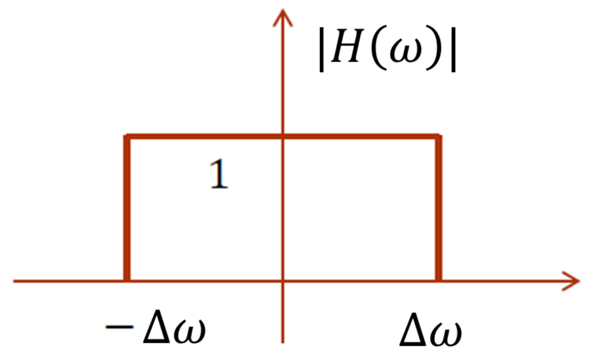
Stochastic Signal Processing

Lesson 11:
Discrete-time process & Spectrum Estimation

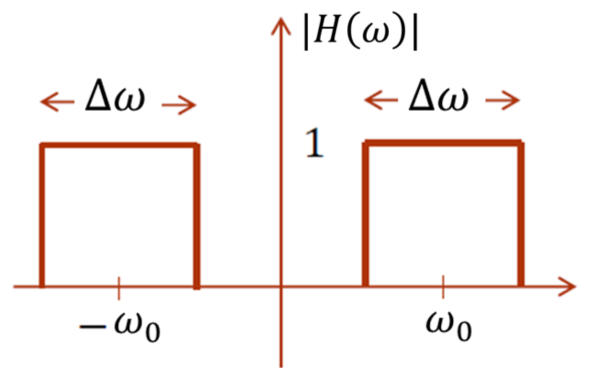
Weize Sun

Examples from last week

- 1: given an input stationary process $X(t)$ with power spectrum $S_X(\omega) = \frac{4\lambda}{4\lambda^2+\omega^2}$, and it is inputted to the ideal low-pass filter with amplitude $|H(\omega)| = 1$ for $-\Delta\omega \le \omega \le \Delta\omega$, calculate the average output power $R_Y(0)$ of the output $Y(t)$. (Hint: $\int_a^b \frac{1}{1+(x)^2} d(x) = \arctan(x)\,|_a^b$)



- 2: given an input white noise $X(t)$ with power spectrum $S_X(\omega) = N_0/2$, and it is inputted to the ideal band-pass filter as below, calculate the average output power of the output $Y(t)$.



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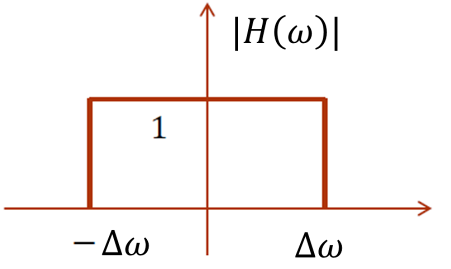


Solution:

We have $R_Y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)\, e^{j\omega\tau} d\omega = \frac{1}{2\pi}\int_{-\Delta\omega}^{\Delta\omega} S_X(\omega)|H(\omega)|^2\, e^{j\omega\tau} d\omega$

let $\tau = 0$ (And note that $\int_a^b \frac{1}{1+(x)^2} d(x) = \arctan(x)\,|_a^b$):

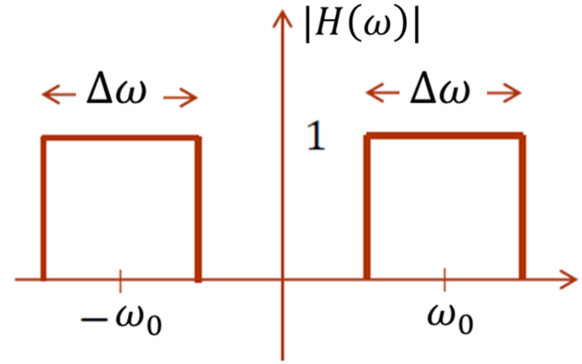
$$R_Y(0) = \frac{1}{2\pi}\int_{-\Delta\omega}^{\Delta\omega} \frac{4\lambda}{4\lambda^2+\omega^2} d\omega = \frac{1}{2\pi}\frac{1}{\lambda}\int_{-\Delta\omega}^{\Delta\omega} \frac{1}{1+(\frac{\omega}{2\lambda})^2} d(\omega)$$

