Stochastic Signal Processing

Lesson 5: Basic of Stochastic Processes

Weize Sun

Basic of Stochastic Processes – outline

- Definitions
- Distributions of Stochastic Processes
  - One-dimensional distribution
  - Two-dimensional and multidimensional distributions
- Statistics of Stochastic Processes
  - Mean, variance, correlation function, covariance function
  - For Discrete time Processes
- Stationary stochastic processes

Definitions

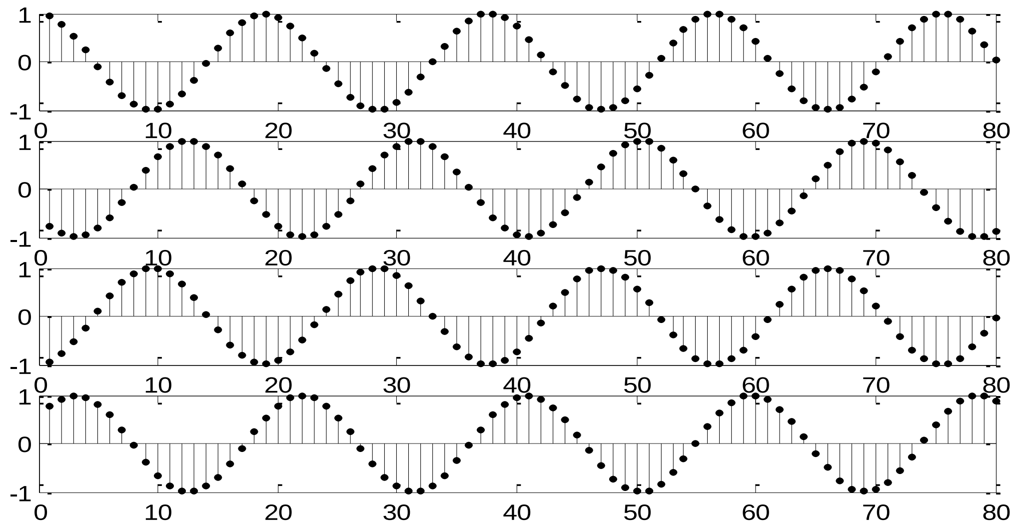
- There are mainly two kinds of processes: deterministic processes and stochastic processes

deterministic processes	The result obtained from each observation is the same: not related to time t.
stochastic processes	The result obtained from each observation might not be the same: related to time t.

Concepts and Definitions of Stochastic Processes

- Examples

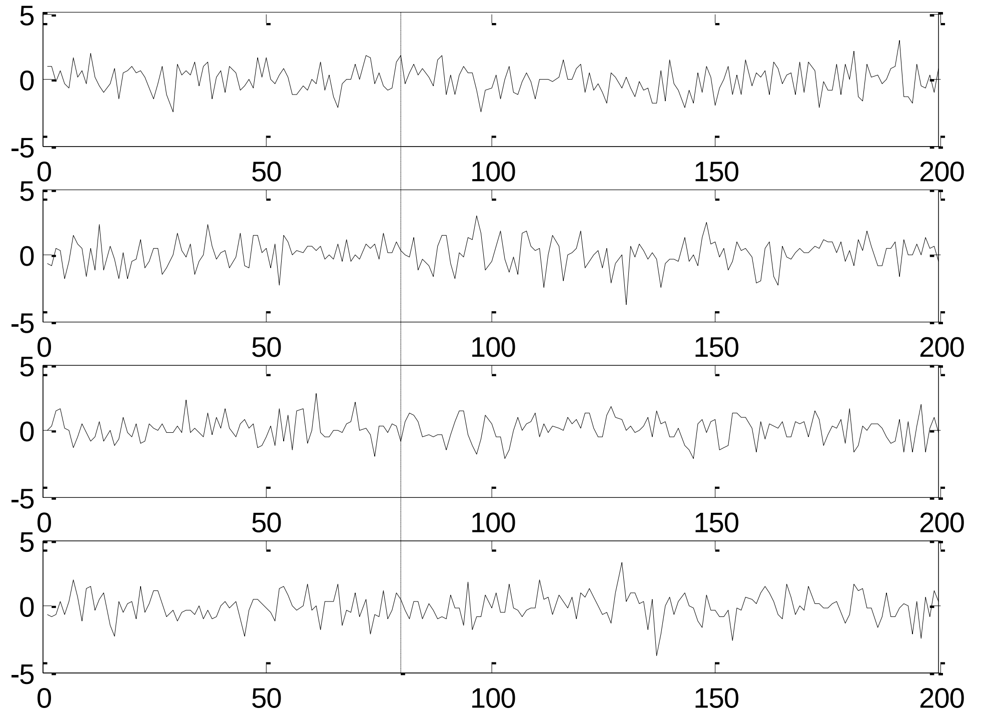
$$X(n) = A\cos(\omega_0 n + \phi)$$



  - Sinusoidal signal
  - Sample（样本）: Each time a sample is taken $X_t(n, \phi_t) = A\cos(\omega_0 n + \phi_t)$, the phase ϕ_t is not necessarily the same and changes randomly according to time
  - This is one typical Stochastic Phase Signal

Concepts and Definitions of Stochastic Processes

- Examples
  - receiver noise



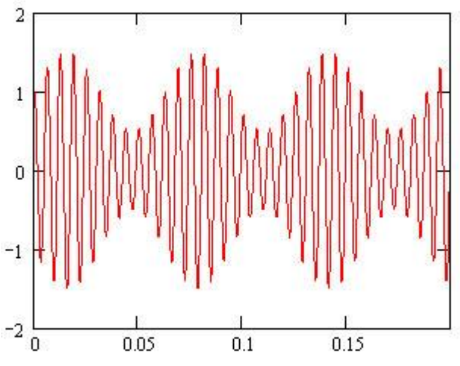
- In each time, when sampling this Stochastic Process, we get a random variable that changes according to time
- In this case, the Stochastic Process can be regarded as a collection of random variables

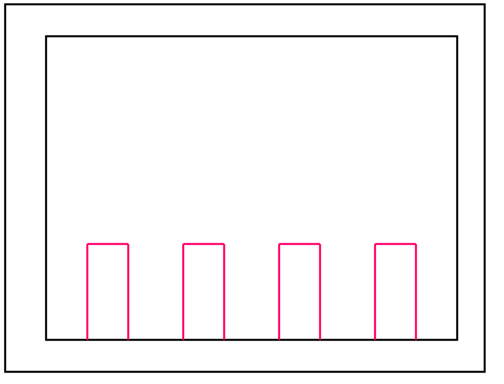
Concepts and Definitions of Stochastic Processes

