Principles of Communications

Chapter 2 — Signal

Zhen Chen

Classification of Signals

Signal can be classified according to various criteria

- Different Criteria
  - Feature of Values
    - Analog signals
    - Digital signals
    - (Introduced in Chapter 1)
  - Signal Determinacy
    - Deterministic signals
    - Random signals
  - Signal Strength
    - Energy signals
    - Power signals

Classification of Signals

Signal Determinacy

Deterministic signals

The values of the signal at any time are deterministic and predictable.

- The values at any time can generally be calculated.
- Example: a definite segment of a sinusoidal wave.
- Classification: periodic signals (e.g. sinusoidal wave with infinite length)

  & nonperiodic signals (e.g. rectangular pulse)

Random signals

The values of the signal are indeterministic (unpredictable).

- The values at any time can't be accurately calculated.
- Follow a certain statistic rule, can find their statistic characteristics.
- In general,these signals are regarded as random processes.

Classification of Signals

Signal Strength s(t) denotes the signal (vary with time)

→ Signal Energy $E = \int s^2(t)dt$ Signal Power $P = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} s^2(t)dt$

Energy signals

If signal energy satisfies: $0 < E = \int s^2(t)dt < \infty$

The average power is 0, since the energy is finite: $P = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} s^2(t)dt = 0$

The signal is termed Energy signal

Power signals

In practice, signals that transmit from base station have limited power.

The signal can have long duration (approximate infinity duration)

→ The energy is infinite

The signal is termed Power signal

Characteristics of Deterministic Signal

Characteristics in Frequency Domain

Frequency spectrum of power signal

Let s(t) denote a periodic power signal, T be the period

The spectrum can be found according to Fourier Transform:

$$C\left(jn\omega_0\right)=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} s(t)\mathrm{e}^{-\mathrm{j}n\omega_0 t}\mathrm{d}t=|C_n|e^{j\theta_n}$$

The periodic power signal s(t) can be expressed by its Fourier series (discrete) as

$$s(t)=\sum_{n=-\infty}^{\infty} C\left(jn\omega_0\right)\mathrm{e}^{jn\omega_0 t}$$

Characteristics of Deterministic Signal

Characteristics in Frequency Domain

Frequency spectral density of energy signal

Let s(t) denote an energy signal

The spectral density can be found according to Fourier Transform:

$$S(\omega) = \int_{-\infty}^{\infty} s(t) e^{-j\omega t} dt$$

Inverse Fourier Transform can obtain the original signal:

$$s(t) = \int_{-\infty}^{\infty} S(\omega) e^{j\omega t} dt$$

Characteristics of Deterministic Signal

Characteristics in Frequency Domain

Energy Spectral Density

Let s(t) denote an energy signal

The energy of the signal can be determined by: $E = \int_{-\infty}^{\infty} s^2(t)dt$

According to Parseval's theorem, the energy can also be determined by:

$$E = \int_{-\infty}^{\infty} |S(f)|^2 \, df$$

$S(\omega)$ is the frequency density of the signal $s(t)$

The energy spectral density is defined as:

$$G(f) = |S(f)|^2$$

Since $s(t)$ is a real function, $|S(f)|$ is an even function ,thus the energy can be revised as:

$$E = 2\int_{0}^{\infty} G(f)df$$

Characteristics of Deterministic Signal

Characteristics in Frequency Domain   Power Spectral Density

Truncate the power signal s(t) to be $s_T(t) \quad -T/2 < t < T/2$

$s_T(t)$ is a energy signal

The energy spectral density of $s_T(t)$ is given as: $E = \int_{-T/2}^{T/2} s_T^2(t)\mathrm{d}t = \int_{-\infty}^{\infty} \left|S_\tau(f)\right|^2 \mathrm{d}f$

The power spectral density of the signal is given as: $P(f) = \lim_{T\to\infty} \frac{1}{T}\left|S_T(f)\right|^2$

The power of the signal is given as: $P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \left|S_T(f)\right|^2 \mathrm{d}f = \int_{-\infty}^{\infty} P(f)\mathrm{d}f$

If the power signal has period T0, then the power becomes

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} s^2(t)\mathrm{d}t = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} s^2(t)\mathrm{d}t = \sum_{n=-\infty}^{\infty} \left|C\left(\mathrm{j}n\omega_0\right)\right|^2$$

Characteristics of Deterministic Signal

Characteristics in Time Domain

Autocorrelation function

The autocorrelation function for the energy signal is defined as:

$$R(\tau) = \int_{-\infty}^{\infty} s(t)s(t+\tau)\mathrm{d}t \quad -\infty < \tau < \infty$$

The autocorrelation function for the power signal is defined as:

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} s(t)s(t+\tau)\mathrm{d}t \quad -\infty < \tau < \infty$$

The autocorrelation function represents the correlation between the signal and itself delayed τ

The autocorrelation function is only dependent on τ

$\tau = 0$

- $R(\tau)$ is the energy of the energy signal
- $R(\tau)$ is the average power of the power signal

Characteristics of Deterministic Signal

Characteristics in Time Domain

Cross-correlation function

The cross-correlation function for the two energy signal is defined as:

$$R_{12}(\tau) = \int_{-\infty}^{\infty} s_1(t) s_2(t+\tau) \mathrm{d}t \quad -\infty < \tau < \infty$$

The cross-correlation function for the two power signal is defined as:

$$R_{12}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} s_1(t) s_2(t+\tau) \mathrm{d}t \quad -\infty < \tau < \infty$$

The cross-correlation function represents the correlation between the signal and another leading τ

The cross-correlation function is only dependent on τ

The cross-correlation function is related to the order of the two signal:

$$R_{21}(\tau) = R_{12}(-\tau)$$

Characteristics of Random Signal

Why we consider Random Signal in communication?