$$= \frac{1}{\pi}\int_{-\Delta\omega}^{\Delta\omega} \frac{1}{1+(\frac{\omega}{2\lambda})^2} d\left(\frac{\omega}{2\lambda}\right) = \frac{1}{\pi}\int_{-\Delta\omega/2\lambda}^{\Delta\omega/2\lambda} \frac{1}{1+(\alpha)^2} d(\alpha) = \frac{1}{\pi}[\arctan(\Delta\omega/2\lambda) - \arctan(-\Delta\omega/2\lambda)]$$

$$= \frac{2}{\pi}\arctan(\Delta\omega/2\lambda)$$

Examples from last week

2: given an input white noise $X(t)$ with power spectrum $S_X(\omega) = N_0/2$, and it is inputted to the ideal band-pass filter as below, calculate the average output power of the output $Y(t)$.



Solution:

$$\sigma_Y^2 = R_Y(0) = \frac{\frac{\Delta\omega}{2} N_0}{\pi} sinc\left(\frac{\Delta\omega}{2}\ 0\right) \cos(\omega_0\ 0) = \frac{N_0 \Delta\omega}{2\pi}$$

or， Starting from $R_Y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega) e^{j\omega\tau}\, d\omega$:

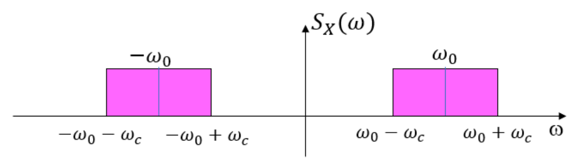
$$\sigma_Y^2 = R_Y(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{N_0}{2} |H(\omega)|^2\, d\omega$$

$$= \frac{1}{2\pi}\frac{N_0}{2}\ 2 \int_{\omega_0-\frac{\Delta\omega}{2}}^{\omega_0+\frac{\Delta\omega}{2}} 1\, d\omega = \frac{N_0}{4\pi}\ 2\,\Delta\omega = \frac{N_0 \Delta\omega}{2\pi}$$

Ideal band-pass process

- The power spectrum of an ideal band-pass process is assumed to be ($\omega_0 \geq \omega_c$):

$$S_X(\omega) = \begin{cases} q, & -\omega_c \leq |\omega| - \omega_0 \leq \omega_c \\ 0, & others \end{cases}$$



The autocorrelation is:

$$R_X(\tau) = \frac{1}{\pi}\int_0^{+\infty} S_X(\omega)\cos\omega\tau\, d\omega = \frac{1}{\pi}\int_{\omega_0-\omega_c}^{\omega_0+\omega_c} q\cos\omega\tau\, d\omega = 2q\frac{\omega_c}{\pi}\frac{\sin\omega_c\tau}{\omega_c\tau}\cos\omega_0\tau$$

- The total average power is $R_X(0) = 2q\omega_C/\pi$

Outline

- Discrete-time process
- Spectrum Estimation

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Discrete-time process

- A digital (or discrete-time) process is a sequence x_n of random variables.

- We use also the notation $x[n]$ where n is an integer. Most results involving analog (or continuous-time) processes can be extended to digital processes.

- The autocorrelation and autocovariance of $\mathbf{x}[n]$ are given by

$$R[n_1, n_2] = E\{\mathbf{x}[n_1]\mathbf{x}^*[n_2]\} \qquad C[n_1, n_2] = R[n_1, n_2] - m_{\mathbf{x}}[n_1]m_{\mathbf{x}}^*[n_2]$$

where $m_{\mathbf{x}}[n] = E\{\mathbf{x}[n]\}$ is the mean of $\mathbf{x}[n]$.

- A process $\mathbf{x}[n]$ is SSS if its statistical properties are invariant to a shift of the origin. It is WSS if $m_{\mathbf{x}}[n] = m_{\mathbf{x}} =$ constant and

$$R[n+m, n] = E\{\mathbf{x}[n+m]\mathbf{x}^*[n]\} = R[m]$$

Discrete-time process

- A process $\mathbf{x}[n]$ is strictly white noise if the random variables $\mathbf{x}[n_i]$ are independent. It is white noise if the random variables $\mathbf{x}[n_i]$ are uncorrelated. (了解此概念即可)

- The autocorrelation of a white-noise process with zero mean is given by

$$R[n_1, n_2] = q[n_1]\delta[n_1 - n_2]$$

Where $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$ and $q[n] = E\{\mathbf{x}^2[n]\}$. If $\mathbf{x}[n]$ is also stationary, then $R[m] = q\delta[m]$ where $m = n_1 - n_2$. Thus a WSS white noise is a sequence of independent and identical distributed (i.i.d., 独立同分布) random variables with variance q.