- Other Stochastic processes

Modulated signal

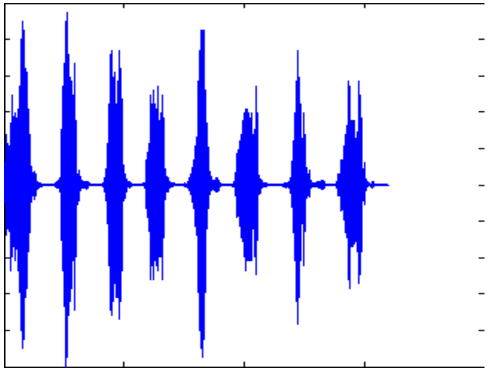
（调制信号）

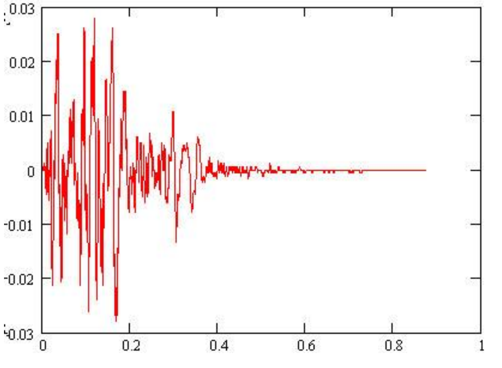




Periodic pulse signal

bird call





blasting signal

（爆破信号）

Concepts and Definitions of Stochastic Processes

- Definition of Stochastic Process:
  - Definition 1: Let E be a random test（随机试验） with $S = \{e\}$, where S is a set. For each $e_i (i = 1,2, \ldots)$, we can use some methods to record a sample（样本）$x(t, e_i)$ then the ensemble（总体）$X(t, e)$ is a Stochastic Processes, abbreviated as $X(t)$.

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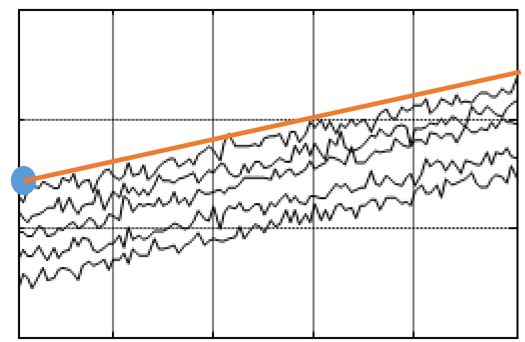
    - From this: a stochastic process is a collection of a lot of samples

Assuming there are 5 values for $\{e\}$, for each value, determine a function, as shown on the right.

At a certain time t, start sampling, for example, the first line is picked.



Continue with time and we will get the first line

- When sampling, one of these five lines will be sampled randomly. The values from two samples might be completely different.
- The results are related to the ‘initial state value’ $\boldsymbol{e}$.

Concepts and Definitions of Stochastic Processes

- Definition of Stochastic Process:
  - Definition 2 (from another perspective): Given a process $X(t)$, for any fixed time $t_j (j = 1,2, \ldots)$, $X(t_j)$ is a random variable, then the $X(t)$ is called a stochastic process.

Concepts and Definitions of Stochastic Processes

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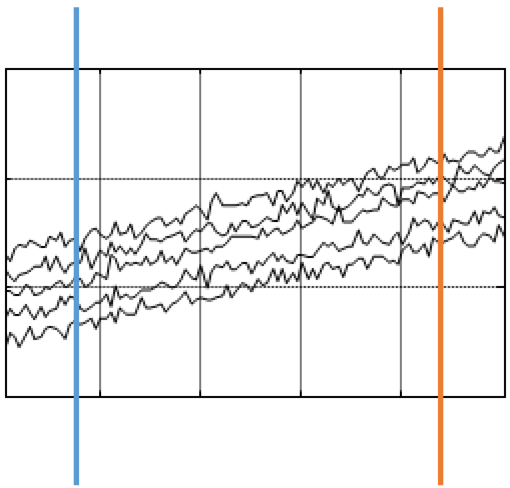
    - From this : A stochastic process is a collection of random variables

At time t_1, perform extensive experiments at the same time, and get all possible values to form a set of "random variables"

At time t_2, do extensive experiments again at the same time and get all possible values to form a set of "random variables"



- When sampling at different times, the set of "random variables" that will be obtained might be completely different.
- Therefore, the stochastic process we finally get is related to the “time” t at the time of sampling.

Concepts and Definitions of Stochastic Processes

- Definition of Stochastic Process:
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  - Definition 2 (from another perspective): Given a process $X(t)$, for any fixed time $t_j (j = 1,2, \dots)$, $X(t_j)$ is a random variable, then the $X(t)$ is called a stochastic process.
    - From this : A stochastic process is a collection of random variables
  - The above two definitions are in fact consistent and complement to each other.
    - In real world observation, definition 1 is usually used (we start observe, and keep observing for a long long time)
    - In theoretical analysis, definition 2 is usually used (we cannot observe a system many times at one moment, but we can analyze it)
  - Later we will talk about 'ergodicity', it can be simply understood as: the equivalence of the above two definitions.

Concepts and Definitions of Stochastic Processes

- Definition of Stochastic Process:
  - Therefore, we generally use $\boldsymbol{X(t,e)}$ to represent a stochastic process:
  - The meaning of random processes $X(t,e)$ in four different situations:

when t and e fixed	X is a number;
when t is fixed, e is variable	$X(e)$ is a random variable equal to the state of the given process at time t
when t is variable, e is fixed	$X(t)$ is a single time function, or a sample of the given process
when t is variable,e is variable	$X(t,e)$ is a stochastic process; in most cases, we write it as $X(t)$

Classification of stochastic processes

- **Classified by state and time:**

	state/ value	time/ time
continuous stochastic process	continuous	continuous
continuous stochastic sequence	continuous	discrete
discrete stochastic process	discrete	continuous
discrete stochastic sequence	discrete	discrete

- Indoor temperature in one day
- The daily closing price of the stock for a month
- Result of a football game
- Number of students taking one course in different years

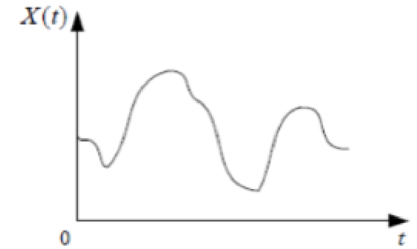
Classification of stochastic processes

- Example 1: classified by state and time:

	state/ value	time/ time	
continuous stochastic process	continuous	continuous	?
continuous stochastic sequence	continuous	discrete	?
discrete stochastic process	discrete	continuous	?
discrete stochastic sequence	discrete	discrete	?

