Stochastic Signal Processing

Lesson 6: Time Domain Analysis of Stochastic Processes

Weize Sun

More examples from last week

1: Suppose that a stochastic process $X(t)$ follows: at any time t_1, the $E(X(t_1)) = 0$, $D[X(t_1)] = \sigma^2 t_1$ for $X(t_1)$, and the$X(t_2) - X(t_1)$ is a Normal r.v with mean 0 and variance $\sigma^2(t_2 - t_1)$, and also independent with $X(t_1)$. Find the autocorrelation $R_X(t_1, t_2)$.

2: Suppose that an stochastic process $Z(t) = Xcos(t) + Ysin(t)$, $-\infty < t < +\infty$. The X & Y are independent r.vs, and are equal to -1 and 2 with probability $2/3$ and $1/3$ independently. Is $Z(t)$ SSS? WSS?

Hint:

$$E(X) = E(Y) = (-1) \times \frac{2}{3} + 2 \times \frac{1}{3} = 0$$

$$E(X^2) = E(Y^2) = (-1)^2 \times \frac{2}{3} + 2^2 \times \frac{1}{3} = \frac{2}{3} + \frac{4}{3} = 2$$

$$E(X^3) = E(Y^3) = (-1)^3 \times \frac{2}{3} + 2^3 \times \frac{1}{3} = -\frac{2}{3} + \frac{8}{3} = 2$$

$$E(XY) = E(YX) = E(X)E(Y) = 0$$

More examples from last week

1: Suppose that a stochastic process $X(t)$ follows: at any time t_1, the $E(X(t_1)) = 0$, $D[X(t_1)] = \sigma^2 t_1$ for $X(t_1)$, and the $X(t_2) - X(t_1)$ is a Normal r.v with mean 0 and variance $\sigma^2(t_2 - t_1)$, and also independent with $X(t_1)$. Find the autocorrelation $R_X(t_1, t_2)$.

Solution:

$$
\begin{aligned}
R_X(t_1, t_2) &= E\big(X(t_1)X(t_2)\big) \\
&= E\{X(t_1) \cdot [X(t_1) + X(t_2) - X(t_1)]\} \\
&= E((X(t_1))^2) + E[X(t_1)(X(t_2) - X(t_1))] \\
&= E((X(t_1))^2) + E[X(t_1)] * E[(X(t_2) - X(t_1))] \\
&= E((X(t_1))^2) = \sigma^2 t_1
\end{aligned}
$$

More examples from last week

2: Suppose that an stochastic process $Z(t) = Xcos(t) + Ysin(t)$, $-\infty < t < +\infty$. The X & Y are independent r.vs, and are equal to -1 and 2 with probability $2/3$ and $1/3$ independently. Is $Z(t)$ SSS? WSS?

Solution:

$$m_Z(t) = E[Z(t)] = E[X]\cos t + E[Y]\sin t = 0$$

$$R_Z(t_1, t_2) = E[Z(t_1)Z(t_2)] = E\{[X\cos t_1 + Y\sin t_1][X\cos t_2 + Y\sin t_2]\}$$

$$= E[X^2]\cos t_1 \cos t_2 + E[Y^2]\sin t_1 \sin t_2\, Z(t) + E[XY]\cos t_1 \sin t_2 + E[YX]\sin t_1 \cos t_2$$

$$= 2\cos t_1 \cos t_2 + 2\sin t_1 \sin t_2 = 2\cos(t_1 - t_2) = 2\cos\tau\,,\quad \tau = t_1 - t_2$$

→WSS

$$E[Z^3(t)] = E\{[X\cos t + Y\sin t]^3\}$$

$$= E[X^3\cos^3 t + Y^3\sin^3 t + 3X^2Y\cos^2 t\sin t + 3Y^2X\cos t\sin t] = 2\cdot(\cos^3 t + \sin^3 t)$$

→not SSS

When you are required to prove it is or not SSS, you can try some moments, like the 3rd moment, if related to t, not SSS; in fact, usually not SSS.

Reminder!

- Trigonometric functions(三角函数)

$$\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)$$

$$\sin(A+B)=\sin(A)\cos(B)+\cos(A)\sin(B)$$

$$\cos(A)\cos(B)=\frac{1}{2}(\cos(A+B)+\cos(A-B))$$

$$\sin(A)\sin(B)=\frac{1}{2}(\cos(A-B)-\cos(A+B))$$

$$\sin(A)\cos(B)=\frac{1}{2}(\sin(A+B)+sin(A-B))$$

$$\cos(A)\sin(B)=\frac{1}{2}(\sin(A+B)-sin(A-B))$$

Time Analysis of Stochastic Processes – outline

- Characteristics of Autocorrelation of Stationary Stochastic Processes
- Correlation Coefficient and Correlation Time
- Joint Distributions and Cross-Correlation of Stochastic Processes
  - Joint Distributions
  - Cross-Correlation, Cross-Covariance and Jointly WSS