1 The transmitted signal has certain uncertainties for receiver.

2 The noise is randomly varying and contaminate the signal

3 The channel is unstable and varies with time.

$\Downarrow$

The received signals are random and unpredictable, which is a random process.

For an instant, the signal is a random variable.

$\Downarrow$

In spite of randomness, the received signal has statistic rule after observing a long time.

Characteristics of Random Signal

Random Variable

Distributed Function / Cumulative Density Function (CDF)

The CDF of a random variable is defined as: $F_X(x) = P(X \leqslant x)$

The probability of the value of the random variable in interval $(a,b]$ is calculated as

$$P(a < X \leqslant b) = F_X(b) - F_X(a)$$

The important properties of the CDF:

1 impossible event: $F_X(-\infty) = 0$

2 certain event: $F_X(+\infty) = 1$

3 monotonic: if x1< x2, then $F_X(x_1) \leqslant F_X(x_2)$

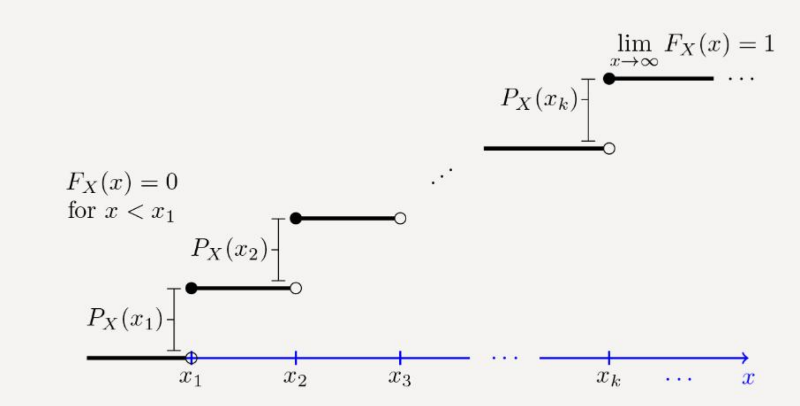
Characteristics of Random Signal

Random Variable

Distributed Function / Cumulative Density Function (CDF)

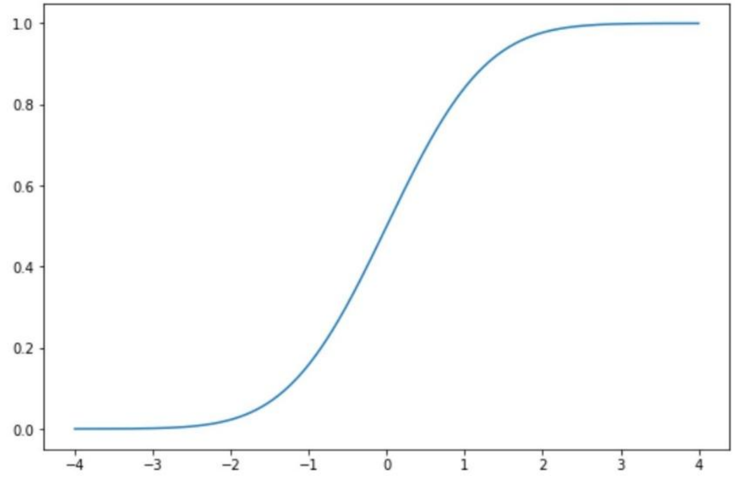
For the discrete random variable, the CDF can be written as

$$F_X(x) = \begin{cases} 0 & x < x_1 \\ \sum_{k=1}^{i} p_k & x_1 \leqslant x < x_{i+1} \\ 1 & x \geqslant x_n \end{cases}$$



For the continuous random variable,

the CDF can be plotted like



Characteristics of Random Signal

Random Variable

Probability Density Function (PDF)

The PDF of a random variable is defined as: $p_X(x) = \dfrac{\mathrm{d}F_X(x)}{\mathrm{d}x}$

The probability of the value of the random variable in interval $(a,b]$ is calculated as

$$P(a < X \le b) = \int_a^b p_x(x)\mathrm{d}x$$

The important properties of the PDF:

1 Relationship of PDF & CDF & probability $\quad P(X \leqslant x) = \int_{-\infty}^{x} p_x(y)\mathrm{d}y \qquad F_X(x) = \int_{-\infty}^{x} p_X(y)\mathrm{d}y$

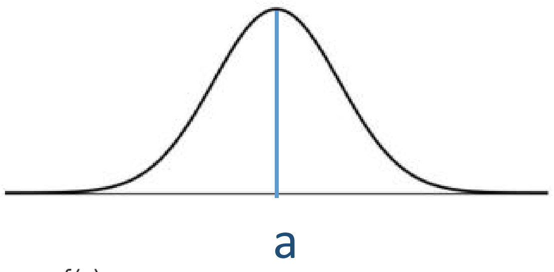
2 The PDF is nonnegative: $\quad p_X(x) \geqslant 0$