?

A



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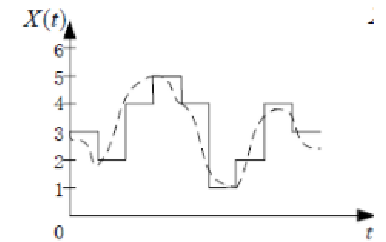

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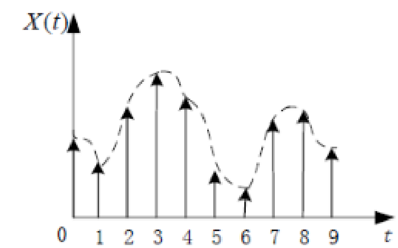

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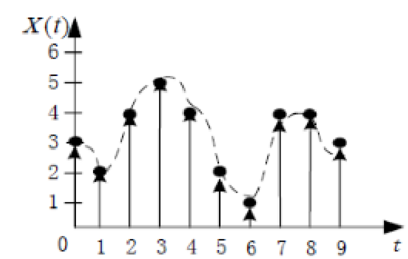

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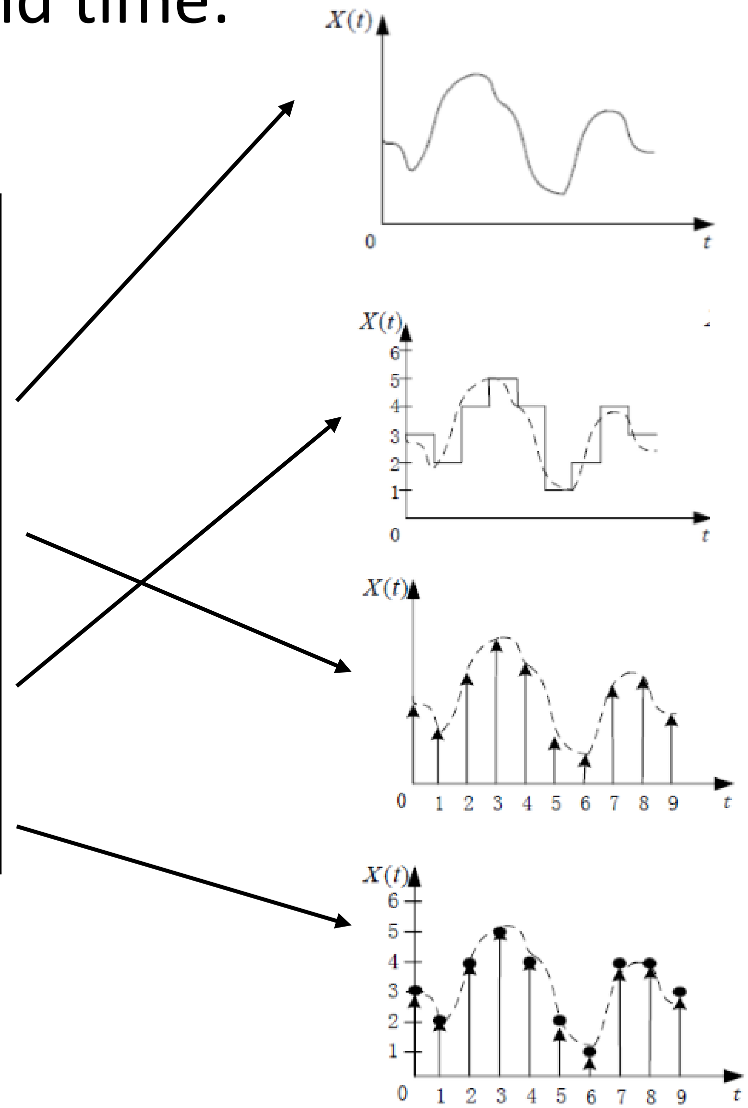


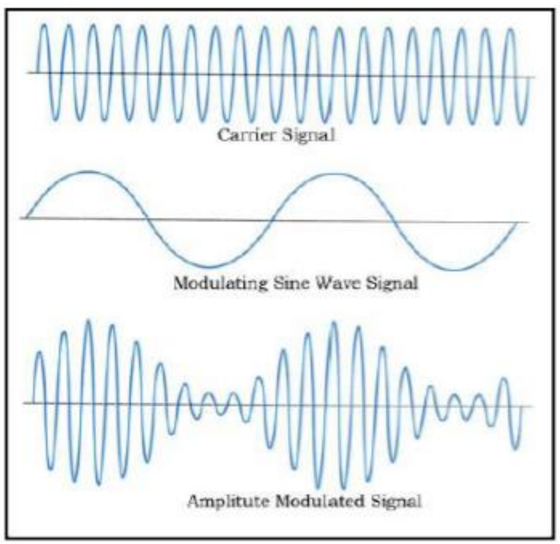
Example

- The sinusoidal signal: $X(t) = A\cos(\omega t + \varphi)$
  - A is a random variable $A(t)$: $X(t) = A(t)\cos(\omega t + \varphi)$;

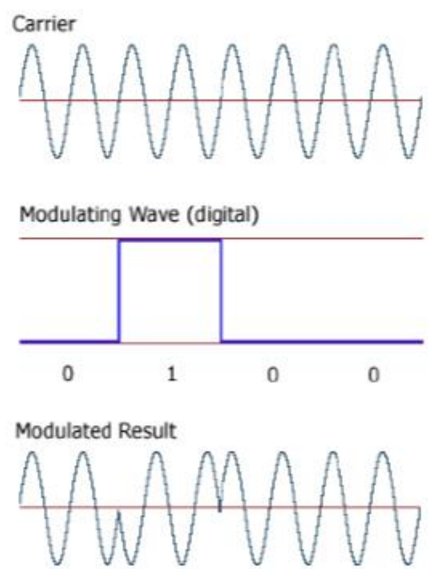
  Amplitude modulation signal (AM)
  - $\varphi(t)$ is a random variable $\varphi(t)$: $X(t) = Acos(\omega t + \varphi(t))$;

  phase modulation signal (PM)
  - ω is a random variable $\omega(t)$: $X(t) = Acos(\omega(t)t + \varphi)$;