Characteristics of Autocorrelation of Stationary Stochastic Processes

For real valued Stationary Stochastic Processes:

$$R_X(\tau) = E[X(t+\tau)X(t)] = E[X(t)X(t+\tau)] = R_X(-\tau)$$

- The autocorrelation is an even function（偶函数）
- And the autocovariance is also an even function:

$$C_X(\tau) = E[(X(t+\tau) - m_X)(X(t) - m_X)] = C_X(-\tau)$$

Characteristics of Autocorrelation of Stationary Stochastic Processes

For real valued Stationary Stochastic Processes:

$$R_X(\tau) = E[X(t+\tau)X(t)] = E[X(t)X(t+\tau)] = R_X(-\tau)$$

- The autocorrelation is an even function（偶函数）
- And the autocovariance is also an even function:

$$C_X(\tau) = E[(X(t+\tau) - m_X)(X(t) - m_X)] = C_X(-\tau)$$

Characteristics:

- 0-point is maximum: $R_X(0) \geq |R_X(\tau)|, C_X(0) \geq |C_X(\tau)|$
- 0-point : $R_X(0) = E[X(t)X(t)] = \sigma_X^2 + m_X^2 \geq 0$
- If a stationary stochastic process does not contain periodic components, then

$$R_X(\infty) = \lim_{\tau\to\infty} R_X(\tau) = \lim_{\tau\to\infty} E[X(t+\tau)]E[X(t)] = m_X^2$$

For periodic components, see experiment for example

Characteristics of Autocorrelation of Stationary Stochastic Processes

For real valued Stationary Stochastic Processes:

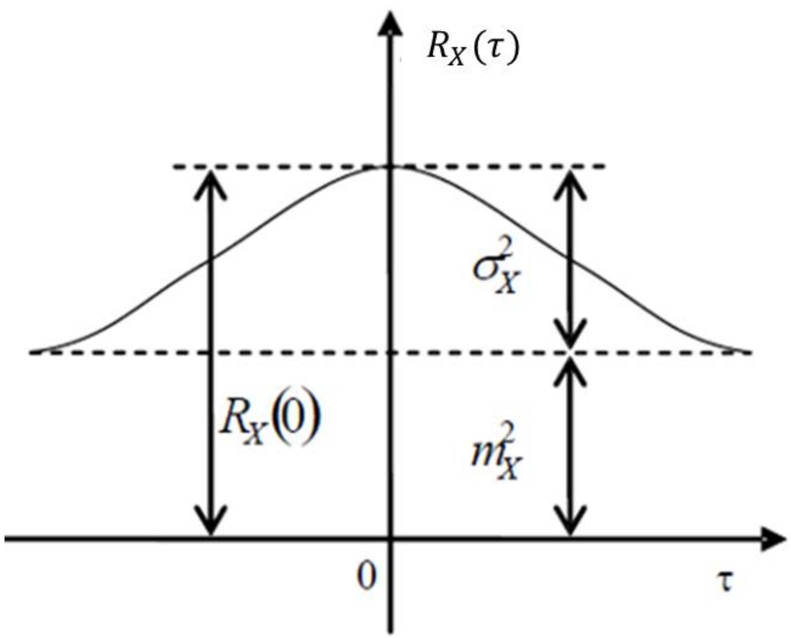
$$R_X(\tau) = E[X(t+\tau)X(t)] = E[X(t)X(t+\tau)] = R_X(-\tau)$$

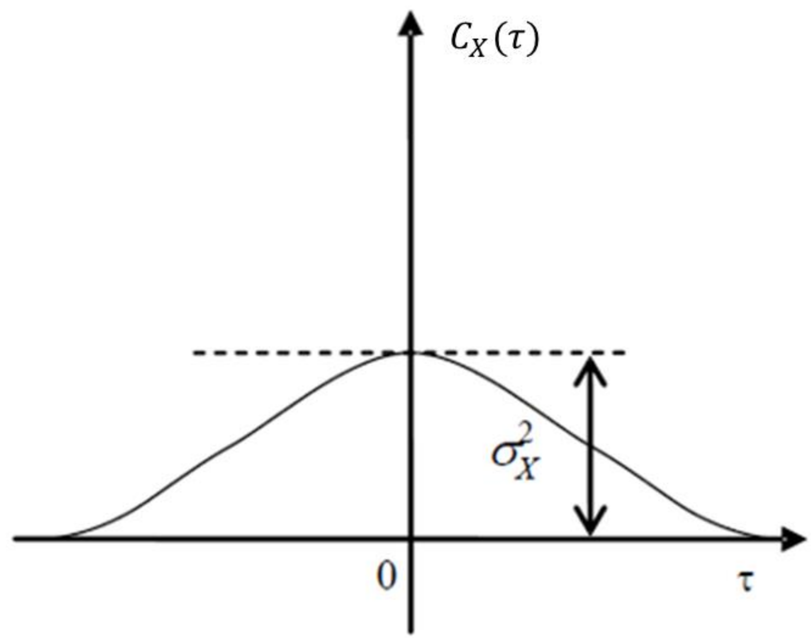
$$R_X(0) \geq |R_X(\tau)|, C_X(0) \geq |C_X(\tau)|$$

$$R_X(0) = E[X(t)X(t)] = \sigma_X^2 + m_X^2 \geq 0$$

$$R_X(\infty) = \lim_{\tau \to \infty} R_X(\tau) = \lim_{\tau \to \infty} E[X(t+\tau)]E[X(t)] = m_X^2$$

- Thus a typical Autocorrelation can be plotted as:





Characteristics of Autocorrelation of Stationary Stochastic Processes

For real valued Stationary Stochastic Processes:

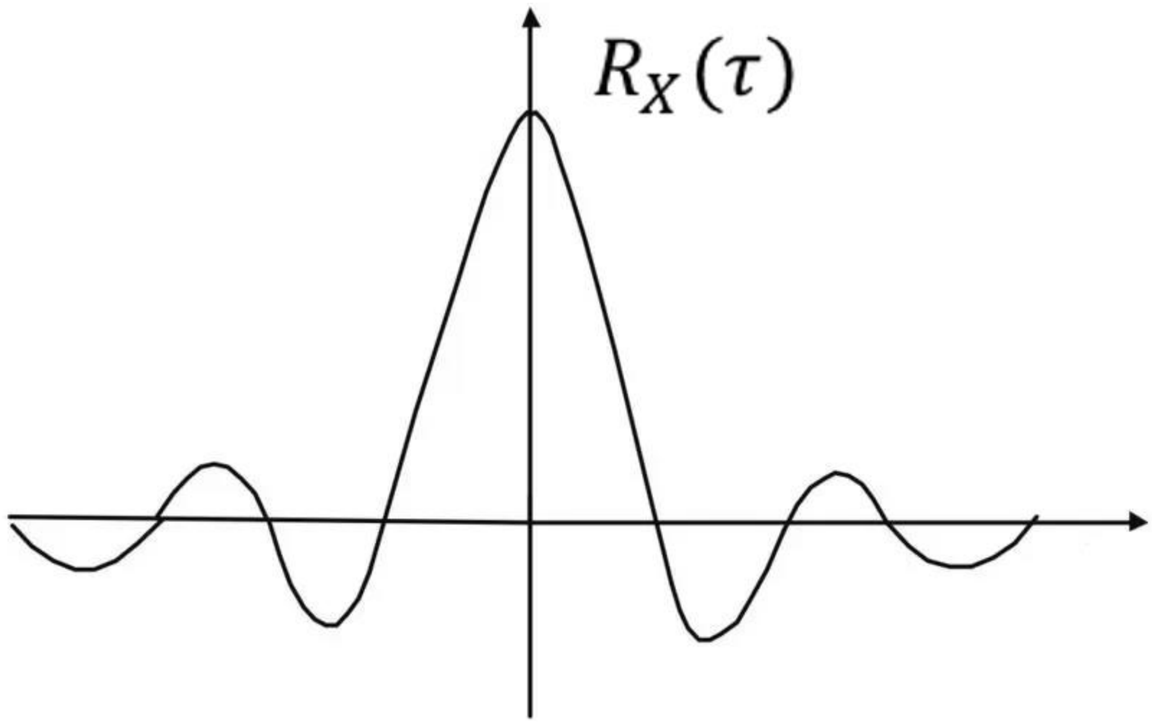
$$R_X(\tau) = E[X(t+\tau)X(t)] = E[X(t)X(t+\tau)] = R_X(-\tau)$$

$$R_X(0) \geq |R_X(\tau)|, C_X(0) \geq |C_X(\tau)|$$

$$R_X(0) = E[X(t)X(t)] = \sigma_X^2 + m_X^2 \geq 0$$

$$R_X(\infty) = \lim_{\tau\to\infty} R_X(\tau) = \lim_{\tau\to\infty} E[X(t+\tau)]E[X(t)] = m_X^2$$

Note: $R_X(\tau)$ can be less than 0 for $\tau \neq 0$! Therefore, you might see



Periodic Signal / Processes: $X(t) = X(t+T)$

- If $X(t) = X(t+T)$, and it is Stationary, then:

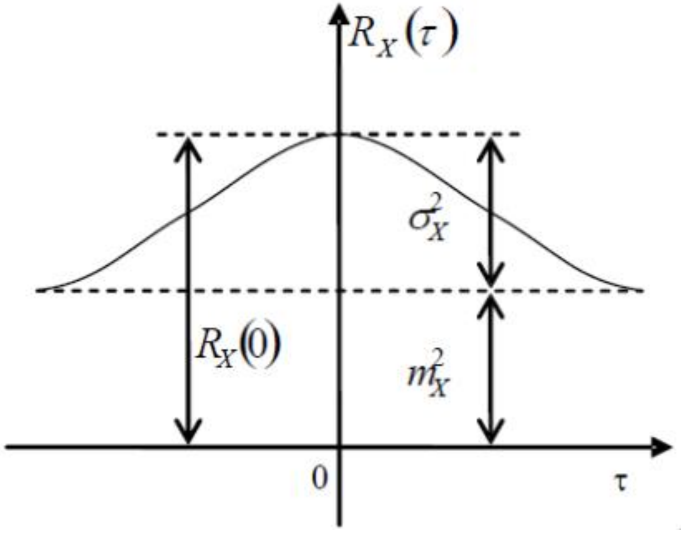
$$
\begin{aligned}
R_X(\tau) &= E[X(t+\tau)X(t)] \\
&= E[X(t+T+\tau)X(t)] = R_X(\tau+T)
\end{aligned}
$$

- If the stochastic process has a periodic component, the autocorrelation function also has a periodic component
- It is called cyclostationary(循环平稳) stochastic process

- Example 1: The autocorrelation of a stationary stochastic process $X(t)$ is:

$$R_X(\tau) = 36 + \frac{4}{1 + 5\tau^2}$$

Find the mean and variance of $X(t)$.



