Principles of Communications

Chapter 4 — Digitization of Analog Signal

Zhen Chen

Introduction

What is digitization of analog signal and Why ?

Analog signal → ADC → Digital signal

ADC: Analog-to-digital converter

Analog signal

Concept: Continuously and infinitely varies with some time-varying parameter

Examples: Original speech signal

Image signal

Characteristics:

1 low immunity to noise and interference

2 unfriendly to long distance propagation

3 Hard for computer to process continuous value

Digital signal

Represents data as a sequence of discrete values

binary signal

multivalued logic signal

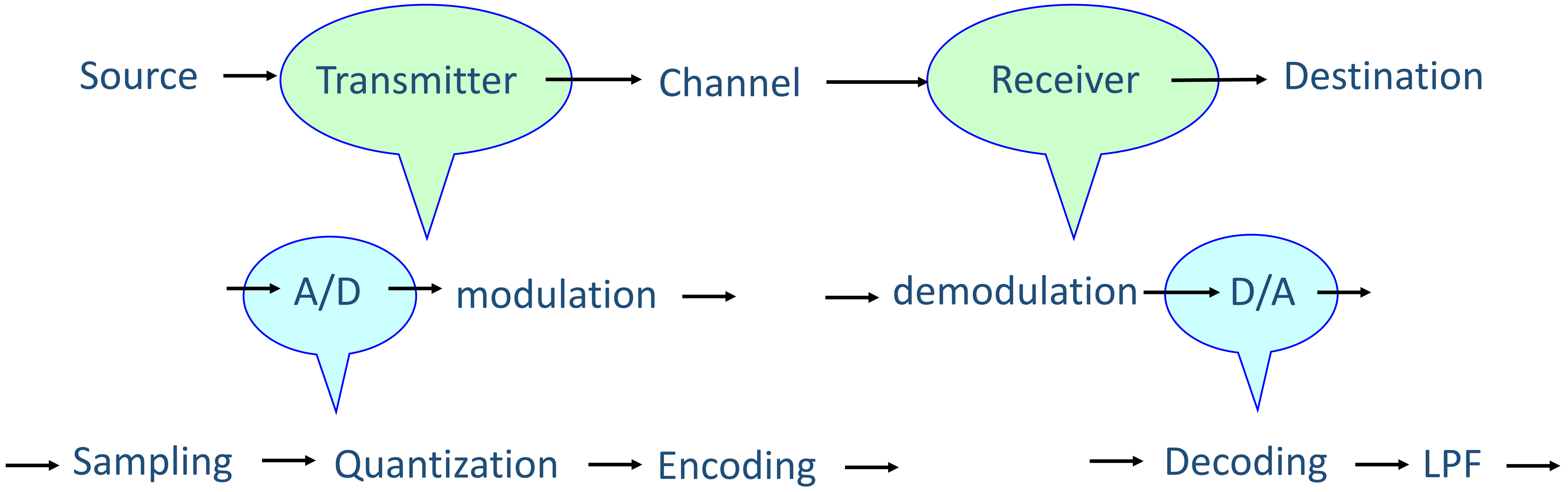
1 convey information with less noise, distortion, and interference

2 Easy to encrypt and compress

3 Easy to be processed in computer

Introduction

The framework of digital communication system



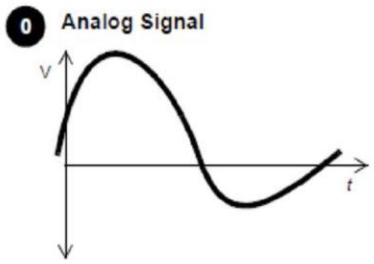
Introduction

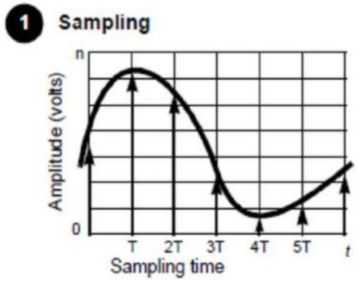
ADC includes 3 steps, including sampling, quantization and encoding

Sampling：Analog signal with continuous time and continuous value

↓

Time-discrete, value-continuous PAM signal

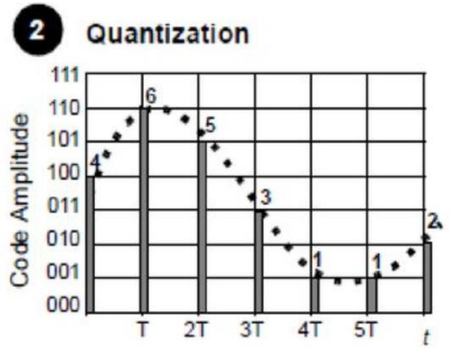




Quantization：Time-discrete, value-continuous PAM signal

↓

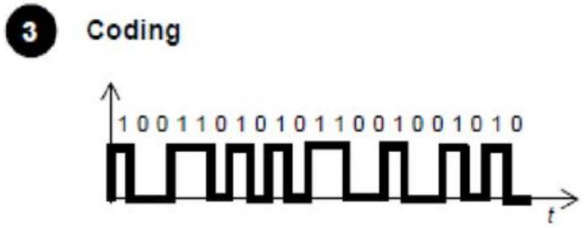
Time-discrete, value-discrete multilevel PAM signal



Coding: time-discrete, value-discrete multi-level PAM signal

↓

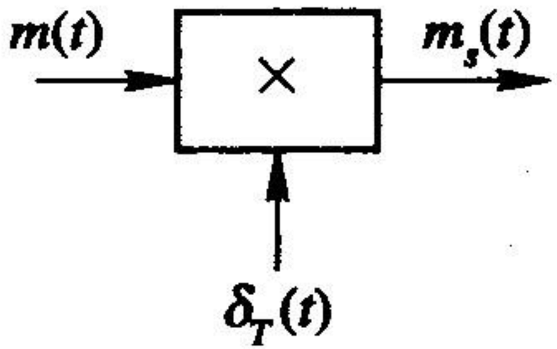
binary PCM signal



Sampling of Analog Signal

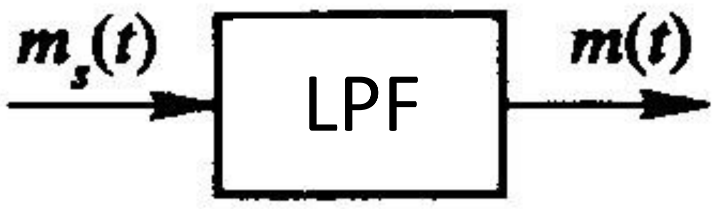
Sampling: Uniform sampling theorem for low-pass signals

Sampling Process



The original analog signal is multiplied with periodic unit impulse (impulse repeated frequency fs)

Signal Recovery



The original signal can be recovered from the sampled signal by applying a low-pass filter.

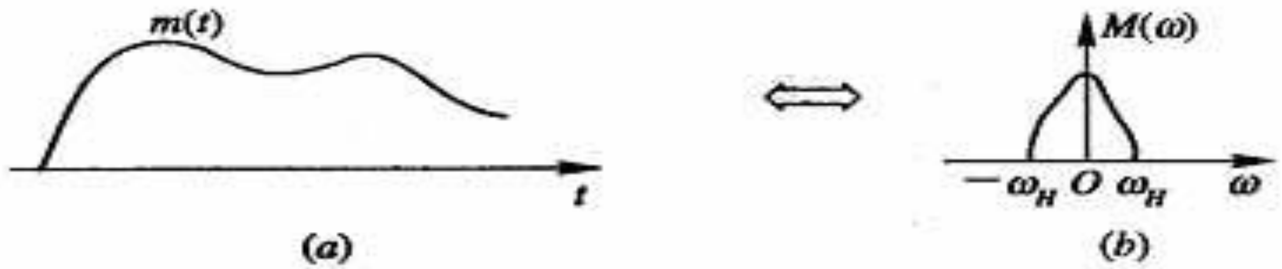
Sampling of Analog Signal

Sampling: Uniform sampling theorem for low-pass signals

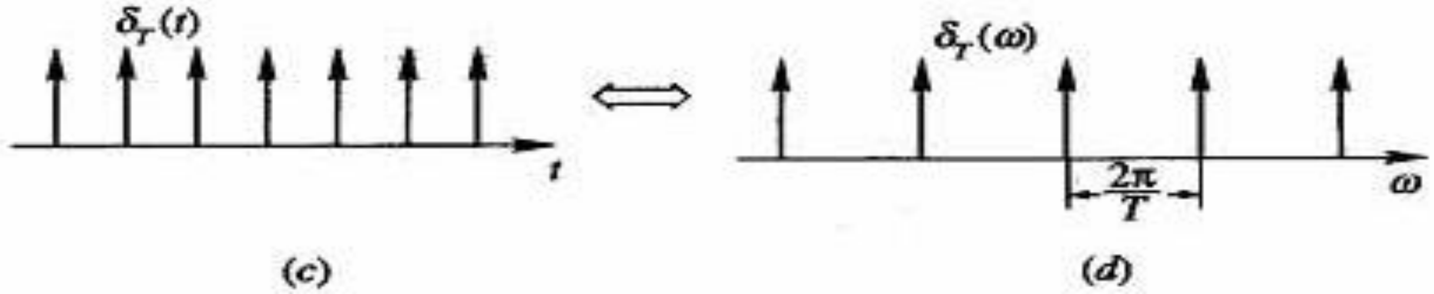
- Assume that a low-pass signal m(t) has frequencies varying from 0 to fh , uniformly sample the signal with the sampling rate of fs, if the sampling rate fs satisfies the condition fs>=2fh, then m(t) can be determined completely by these samples
- The lowest sampling frequency fs=2fh is called Nyquist sampling rate, the corresponding sampling time interval 1/2fh is called Nyquist sampling interval