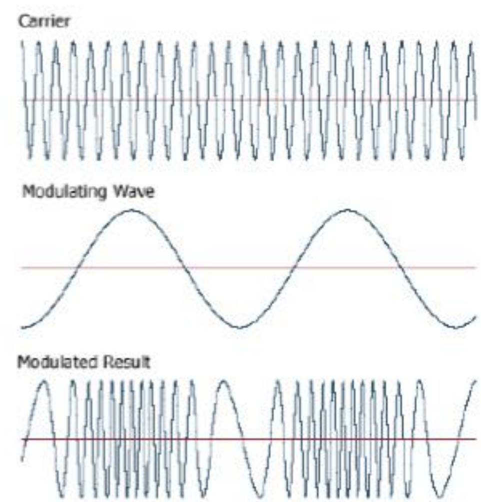
  frequency modulation signal (FM)



AM



PM



FM

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Distributions of Stochastic Processes

- By definition, a stochastic process can be regarded as a set of random variables that change according to time, therefore its statistics as the same as those from random variables:

	state/ value	time/ time
continuous stochastic process	continuous	continuous
continuous stochastic sequence	continuous	discrete
discrete stochastic process	discrete	continuous
discrete stochastic sequence	discrete	discrete

- One-dimensional (1D) CDF (left) and pdf (right)
  - For continuous processes:

$$F_X(x,t) = P\{X(t) \le x\} \qquad f_X(x,t) = \frac{\partial F_X(x,t)}{\partial x}$$

  - For discrete sequences:

$$F_X(x,n) = P\{X(n) \le x\} \qquad f_X(x,n) = \frac{\partial F_X(x,n)}{\partial x}$$

- 1D distribution can only describe of a certain time moment of the stochastic process $X(t)$, and the relationship between different time moments cannot be seem.

Distributions of Stochastic Processes

- Example 2: Given $X(t) = Y \cos \omega_0 t$, where ω_0 is a constant, Y is a normal variable with zero mean and variance 1, find the pdf when $t = 0, \frac{2\pi}{3\omega_0}, \frac{\pi}{2\omega_0}$.

Distributions of Stochastic Processes

- Example 2: Given $X(t) = Y\cos\omega_0 t$, where ω_0 is a constant, Y is a normal variable with zero mean and variance 1, find the pdf when $t = 0, \frac{2\pi}{3\omega_0}, \frac{\pi}{2\omega_0}$.

Solution:

1. $X(0) = Y$ therefore:

$$f_X(x,0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

2. $X\left(\frac{2\pi}{3\omega_0}\right) = -\frac{1}{2}Y$, $\quad f_X(x, \frac{2\pi}{3\omega_0}) = f_Y(y)|J||_{Y=-2X}$, $\quad |J| = 2$

$$\rightarrow \quad f_X\left(x, \frac{2\pi}{3\omega_0}\right) = \sqrt{\frac{2}{\pi}} e^{-2x^2}$$

3. $X\left(\frac{\pi}{2\omega_0}\right) = 0 \quad \rightarrow \quad f_X\left(x, \frac{\pi}{2\omega_0}\right) = \delta(x)$

Distributions of Stochastic Processes

- Two-dimensional(2D) distributions:
  - The distributions of a two-dimensional random variable $[X(t_1), X(t_2)]$ corresponding to any two moments t_1 & t_2 :

$$F_X(x_1, x_2, t_1, t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\}$$

$$f_X(x_1, x_2, t_1, t_2) = \frac{\partial^2 F_X(x_1, x_2, t_1, t_2)}{\partial x_1 \partial x_2}$$

  - Note: $X(t_1)$and $X(t_2)$ are random variables, refer to two different moments t_1 & t_2 of the same stochastic process

Distributions of Stochastic Processes

- n-dimensional(n-D) distributions:
  - For $X(t_1), X(t_2), \dots, X(t_n)$:

$$F_n(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = P\{X(t_1) \le x_1, X(t_2) \le x_2, \dots, X(t_n) \le x_n\}$$

(n-D CDF)

$$f_n(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = \frac{\partial F_n(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n)}{\partial x_1 \partial x_2 \dots \partial x_n}$$

(n-D pdf)

  - When n is large, it can describe the stochastic processes in more detail, but will increase the complexity of the analysis.
  - In theory, $n = \infty$ can fully describe the statistical properties of a stochastic process.
  - In practice, only 2-D distributions are used, it is a trade-off between accuracy and efficiency
  - If $X(t_1), X(t_2), \dots, X(t_n)$ are statistically independent, then

$$\boxed{f_n(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = f_1(x_1, t_1) f_1(x_2, t_2) \dots f_1(x_n, t_n)}$$

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Statistics of Stochastic Processes

- Mean of a stochastic process:

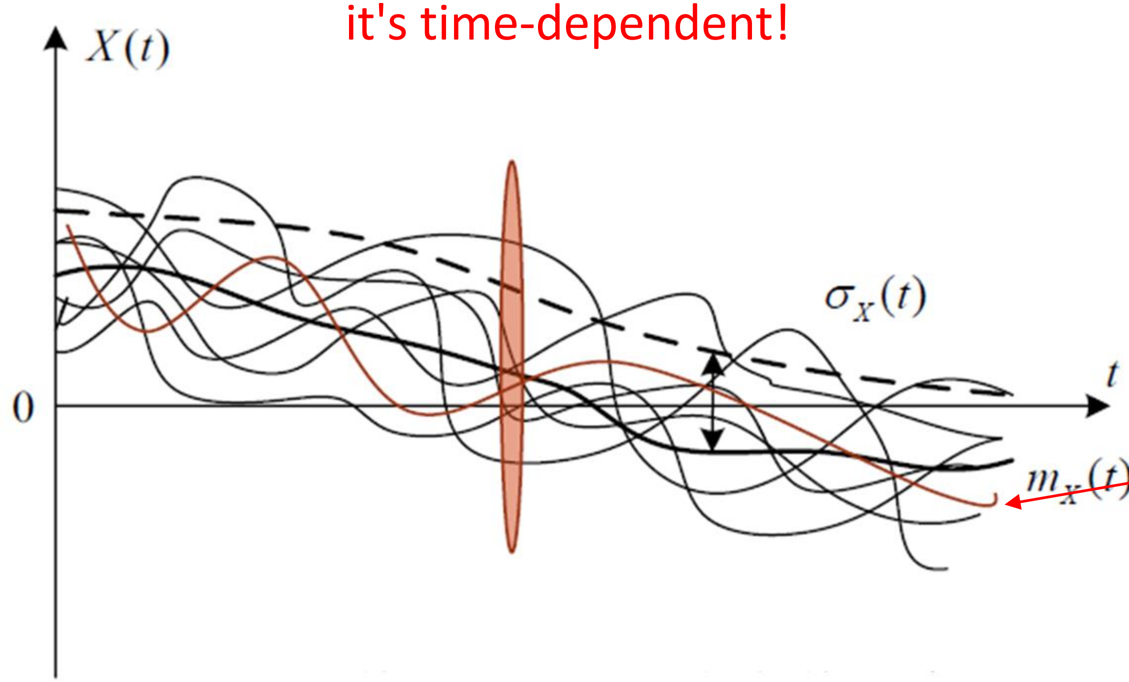
$$m_X(t) = E\{X(t)\} = \int_{-\infty}^{+\infty} x f_X(x,t)dx$$

