Stochastic Signal Processing

Lesson 13-1: Ergodicity

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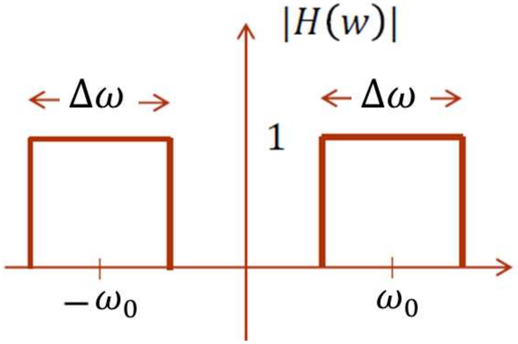
Example explained last week (回顾)

It is well known that if feeding an input Gaussian process to a stationary linear system (any deterministic system is stationary), the output is also a **Gaussian process**.

- 2: given an input Gaussian white noise $X(t)$ with power spectrum $S_X(\omega) = N_0/2$, and it is inputted to the ideal band-pass filter as below, calculate the 1-D pdf of the output.



Solution:

$$\sigma_Y^2 = R_Y(0) = \frac{\frac{\Delta\omega}{2} N_0}{\pi} sinc\left(\frac{\Delta\omega}{2}\ 0\right) \cos(\omega_0\ 0) = \frac{N_0 \Delta\omega}{2\pi}$$

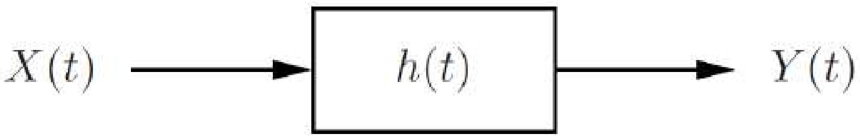
And $E[Y(t)] = 0$

Therefore, the output $Y(t)$ is a Gaussian process with mean 0, and its 1-D variance is $\frac{N_0\Delta\omega}{2\pi}$, thus

$$f_Y(y,t) = f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{y^2}{2\sigma_y^2}} = \frac{1}{\sqrt{N_0\Delta\omega}} e^{-\frac{\pi y^2}{N_0\Delta\omega}}$$

Distributions of Stochastic Processes at the Output （知识点补充）

- Based on the example from last week, we conclude the distributions of Stochastic Processes at the Output of a linear system.
- Here we only discuss the case when the input is a Gaussian process.
- Assume the system is