Discrete-time process – linear system

- The delta response(or impulse response, 冲激响应) $h[n]$ of a linear system is its response to the delta sequence $\delta[n]$. Its system transfer function（系统传输函数，simplified called system function） is the z transform of $h[n]$:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

Where $z = e^{j\omega}$ $(H(\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n})$, ω is the (angular) frequency

- If $\mathbf{x}[n]$ is the input to a digital system, the resulting output is the digital convolution $*$ of $\mathbf{x}[n]$ with $h[n]$:

$$\mathbf{y}[n] = \sum_{k=-\infty}^{\infty} \mathbf{x}[n-k]h[k] = \mathbf{x}[n] * h[n]$$

Discrete-time process – linear system

- Therefore $m_y[n] = m_x[n] * h[n]$ ($*$ is the convolution process), and

$$R_{xy}[n_1, n_2] = \sum_{k=-\infty}^{\infty} R_{xx}[n_1, n_2 - k]h^*[k]$$

$$R_{yy}[n_1, n_2] = \sum_{r=-\infty}^{\infty} R_{xy}[n_1 - r, n_2]h[r]$$

- If $\mathbf{x}[n]$ is WSS, then $\mathbf{y}[n]$ is also WSS with $m_y = m_x H(z)|_{z=1}$ (or written as $H(1)$):

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}, \text{ thus } H(1) = \sum_{n=-\infty}^{\infty} h[n]1^{-n},$$

$$\text{as } z = e^{j\omega} = 1, \text{ it means } \omega = 0$$

- And $\quad R_{xy}[m] = R_{xx}[m] * h^*[-m]\,; \qquad R_{yy}[m] = R_{xy}[m] * h[m]$

$$R_{yy}[m] = R_{xx}[m] * h^*[-m] * h[m]$$

Discrete-time process – power spectrum

- Given a WSS process $\mathbf{x}[n]$, the power spectrum is:

$$S(\omega) = S\left(e^{j\omega}\right) = \sum_{m=-\infty}^{\infty} R[m]e^{-jm\omega} \geq 0$$

- Thus $S(\omega)$ is the discrete Fourier transform (DFT) of $R[m]$. The function $S(\omega)$ is periodic with period 2π and

$$R[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S\left(e^{j\omega}\right)e^{jm\omega}d\omega$$

- For real process, we have $R[m] = R[-m]$ thus:

$$S(\omega) = R[0] + 2\sum_{m=1}^{\infty} R[m]\mathrm{cos}m\omega$$

This shows that the power spectrum of a real process is a cosine function, and ≥ 0.

注：power spectrum 两种写法$S(\omega)$和$S\left(e^{j\omega}\right)$都在一些书里面出现

Discrete-time process

- Example 1: If $R[m] = a^{|m|}$, find the power spectrum $S(\omega)$.

Discrete-time process

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Solution:

$$S(\omega) = \sum_{m=-\infty}^{\infty} a^{|m|} e^{-jm\omega}$$

$$= \sum_{m=-\infty}^{-1} a^{-m} e^{-jm\omega} + \sum_{m=0}^{\infty} a^{m} e^{-jm\omega} = \frac{ae^{-j\omega}}{1 - ae^{-j\omega}} + \frac{e^{-j\omega}}{e^{-j\omega} - a}$$

$$= \frac{a^{-1} - a}{(a^{-1} + a) - (e^{j\omega} + e^{-j\omega})}$$

$$= \frac{a^{-1} - a}{a^{-1} + a - 2\cos\omega}$$

Discrete-time process

- As

$$R_{xy}[m] = R_{xx}[m] * h^*[-m]; R_{yy}[m] = R_{xy}[m] * h[m];$$

$$R_{yy}[m] = R_{xx}[m] * h^*[-m] * h[m]$$

According to the convolution theorem, it follows that if $y[n]$ is the output of a linear system with input $x[n]$, then

$$S_{xy}(e^{j\omega}) = S_{xx}(e^{j\omega})H^*(e^{j\omega})$$

$$S_{yy}(e^{j\omega}) = S_{xy}(e^{j\omega})H(e^{j\omega})$$

$$S_{yy}(e^{j\omega t}) = S_{xx}(e^{j\omega})\left|H(e^{j\omega})\right|^2$$

- If $h[n]$ is real, $\mathbf{H}^*(e^{j\omega}) = \mathbf{H}(e^{-j\omega})$. In this case,

$$\mathbf{S}_{yy}(e^{j\omega}) = \mathbf{S}_{xx}(e^{j\omega})\mathbf{H}(e^{j\omega})\mathbf{H}(e^{-j\omega})$$