  - The mean is a function of time t, also known as the mean function, this is the probability-weighted average of all the values of all samples of the stochastic process at time t, so it is also called the **ensemble average**(总体平均，也有称为集合平均的).
  - The mean value of a stochastic process can be intuitively understood as the center of all sample at time t (see definition 2 in page 8 of this ppt), the change of mean reflects the statistical average value of the stochastic process along time.
  - Another important concept or definition of mean is time average (时间平均, see page 487 of text book), which is the average of a specific sample for a long time (see definition 1 in page 7 of this ppt), will explain later.

• Mean of a stochastic process:

$m_X(t)$: **ensemble average**

it's time-dependent!



Time average

- It is shown that the time average and the ensemble average are different.
- In most real world applications, we can only record the time average, then, is the time average = ensemble average?
- This is the problem of Ergodicity(各态历经性).

Statistics of Stochastic Processes

- Variance

$$\sigma_X^2(t) = E\{[X(t) - m_X(t)]^2\} = E\{X^2(t)\} - m_X^2(t)$$

- An example of the physical meaning of mean and variance:

$X(t)$-----Voltage on unit resistance

$X^2(t)/1$-----Instantaneous power consumed on unit resistance

$[X(t) - m_{x(t)}]^2/1$-----Instantaneous AC power consumed on unit resistance

$E\{[X(t) - m_{x(t)}]^2/1\}$-----Statistical average value of instantaneous AC power consumed on unit resistance

$$E\{X^2(t)\} = \sigma_X^2(t) + m_X^2(t)$$

The total average power consumed on unit esistance.

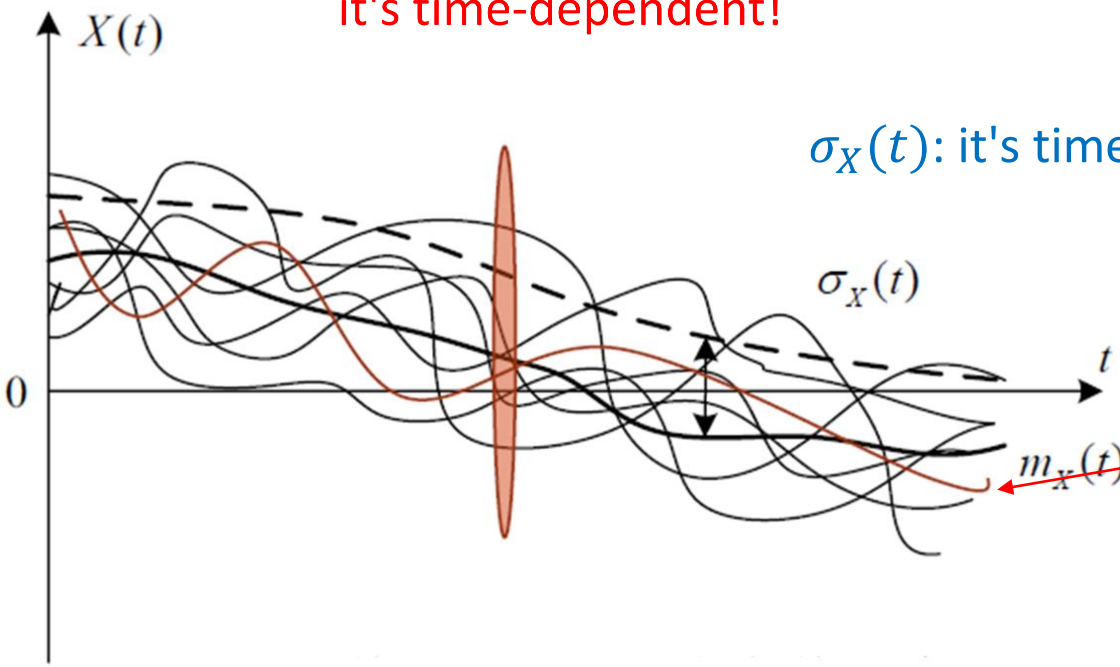
Average DC power

Average AC power

- Mean and variance

$m_X(t)$: **ensemble average**
it's time-dependent!

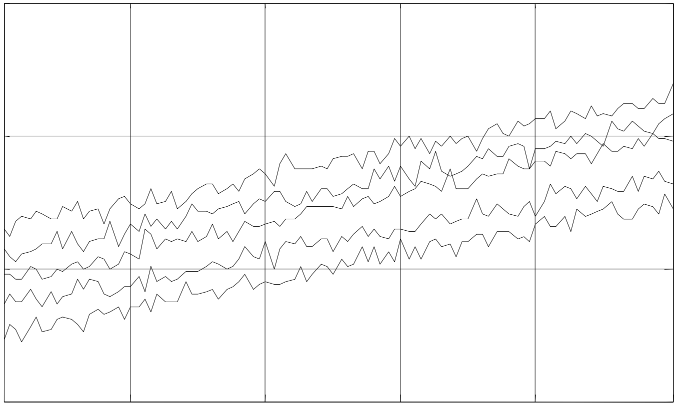
$\sigma_X(t)$: it's time-dependent!