$$Y(t) = \int_{-\infty}^{\infty} X(\tau)h(t-\tau)d\tau$$

- Then we have

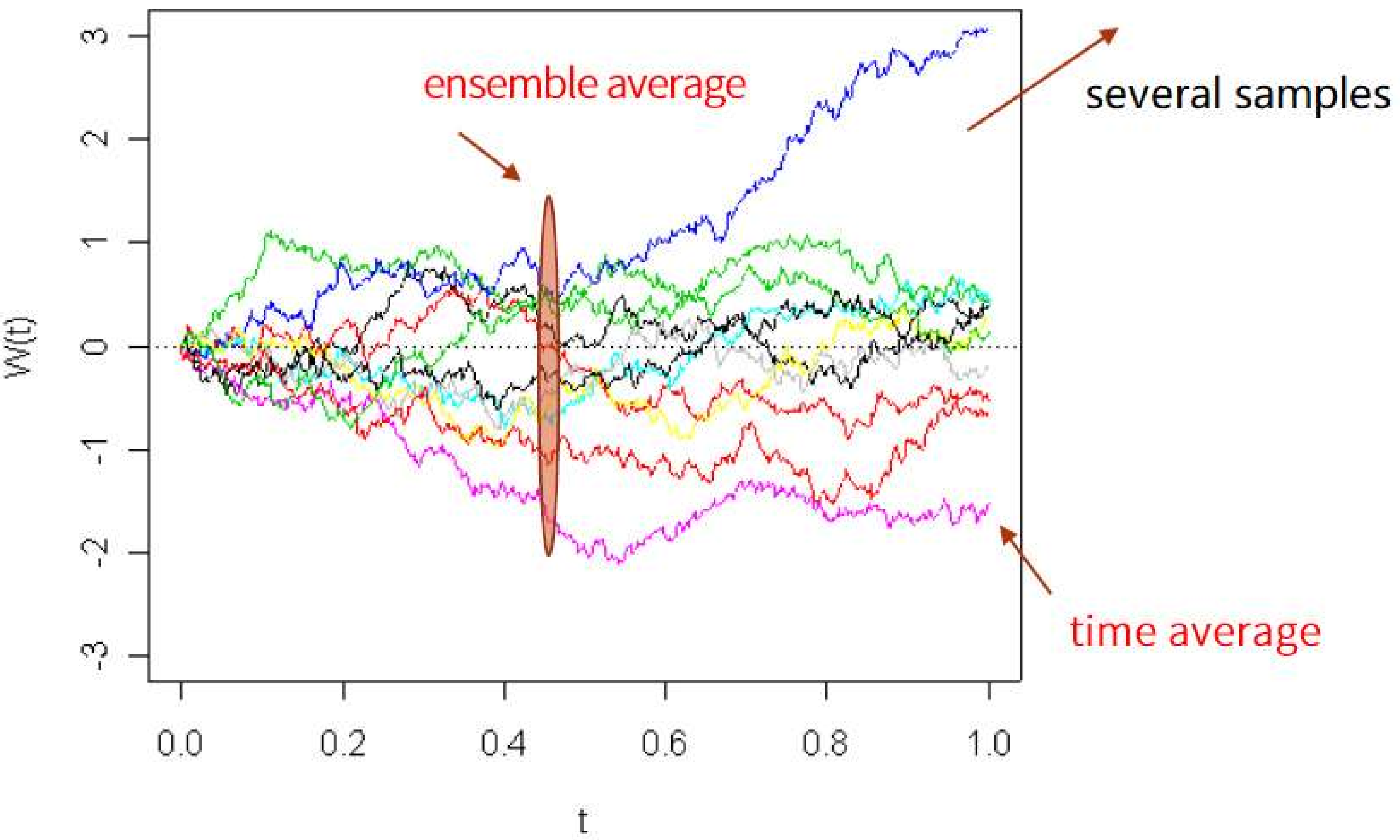
$$Y(t) = \lim_{\Delta\tau \to 0, n \to \infty} \sum_{i=1}^{n} X(\tau_i)h(t-\tau_i)\,\Delta\tau$$

- Therefore, when the input $X(t)$ is a Gaussian process, for any t, $Y(t)$ is the sum of an infinite number of Gaussian variables $X(\tau_i)$, which is also a Gaussian variable.
- In fact, $X(t)$ and $Y(t)$ form a joint Gaussian processes.
- Conclusion: if the input to a stationary linear system is a Gaussian process, the output is also a **Gaussian** process. Thus we can write the pdf/CDF.

Outline

- Ergodicity

Ergodicity – starting from the ‘average’ of a process



Ergodicity – for a stationary process

- Ensemble average: Average the values of all samples at time t

  $E[X(t)] = \int_{-\infty}^{\infty} xf(x)\,dx = m_X$ (for stationary process, the mean is time independent)

- The 'ensemble' autocorrelation:

$$R_X(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2, \tau)\,dx_1\,dx_2 = R_X(\tau)$$

- It requires the system to take a great deal of samples (n samples) of a process at time t:

$$m_X(t) \approx \frac{1}{n}\sum_{i=1}^{n} x_i(t)$$

$$R_X(t+\tau, t) \approx \frac{1}{n}\sum_{i=1}^{n} x_i(t+\tau)x_i(t)$$

  - for a large n, these average and autocorrelation are called 'accurate estimates of the true average and autocorrelation'.
  - However, this ensemble average and autocorrelation cannot be computed in real world experiment, because for any real world system, we can only get 1 sample at any time t.