2 The integral of the whole PDF is 1: $\quad \int_{-\infty}^{\infty} p_x(x)\mathrm{d}x = 1$

Examples of Frequently Used Random Variables

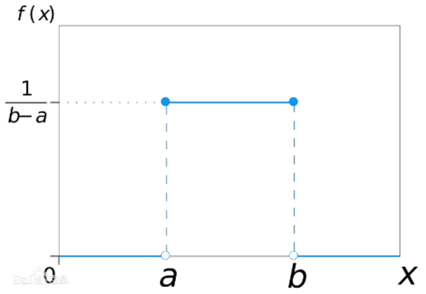
Random Variable with Normal Distribution (Gaussian Distribution)

PDF: $$p_x(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-a)^2}{2\sigma^2}\right]$$



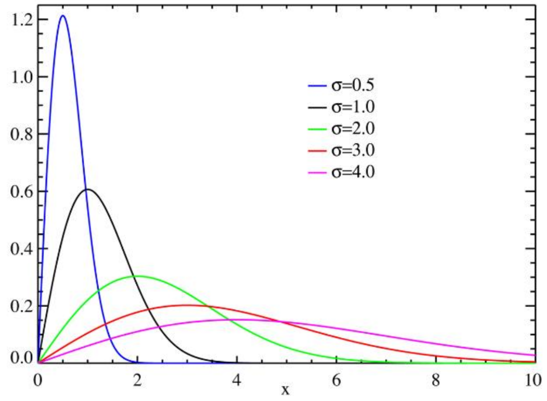
Random Variable with Uniform Distribution

PDF: $$p_X(x) = \begin{cases} 1/(b-a) & a \leqslant x \leqslant b \\ 0 & \text{else} \end{cases}$$



Random Variable with Rayleigh Distribution

PDF: $$p_X(x) = \frac{2x}{\sigma}\exp\left(+\frac{x^2}{\sigma}\right) \quad x \geqslant 0$$



Numerical Characteristics of Random Variable

Expectation (Statistic Mean) $E(X)=\int_{-\infty}^{\infty} x p_X(x)\mathrm{d}x$

Properties:

1 For a constant, the expectation is $E(C)=C$

2 Homogeneity $E(CX)=CE(X)$

3 Additive $E(X_1+X_2+\cdots+X_n)=E(X_1)+E(X_2)+\cdots+E(X_n)$

4 If two random variables are independent, $E(XY)=E(X)E(Y)$

First origin moment

Variance $D(X)=\sigma_X^2=E\left[(X-\overline{X})^2\right]$

Properties:

1 For a constant, the variance is $D(C)=0$

2 Assume C is a constant, then $D(CX)=C^2D(X)$

3 Can also be calculated with $D(X)=E\left[(X-\overline{X})^2\right]=\overline{X^2}-\overline{X}^2$

4 If two random variables are independent, $D(X+Y)=D(X)+D(Y)$

Second central moment

Random Process

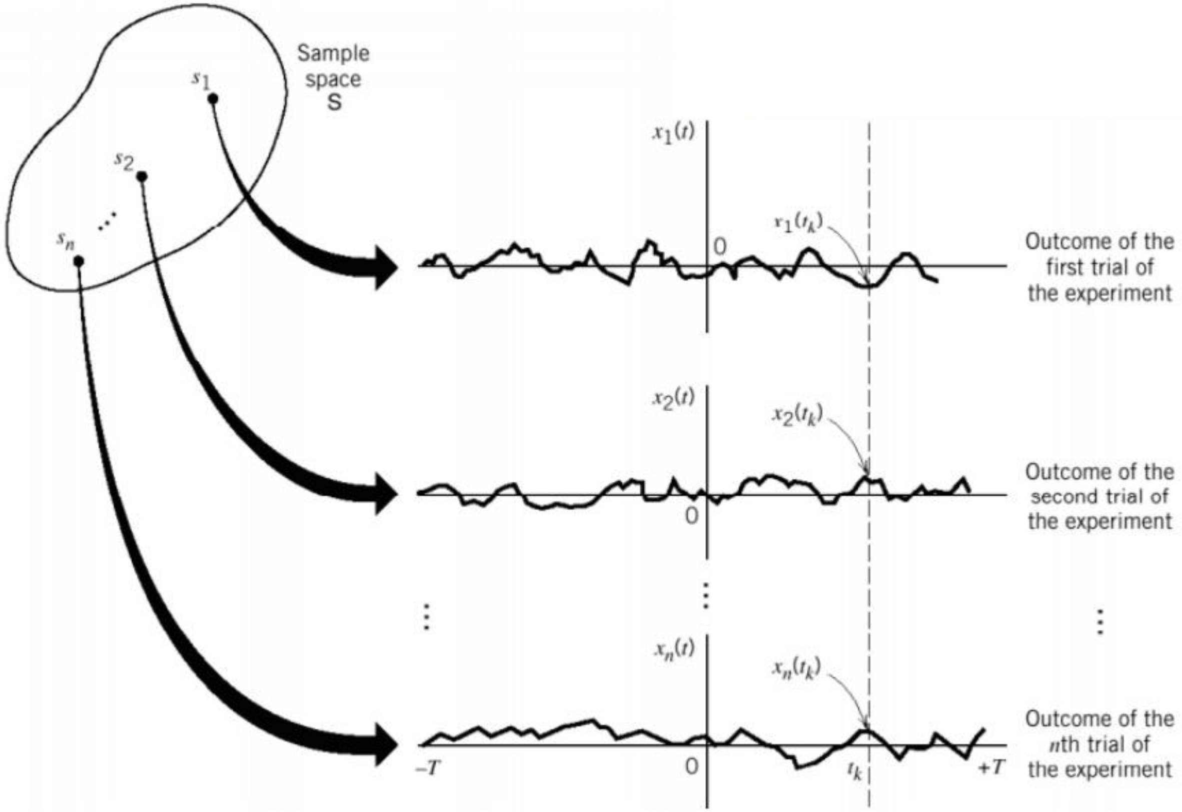
The concept of random process:

The process is a real functionof time t, but its value observed at an arbitrary instant is a random variable

1 There are numerous samples in the sample space

2 Every sample in the sample space varies with time t

3 When given a certain time t, the observation value is random variable



Random Process

Example: The random process is X(t)=Acos(t), -∞<t<∞, where A is the random variable, whose distribution is

A	1	2	3
p	1/3	1/3	1/3

Please find the CDF $F\left(x,\frac{\pi}{4}\right)$

Solution: At $t=\frac{\pi}{4}$

It is a random variable $X\left(\frac{\pi}{4}\right)=A\cos\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}A$

The possible values are $\frac{\sqrt{2}}{2},\ \sqrt{2},\ \frac{3\sqrt{2}}{2}$

Random Process

Example: The random process is X(t)=Acos(t)， -∞<t<∞, where A is the random variable, whose distribution is

A	1	2	3
p	1/3	1/3	1/3

Please find the CDF $F\left(x, \frac{\pi}{4}\right)$

Solution:

Obtain the probabilities $P\left(X\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\right)=P\left(A\cos\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\right)=P\{A=1\}=\frac{1}{3},$

$$P\left(X\left(\frac{\pi}{4}\right)=\sqrt{2}\right)=P\{A=2\}=\frac{1}{3},\quad P\left(X\left(\frac{\pi}{4}\right)=\frac{3\sqrt{2}}{2}\right)=P\{A=3\}=\frac{1}{3}$$

So the CDF is

$$F\left(x,\frac{\pi}{4}\right)=\begin{cases} 0, & x<\sqrt{2}/2 \\ 1/3, & \sqrt{2}/2\le x<\sqrt{2} \\ 2/3, & \sqrt{2}\le x<3\sqrt{2}/2 \\ 1, & x\ge 3\sqrt{2}/2 \end{cases}$$

Random Process

The random process is described by statistic characteristic

Important numerical characteristic: Mean & Variance & Autocorrelation function

Statistic Mean of a random process: at any instant t_i,

$$E\left[X\left(t_i\right)\right]=\int_{-\infty}^{\infty} x p_{X_i}(x)\mathrm{d}x = m_X\left(t_i\right)$$

Variance of a random process:

$$D\left[X\left(t_i\right)\right]=E\left\{X\left(t_i\right)-E\left[X\left(t_i\right)\right]\right\}^2$$

The autocorrelation of a random process:

$$R_X\left(t_1,t_2\right)=E\left[X\left(t_1\right)X\left(t_2\right)\right]$$

This represents the correlation degree of two random values at two instants.

Random Process

Special random process: Stationary random process

Strict stationary random process: the statistic characteristics are independent of the time origin

⇓

Strict stationary random process is almost impossible in reality

⇓

Generalized stationary random process: the mean & variance & autocorrelation function are independent of the time origin

i.e.

$$E[X(t)] = m_X = \text{ constant}$$

$$D[X(t)] = E\{X(t) - E[X(t)]\}^2 = \sigma_X^2 = \text{ constant}$$

$$R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau)$$

The autocorrelation function of a generalized stationary random process is related only to the interval between t1 and t2.

Random Process

Special random process: Stationary random process

Strict stationary random process must be a Generalized stationary random process

Generalized stationary random process might not be a Strict stationary random process

Mean & variance & autocorrelation function are some of the statistic

Normally we only consider generalized stationary random process (short for stationary random process)

Random Process

Ergodicity of Stationary random process: What and Why?

To find the statistic of a stationary random process, need to find the statistical mean for all realizations

(Impossible)

↓

Alternative: if a random process has ergodicity ,then its statistic mean is replaced by its time average

↓

Ergodicity: a realization of a stationary random process can go through all states of the process

statistic mean is equal to its time average

Benefit:

With ergodicity, not necessary to make infinite observations ,but make one observation for a long time. hence the calculation is reduced tremendously

In most communication systems, it is always supposed that means and autocorrelation are ergodic

Random Process

Ergodicity of Stationary random process:

Let Xi(t) denote an arbitrary realization of an ergodicity process

Ergodic with respect to the mean:

$$m_X = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} X_i(t) \mathrm{d}t$$

Expectation

Time Average

Ergodic with respect to the autocorrelation function:

$$R_X(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} X_i(t) X_i(t+\tau) \mathrm{d}t$$