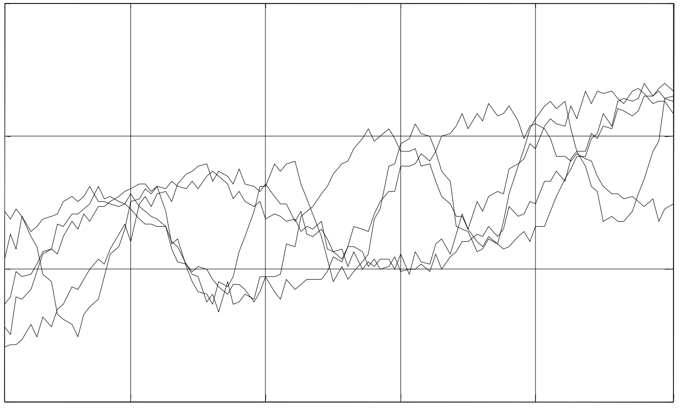


Time average

Statistics of Stochastic Processes

- Autocorrelation(自相关/自相关函数):
  - Two stochastic processes might have the same mean and variance





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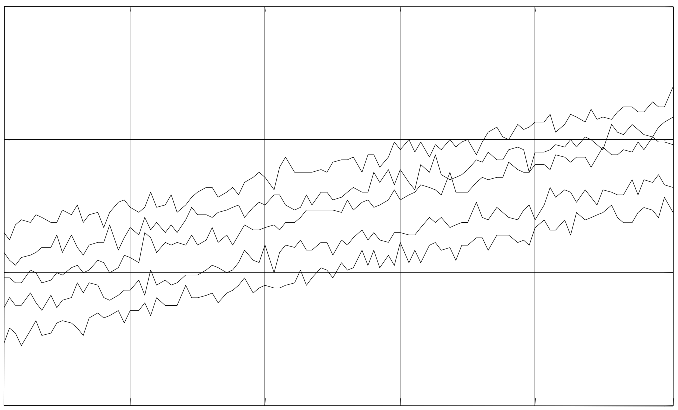

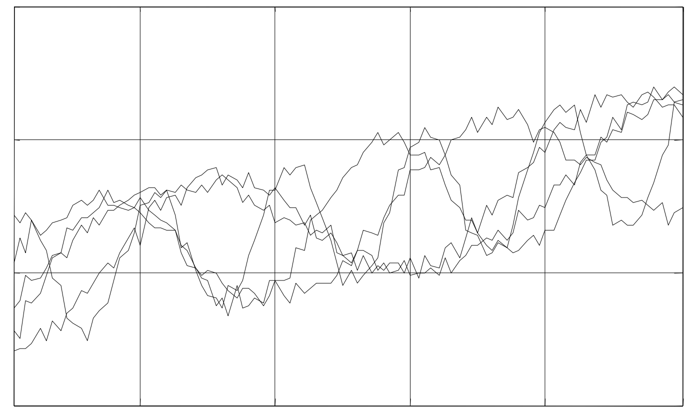

  - For any two random variables $X(t_1)$ & $X(t_2)$ draw from the complex stochastic processe $X(t)$, define the expectation of $X(t_1)X^*(t_2)$ as the **autocorrelation**:

$$R_X(t_1, t_2) = E\{X(t_1)X^*(t_2)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2^* f(x_1, x_2, t_1, t_2) dx_1\, dx_2$$

    - Reflects the average degree of correlation between values taken at any two moments of the $X(t)$.
    - $X^*(t_2)$ is the conjugate（共轭） of $X(t_2)$
    - Note that, in this case, $R_X(t_1, t_2) = E\{X(t_1)X^*(t_2)\} \neq R_X(t_2, t_1) = E\{X(t_2)X^*(t_1)\}$

Statistics of Stochastic Processes

- Autocorrelation:
  - For any two random variables $X(t_1)$ & $X(t_2)$ draw from the stochastic processe $X(t)$, define the expectation of $X(t_1)X^*(t_2)$ as the **autocorrelation**:

  Complex stochastic processe:

$$R_X(t_1, t_2) = E\{X(t_1)X^*(t_2)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2^* f(x_1, x_2, t_1, t_2) dx_1\, dx_2$$

  Real stochastic processe:

$$R_X(t_1, t_2) = E\{X(t_1)X(t_2)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 f(x_1, x_2, t_1, t_2) dx_1\, dx_2$$

- The autocorrelation can be complex or real valued, and can be positive or negative
- In general, the farther apart the time is, the weaker the correlation becomes, and the weaker the absolute value of the autocorrelation is.
- When the two moments coincide(重合), the strongest correlation (of this process) is obtained, therefore, $|R_X(t, t)|$ is the maximum

Statistics of Stochastic Processes

- Autocorrelation:
  - For any two random variables $X(t_1)$ & $X(t_2)$ draw from the stochastic processe $X(t)$, define the expectation of $X(t_1)X^*(t_2)$ as the **autocorrelation**:

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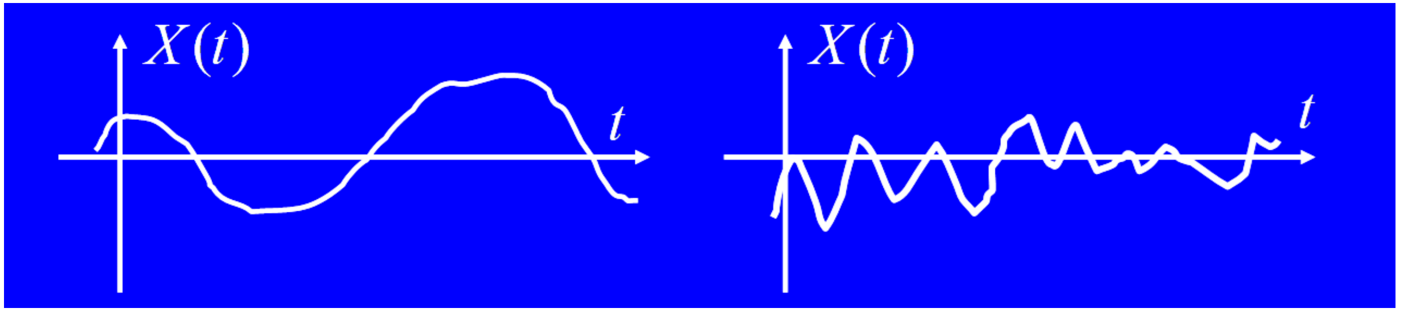
- The figure below is an example of 2 different therefore the correlation is stronger under the same conditions (same $t_1\&t_2$).



Statistics of Stochastic Processes

- Autocorrelation:
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  Real stochastic processe:

$$R_X(t_1,t_2) = E\{X(t_1)X(t_2)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 f(x_1,x_2,t_1,t_2)dx_1\, dx_2$$

- Application examples:
  - The channel information in the communication system changes with time. The information estimated at time t_1 (for example, the distribution of mobile phones in a cellular grid) is not necessarily consistent with the information at time t_2.
  - The farther the two moments are, the higher the degree of inconsistency.