Sampling of Analog Signal

The proof of uniform sampling theorem

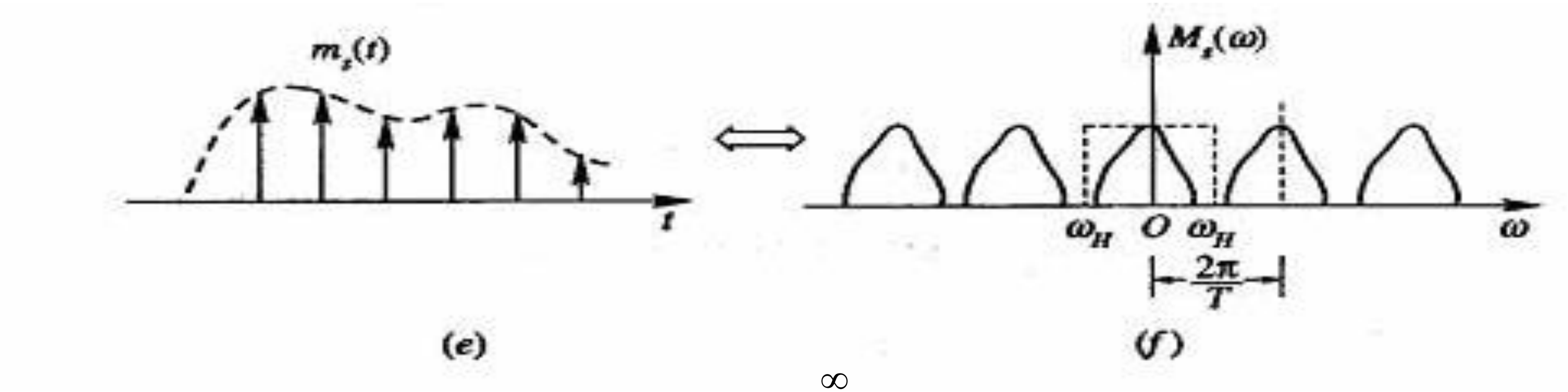


$$m(t) \Longleftrightarrow M(\omega)$$



$$\delta_T(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_s) \Longleftrightarrow \delta_T(\omega)=\frac{2\pi}{T_s}\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_s)$$

Sampling of Analog Signal



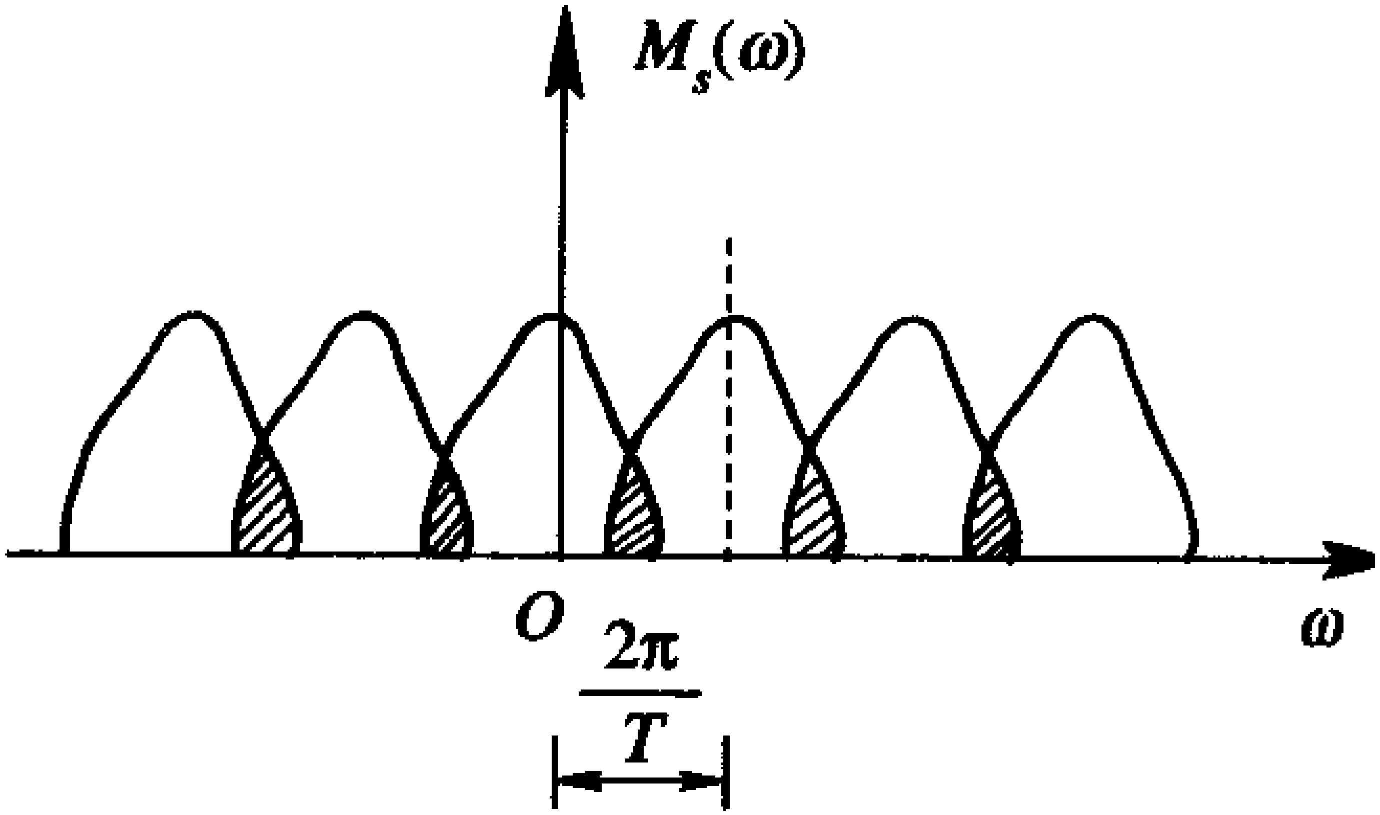
$$m_s(t) = m(t)\delta_T(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\delta(t - nT_s)$$

$$\Longleftrightarrow \quad M_s(\omega) = \frac{1}{2\pi}\left[M(\omega) * \delta_T(\omega)\right] = \frac{1}{T_s}\left[M(\omega) * \sum_{N=-\infty}^{\infty} \delta(\omega - n\omega_s)\right]$$

$$= \frac{1}{T_s}\left[\sum_{N=-\infty}^{\infty} M(\omega - n\omega_s)\right]$$

Sampling of Analog Signal

Sampling spectral aliasing



Sampling of Analog Signal

Uniform sampling theorem-The reconstructed signal

LPF in frequency domain: $M(\omega)=T_s[M_s(\omega)\times D_{\omega_H}(\omega)]$

Gate function in frequency domain

LPF in time domain: $$m(t)=T_s\left[m_s(t)*\frac{\omega_H}{\pi}\mathrm{Sa}(\omega_H t)\right]=m_s(t)*\mathrm{Sa}(\omega_H t)$$

$$=\sum_{n=-\infty}^{\infty}m(nT_s)\delta(t-nT_s)*\mathrm{Sa}(\omega_H t)$$

$$=\sum_{n=-\infty}^{\infty}m(nT_s)\mathrm{Sa}\left[\omega_H(t-nT_s)\right]$$

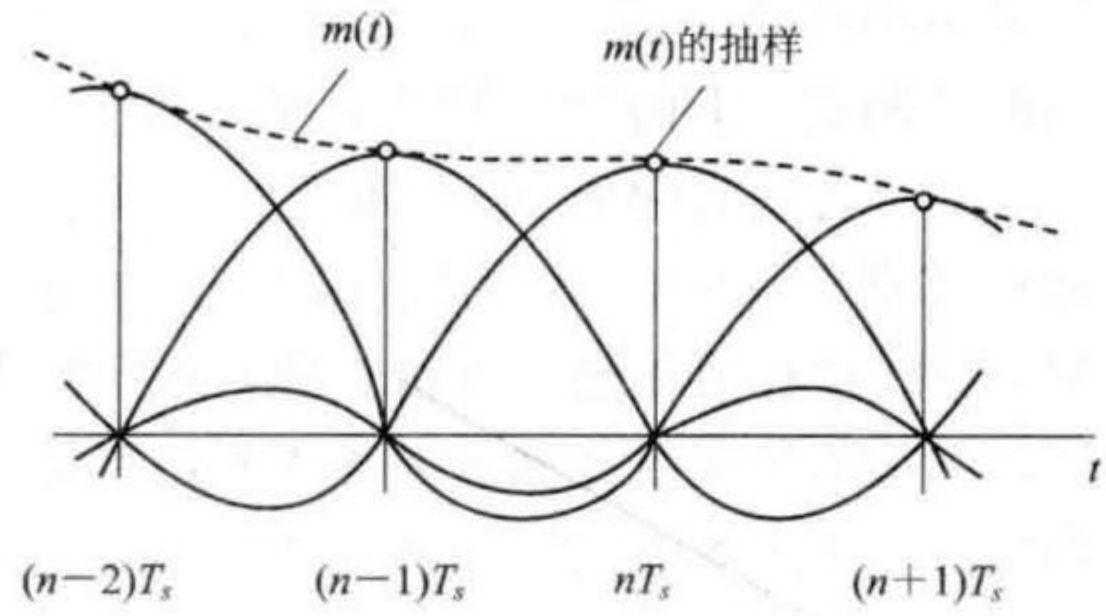
$$=\sum_{n=-\infty}^{\infty}m(nT_s)\frac{\sin\omega_H(t-nT_s)}{\omega_H(t-nT_s)}$$

Interpolation formula

Sampling of Analog Signal

Uniform sampling theorem-The reconstructed signal

A band-limited signal m(t) sampled at the Nyquist rate can be reconstructed from its samples using an interpolation formula