Ergodicity – for a stationary process

- What we can calculated in real world system is to sample a process for a time period of $2T$, and get $X(t)$ for time interval $[-T,\ T]$, and calculate the average and autocorrelation:

$$\overline{m_X} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)dt$$
$$\overline{R_X(\tau)} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t+\tau)X(t)dt$$

- These are called time average and time autocorrelation.
- Given a long time $2T$, we can calculate an accurate time average and time autocorrelation.
- The problem is, is the **computable** time average $\overline{m_X}$ and time autocorrelation $\overline{R_X(\tau)}$ equals to the **incomputable** ensemble average m_X and ensemble autocorrelation $R_X(\tau)$?

This is called the ergodicity problem

Ergodicity

- If the time average of a stationary stochastic process $X(t)$ equals to the ensemble average of $X(t)$, then the $X(t)$ is average/mean ergodicity:

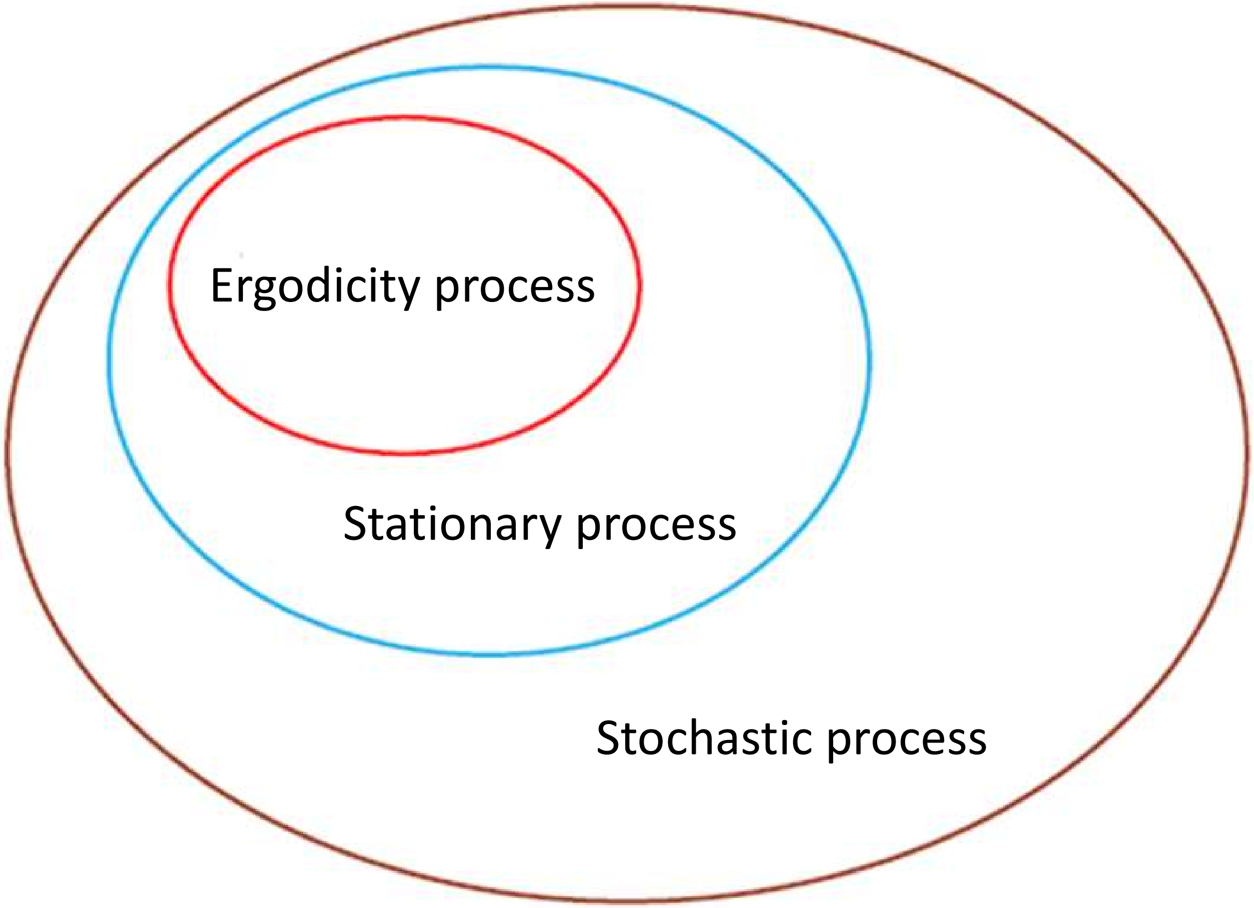
$$\overline{m_X} = m_X$$

- If the time autocorrelation of a stationary stochastic process $X(t)$ equals to the ensemble autocorrelation of $X(t)$, then the $X(t)$ is autocorrelation ergodicity:

