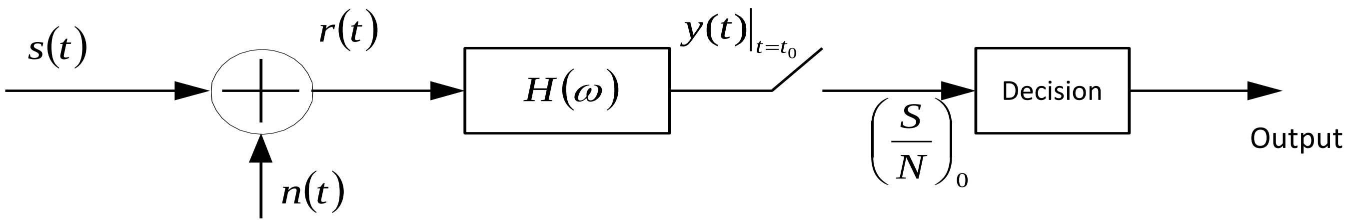
$$V_d^* = \frac{a}{2} + n_0 \ln \lambda_o = \frac{a}{2} + n_0 \ln \frac{p(H_0)}{p(H_1)}$$

All formulas apply not only to baseband signals but also to bandpass signals

Optimal Detector



Intersymbol interference problem $H(f)=G_T(f)C(f)G_R(f)$ $\Longrightarrow$ $\sum_{m=-\infty}^{\infty} H(f+m/T_b)=T_b$



Noise Problem $H(\omega)$ is designed based on minimal error criterion ⟹ Match filter

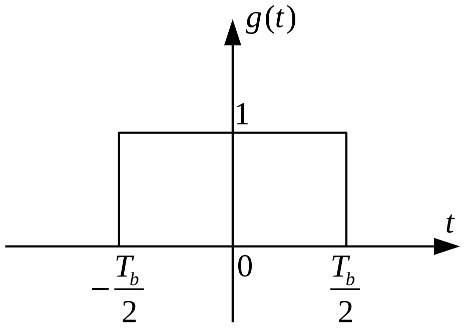
Optimal baseband transmit system:

Eliminate the intersymbol interference and have the minimal probability of error

Design problem: how to design the transmitting filter $G_T(f)$ and the receiving filter $G_R(f)$

Optimal Detector

Example: Assume the baseband transmit system transmits a bipolar signal, where g1(t)=–g2(t)=g(t), please find the optimal baseband system and its error rate.



Solution:

When transmit "0", the transmitted waveform is $s_0(t)=g_T(t)$

When transmit "1", the transmitted waveform is $s_1(t)=-g_T(t)$

The optimal receiver: $g_R(t)=g_T(T_b-t)$ $\quad G_R(f)=G_T^*(f)e^{-j\omega T_b}$

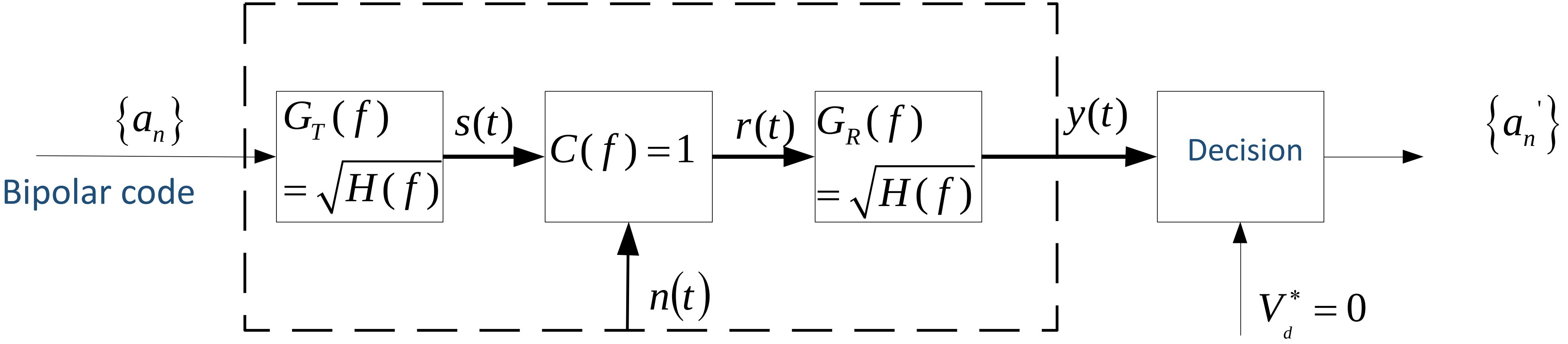
Correlation coefficient ρ=–1

Assume the channel is ideal, i.e. C(f)=1

$$\begin{cases} H(f)=G_T(f)\cdot G_R(f) \\ G_R(f)=G_T^*(f)e^{-j\omega T_b} \end{cases} \Longrightarrow G_R(f)=G_T(f)=\sqrt{H(f)}$$

Optimal Detector

The optimal baseband transmit system for the bipolar signal of equal probability



$$p_e = \frac{1}{2} erfc\left[\sqrt{\frac{E_b}{n_0}}\right]$$

Optimal Detector

The design of the optimal baseband transmit system:

- Find an overall transmit system H (f) without intersymbol interference
- Split H (f) into two identical part.
- One part is used as transmit filter, while the other is used as receiving filter.
- The baseband system designed with the above method is an optimal baseband transmit system with minimum error rate when the transmitting power is fixed.

Thank you！

Exercise

Fill in the blank

(1) The match filter refers to an optimal linear filter with the Gaussian white noise interference under the sense of ______________ .

(2) The maximum output SNR of the match filter is ____________, and the maximum SNR appears when _________________. The match filter in time domain and frequency domain can be expressed as ___________ and _____________.

(3) The optimal receiver includes the design of ___________ and _____________.

(4) The purpose of the optimal receiver is to ________________.

(5) When deriving the symbol error rate for the binary digital transformation system, three steps are commonly used, including __________、 _____________、 ______________.

Exercise

Ex1: For unipolar baseband waveform, please prove that the optimal decision threshold is $V_d^* = \frac{E_b}{2} + \frac{n_0}{2}\ln\frac{P(0)}{P(1)}$. Furthermore, when P(1)=P(0), the system symbol error rate is $P_e = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{E_b}{4n_0}}\right)$

MATLAB

(1) Generate a random 0/1 sequence (10 digits) and transform it to a bipolar symbol sequence (symbol duration time is Tb=0.001), plot the generated symbol sequence.

(2) Add noise to the symbol sequence as received signal (for simplicity, can use randn() directly), plot the received signal.

(3) Match filter the received signal and plot the filtered signal

(4) Based on the filtered signal, decide on the signal and compared it to the transmitted signal.