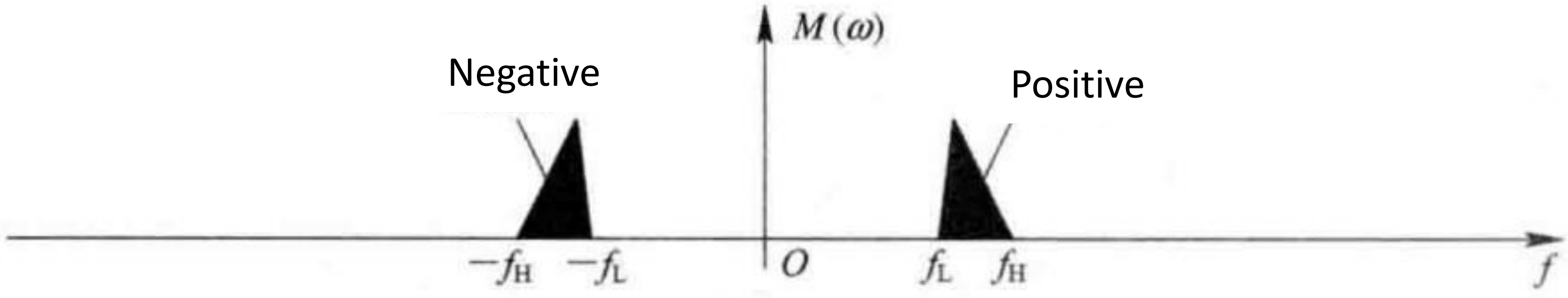


A waveform of Sa function is drawn with each sample as the peak value, and the synthesized waveform is m(t). Since the Sa function is closely related to the recovery of the signal after sampling, the Sa function is also called the sampling function.

Sampling of Analog Signal

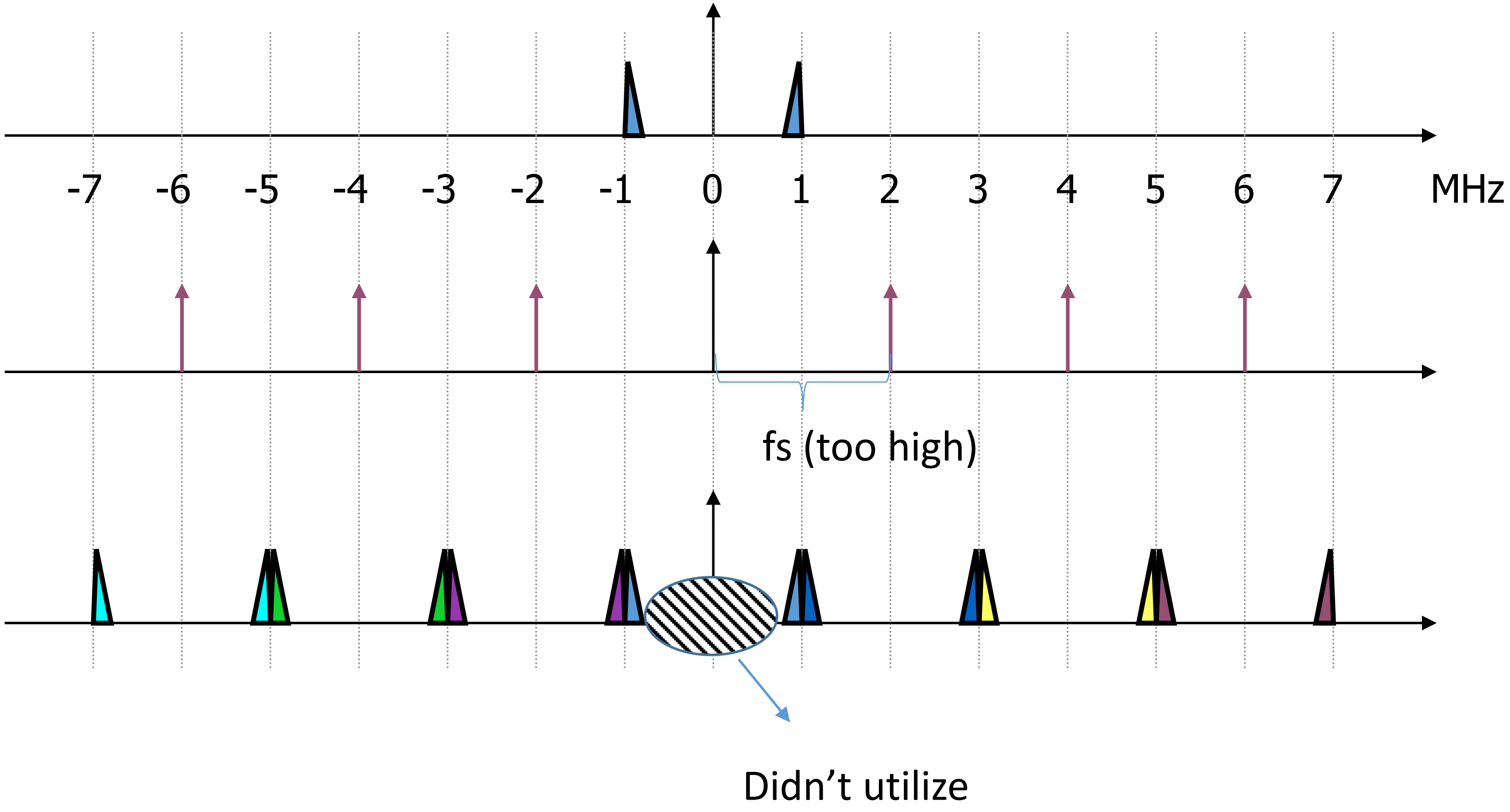
Sampling Bandpass Signals

- Assume that a band-pass signal m(t) has frequencies varying from fL to fH, the bandwidth is B = fH-fL. (Assume that B<<fH, then the signal can be viewed as a narrow band signal)
- The bandpass signal sampling theorem should be different from that of low-pass signal sampling theorem.



Sampling of Analog Signal

Sampling Bandpass Signals Using Lowpass Sampling Theorem (fs=2fH)



Sampling of Analog Signal

Sampling: Uniform sampling theorem for band-pass signals

m(t) can be completely determined by its sampled value with sampling rate of fs:

1、If the highest frequency fH is an integer multiple of the bandwidth, that is fH =nB, then fS = 2B

2、If the highest frequency fH=(n +k)B is not an integer multiple of B. For high-frequency signals, n>>k, then directly take twice the bandwidth

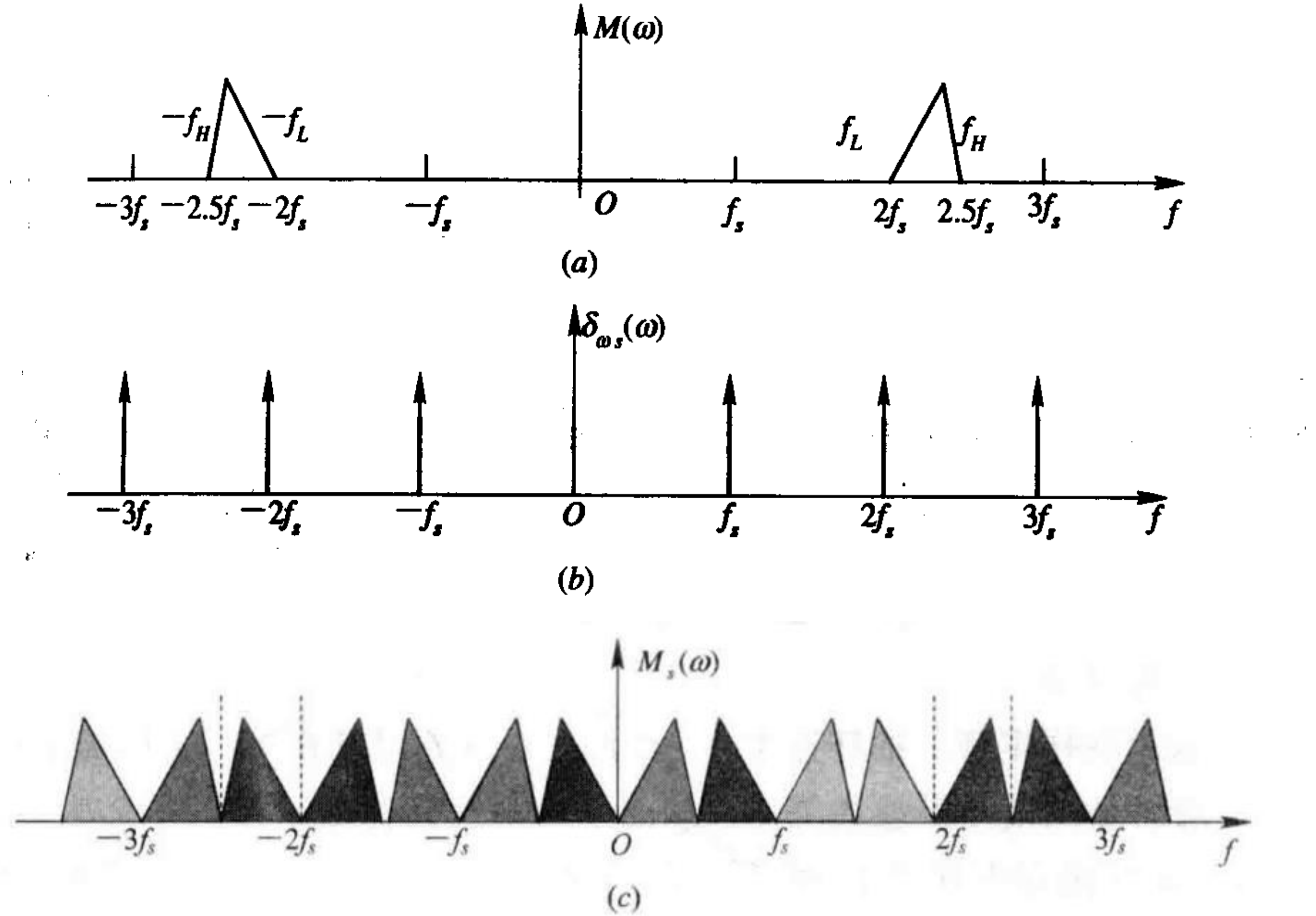
Sampling of Analog Signal

An illustration of uniform sampling theorem for bandpass signal

spectrum diagram when fH=5B

Sample the signal with the sampling rate 2B (2B<10B)