$$\overline{R_X(\tau)} = R_X(\tau)$$

- If both the mean and the autocorrelation of a $X(t)$ stationary stochastic process $X(t)$ are ergodicity, it is called an ergodicity process.
- Physical meaning:
  - All possible states of a stochastic process appear in any sample given a long time (任选一个样本，记录足够长的时间，所有的状态/取值都会出现)
  - Any sample can be used as a typical sample which is fully representative of the stochastic process （任选一个样本都是具有代表性的样本）

- Relationship between ergodicity and stationarity:



Ergodicity process must be stationary!

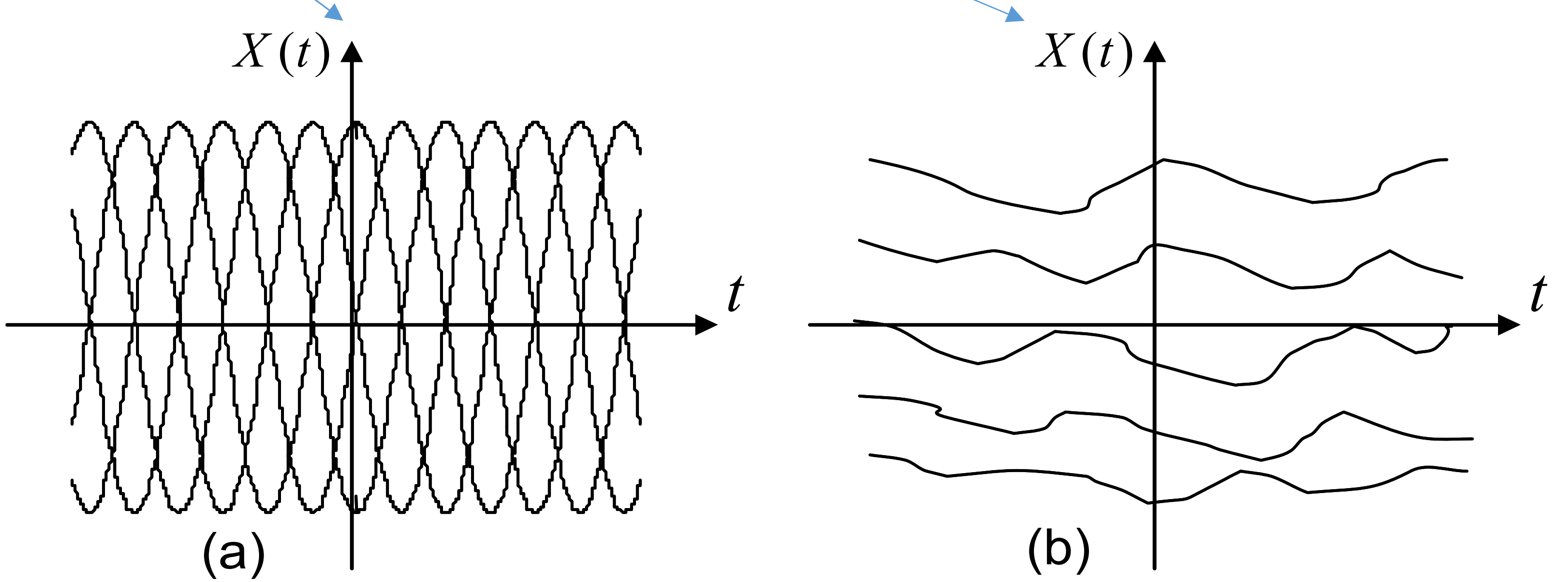
Ergodicity

- Another method to determine the Ergodicity of a process:
  - For the most common zero mean stationary normal stochastic process, it can be proved that if:

$$\int_0^\infty |R_X(\tau)|\, d\tau < \infty$$

It is an ergodicity process.