"Statistic Mean" of autocorrelation function

"Time Average" of autocorrelation function

Random Process

Autocorrelation function & Power Spectral Density of Stationary Random Process

The autocorrelation function of a stationary random process is

$$R_X(t_1,t_2)=E\left[X(t_1)X(t_2)\right]=R_X(t_1-t_2)=R_X(\tau)$$

The properties of autocorrelation function

1 Second origin moment is average normalized power $R(0)=E\left[X^2(t)\right]=P_X$

2 It's an even function $R(\tau)=R(-\tau)$

3 The average normalized power is the upper bound $|R(\tau)|\leqslant R(0)$

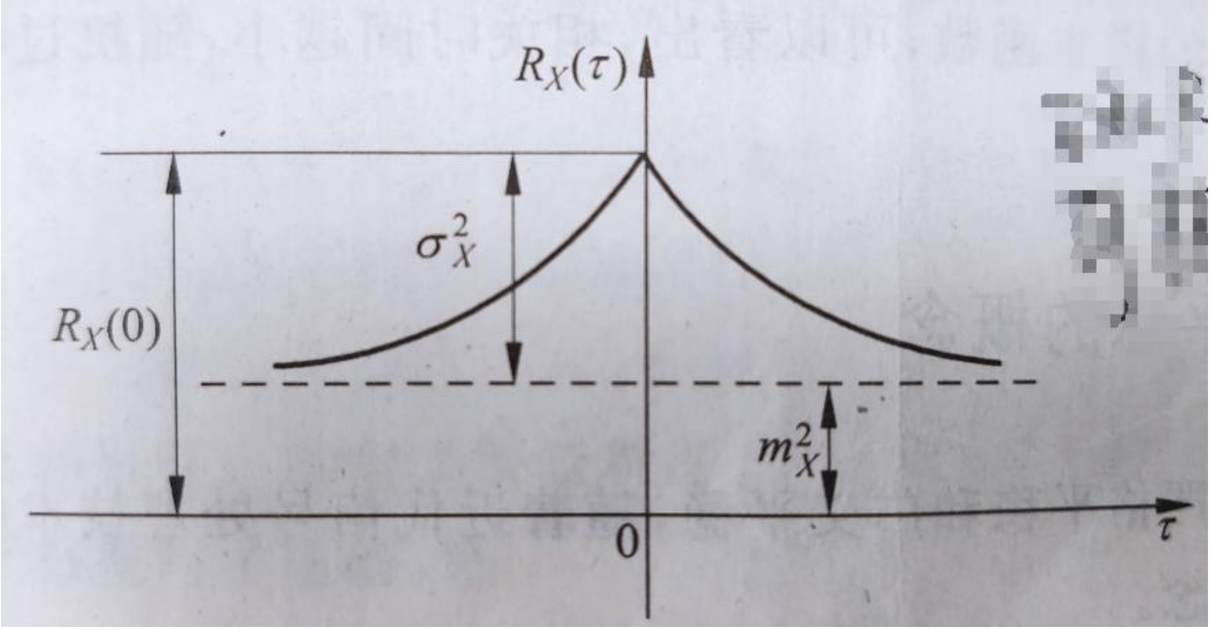
4 The normalized power of DC component $R(\infty)=E^2[X(t)]$

5 The variance $R(0)-R(\infty)=\sigma_X^2$

Random Process

Autocorrelation function & Power Spectral Density of Stationary Random Process

According to the properties, a typical autocorrelation function of a stationary random process is plotted like



Random Process

Autocorrelation function & Power Spectral Density of Stationary Random Process

The power spectral density of a stationary random process $X(t)$ is defined as

$$P_X(f) = E[P(f)] = \lim_{T\to\infty} \frac{E\left|S_T(f)\right|^2}{T}$$

$P(f)$ power spectral density of a deterministic signal

$X_T(t)$ the truncated function of a realization of $X(t)$

$S_T(f)$ the Fourier transform of $X_T(t)$

It is regarded as the statistic mean of the power spectral density of each possible realization

The average power of $X(t)$

$$P_X = \int_{-\infty}^{\infty} P_X(f)\mathrm{d}f = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E\left[\left|S_T(f)\right|^2\right]}{T}\mathrm{d}f$$

Random Process

The relationship between Autocorrelation function & Power Spectral Density

The power spectral density $P_X(f)$ and the autocorrelation function $R(\tau)$ are a pair of Fourier transform

$$P_X(f) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau$$

$$R(\tau) = \int_{-\infty}^{\infty} P_X(f) e^{j\omega\tau} df$$

The characteristic of power spectral density

1 $P_X(f) \geq 0$ and $P_X(f)$ is a real function because $R(\tau)$ is positive definite.