Quick example:

- The first difference $y[n] = x[n] - x[n-1]$ of a process $x[n]$ can be considered as the output of a linear system with input $x[n]$ and system function $H(\omega) = 1 - e^{-j\omega}$. Then $S_{yy}(\omega) = ?$

Quick example:

- The first difference $y[n] = x[n] - x[n-1]$ of a process $x[n]$ can be considered as the output of a linear system with input $x[n]$ and system function $H(\omega) = 1 - e^{-j\omega}$. Then $S_{yy}(\omega) = ?$

$$S_{yy}(\omega) = S_{xx}(\omega)\left(1 - e^{-j\omega}\right)\left(1 - e^{j\omega}\right)$$

$$= S_{xx}(\omega)\left(2 - e^{j\omega} - e^{-j\omega}\right)$$

$$= 2S_{xx}(\omega)(1 - \cos\omega)$$

Note: $R_{yy}[m] = -R_{xx}[m+1] + 2R_{xx}[m] - R_{xx}[m-1]$

Outline

- Discrete-time process
- Spectrum Estimation

Spectrum Estimation

- We wish to estimate the power spectrum $S(\omega)$ of a real process $\mathbf{x}(t)$ in terms of a single realization of a finite segment

$$\mathbf{x}_T(t) = \mathbf{x}(t)p_T(t)\,, \qquad \text{where} \quad p_T(t) = \begin{cases} 1 & |t| < T \\ 0 & |t| > T \end{cases}$$

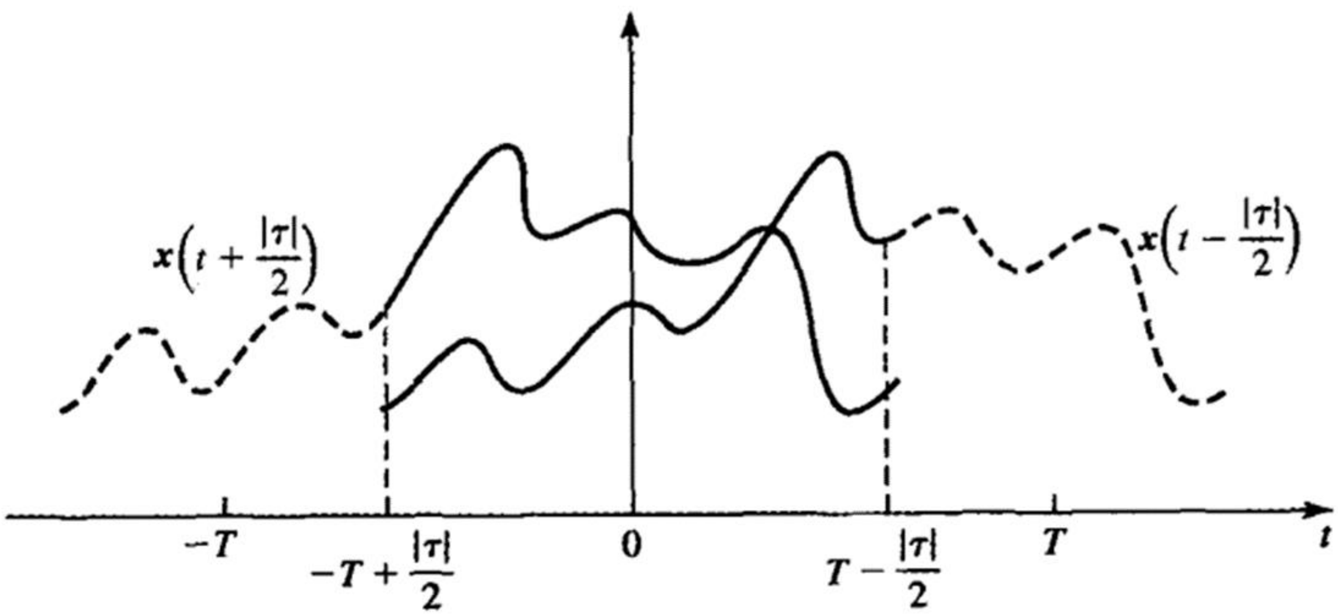
Note that the spectrum $S(\omega)$ is in fact the Fourier transform of the autocorrelation

$$R(\tau) = E\left\{\mathbf{x}\left(t + \frac{\tau}{2}\right)\mathbf{x}\left(t - \frac{\tau}{2}\right)\right\}$$