- Typical ergodicity and non-ergodicity process



Ergodicity

- Example 1: Determine whether the following process is rgodicity or not, the ϕ is uniformly distributed in $(0,\ 2\pi)$.

$$X(t) = A\cos(\omega_0 t + \phi)$$

Reminder:

$$\overline{m_X} = m_X \ ?$$

$$\overline{R_X(\tau)} = R_X(\tau) \ ?$$

Ergodicity

- Example 1: Determine whether the following process is rgodicity or not, the ϕ is uniformly distributed in $(0,\ 2\pi)$.

$$X(t) = A\cos\left(\omega_0 t + \phi\right)$$

Solution:

Ergodicity

- Example 1: Determine whether the following process is rgodicity or not, the ϕ is uniformly distributed in (0, 2π).

$$X(t) = A\cos\left(\omega_0 t + \phi\right)$$

Solution:

Here, ϕ is regarded as a constant (that is, a fixed "state"); the cos function changes with the value of t, and the integral is 0 for one period; when T tends to infinity, which is equivalent to infinite periods, The final result is 0.

$$\overline{\mathrm{m_X}} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A\cos(\omega_0 t + \phi)\, dt = 0$$

$$E[X(t)] = \frac{1}{2\pi} \int_{0}^{2\pi} A\cos(\omega_0 t + \phi)\, d\phi = 0$$

Here, t is regarded as a constant (that is, a fixed "time"); note that ϕ is uniformly distributed in (0, 2π), it is one period; integral is 0 for any value t.

- Note: If the ensemble average is dependent to t, it is non-stationary thus not rgodicity.

Ergodicity

- Example 1: Determine whether the following process is rgodicity or not, the ϕ is uniformly distributed in $(0,\ 2\pi)$.

$$X(t) = A\cos\left(\omega_0 t + \phi\right)$$

Solution:

$$\overline{R_X(\tau)} = \lim_{T\to\infty} \frac{1}{2T}\int_{T}^{T} A\cos[\omega_0(t+\tau)+\phi]A\cos(\omega_0 t+\phi)d\phi$$

$$= \frac{A^2}{2}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[\cos[2\omega_0 t+\omega_0\tau+2\phi]+\cos\omega_0\tau]d\phi$$

$$= \frac{A^2}{2}\cos\omega_0\tau$$

Note that this part is a constant for any t.

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$$R_X(t+\tau,t) = E[X(t+\tau)X(t)] = A^2E\{\cos[\omega_0(t+\tau)+\Phi]\cos[\omega_0 t+\Phi]\}$$

$$= \frac{A^2}{2}E[\cos[2\omega_0 t+\omega_0\tau+2\Phi]+\cos\omega_0\tau]$$

$$= \frac{A^2}{2}\cos\omega_0\tau + \frac{A^2}{2}\frac{1}{2\pi}\int_0^{2\pi}\cos(2\omega_0 t+\omega_0\tau+2\phi)d\phi$$

$$= \frac{A^2}{2}\cos\omega_0\tau$$

Note that this part is a constant for any ϕ.

Therefore, it is an ergodicity process.

Ergodicity

- Example 2: The power spectrum of a zero mean stationary normal stochastic process $X(t)$ is $S(\omega) = 1/(\omega^2 + 1)$, determine its ergodicity.

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Solution:

$$R_X(\tau) = e^{-\frac{1}{2}|\tau|}$$

use the property ' if $\int_0^\infty |R_X(\tau)|\, d\tau < \infty$, it is an ergodicity process':

$$\int_0^\infty |R_X(\tau)|\, d\tau = \int_0^\infty e^{-\frac{1}{2}|\tau|}\, d\tau = \frac{1}{2} - 0 = \frac{1}{2}$$

therefore it is an ergodicity process

Reading

- Next week:
  - Random walk (12.1)

Others

- Go on with Experiment 3