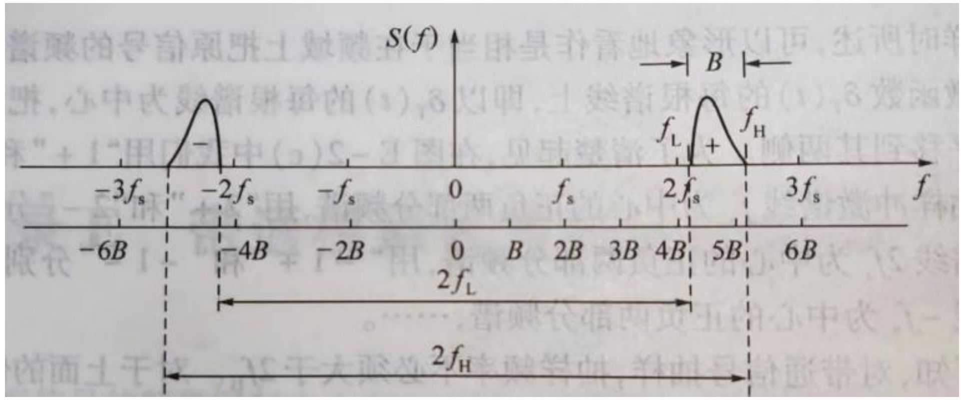
the spectrum M (f) of the sampled signal has neither aliasing nor gaps. signal can be restored without distortion by using a band-pass filter



Sampling of Analog Signal

An illustration of uniform sampling theorem for bandpass signal

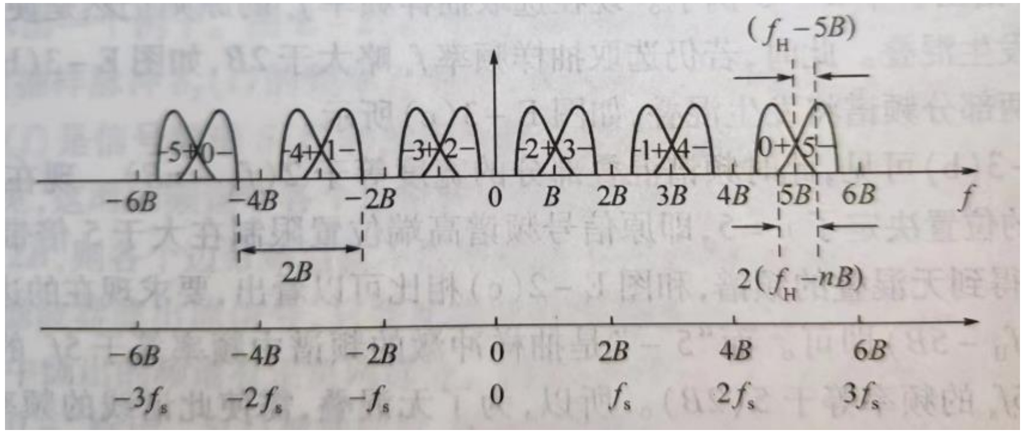
When fH=nB+kB (0<k<1)



if the sampling frequency fs is still 2B,

The positive and negative spectrum

of the sampled signal will be aliased



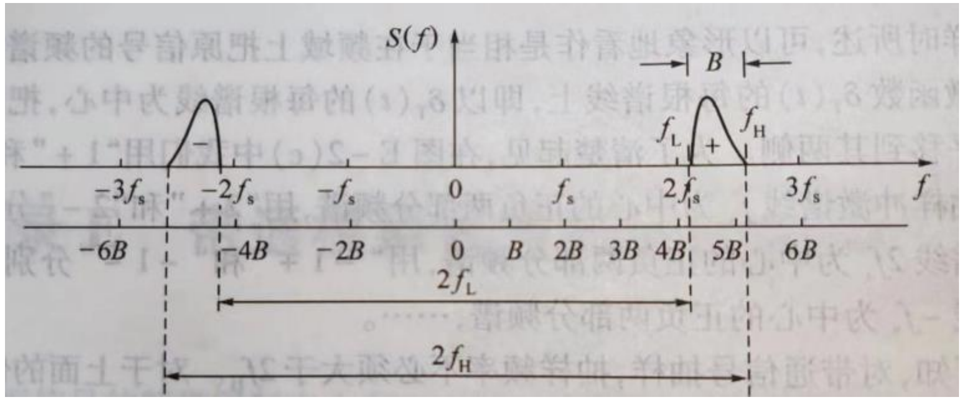
the width of the spectrum

aliasing part is equal to 2(fh-nB).

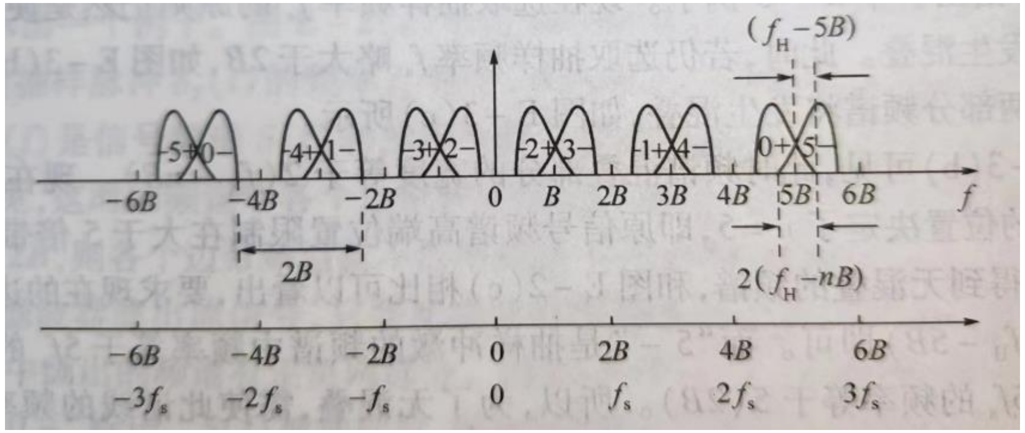
Sampling of Analog Signal

An illustration of uniform sampling theorem for bandpass signal

When fH=nB+kB (0<k<1)



To have no aliasing part, the frequency of this spectrum needs to be increased by 2(fH-nB)



The sampling rate need to satisfy:

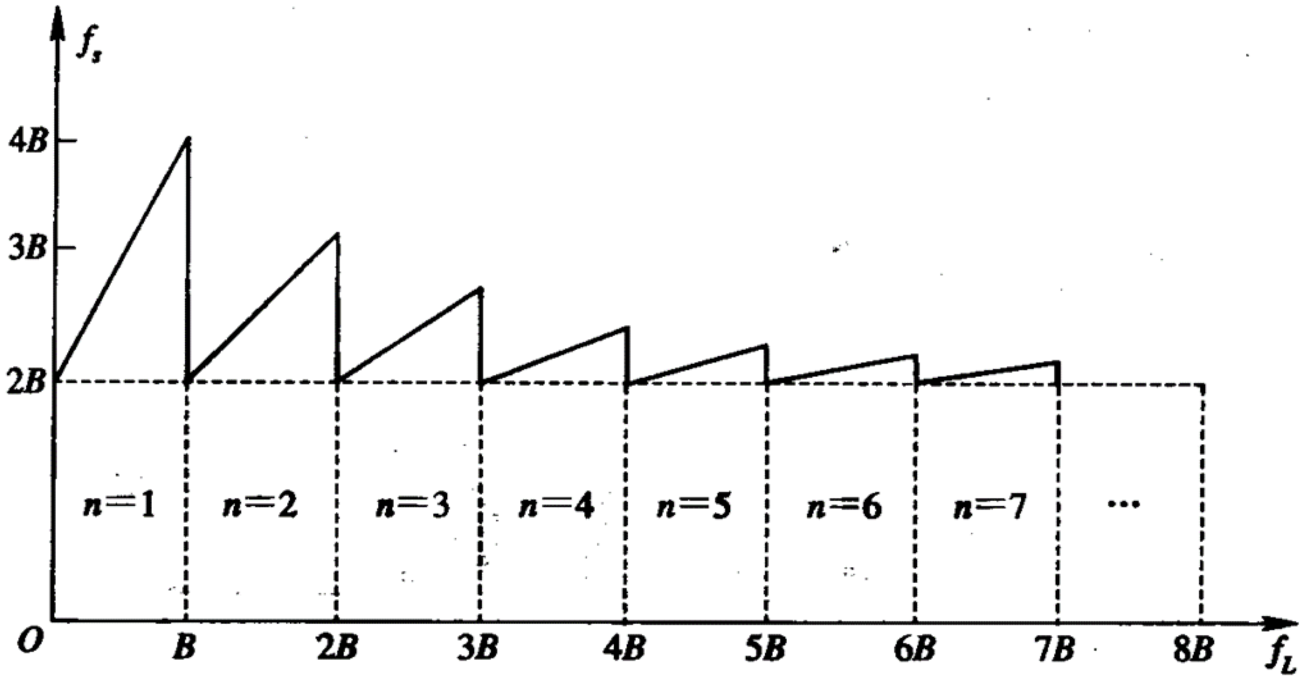
$$nf_s = n(2B) + 2\left(f_H - nB\right)$$