Multiple choices?(m_X =?; σ_X^2 =?)

A 36, 4

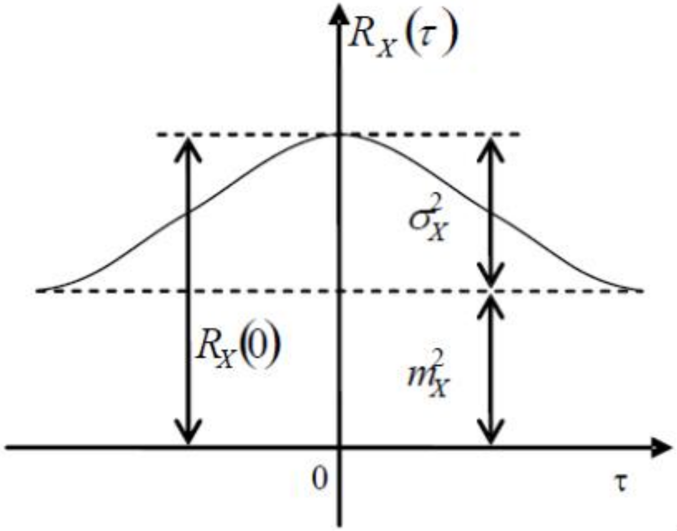
B 36, 2

C ± 6, 4

D ± 6, 2

- Example 1: The autocorrelation of a stationary stochastic process $X(t)$ is:

$$R_X(\tau) = 36 + \frac{4}{1 + 5\tau^2}$$



Find the mean and variance of $X(t)$.

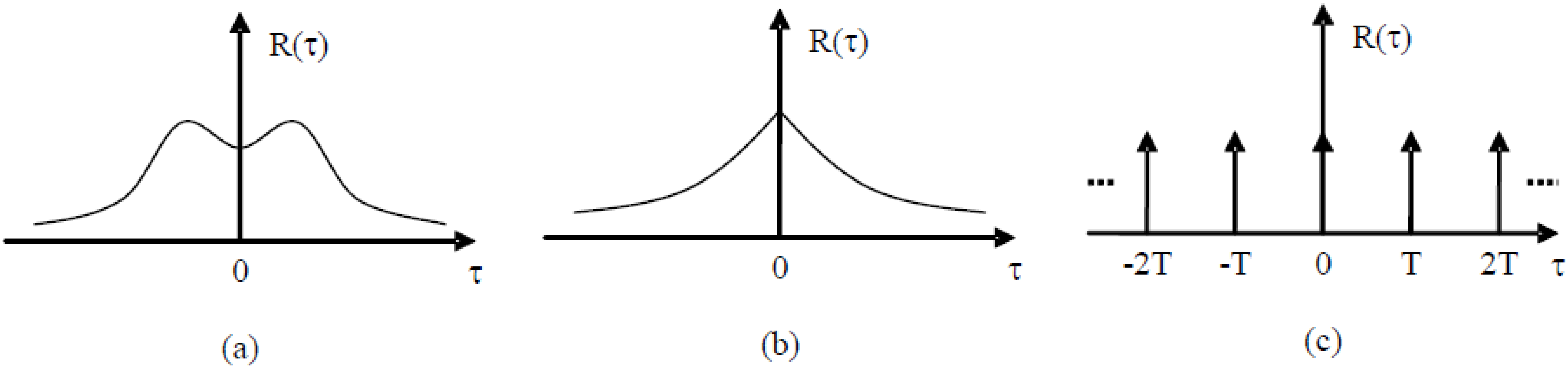
Solution:

$$m_X^2 = R_X(\infty) = 36$$

$$m_X = \pm 6$$

$$\sigma_X^2 = R_X(0) - R_X(\infty) = 40 - 36 = 4$$

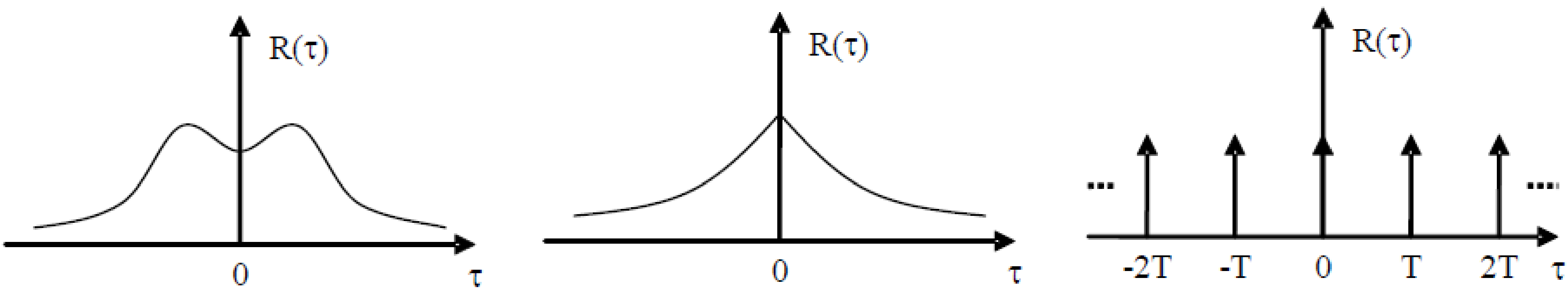
- Example 2: Please judge whether the autocorrelation of the following stationary stochastic process are correct or not? If it is correct, is it periodic(有周期性的)? If it is periodic, what is the period(周期)?