2 $P_X(f) = P_X(-f)$ i. e., $P_X(f)$ is an even function because $R(\tau)$ is an even function of τ

Random Process

Example: Find the mean, variance and the autocorrelation function of $X(t) = acos(\omega t + \Theta) \quad -\infty < t < +\infty,$

where Θ follows uniform distribution in $(0, 2\pi)$

Solution: The PDF of Θ

$$f(\Theta) = \begin{cases} \dfrac{1}{2\pi} & 0 < \Theta < 2\pi \\ 0 & \text{others} \end{cases}$$

Then, the mean is

$$m_X(t) = E[X(t)] = E\left[acos\left(\omega t + \Theta\right)\right] = \int_0^{2\pi} acos\left(\omega t + \Theta\right) \cdot \frac{1}{2\pi} d\Theta = 0$$

The autocorrelation function is

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = E[a^2 cos(\omega t_1 + \Theta) cos(\omega t_2 + \Theta)]$$

$$= a^2 \int_0^{2\pi} cos(\omega t_1 + \Theta) cos(\omega t_2 + \Theta) \cdot \frac{1}{2\pi} d\Theta = \frac{a^2}{2} cos\omega(t_2 - t_1) \overset{t = t_2 - t_1}{=\!=\!=} \frac{a^2}{2} cos\omega t$$

The variance is

$$\sigma_X^2(t) = R_X(t,t) - \mu_X^2(t) = R_X(t,t) = \frac{a^2}{2}$$

This is a stationary random process

Random Process

Example: A, B are 2 random variables, please determine the mean and the autocorrelation function of

$$X(t) = At + B, t \in T = (-\infty, +\infty)$$

If A, B are independent, and $A \sim N(0,1), B \sim U(0,2)$, what is the mean and autocorrelation

Solution:

$$\mu_X(t) = E[X(t)] = tE(A) + E(B)$$

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = t_1 t_2 E(A^2) + (t_1 + t_2)E(AB) + E(B^2) \quad t_1, t_2 \in T$$

when $A \sim N(0,1), B \sim U(0,2)$, $\quad E(A) = 0, E(A^2) = 1, E(B) = 1, E(B^2) = \frac{4}{3}$

A, B independent $\quad E(AB) = E(A)E(B) = 0$

$$\Rightarrow \mu_X(t) = 1, R_X(t_1, t_2) = t_1 t_2 + \frac{4}{3} \quad t_1, t_2 \in T$$

Random Process

Example: The autocorrelation function of a stationary process is

$$R_X(\tau) = S_0\delta(\tau), S_0 > 0$$

Find the power spectral density

Solution: the power spectral density

$$S_X(\omega) = S_0\int_{-\infty}^{\infty} e^{-i\omega\tau}\delta(\tau)d\tau = S_0 e^{-i\omega 0} = S_0$$

This is the typical characteristic of white noise

Random Process

Example: The power spectral density of a stationary process is known as

$$S_X(\omega)=\frac{\omega^2+1}{\omega^4+10\omega^2+9}$$

Find the autocorrelation function and average power

Solution: the autocorrelation function $$R_X(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\omega^2+1}{\omega^4+10\omega^2+9}e^{i\omega\tau}d\omega$$

Fourier transform pair

$$\exp(-\alpha|\tau|)\Leftrightarrow\frac{2\alpha}{\alpha^2+\omega^2}$$

1	$2\pi\delta(\omega)$
$\delta(\tau)$ 白噪声	1
$e^{-\alpha\lvert\tau\rvert}$	$\frac{2\alpha}{\alpha^2+\omega^2}$
$e^{-\alpha\lvert\tau\rvert}\cos\omega_0\tau$	$\frac{\alpha}{\alpha^2+(\omega-\omega_0)^2}+\frac{\alpha}{\alpha^2+(\omega+\omega_0)^2}$
$\Delta(\tau/T)$	$\frac{T}{2}\cdot\frac{\sin^2(\omega T/4)}{(\omega T/4)^2}$
$\frac{\Omega}{\pi}\mathrm{sinc}(\Omega\tau)$	$\mathrm{rect}(\omega/2\Omega)$
$\frac{\Omega}{2\pi}\mathrm{sinc}^2(\Omega\tau/2)$	$\Delta(\omega/2\Omega)$
$e^{-\tau^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$

Random Process

Example: The power spectral density of a stationary process is known as

$$S_X(\omega)=\frac{\omega^2+1}{\omega^4+10\omega^2+9}$$

Find the autocorrelation function and average power

Solution: the autocorrelation function (residue method)

$$R_X(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\omega^2+1}{\omega^4+10\omega^2+9}e^{i\omega\tau}d\omega=\frac{1}{48}\left(9e^{-|\tau|}+5e^{-3|\tau|}\right)$$

the average power is

$$R_X(0)=\frac{14}{48}=\frac{7}{24}$$

Random Process

Example: The random process $X(t) = a\cos(\omega t + \Phi)$ a, ω are constant

Φ follows (0,2pi) uniform distribution.

Discuss about the ergodicity of the random process

Solution: $X(t)$ is obviously a stationary process $m_X(t) = 0$ $R_X(t_1,t_2) = \frac{a^2}{2} cos\,\omega t$