$$f_s = 2B + 2\left(f_H - nB\right)/n$$

$$\Rightarrow \quad = 2B + \frac{2kB}{n} = 2B\left(1 + \frac{k}{n}\right)$$

Sampling of Analog Signal

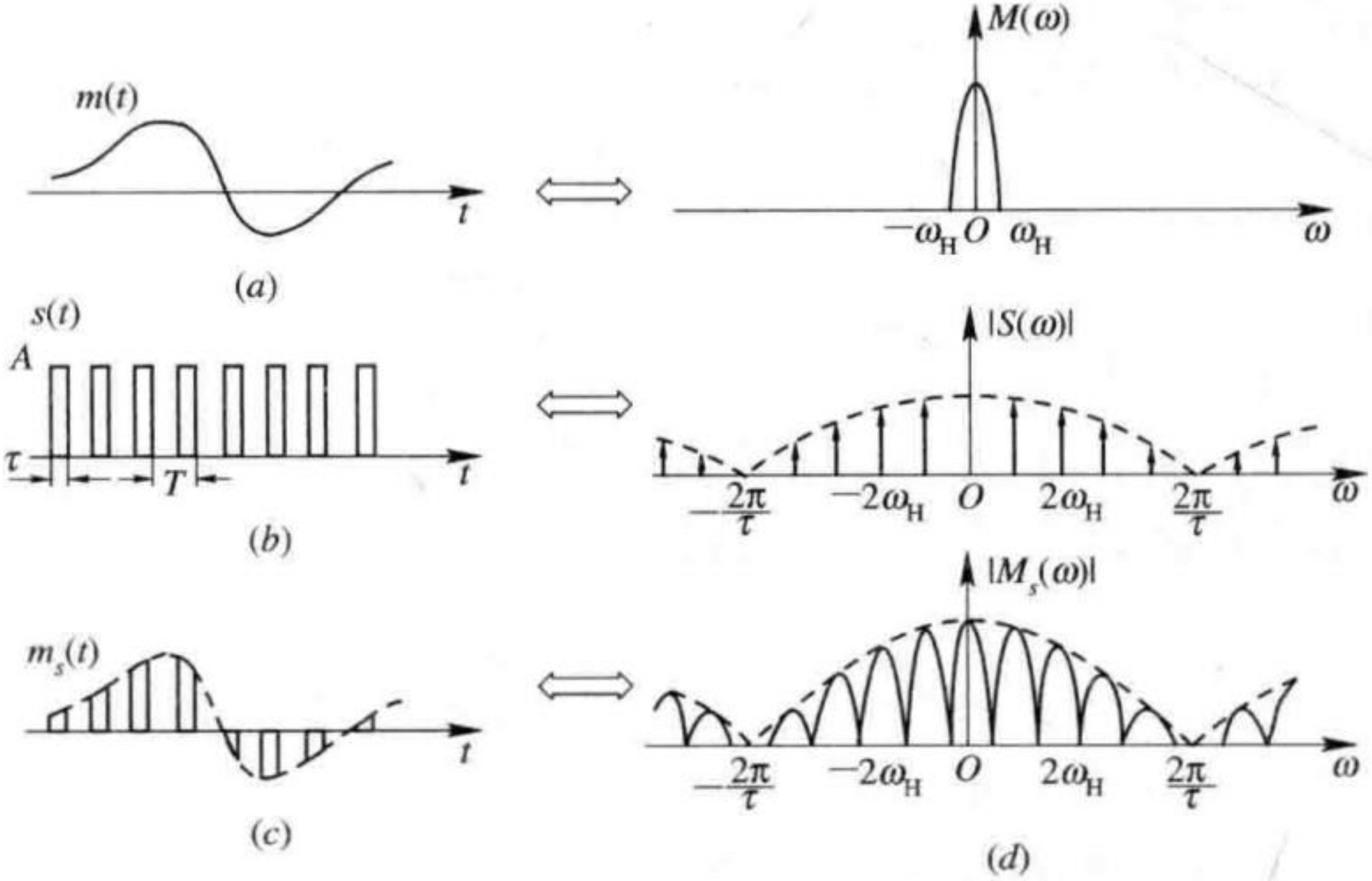
Sampling theorem figure



当 $f_H >> B$时，$f_s \approx 2B$

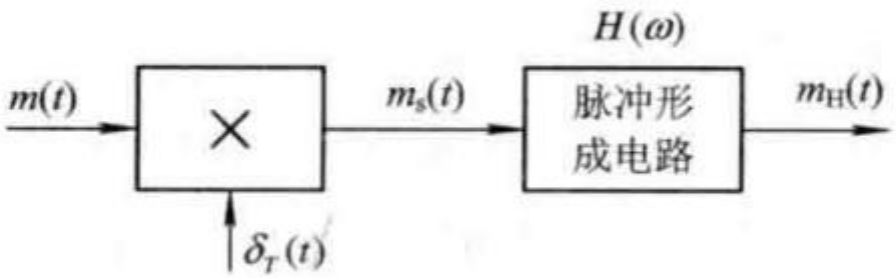
Sampling of Analog Signal

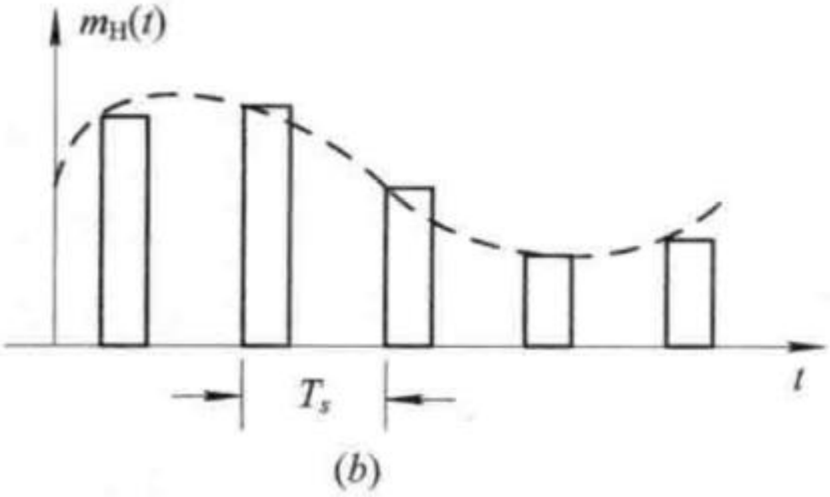
Analog Pulse Modulation : Natural sampling of PAM



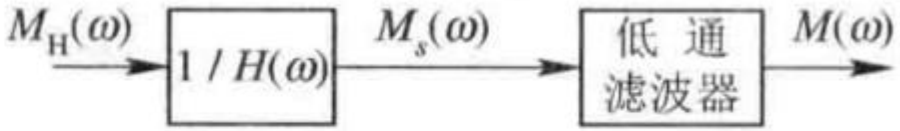
Sampling of Analog Signal

Analog Pulse Modulation : flat top sampling of PAM





Demodulation



Quantization of Analog Signal

Quantization: Principles of Quantization

Representing the sampled value of an analog signal with a predetermined finite number of levels is called quantization.

Sampling: Turning a time-continuous analog signal into a time-discrete analog signal (also known as a discrete signal);

$M = 2^N$ level samples ← An N-bit binary sequence $\xrightarrow{\text{represent}}$ the sample value

↓

transmitted by a digital transmission system

Quantization of Analog Signal

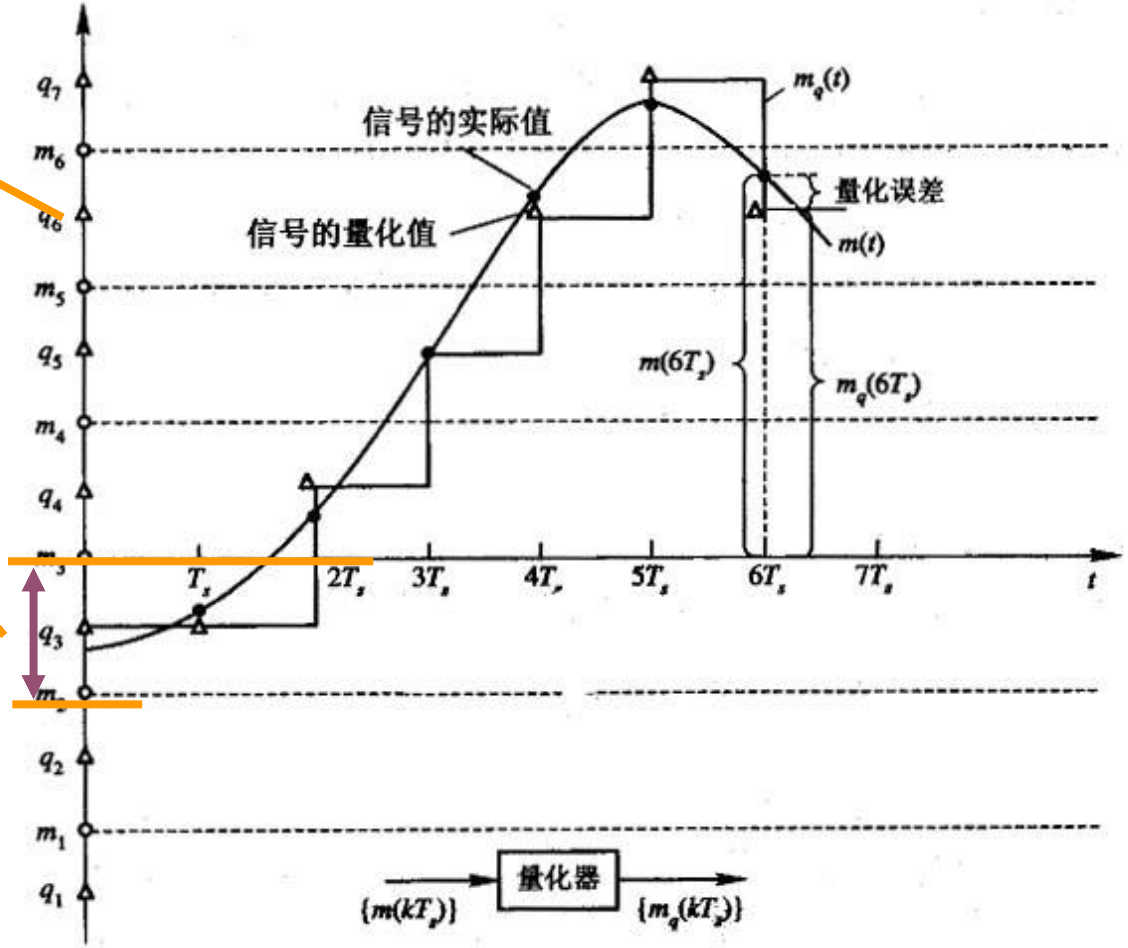
Quantization: Uniform Quantization

Quantization with the same quantization steps is called uniform quantization.

