homework 3

1. For a zero mean Gaussian stationary process $X(t)$, whose power spectrum is:

$$S_X(\omega) = \begin{cases} A, & ||\omega| - \omega_0| < \frac{\Delta\omega}{2} \\ 0, & \text{others} \end{cases}$$

Where $\omega_0 > \Delta\omega$. Calculate the 1-D pdf of it. (10 points)

$$R_X(\tau) = 2A \frac{\frac{\Delta w}{2}}{\pi} \cdot \frac{\sin\left(\frac{\Delta w}{2} \cdot \tau\right)}{\frac{\Delta w}{2} \cdot \tau} \cos w_0 \tau$$

$$= \frac{2A\Delta w}{\pi} \cdot \frac{\sin\left(\frac{1}{2}\Delta w \cdot \tau\right) \cos w_0 \tau}{\Delta w \cdot \tau}$$

$$R_X(0) = \frac{2A \cdot \frac{\Delta w}{2}}{\pi} = \frac{A\Delta w}{\pi} = m_X^2 + \sigma_X^2 = \sigma_X^2$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sqrt{\frac{A\Delta w}{\pi}}} \exp\left| \frac{-x^2}{\frac{2A\Delta w}{\pi}} \right|$$

2. Assuming that a stationary Gaussian white noise with power spectrum A is fed into a filter $H(\omega) = \frac{2}{j\omega+1}$, calculate the one-dimensional pdf of the output. (20 points)

$$S_Y(w) = S_X(w) \cdot |H(w)|^2 = A \left| \frac{2}{jw+1} \right|^2 = \frac{4A}{w^2+1}$$

$$R_Y(\tau) = 2A e^{-|\tau|} \qquad R_Y(0) = \sigma_Y^2 = 2A$$

$$f_Y(y) = \frac{1}{\sqrt{4\pi A}} \exp\left| \frac{-y^2}{4A} \right|$$

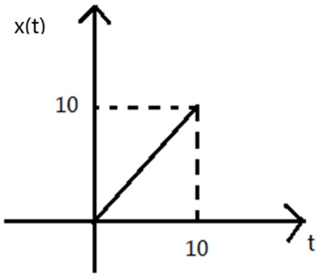
3. A stationary stochastic process $X(t)$ is fed into a low-pass filter $h(t) = \alpha e^{-\alpha t} U(t)$. The autocorrelation of $X(t)$ is $\delta(\tau)$. Calculate the output autocorrelation $R_Y(\tau)$. (20 points)

$$H(w) = \frac{\alpha}{\alpha + jw} \qquad S(w) = 1 \qquad \text{Hint: } e^{-a|\tau|} \quad \frac{2a}{\alpha^2 + w^2}$$

$$S_Y(w) = 1 \cdot \left| \frac{\alpha}{\alpha + jw} \right|^2 = \frac{\alpha^2}{\alpha^2 + w^2}$$

$$R_Y(\tau) = \frac{\alpha}{2} e^{-\alpha|\tau|}$$

4. Given an input $z(t) = x(t) + n(t)$, where $x(t)$ is a deterministic signal as follows (triangular wave), and $n(t)$ is stationary Gaussian white noise with power spectrum of q. (30 points)



a) Calculate the maximum signal-to-noise ratio of the output if this $z(t)$ is fed into its matched filter. (10 points)

b) There is another input $z_1(t) = \frac{1}{3}x(t) + n(t)$, calculate the maximum signal-to-noise ratio of the output of this input if it is fed into its matched filter. (10 points)

c) Calculate the matched filter $H(\omega)$ of the signal. (10 points)

a. $E_x = \int_0^{10} x^2(t)\,dt = \frac{1000}{3}$

$SNR = \frac{E_x}{E_n} = \frac{1000}{3q}$

b. $E_x = \int_0^{10} \left(\frac{1}{3}x(t)\right)^2 dt = \frac{1000}{27}$

$SNR = \frac{1000}{27q}$

c. $H(\omega) = X^*(\omega)$

$X(\omega) = \frac{10\sin^2\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)^2}$ $\quad T = 10$

$H(\omega) = \frac{10\sin^2(5\omega)}{25\omega^2}$

5. The stochastic process $Y(t) = X\cos(\omega_0 t + \theta)$, where ω_0 is a constant, X and θ are independent random variables, X is zero mean white Gaussian variable with variance c^2 where c is a constant, and θ follows uniformly distributed in $(-\pi, \pi)$. (20 points)

a) Calculate the power spectrum of $Y(t)$.

b) Is $Y(t)$ an ergodicity process? Prove it.

a) $R_Y(\tau) = E\{X^2\cos(\omega_0 t + \theta)\cos[\omega_0(t+\tau)+\theta]\}$

$= E\{X^2\}\frac{1}{2}E\{\cos(-\omega_0\tau) + \cos[\omega_0(2t+\tau)+2\theta]\}$

$= \frac{E\{X^2\}}{2}\cos\omega_0\tau \qquad E\{X^2\} = D(x) + m^2(x) = c^2$

$= \frac{c^2}{2}\cos\omega_0\tau$

$S_Y(\omega) = \frac{c^2}{2}\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$

b) $E[Y] = E[X\cos(w_0 t + \theta)] = 0$

$\frac{1}{T}\int_0^T Y(t)dt = 0$

$\therefore \lim_{T\to\infty} \frac{1}{T}\int_0^T Y(t)\,dt = E[Y(t)]$ $\therefore Y(t)$ ergodicity process