Now calculate the time average and autocorrelation

$$\langle X(t) \rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} a\cos\left(\omega_0 t + \Phi\right) dt$$

$$= \lim_{T\to\infty} \frac{a}{2T}\int_{-T}^{T} \left[\cos\omega_0 t\cos\Phi - \sin\omega_0 t\sin\Phi\right] dt$$

$$= \lim_{T\to\infty} \frac{a\cos\Phi}{2T}\int_{-T}^{T} a\cos\left(\omega_0 t\right) dt = \lim_{T\to\infty} \frac{a\cos\Phi\sin\omega_0 T}{\omega_0 T} = 0$$

$$\langle X(t)X(t+\tau) \rangle$$

$$= \lim_{T\to\infty} \frac{a^2}{2T}\int_{-T}^{T} \cos\left(\omega_0 t + \Phi\right)\cos\left(\omega_0(t+\tau) + \Phi\right) dt$$

$$= \mathrm{im}_{T\to\infty} \frac{a^2}{2T}\cdot\frac{1}{2}\int_{-T}^{T} \left[\cos\left(2\omega_0 t + \omega_0\tau + 2\Phi\right) + \cos\left(\omega_0\tau\right)\right] dt$$

$$= \frac{1}{2}a^2\cos\left(\omega_0\tau\right)$$

Therefore $\langle X(t) \rangle = m_X, \quad \langle X(t)X(t+\tau) \rangle = R_X(\tau)$ The random process is ergodic

Gaussian Process (Normal random Process)

A normal process is a special second-order moment process, can be determined by expectation and variance

One dimensional probability density function of Gaussian process

$$p_x(x,t_1)=\frac{1}{\sqrt{2\pi\sigma}}\exp\left[-\frac{(x-a)^2}{2\sigma^2}\right]$$

where $a=E[X(t)]$ is the mean $\sigma^2=E[X(t)-a]^2$ is the variance

N dimensional probability density function of Gaussian process

$$f(x_1,x_2,\ldots,x_n;t_1,t_2,\ldots,t_n)=\frac{1}{(2\pi)^{n/2}\,|\mathbf{C}^{1/2}|}\exp\left\{-\frac{1}{2}(\mathbf{x}-\mathbf{m}_\mathbf{x})\mathbf{C}^{-1}(\mathbf{x}-\mathbf{m}_\mathbf{x})^T\right\}$$

where $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, $\mathbf{m}_\mathbf{x}=(m_X(t_1),m_X(t_2),\ldots,m_X(t_n))$

$$\mathbf{C}=\begin{bmatrix} C_X(t_1,t_1) & C_X(t_1,t_2) & \ldots\ldots & C_X(t_1,t_n) \\ C_X(t_2,t_1) & C_X(t_2,t_2) & \ldots. & C_X(t_2,t_n) \\ \ldots\ldots & & & \\ C_X(t_n,t_1) & C_X(t_n,t_2) & \ldots. & C_X(t_n,t_n) \end{bmatrix}$$ is the covariance matrix

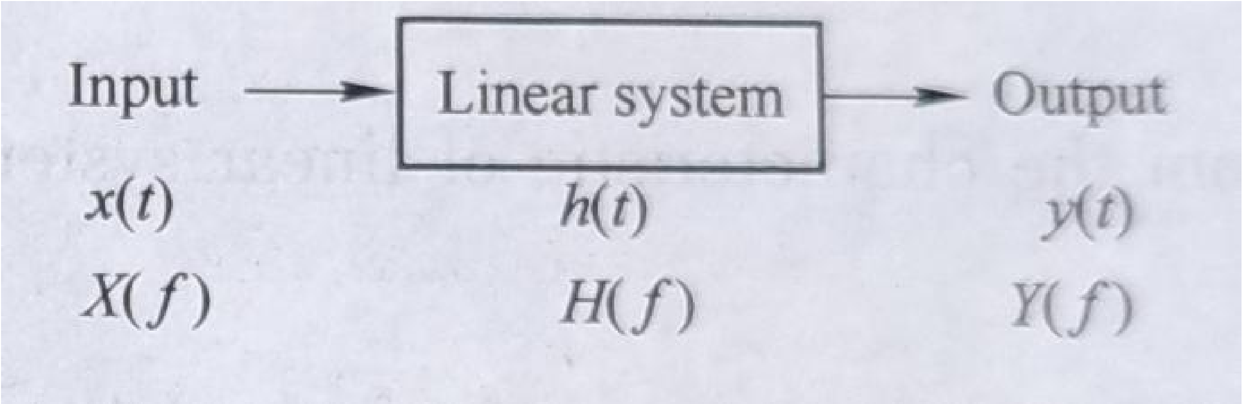
Signal Transfer through Linear Systems

Linear System

The input signal and the output signal satisfy the superposition principle

When the system input is $x(t)=a_1x_1(t)+a_2x_2(t)$

the system output is $y(t)=a_1y_1(t)+a_2y_2(t)$



Why discuss linear system

Most parts of communication systems have linear characteristics.

Linear system is the most basic and simplest network

Signal Transfer through Linear Systems

Deterministic Signal transfer through linear system

Time domain analysis

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\mathrm{d}\tau$$

For physically realizable systems, the condition should be satisfied

no output before the signal inputs $h(t) = 0 \quad t < 0$

finit energy of the impulse response $\int_{-\infty}^{\infty} |h(t)| \,\mathrm{d}t < \infty$

Frequency domain analysis

$$Y(f)=X(f)\,H(f)$$

The output signal can be found from the inverse Fourier transform

$$y(t) = \int_{-\infty}^{\infty} Y(f)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}f$$

Signal Transfer through Linear Systems

Random Signal transfer through linear system

input signal is a random process X(t) ,the output randomprocess Y(t) can be written as

$$Y(t)=\int_0^{\infty} h(\tau)X(t-\tau)\mathrm{d}\tau$$

Mathematical Expectation E[Y(t)] of the Output Random Process Y(t)

$$E[Y(t)]=E\left[\int_0^{\infty} h(\tau)X(t-\tau)\mathrm{d}\tau\right]=\int_0^{\infty} h(\tau)E[X(t-\tau)]\mathrm{d}\tau$$