Choose some students to answer the questions

Characteristics of Autocorrelation of Stationary Stochastic Processes

- Example 2: Please judge whether the autocorrelation of the following stationary stochastic process are correct or not? If it is correct, is it periodic(有周期性的)? If it is periodic, what is the period(周期)?



(a) (b) (c)

Choose some students to answer the questions

- A is not correct, 0 point should be the maximum.
- B is correct, not periodic.
- C is correct, and periodic, the period is T.

- Characteristics of Autocorrelation of Stationary Stochastic Processes
- Correlation Coefficient and Correlation Time
- Joint Distributions and Cross-Correlation of Stochastic Processes
  - Joint Distributions
  - Cross-Correlation, Cross-Covariance and Jointly WSS

Correlation Coefficient and Correlation Time

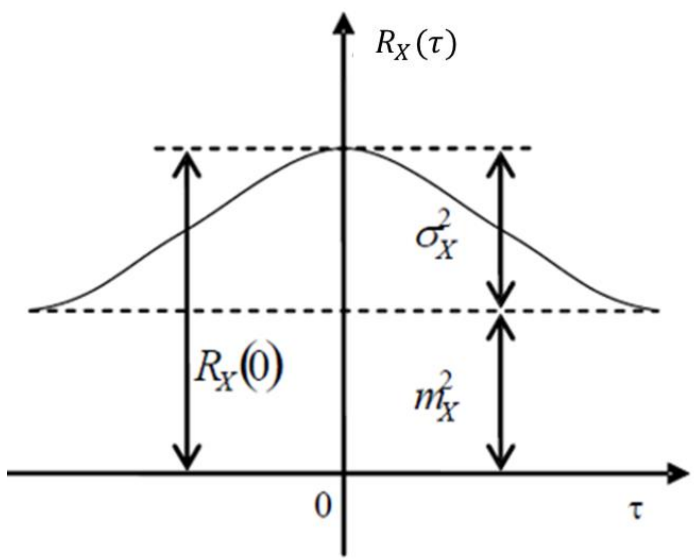
- Correlation Coefficient (page 348 of text book):

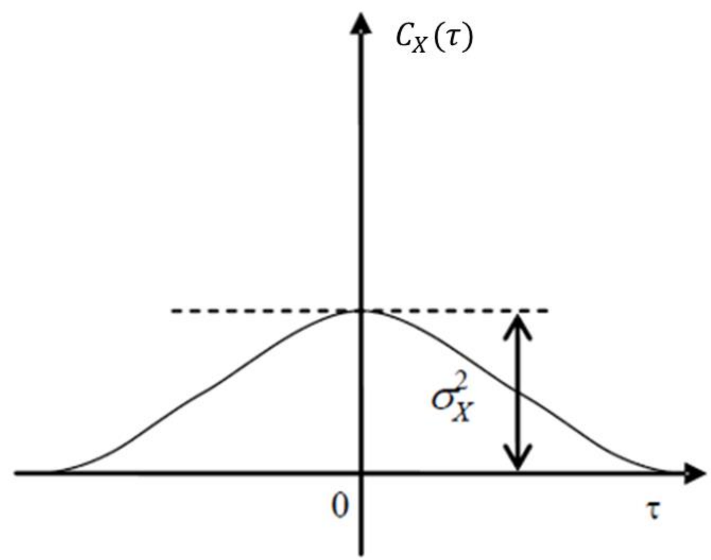
$$r_X(t_1,t_2) = \frac{C_X(t_1,t_2)}{[C_X(t_1,t_1)C_X(t_2,t_2)]^{1/2}} = \frac{C_X(\tau)}{\sigma_X(t_1)\sigma_X(t_2)}$$

As $C_X(t,t) = \sigma_X^2(t)$

- For Stationary Processes:

$$r_X(\tau) = \frac{C_X(\tau)}{[C_X(0)C_X(0)]^{1/2}} = \frac{C_X(\tau)}{C_X(0)} = \frac{R_X(\tau) - m_X^2}{\sigma_X^2}$$





Correlation Coefficient and Correlation Time

- Correlation Coefficient of a Stationary Processe:

$$r_X(t_1, t_2) = \frac{C_X(\tau)}{[C_X(0)C_X(0)]^{1/2}} = \frac{C_X(\tau)}{C_X(0)} = \frac{R_X(\tau) - m_X^2}{\sigma_X^2}$$

  - Range: [-1, 1]
  - It reflects the degree of linear correlation between the two moments of the stochastic process:
    - $r_X(\tau) = 1$, fully related
    - $r_X(\tau) = 0$, uncorrelated
    - $r_X(\tau) > 0$, positive correlated
    - $r_X(\tau) < 0$, negatively correlated
  - Generally speaking, for a stochastic process without a periodic component, $|r_X(\tau)|$ decreases when $|\tau|$ increases, and $r_X(\infty) = 0$, which is, r.vs draw from two moments of a stochastic process will be uncorrelated if the time interval is sufficiently large
  - White noise: $r_X(\tau) = \delta(\tau)$ (only $\tau = 0$ gives non-zero value)