Spectrum Estimation

- $S(\omega)$ will be determined in terms of the estimate of $R(\tau)$. Note that the product $\mathbf{x}(t+\tau/2)\mathbf{x}(t-\tau/2)$ is available only for t in the interval $(-T+|\tau|/2, T-|\tau|/2\ )$ (Fig. below). Thus, changing $2T$ to $2T-|\tau|$, we obtain the estimate

$$\mathbf{R}^T(\tau) = \frac{1}{2T-|\tau|}\int_{T+|\tau|/2}^{T-|\tau|/2} \mathbf{x}\left(t+\frac{\tau}{2}\right)\mathbf{x}\left(t-\frac{\tau}{2}\right)dt$$



Spectrum Estimation

- This integral specifies $\mathbf{R}^T(\tau)$ for $|\tau| < 2T$; for $|\tau| > 2T$ we set $\mathbf{R}^T(\tau) = 0$.

- This estimate is unbiased; however, its variance increases as $|\tau|$ increases because the length $2T - |\tau|$ of the integration interval decreases. Instead of $\mathbf{R}^T(\tau)$, we shall use the product

$$\mathbf{R}_T(\tau) = \left(1 - \frac{|\tau|}{2T}\right)\mathbf{R}^T(\tau)$$

- This estimator is biased; however, its variance is smaller than the variance of $\mathbf{R}^T(\tau)$. The main reason we use it is that its transform is proportional to the energy spectrum of the segment $\mathbf{x}_T(t)$ of $\mathbf{x}(t)$.

The Periodogram

- The periodogram of a process $\mathbf{x}(t)$ is defined ad

$$\mathbf{S}_T(\omega) = \frac{1}{2T}\left|\int_{-T}^{T} \mathbf{x}_T(t)e^{-j\omega t}dt\right|^2 \qquad (11\text{-}1)$$

- This integral is the Fourier transform of the known segment $\mathbf{x}_T(t)$ of $\mathbf{x}(t)$:

$$\mathbf{S}_T(\omega) = \frac{1}{2T} \mid \mathbf{X}_T(\omega) \mid^2$$

where $$\mathbf{X}_T(\omega) = \int_{-T}^{T} \mathbf{x}(t)e^{-j\omega t}dt$$

- $\mathbf{S}_T(\omega)$ can also be expressed in terms of the estimator $\mathbf{R}_T(\tau)$ of $R(\tau)$:

$$\mathbf{S}_T(\omega) = \int_{-2T}^{2T} \mathbf{R}_T(\tau)e^{-j\omega\tau}d\tau \qquad (11\text{-}2)$$

The Periodogram

Property （了解）：

$$\mathbf{S}_T(\omega) = \int_{-2T}^{2T} \mathbf{R}_T(\tau) e^{-j\omega\tau} d\tau \qquad (11\text{-}2)$$

Proof. The integral here is the convolution of $\mathbf{x}_T(t)$ with $\mathbf{x}_\tau(-t)$ because $\mathbf{x}_\tau(t) =$ 0 for $|t| > T$. Hence

$$\mathbf{R}_T(\tau) = \frac{1}{2T} \mathbf{x}_T(\tau) * \mathbf{x}_T(-\tau)$$

Since $\mathbf{x}_T(t)$ is real, the transform of $x_T(-t)$ equals $X_T^*(\omega)$ and thus the estimated power spectrum is get.

- In the early years of signal analysis, the spectral properties of random processes were expressed in terms of their periodogram.
- This approach yielded reliable results so long as the integrations were based on analog techniques of limited accuracy.