Statistics of Stochastic Processes

- Autocovariance: the covariance of $X(t_1)$ and $X(t_2)$:
  - For complex stochastic processe:

$$C_X(t_1,t_2) = E\{[X(t_1) - m_X(t_1)][X(t_2) - m_X(t_2)]^*\}$$

$$= R_X(t_1,t_2) - m_X(t_1)m_X^*(t_2)$$

$$R_X(t,t) = E\{|X(t)|^2\} \ \& \ C_X(t,t) = \sigma_X^2(t)$$

$$\rightarrow \sigma_X^2(t) = E\{|X(t)|^2\} - |m_X\ (t)|^2$$

  - For real stochastic processe (we will go on with real stochastic processe unless stated):

$$C_X(t_1,t_2) = E\{[X(t_1) - m_X(t_1)][X(t_2) - m_X(t_2)]\}$$

$$= R_X(t_1,t_2) - m_X(t_1)m_X(t_2)$$

$$R_X(t,t) = E\{X(t)^2\} \& C_X(t,t) = \sigma_X^2(t)$$

$$\rightarrow \sigma_X^2(t) = E\{X(t)^2\} - m_X^2(t)$$

Statistics of Stochastic Processes

- Conclusion:

$$R_X(t_1,t_2) = E\{X(t_1)X(t_2)\}$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1x_2f(x_1,x_2,t_1,t_2)dx_1\,dx_2$$
$$C_X(t_1,t_2) = E\{[X(t_1)-m_X(t_1)][X(t_2)-m_X(t_2)]\}$$

Correlated? Orthogonal? Independent?

Given	$X(t_1), X(t_2)$ will be
$C_X(t_1,t_2) = 0$	?
$R_X(t_1,t_2) = 0$	?
$f_X(x_1,x_2,t_1,t_2) = f_X(x_1,t_1)f_X(x_2,t_2)$	?

Statistics of Stochastic Processes

- Conclusion:

$$R_X(t_1, t_2) = E\{X(t_1)X(t_2)\}$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 f(x_1, x_2, t_1, t_2) dx_1\, dx_2$$
$$C_X(t_1, t_2) = E\{[X(t_1) - m_X(t_1)][X(t_2) - m_X(t_2)]\}$$

Correlated? Orthogonal? Independent?

Given	$X(t_1), X(t_2)$ will be
$C_X(t_1, t_2) = 0$	Uncorrelated
$R_X(t_1, t_2) = 0$	Orthogona
$f_X(x_1, x_2, t_1, t_2) = f_X(x_1, t_1) f_X(x_2, t_2)$	Independent

Statistics of Stochastic Processes

- Statistics of Discrete type stochastic processes

$$m_X(t) = \sum_{i=1}^{N} x_i(t) p_i(t)$$

$$\sigma_X^2(t) = \sum_{i=1}^{N} [x_i(t) - m_X(t)]^2 p_i(t)$$

$$R_X(t_1, t_2) = E\{X(t_1)X(t_2)\} = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i(t_1) x_j(t_2) p_{ij}(t_1, t_2)$$

$$C_X(t_1, t_2) = \sum_{i=1}^{N} \sum_{j=1}^{N} [x_i(t_1) - m_X(t_1)][x_j(t_2) - m_X(t_2)] p_{ij}(t_1, t_2)$$

Note that in Matlab, you can only use these equations to do the calculation, but the analysis is still based on the continuous type statistics.

Statistics of Stochastic Processes

- Example 3: Consider a stochastic process that change linearly with time: $Y(t) = at + X$, where X is an uniformly r.v distributed in $[-1,1]$; calculate the mean, variance, autocorrelation and autocovariance of $Y(t)$.

Statistics of Stochastic Processes

- Example 3: Consider a stochastic process that change linearly with time: $Y(t) = at + X$, where X is an uniformly r.v distributed in $[-1,1]$; calculate the mean, variance, autocorrelation and autocovariance of $Y(t)$.

Solution:

$$\mathrm{E}(X) = 0 \qquad \mathrm{D(X)} = \frac{[1-(-1)]^2}{12} = \mathrm{E(X^2)} - \mathrm{E(X)}^2$$

$$m_Y(t) = E(Y(t)) = at + E[X] = at$$

$$D(Y(t)) = E((Y(t) - m_Y(t))^2) = E(X^2) = \frac{1}{3}$$

$$R_Y(t_1, t_2) = E\{Y(t_1)Y(t_2)\} = E\{(at_1 + X)(at_2 + X)\}$$

$$= a^2 t_1 t_2 + E(X^2) = a^2 t_1 t_2 + \frac{1}{3}$$

$$C_Y(t_1, t_2) = R_Y(t_1, t_2) - m_Y(t_1) m_Y(t_2) = a^2 t_1 t_2 + \frac{1}{3} - a^2 t_1 t_2 = \frac{1}{3}$$

Basic of Stochastic Processes – outline

- Definitions
- Distributions of Stochastic Processes
  - One-dimensional distribution
  - Two-dimensional and multidimensional distributions
- Statistics of Stochastic Processes
  - Mean, variance, correlation function, covariance function
  - For Discrete time Processes
- Stationary stochastic processes

Stationary stochastic processes(平稳随机过程)

- Stochastic processes can be divided into two categories: stationary and non-stationary.
  - Strictly speaking, all signals are non-stationary;
  - However, the analysis of stationary signals is much easier. For example, in an electronic system, if the main physical conditions that produce a stochastic process do not change in a certain time, or the change is so tiny that can be ignored, the signal can be considered as stationary:
    - For the noisy voltage signal of the receiver, due to the temperature change of the components in the beginning, the voltage is a transient process (non-stationary). After a short period of time, the temperature becomes stable, thus the signal can be considered stationary.

Stationary stochastic processes

- Strict Sense Stationary (SSS, 严格平稳)
  - If for any n, the n dimensional distribution of a stochastic process does not change with the start time, that is, when the time shifts, its n dimensional CDF/pdf does not change, then it is called Strict Sense Stationary (SSS).
    - The 1-D pdf is time independent:

$$f_X(x,t) = f_X(x,0) = f_X(x)$$

    - The 2-D joint pdf is related to the time difference, not the absolute value of time:

$$f_X(x_1,x_2,t_1,t_2) = f_X(x_1,x_2,\tau,0) = f_X(x_1,x_2,\tau), \tau = t_1 - t_2$$

Note that in some books, it is defined as $\tau = t_2 - t_1$, which will lead to a sign difference (+/-) in many of the theorems/equations we will learn later. However, in this course, the definition of the book will be used (see page 351, text book), and no reminder will be given later.