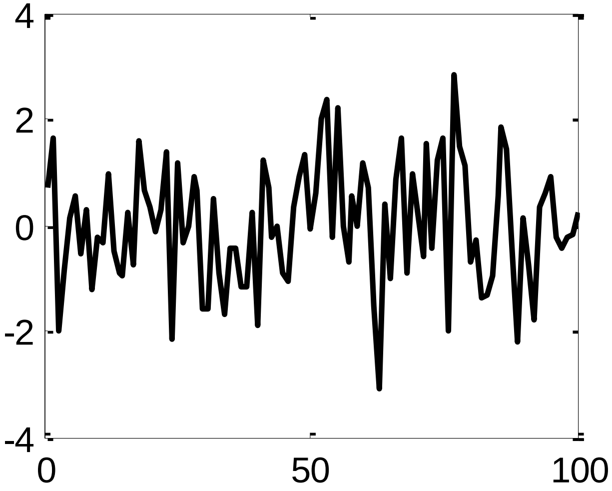
Correlation Coefficient and Correlation Time

- Correlation Time (page 353 of text book) :

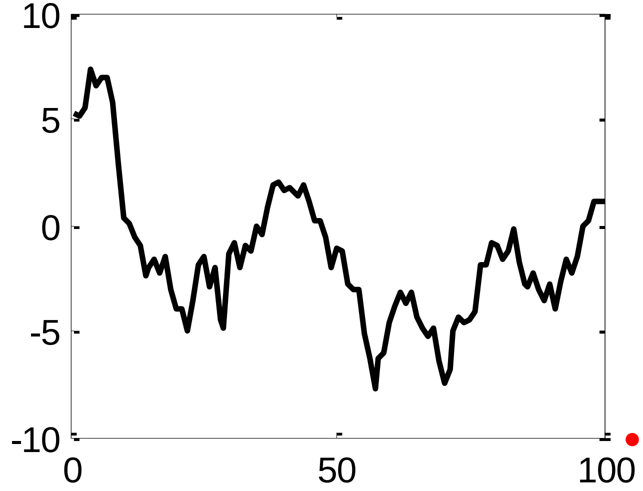
$$\tau_0 = \frac{1}{C(0)}\int_0^{\infty} C_X(\tau)d\tau = \int_0^{\infty} r_X(\tau)d\tau$$

  - In engineering, if $t_2 - t_1 > \tau_0$, then the $X(t_2)$ and $X(t_1)$, the two moments draw from the same process $X(t)$, are assumed to be uncorrelated.
  - It describe the speed of change of a process, i.e.:

$\tau_0 = 1$



$\tau_0 = 100$



- The larger the correlation time is, the stronger the correlation between the adjacent values of a stochastic process, and the slower the change
- Vice versa

Correlation Coefficient and Correlation Time

- Example 3: The autocovariance of process $Y(t)$ is $C_Y(\tau) = \frac{\sin \lambda\tau}{\lambda\tau}$, please

a) Calculate the correlation time of $Y(t)$

b) Calculate the correlation coefficient of $Y(t)$ at time $\tau = \frac{\pi}{\lambda}$

c) Calculate the correlation coefficient of $Y(t)$ at time $\tau = 0$

Hint:

$$\int_0^\infty \frac{\sin x}{x} dx = \int_0^\infty \sin x \int_0^\infty e^{-tx} dt dx = \int_0^\infty \int_0^\infty \sin x e^{-tx} dx dt = \int_0^\infty \frac{1}{1+t^2} dt$$

$$= \mathrm{arctg}(\infty) = \pi/2$$

$$\frac{\sin 0}{0} = 1$$

More examples from the last week

- Example 3: The autocovariance of process $Y(t)$ is $C_Y(\tau) = \frac{\sin \lambda\tau}{\lambda\tau}$, please

a) Calculate the correlation time of $Y(t)$

Solution:

a)

$$\sigma_Y^2 = C_Y(0) = 1,$$

$$r_Y(\tau) = \frac{C_Y(\tau)}{\sigma_Y^2} = \frac{\sin \lambda\tau}{\lambda\tau}$$

$$\tau_{0Y} = \int_0^\infty \frac{\sin \lambda\tau}{\lambda\tau} d\tau = \frac{\pi}{2\lambda}$$

$$\int_0^\infty \frac{\sin x}{x} dx$$
$$= \int_0^\infty \sin x \int_0^\infty e^{-tx} dt dx$$
$$= \int_0^\infty \int_0^\infty \sin x e^{-tx} dx dt$$
$$= \int_0^\infty \frac{1}{1+t^2} dt$$
$$= \operatorname{arctg}(\infty) = \pi/2$$

More examples from the last week

b) Calculate the correlation coefficient of $Y(t)$ at time $\tau = \frac{\pi}{\lambda}$

$$C_Y(\tau) = \frac{\sin \lambda\tau}{\lambda\tau}$$

$$r_Y(\tau) = \frac{C_Y(\tau)}{\sigma_Y^2} = \frac{\sin \lambda\tau}{\lambda\tau}$$