If input is a stationary process

$$E[X(t-\tau)]=E[X(t)]=k,\quad k=\text{ constant}$$

$$\Rightarrow \quad E[Y(t)]=k\int_0^{\infty} h(\tau)\mathrm{d}\tau$$

Since $\quad H(0)=\int_0^{\infty} h(t)\mathrm{d}t$

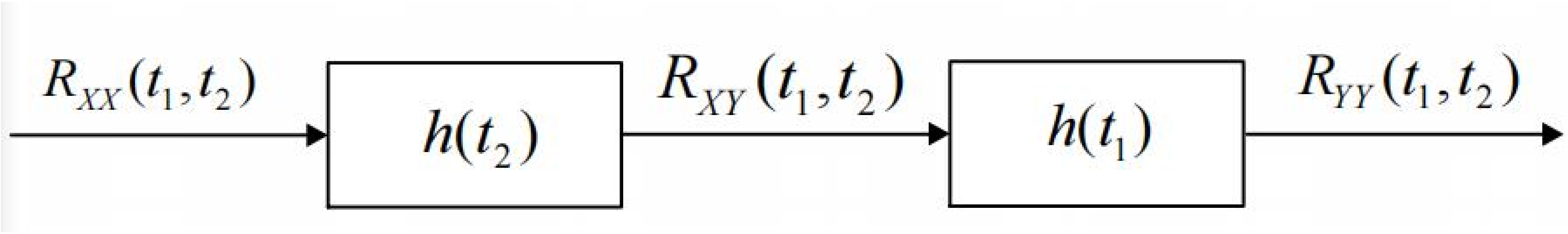
Thus $\quad E[Y(t)]=kH(0)$

Signal Transfer through Linear Systems

Random Signal transfer through linear system

Autocorrelation function of the Output Random Process Y(t)   (t1 is treated as a constant)

$$R_{XY}\left(t_1,t_2\right)=\int_{-\infty}^{\infty} h(\lambda)R_{XX}\left(t_1,t_2-\lambda\right)d\lambda=h\left(t_2\right)*R_{XX}\left(t_1,t_2\right)$$

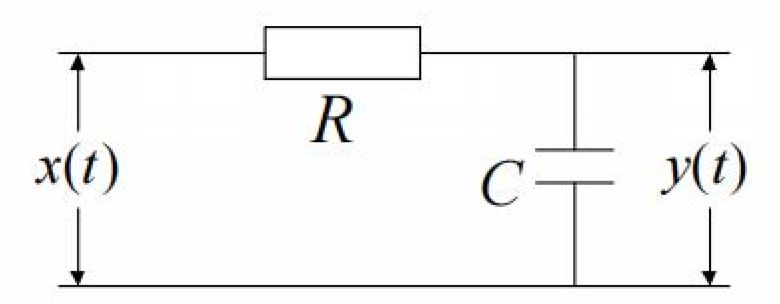


Power spectral density of the Output Random Process Y(t)

$$P_y(f)=H^*(f)H(f)P_X(f)=|H(f)|^2 P_X(f)$$

Signal Transfer through Linear Systems

Example: Assuming the R-C circuit system (as shown in the figure), the input and output voltages are x(t) and y(t), try to find the transfer function and impulse response function of the system and frequency response functions.



Solution: The input and the output has the relationship of

$$RC\frac{dy}{dt}+y(t)=x(t)$$

let $\alpha=-\frac{1}{RC}$ Then $\frac{1}{\alpha}\frac{dy}{dt}+y(t)=x(t)$ Take Laplacian transform $\frac{p}{\alpha}Y(p)+Y(p)=X(p)$

$$Y(p)=\frac{1}{\frac{p}{\alpha}+1}X(p)=\frac{\alpha}{p+\alpha}X(p) \Rightarrow H(p)=\frac{\alpha}{p+\alpha} \Rightarrow h(t)=F^{-1}(H(p))=\begin{cases}\alpha e^{-\alpha t}, & t\geq 0\\ 0, & t<0\end{cases}$$

Thank you！

Exercise

A random process is $X(t)=e^{-At}$, where A is a random variable following uniform distribution in (1,2)

Please find the PDF and the CDF at t=1

Exercise

A random process is $X(t) = A\cos(\omega_0 \cdot t + \Phi)$, where A is a random variable following uniform distribution in (1,2), Φ is a random variable following uniform distribution in (0,2pi),

1 Please find the mean and the autocorrelation function of the random process

2 Discuss about the ergodicity of the random process

Exercise

A stationary random process is $X(t), -\infty < t < \infty$, its spectral density is $S_X(\omega) = \dfrac{\omega^2 + 4}{\omega^4 + 10\omega^2 + 9}$

Please find the autocorrelation function of the random process and $EX^2(t)$

Exercise

A stationary random process is $X(t), -\infty < t < \infty$, its autocorrelation function is $R_X(\tau) = e^{-\beta|\tau|}$,

There is an output satisfying the relationship of $Y'(t) + \alpha Y(t) = X(t)$

Please find the autocorrelation function and the spectral density of $Y(t)$