    - Generalize to multidimensional:

$$f_X(x_1,\dots,x_n,t_1+c,\dots,t_n+c) = f_X(x_1,\dots,x_n,t_1,\dots,t_n) \qquad (5\text{-}1)$$

Stationary stochastic processes

- Strict Sense Stationary (SSS, 严格平稳)
    - For an SSS process, its mean and variance are time-independent constants, and the autocorrelation function is only related to the difference between t_1 and t_2, instead of their absolute value.
    - The most basic feature of SSS is that the change of the start time will not affect its statistical properties, that is, it $X(t)$ has the same statistical properties as $X(t + \Delta t)$.
    - It is the strongest form of Stationary:
        - SSS is most likely not true for most real world applications
        - However, in most real world applications, what we care about are the first and second order moments (一二阶矩): mean and autocorrelation

Stationary stochastic processes

- Generalize to multidimensional:

$$f_X(x_1,\dots,x_n,t_1+c,\dots,t_n+c)=f_X(x_1,\dots,x_n,t_1,\dots,t_n) \qquad \text{(5-1)}$$

- Special case: Nth-order stationary:
  - If (5-1) holds for any $n \le N$ only, it is called Nth-order stationary
  - SSS means that for any N, it is Nth order stationary
- Example: 1st order stationary
  - $f_1(x_1;t_1)=f_1(x_1;t_1+\varepsilon)=f_1(x_1;0)$
  - That is, both expectation and variance are time-independent constants

$$E[X(t)]=\int_{-\infty}^{\infty}xf(x,0)dx=m_X$$

$$D[X(t)]=E\{[X(t)-m_X]^2\}$$

$$=\int_{-\infty}^{\infty}[x-m_X]^2f(x,0)dx=\sigma_X^2 \text{ (real valued)}$$

Stationary stochastic processes

- **Wide Sense Stationary (WSS, 广义平稳)**
  - $E[X(t)] = m_X$ Mean is time-independent
  - $R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau)$ Autocorrelation only related to τ

Note:

- For complex valued WSS processes:

$$R_X(\tau) = E\{X(t+\tau)X^*(t)\}$$
$$R_X(-\tau) = E\{X(t-\tau)X^*(t)\} = E\{X^*(t+\tau)X(t)\} \neq R_X(\tau)$$

- For real valued WSS processes:

$$R_X(\tau) = E\{X(t+\tau)X(t)\}$$
$$= E\{X(t)X(t+\tau)\} = E\{X(t-\tau)X(t)\} = R_X(-\tau)$$

$$\text{SSS} \;\underset{\text{Might not}}{\overset{\text{is}}{\rightleftarrows}}\; \text{WSS}$$

- When the stochastic process is Gaussian distributed, the two equivalent.

In many practical problems, WSS is enough for analysis.

Therefore, we will mainly focus on WSS later

Stationary stochastic processes

- Example 4: A stochastic process $Z(t) = Acos(\omega t) + n(t)$, where A and ω are constants, and $n(t)$ is a WSS normal processes with mean 0 and variance σ^2. What is the one-dimensional probability density of $Z(t)$? Is $Z(t)$ WSS? SSS?

Multiple choices?

A neither SSS nor WSS

B SSS but not WSS

C WSS but not SSS

D SSS and WSS

Stationary stochastic processes

- Example 4: A stochastic process $Z(t) = Acos(\omega t) + n(t)$, where A and ω are constants, and $n(t)$ is a WSS normal processes with mean 0 and variance σ^2. What is the one-dimensional probability density of $Z(t)$? Is $Z(t)$ WSS? SSS?

Solution:

$$Z(t) \sim \mathcal{N}(A\cos \omega t, \sigma^2) \Rightarrow f_1(x,t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x - A\cos \omega t)^2}{2\sigma^2}\right)$$

$$E(Z(t)) = A\cos \omega t$$

$\Rightarrow$ The mean is correlated with time t, obviously neither SSS nor WSS

Reading

- This week:
  - Text book: 7.1
  - Red book: 2.1,2.2
  - Blue book: 6.1 - 6.4
- Next week:
  - Text book: 7.2
  - Red book: 2.3, 2.4, 3.1, 3.2.1
  - Blue book: 6.5 – 6.7

More examples:

- 1: Suppose that a stochastic process $X(t)$ follows: at any time t_1, the $E(X(t_1)) = 0$, $D[X(t_1)] = \sigma^2 t_1$ for $X(t_1)$, and the $X(t_2) - X(t_1)$ is a Normal r.v with mean 0 and variance $\sigma^2(t_2 - t_1)$, and also independent with $X(t_1)$. Find the autocorrelation $R_X(t_1, t_2)$.

- 2: Suppose that an stochastic process $Z(t) = Xcos(t) + Ysin(t), -\infty < t < +\infty$. The X & Y are independent r.vs, and are equal to -1 and 2 with probability $2/3$ and $1/3$ independently. Is $Z(t)$ SSS? WSS?

Hint:

$$E(X) = E(Y) = (-1) \times \frac{2}{3} + 2 \times \frac{1}{3} = 0$$

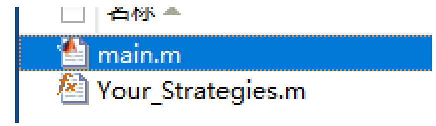
$$E(X^2) = E(Y^2) = (-1)^2 \times \frac{2}{3} + 2^2 \times \frac{1}{3} = \frac{2}{3} + \frac{4}{3} = 2$$

$$E(X^3) = E(Y^3) = (-1)^3 \times \frac{2}{3} + 2^3 \times \frac{1}{3} = -\frac{2}{3} + \frac{8}{3} = 2$$

$$E(XY) = E(YX) = E(X)E(Y) = 0$$

Experiment

- Go on for your Experiment 1 today
  - 关于extra的一些建议：



    - 满分是100分
    - 前面做好才是最重要的保障
    - 所谓的“submit the whole system with your strategy”，意思就是整个main函数，是需要自己写的
    - 所以，这个“whole system”，里面大部分的内容，大家均可以自由发挥，基本没有限制

Experiment

- Go on for your Experiment 1 today
  - 对experimental report 1的basic 2部分，请每个同学完成了8个图之后，叫译哲看一下。
  - 画4个分布的一共8个图，这4个分布都采用什么参数设置？
    - 大家自选参数设置就行（需要在实验报告里面说明选了什么参数）。只要运行能通过，就说明大家所自己选的参数是没有问题的。
    - 建议大家对程序进行仔细的调整，以画出一个比较漂亮的图。