→

$$r_Y(\frac{\pi}{\lambda}) = \frac{\sin \pi}{\pi} = 0$$

c) Calculate the correlation coefficient of $Y(t)$ at time $\tau = 0$

$$r_Y(0) = 1$$

- Characteristics of Autocorrelation of Stationary Stochastic Processes
- Correlation Coefficient and Correlation Time
- Joint Distributions and Cross-Correlation of Stochastic Processes
  - Joint Distributions
  - Cross-Correlation, Cross-Covariance and Jointly WSS

Joint Distributions

- In many real world applications, there are two or more signals (Stochastic Processes).
  - The signals received by the communication and radar antenna include: 1 or 2 or more target echo signal + noise
  - Analysis of the relationship between the emitted signal and the received signal, even excluding noise
- Given the N-D distribution of the stochastic process $X(t)$ and M-D distribution of the stochastic process $Y(t)$, the $(N+M)$-dimensional joint CDF is:

$$F_{XY}(x_1,\ldots,x_n,t_1\ldots,t_n,y_1,\ldots,y_m,t_1',\ldots,t_m')$$
$$= P\{X(t_1)\le x_1,\cdots,X(t_n)\le x_n,Y(t_1')\le y_1,\cdots,Y(t_m')\le y_m\}$$

- The joint pdf is:

$$f_{XY}(x_1,\ldots,x_n,t_1\ldots,t_n,y_1,\ldots,y_m,t_1',\ldots,t_m')$$
$$= \frac{\partial^{n+m}F_{XY}(x_1,\ldots,x_n,t_1\ldots,t_n,y_1,\ldots,y_m,t_1',\ldots,t_m')}{\partial x_1\ldots\partial x_n\partial y_1\ldots\partial y_m}$$

Cross-Correlation, Cross-Covariance and Jointly WSS

- In many applications, we need to consider the characteristics between two or more stochastic processes simultaneously. For example, signal detected from two microphones.
- The cross-correlation of two processes $X(t)$ & $Y(t)$is:
  - Real valued case:

$$R_{XY}(t_1,t_2) = E\{X(t_1)Y(t_2)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy f_{XY}(x,y,t_1,t_2)dxdy$$

  - Complex valued case:

$$R_{XY}(t_1,t_2) = E\{X(t_1)Y^*(t_2)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy^* f_{XY}(x,y,t_1,t_2)dxdy$$

- The cross-covariance is:
  - Real valued case:

$$C_{XY}(t_1,t_2) = E\{[X(t_1)-m_X(t_1)][Y(t_2)-m_Y(t_2)]\}$$
$$= R_{XY}(t_1,t_2) - m_X(t_1)m_Y(t_2)$$

  - Complex valued case: $C_{XY}(t_1,t_2) = R_{XY}(t_1,t_2) - m_X(t_1)m_Y^*(t_2)$
- If $R_{XY}(t_1,t_2) = 0$, $X(t_1)$ and $Y(t_2)$ are orthogonal
- If $C_{XY}(t_1,t_2)$ =0, $X(t_1)$ *and* $Y(t_2)$ are uncorrelated

Cross-Correlation, Cross-Covariance and Jointly WSS

- Two processes $X(t)$ and $Y(t)$ are called jointly WSS if both are WSS and their cross-correlation depends only on $\tau = t_1 - t_2$:

$$m_X(t) = m_X, m_Y(t) = m_Y, R_X(t_1, \mathrm{t}_2) = R_X(\tau), R_Y(t_1, \mathrm{t}_2) = R_Y(\tau)$$

and

$$R_{XY}(t_1, t_2) = R_{XY}(\tau), \tau = t_1 - t_2$$

Note:

- For complex valued process:
  - $R_{XY}(\tau) = E\{X(t+\tau)Y^*(t)\}, C_{XY}(\tau) = R_{XY}(\tau) - m_X m_Y^*$
- $R_{XY}(0)$ might be negative for real valued process

Cross-Correlation, Cross-Covariance and Jointly WSS

Properties of jointly WSS:

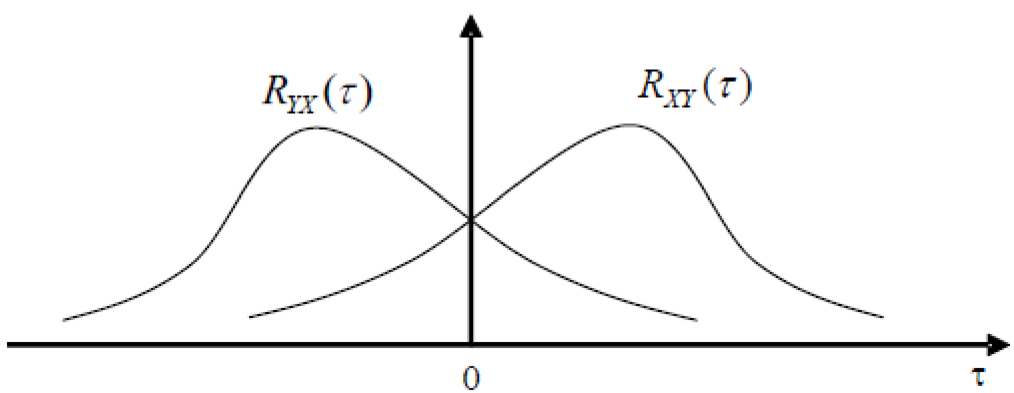
- Property 1: for real valued process

$$R_{XY}(-\tau) = R_{YX}(\tau)$$
$$C_{XY}(-\tau) = K_{YX}(\tau)$$

$$R_{XY}(\tau) = E[X(t)Y(t-\tau)]$$
$$R_{XY}(-\tau) = E[X(t)Y(t+\tau)]$$

$\rightarrow R_{XY}(\tau) \neq R_{XY}(-\tau) \quad R_{XY}(\tau) \neq -R_{XY}(-\tau)$