quantization level

quantization interval

Constant



Quantization Area (Dynamic Range)

Quantization of Analog Signal

Quantization: Uniform Quantization

Suppose the input signal ranges between a and b   quantization level is M

The quantization interval of uniform quantization is

$$\Delta V = \frac{b-a}{M}$$

The boundary points of the quantization intervals are

$$m_i = a + i\Delta V, \quad i = 0,1,\cdots,M$$

The quantized output level q_i is the midpoint of the quantization interval:

$$q_i = \frac{m_i + m_{i-1}}{2}, \quad i = 0,1,\cdots,M$$

Quantization of Analog Signal

Quantization: Uniform Quantization

Example: the signal is $m(t) = 9 + 10\cos\omega t$

If we quantizes it uniformly and the quantization interval is 0.1, what is the number of quantization levels M

Solution:

$$\Delta V = 0.1$$

$$b = 9 + 10 = 19$$

$$a = 9 - 10 = -1$$

$$M = \frac{b-a}{\Delta V} = \frac{19-(-1)}{0.1} = 200$$

Quantization of Analog Signal

Quantization: Uniform Quantization

Quantization noise: the difference between the quantized output level and the sampled value of the signal before quantization

Quantization SNR: The ratio of signal power to quantization noise power (the main metric of the quantizer)

In general, when given the maximum magnitude of the signal, the more quantization levels, the less quantization noise, the higher quantization SNR

Quantization of Analog Signal

Quantization: Uniform Quantization

Average quantization SNR

Numbers of quantization levels

$$\frac{S}{N_q} = M^2$$

Conflict！

Easy to find：

1) M greater, higher quantization SNR, more reliably。

2) M greater, each quantization value needs more bits to represent, less efficient！

Balance！

3) Select M properly to tradeoff the reliability and efficiency

Quantization of Analog Signal

Quantization: Uniform Quantization

Quantization SNR definition:

$$\frac{S}{N_q} = \frac{E[m^2]}{E[(m-m_q)^2]}$$

Signal average power:

$$S = E[m_k^2] = \int_a^b m_k^2 f(m_k)dm_k = \int_a^b m_k^2 \frac{1}{b-a} dm_k = \frac{M^2}{12}(\Delta V)^2$$

Noise power:

$$N_q = E\left[(m-m_q)^2\right] = \int_a^b (x-m_q)^2 f(x)dx = \sum_{i=1}^{M} \int_{m_{i-1}}^{m_i} (x-q_i)^2 f(x)dx = \sum_{i=1}^{M} p_i \int_{m_{i-1}}^{m_i} (x-q_i)^2 dx = \frac{(\Delta V)^2}{12} \sum_{i=1}^{M} p_i \Delta V = \frac{(\Delta V)^2}{12}$$

$$\boxed{\begin{aligned} q_i &= a + i(\Delta V) - \frac{(\Delta V)}{2} \\ m_i &= a + i(\Delta V) \end{aligned}}$$

Quantization SNR:

$$\left(\frac{S}{N_q}\right)_{dB} = 10\log M^2 = 20\log M = 20\log 2^N \approx 6N(dB)$$

Quantization of Analog Signal

Quantization: Uniform Quantization

Examples: Suppose the spectrum range of the voice signal is (0,4000) Hz, and the amplitude is evenly distributed in (-2V, 2V).

Q1: What is the minimum sampling frequency?

Q2: If it is uniformly quantized, and each quantized value can be encoded into 8 bits, What is the quantization interval? What is the quantization signal-to-noise ratio?

Quantization of Analog Signal

Quantization: Uniform Quantization

Solutions:

Minimum sampling rate:

$$f_s = 2f_H = 8kHz$$

Quantization level number:

$$256 = 2^8$$

Quantization interval:

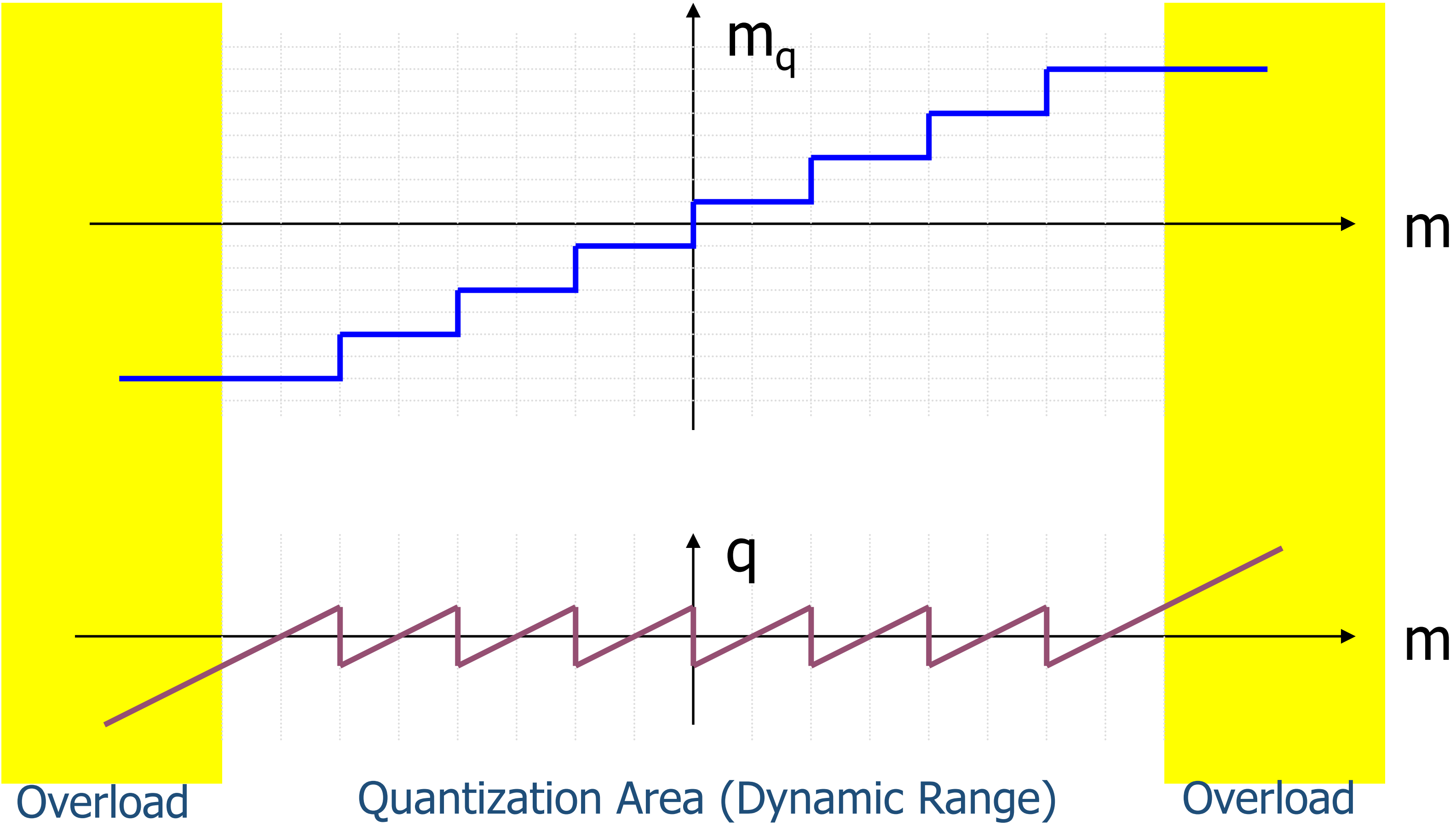
$$\frac{2-(-2)}{256} = \frac{1}{64}$$

Quantization SNR:

$$6N = 6 \times 8 = 48(dB)$$

Quantization of Analog Signal

Quantization: Uniform Quantization



Quantization of Analog Signal

Quantization: Uniform Quantization

Some conclusions about uniform quantization

1. Quantization noise has nothing to do with the signal strength, only with the quantization interval

2. The number of coding bits increases by 1 bit, the quantization noise decreases by 6dB, and the quantization signal-to-noise ratio increases by 6dB

3. The quantization signal-to-noise ratio decreases as the decrease of signal power.

4. Uniform quantization is generally used in ADC interfaces with small signal dynamic ranges, such as digital interfaces in the systems like computers, telemetry and remote sensing, instruments, and image communications.

Quantization of Analog Signal

Quantization: Nonuniform Quantization

The Problems of Uniform Quantization and Solutions

Problems：

1. The quantization step is unrelated to signal size and is not good for small signals
2. The quantization step is unrelated to the signal distribution, which is not good for speech, etc.

