第一次作业答案

1. Multiple choices。(5*2=10 points)
   1) If $X = c$， where c is a constant, which is correct?
      A. $f_x(x) = U(x-c)$
      B. $f_x(x) = \delta(x-c)$
      C. $f_x(x) = \frac{1}{2}\delta(x-c) + \frac{1}{2}\delta(x+c)$
      D. $f_x(x) = \delta(x)$
      E. *None of the above*

   B it is a basic concept of the advanced mathematics

   2) *If* $Y = 2X$， which is correct?
      A. $P_Y(y) = \frac{1}{2}P_X\left(\frac{1}{2}y\right)$
      B. $P_Y(y) = 2P_X(2y)$
      C. $P_Y(y) = P_X(2y)$
      D. $P_Y(y) = P_X\left(\frac{1}{2}y\right)$
      E. *None of the above*

   D $P_Y(y)$ is the CDF $F_Y(y)$， see lesson 3

2. (30 points) the pdf of the r.v X is：

$$f(x) = \begin{cases} c(x^2+2x-2), & 0 \le x \le 1 \\ 0, & \text{其他} \end{cases}$$

   1) Find c。
   2) Calculate the CDF of X, which is $F_X(x)$.

Sol：

1) $\int_0^1 c(x^2+2x-2)dx = c\left(\frac{1}{3}x^3 + x^2 - 2x\right)|_0^1 = -\frac{2}{3}c = 1$ therefore， $c = -\frac{3}{2}$

2) $f(x) = \begin{cases} -\frac{3}{2}(x^2+2x-2) = \frac{3}{2}(2-2x-x^2), & 0 \le x \le 1 \\ 0, & \text{others} \end{cases}$

   Integration of the pdf is the CDF：

$$F_X(x) = \begin{cases} 0, & x < 0 \\ -\frac{3}{2}\left(\frac{1}{3}x^3 + x^2 - 2x\right) = \frac{3}{2}\left(2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right), & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

3. (20 points) Given the r.v X uniformly distributed in $[a,b]$, $a \geq 0$, and r.v Y uniformly distributed in $[0,X]$, find:
   1) $E(Y|X=x), a \leq x \leq b$
   2) $E(Y)$
   Hint: $E(Y)=E(E(Y|X))$

Sol:

1) 由条件得到$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x-0} = \frac{1}{x}, & 0 \leq y \leq x, a \leq x \leq b \\ 0, & 其他 \end{cases}$

即有：$E(Y|X=x) = \int_0^x y f_{Y|X}(y|x) dy = \frac{1}{x}\int_0^x y dy = \frac{x}{2}$

2) $E(X) = \frac{a+b}{2}$

$$E(Y) = E\big(E(Y|X)\big) = E\left(\frac{X}{2}\right) = \frac{a+b}{4}$$

4. (20 points) Two boxes B_1 and B_2 both contain 100 balls. The first box (B_1) has 70 red balls and 30 blue balls, while the second box (B_2) has 80 red balls and 20 blue balls. Suppose a box is selected randomly (50% B_1 and 50% B_2), and one ball is picked out:
   1) What is the probability that it is a red ball?
   2) Suppose that the ball picked out is red, what is the probability that it come from box B_1?

Solution:

1) Let R= "red ball is picked out", then

$P(R|B_1)=0.7$, $P(R|B_2)=0.8$ $P(B_1)=P(B_2)=0.5$

Therefore, $P(R)=P(B_1)P(R|B_1)+P(B_2)P(R|B_2)=0.75$

2) $P(B_1|R) = \frac{P(R|B_1)P(B_1)}{P(R)} = \frac{0.7\times 0.5}{0.75} = \frac{7}{15}$

5. (20 points) Assume that the Θ is a r.v X uniformly distributed in $(0,2\pi)$, and $X=cos2\Theta$，$Y=sin2\Theta$, then:
   1) Is X and Y orthogonal? Prove it
   2) Calculate the correlation coefficient of X and Y

Sol:

1) $E(XY) = E(cos2\Theta sin2\Theta) = \int_0^{2\pi} cos2\theta sin2\theta f_\Theta(\theta) d\theta =$

$\int_0^{2\pi} cos2\theta sin2\theta \frac{1}{2\pi} d\theta$

$$= \frac{1}{2\pi}\,\frac{1}{2}\int_0^{2\pi} sin4\theta d\theta = 0$$

所以是正交的
（注：三角函数的一个或者多个周期之内的积分为 0。）

2) $E(X)=\int_0^{2\pi} cos2\theta\frac{1}{2\pi}d\theta=0$

$$E(Y)=\int_0^{2\pi} sin2\theta\frac{1}{2\pi}d\theta=0$$

$cov(X,Y)=E(XY)-E(X)E(Y)=0$
所以是不相关的，相关系数为 0