Topical curve of Cross-Correlation of Jointly WSS processes



Cross-Correlation, Cross-Covariance and Jointly WSS

Properties of jointly WSS:

- Property 2: If $X(t)$ and $Y(t)$ jointly WSS, then $Z(t) = X(t) + Y(t)$ is WSS, and

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{YX}(\tau)$$

- When uncorrelated, $C_{XY}(t_1, t_2) = 0$, then

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau) + 2m_X m_Y$$

- when orthogonal, $R_{XY}(t_1, t_2) = 0$, then

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau)$$

Cross-Correlation, Cross-Covariance and Jointly WSS

- The cross-correlation coefficient, also known as the normalized cross-covariance of two real valued processes is:

$$r_{XY}(\tau) = \frac{C_{XY}(\tau)}{\sqrt{C_X(0)C_Y(0)}} = \frac{R_{XY}(\tau) - m_X m_Y}{\sigma_X \sigma_Y}$$

- If $r_{XY}(\tau) = 0$ for any τ (in this case, any $C_{XY}(t_1, t_2) = 0$), then $X(t)$ and $Y(t)$ are uncorrelated:

$$R_{XY}(\tau) = m_X m_Y$$

- Example 4: given $X(t) = \sin(\omega_0 t + \Phi)$, $Y(t) = \cos(\omega_0 t + \Phi)$, where ω_0 is a constant, Φ uniformly distributed in $(-\pi, \pi)$, calculate the cross-covariance

Solution:

Cross-Correlation, Cross-Covariance and Jointly WSS

- Example 4: given $X(t) = \sin(\omega_0 t + \Phi)$, $Y(t) = \cos(\omega_0 t + \Phi)$, where ω_0 is a constant, Φ uniformly distributed in $(-\pi, \pi)$, calculate the cross-covariance

Solution:

$$E\{X(t)\} = E\{\sin(\omega_0 t + \Phi)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(\omega_0 t + \phi) d\phi = 0$$

$$E\{Y(t)\} = E\{\cos(\omega_0 t + \Phi)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega_0 t + \phi) d\phi = 0$$

$$C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - m_X m_Y$$

$$= \frac{1}{2} E\{\sin(\omega_0 t_1 + \omega_0 t_2 + 2\Phi) + \sin \omega_0 (t_1 - t_2)\}$$

$$= \frac{1}{2} \sin \omega_0 \tau$$

$$\tau = t_1 - t_2$$

Cross-Correlation, Cross-Covariance and Jointly WSS

- Example 4: given $X(t) = \sin(\omega_0 t + \Phi)$, $Y(t) = \cos(\omega_0 t + \Phi)$, where ω_0 is a constant, Φ uniformly distributed in $(-\pi, \ \pi)$, calculate the cross-covariance

Solution:

$$E\{X(t)\} = E\{\sin(\omega_0 t + \Phi)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(\omega_0 t + \phi) d\phi = 0$$

$$E\{Y(t)\} = E\{\cos(\omega_0 t + \Phi)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega_0 t + \phi) d\phi = 0$$

$$C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - m_X m_Y$$

$$= \frac{1}{2} E\{\sin(\omega_0 t_1 + \omega_0 t_2 + 2\Phi) + \sin \omega_0 (t_1 - t_2)\}$$

$$= \frac{1}{2} \sin \omega_0 \tau$$

$$\tau = t_1 - t_2$$

- Note that here t, t_1, t_2 are regarded as constants (i.e. fixed "time");
- For any time t, integrate the "state“, which is Φ.
- Since Φ is uniformly distributed in $(-\pi, \ \pi)$, the integration result is 0!

The Application of Autocorrelation and Cross Correlation

- We can still see it in papers:

The Application of Autocorrelation and Cross Correlation

- We will use it when doing experimental 2, using time domain microphone sound synthesis (声音合成) as an example. This is the advance part of experimental report 2
- Will introduce the experimental report 2 next week

Reading

- This week:
  - Text book: 7.1, 8.4
  - Red book: 2.3.1-2.3.4, 2.4
  - Blue book: 6.5
- Next week:
  - Text book: 7.3 (power spectrum) (we will go back to 7.2 the week after next)
  - Red book: 2.5

Experiment

- Go on for your Experiment 1 today

More examples

1: given a stochastic phase signal $X(t) = Acos(\omega t + \varphi)$, where A and ω are constants, and φ is a r.v uniformly distributed in $(0{,}2\pi)$. Is $X(t)$ Wide-Sense Stationary? Strict-Sense Stationary?

2: The autocovariance of a stationary stochastic process $X(t)$ is:

$$C_X(\tau) = \frac{9}{1 + \tau^2}$$

If an 'engineering correlation time' is defined as the value t_e where $r_X(t_e) \leq 0.1 * r_X(0)$ with the positive and minimum value t_e (if there are a lot values satisfying this rule), calculate t_e?

Multiple choices?

A 9

B 3

C 89

D $(89)^{1/2}$