Solutions：

1. Quantization steps vary with signal strength
2. Quantization steps vary with signal distribution

→ $\boxed{\text{Nonuniform Quantization}}$

Quantization of Analog Signal

Quantization: Nonuniform Quantization

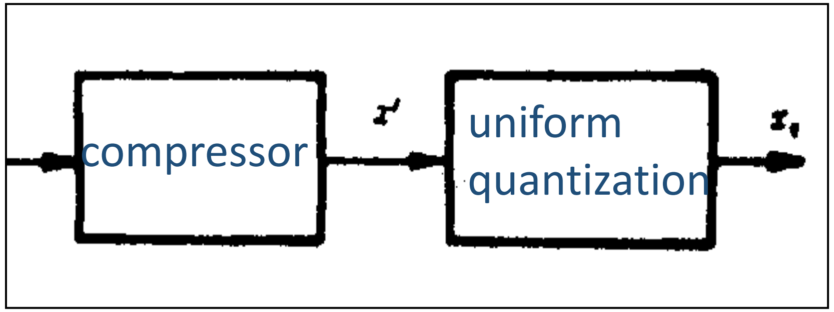
Non-uniform quantization determines the quantization interval according to different intervals of the signal. Its advantages are:

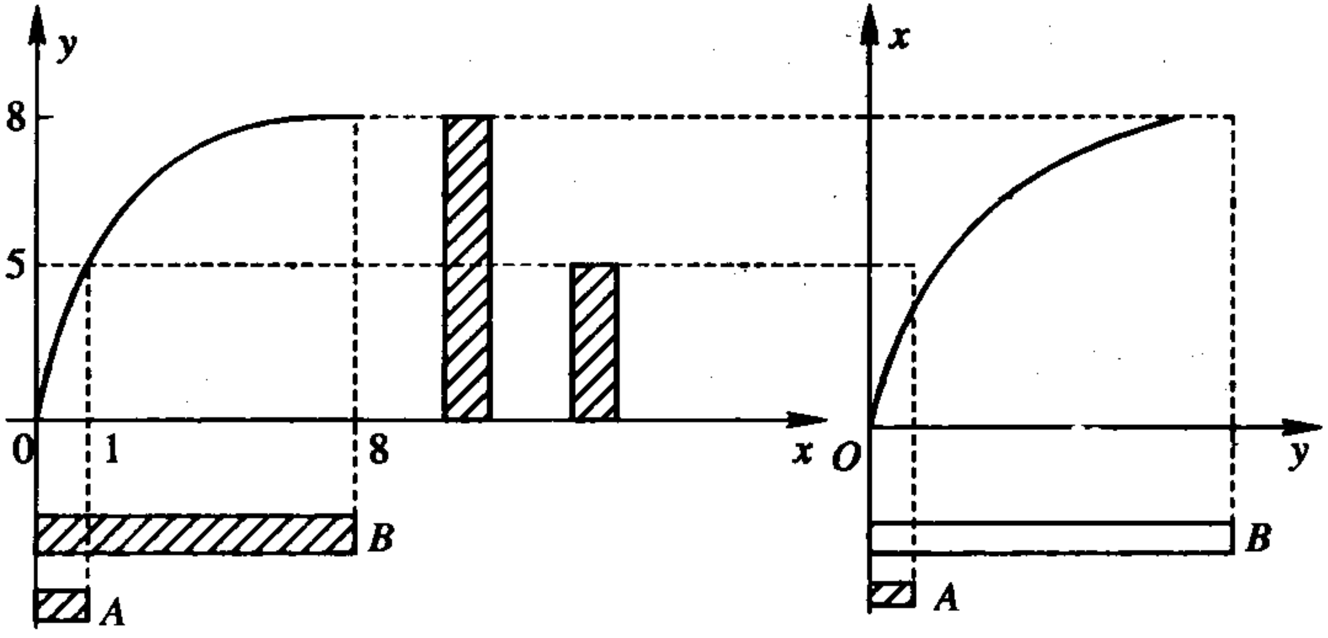
（1）When the signal input to the quantizer has a non-uniform distribution of probability density, the output of the non-uniform quantizer can get a higher average quantization noise power ratio

（2） Improved quantization signal-to-noise ratio for small signals

Quantization of Analog Signal

Quantization: Nonuniform Quantization





(a)

(b)

Quantization of Analog Signal

Quantization: Nonuniform Quantization

The implementation method of non-uniform quantization :

1. uniformly quantize the sample value after passing through the compressor
2. the common compressors are μ-law compression and A-law compression.

The United States and Japan use μ-law compression

Our country and European countries use A-law compression

Quantization of Analog Signal

Quantization: Nonuniform Quantization

Compression characteristic

μ-law compression characteristic：

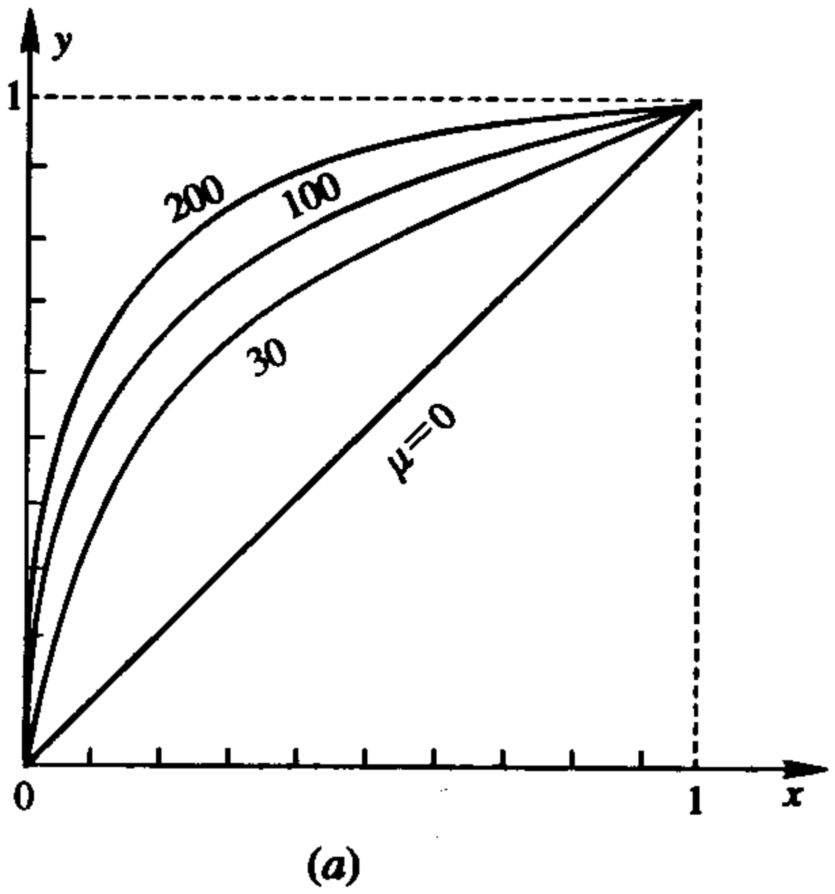
$$y = \frac{\ln\left(1+\mu x\right)}{\ln(1+\mu)} \ldots\ldots\ldots\ldots 0 \le x \le 1$$

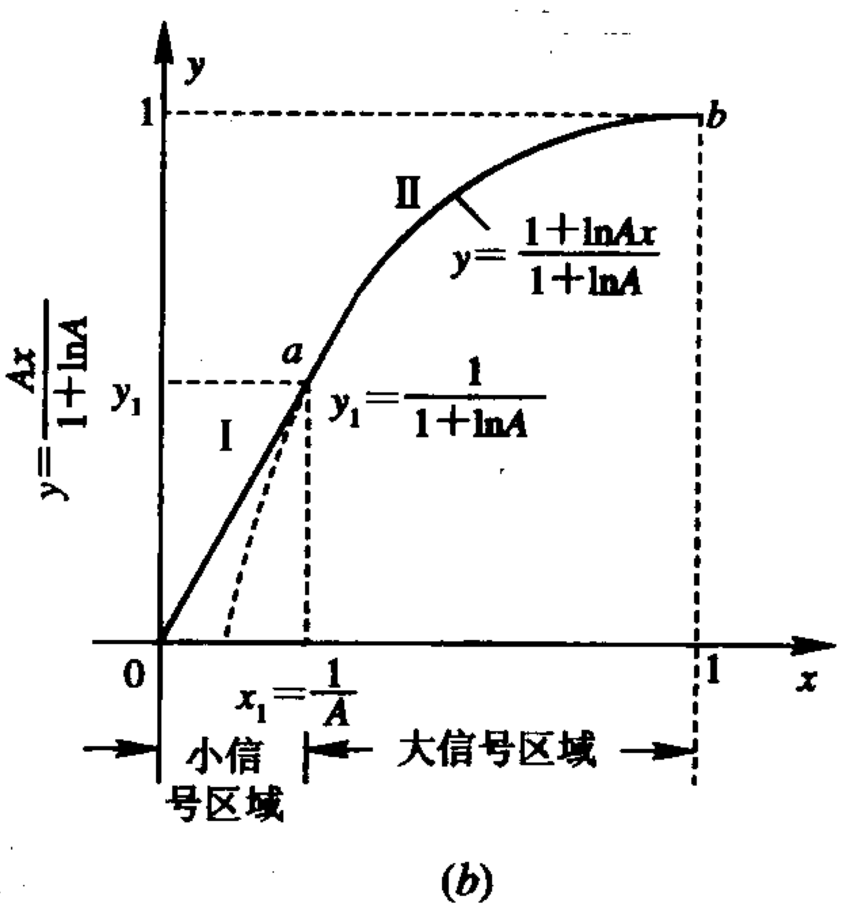
A – law compression characteristic：

$$y = \begin{cases} \dfrac{Ax}{1+\ln A} \ldots\ldots\ldots\ldots\ldots 0 \le x \le \dfrac{1}{A} \\ \dfrac{1+\ln Ax}{1+\ln A} \ldots\ldots\ldots\ldots \dfrac{1}{A} \le x \le 1 \end{cases}$$