The Periodogram

- With the introduction of digital processing, the accuracy was improved and, paradoxically, the computed spectra exhibited noisy behavior.
- This apparent paradox can be readily explained in terms of the properties of the periodogram: The integral in (11-2) depends on all values of $\mathbf{R}_T(\tau)$ for τ large and small. The variance of $\mathbf{R}_T(\tau)$ is small for small τ only, and it increases as $\tau \to 2T$. As a result, $S_T(\omega)$ approaches a white-noise process with mean $S(\omega)$ as T increases.
- To overcome this behavior of $\mathbf{S}_T(\omega)$, we can replace in (11-2) the term $\mathbf{R}_T(\tau)$ by the product $w(\tau)\mathbf{R}_T(\tau)$, where $w(\tau)$ is a function (window) close to 1 near the origin, approaching 0 as $\tau \to 2T$. This deemphasizes the unreliable parts of $\mathbf{R}_T(\tau)$, thus reducing the variance of its transform.

More examples

- Example 2: given joint stationary processes $X(t)$ and $Y(t)$: $\alpha Y(t)+\beta\frac{d^2Y(t)}{d^2t}-\frac{d^3Y(t)}{d^3t}=\frac{dX(t)}{dt}+X(t)$，and the power spectrum of X(t) is $S_X(\omega)$

1. Calculate $H(\omega)$、$S_{YX}(\omega)$and $S_{XY}(\omega)$（written as $\alpha,\beta,S_X(\omega)$）
2. Assume that $X(t)$ is a white noise of power q, and $S_Y(\omega)=\frac{1}{\omega^4-\omega^2+1}$，calculate α，β，$q$（numerical values only, units not required）

More examples

Solution:

1. We get

$$\alpha Y(\omega) + \beta(j\omega)^2 Y(\omega) - (j\omega)^3 Y(\omega)$$
$$= \alpha Y(\omega) - \beta\omega^2 Y(\omega) + j\omega^3 Y(\omega) = j\omega X(\omega) + X(\omega)$$

Thus $$H(\omega) = \frac{1+j\omega}{\alpha-\beta\omega^2+j\omega^3}$$

Then

$$S_{YX}(\omega) = S_X(\omega)H(\omega) = \frac{1+j\omega}{\alpha-\beta\omega^2+j\omega^3} S_X(\omega)$$
$$S_{XY}(\omega) = S_X(\omega)H^*(\omega) = \frac{1-j\omega}{\alpha-\beta\omega^2-j\omega^3} S_X(\omega)$$

More examples

2. We have

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega) = \frac{1+\omega^2}{(\alpha-\beta\omega^2)^2+\omega^6} S_X(\omega) = \frac{\omega^2+1}{\omega^6+\beta^2\omega^4-2\alpha\beta\omega^2+\alpha^2} S_X(\omega)$$

When $S_X(\omega) = q$, we get

$$S_Y(\omega) = \frac{\omega^2+1}{\omega^6+\beta^2\omega^4-2\alpha\beta\omega^2+\alpha^2} q = \frac{1}{\omega^4-\omega^2+1}$$

$\rightarrow$ $(\omega^6+\beta^2\omega^4-2\alpha\beta\omega^2+\alpha^2) = q(\omega^2+1)(\omega^4-\omega^2+1) = q(\omega^6+1)$

$\rightarrow$ $$q = 1,\ \ \beta^2 = 2\alpha\beta = 0,\ \ \alpha^2 = q$$

Thus

$$\beta = 0$$

$$\alpha = \pm 1$$

Reading

- Next week:
  - Optimal linear filter (& Matched filter)
  - Text book: 10.1, 10.2
  - Red book: 3.5 (最佳线性滤波器) 注：最佳线性滤波器的讲解将以此书为主导
  - And more examples
- After:
  - Random walk (Text book 12.1)
  - Markov Chain (Text book 15)
  - Spectrum Estimation II (if there enough time)

Experiment

- Basic of Experiment 3
- Go on with Experiment 2

More examples for next week

- 1: given input signal $X(t)$, output signal $Y(t)$ and their relationship:
$$\frac{4dY(t)}{dt} + 2Y(t) = X(t)$$

When $X(t) = \delta(t)$, find $R_Y(\tau)$ and $R_{YX}(\tau)$.

It is well known that if feeding an input Gaussian process to a stationary linear system (any deterministic system is stationary), the output is also a Gaussian process.

- 2: given an input Gaussian white noise $X(t)$ with power spectrum $S_X(\omega) = N_0/2$, and it is inputted to the ideal band-pass filter as below, calculate the 1-D pdf of the output.