Quantization of Analog Signal

Quantization: Nonuniform Quantization





Quantization of Analog Signal

Quantization: Nonuniform Quantization

A-law and μ-law curves are continuous smooth curves：

1. They used to be designed based on analog circuit.

2. The accuracy and stability are not good for analog.

↓

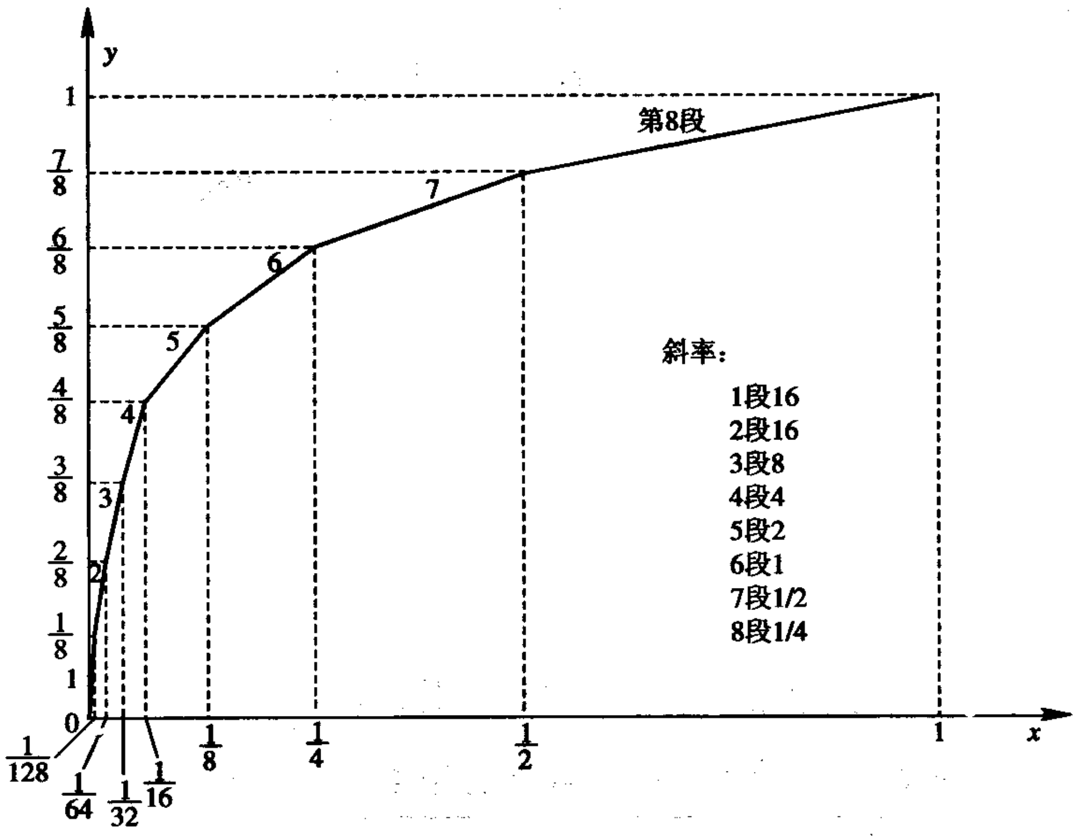
To use digital compressor:

1.Use digital circuit to generate multiple broken lines to approximate the curves

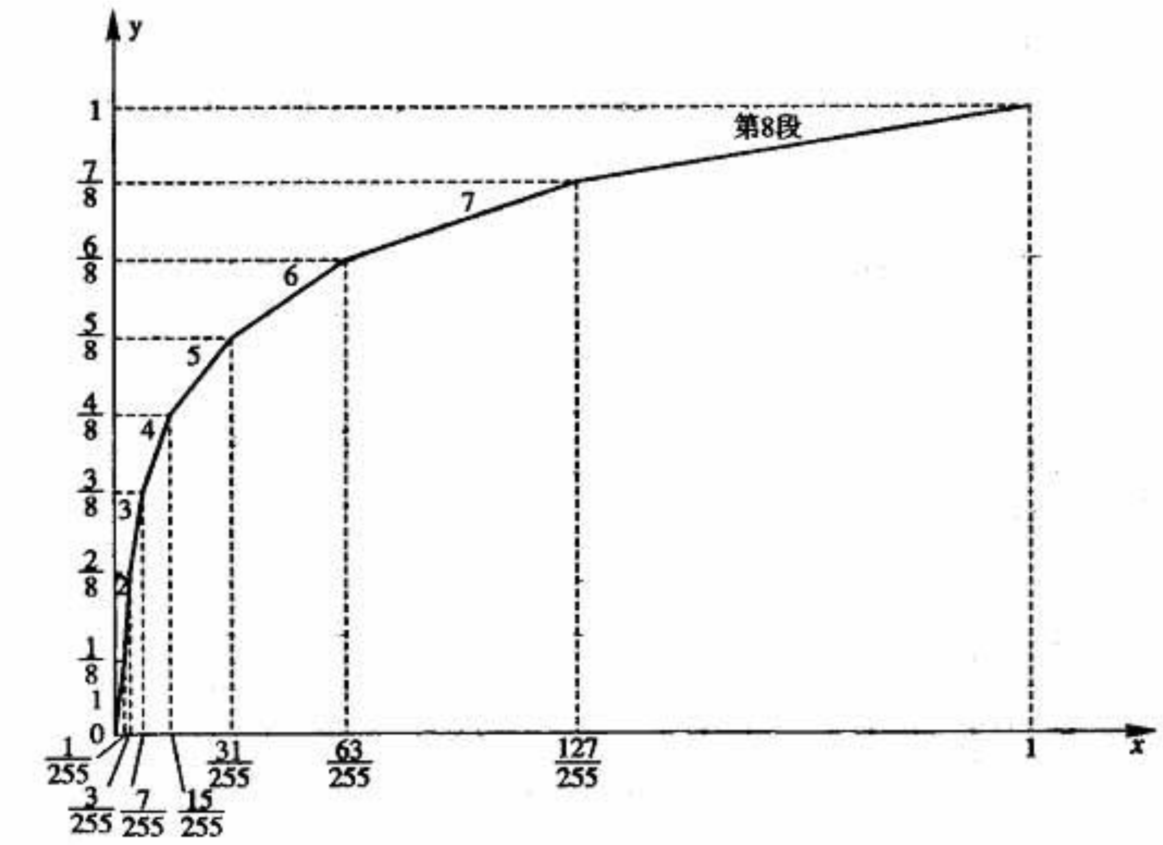
2. A-law can be approximated by 13 broken lines; μ-law can be approximated by 15 broken lines

Quantization of Analog Signal

Quantization: Nonuniform Quantization



A-law 13 broken lines



μ-law 15 broken lines

Quantization of Analog Signal

Quantization: Nonuniform Quantization

y	0	1/8	2/8	3/8	4/8	5/8	6/8	7/8	1
x	0	1/128	1/60.6	1/30.6	1/15.4	1/7.79	1/3.93	1/1.98	1
x of the broken lines	0	1/128	1/64	1/32	1/16	1/8	1/4	1/2	1
Segment	1	2	3	4	5	6	7	8	
slope	16	16	8	4	2	1	1/2	1/4	

Thank you！

Exercise

Ex1: Assume that a low-pass signal has the spectrum such that

$$M(f)=\begin{cases}1-\dfrac{|f|}{200} & |f|<200Hz\\ 0 & others\end{cases}$$

(1) If the sampling rate is 300Hz, draw the spectrum after sampling

(2) If the sampling rate is 400Hz, draw the spectrum after sampling

Exercise

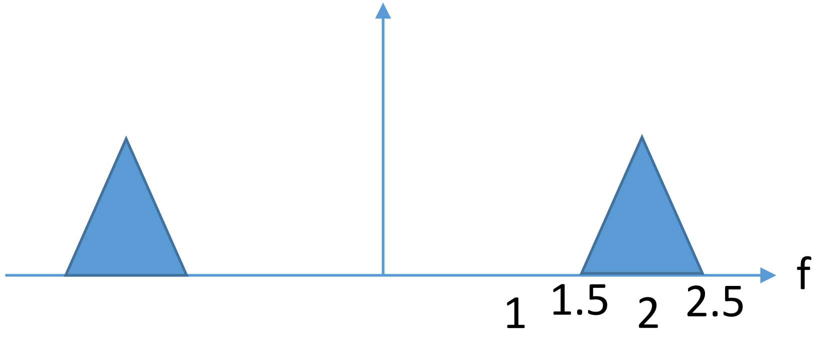
Ex2: Sample the bandpass signal whose spectrum is as the right figure

Explain:

(1) When the sampling rate is fs=2.5B, the signal can be recovered

(2) When the sampling rate is fs>5B, the signal can be recovered

(3) When the sampling rate is fs=3.5B, the signal can't be recovered



Exercise

Ex3: the signal is m(t)=9+Acos(2*pi*f*t), where A<=10, if the number of quantization levels is 40

(1) How many bits N does it need for binary coding?

(2) Determine the quantization interval.

Exercise

Ex4: there is an 8-bit uniform quantizer, ranging from -1V to 1V

(1) Determine the quantization interval

(2) If a sine signal (varying from -1V to 1V) is input into this quantizer, calculate its quantization signal-to-noise ratio.

Exercise

Ex5: For an A-law compressor (A=90), whose maximum input voltage is 1V, please find

(1) When the input voltage is 0.1V, what is the output voltage?

(2) When the input voltage is 0.01V, what is the output voltage?

MATLAB

Assume the signal is $m(t) = cos(90\pi t)$

(1) Let the sampling rate be fs=2000, plot the original signal and the sampled signal

(2) Plot the quantized signal (uniform quantization and the quantization level number is 32)