Stochastic Signal Processing

Experiment 3 Basic

- The delta response(or impulse response, 冲激响应) $h[n]$ of a linear system is its response to the delta sequence $\delta[n]$. Its system transfer function（系统传输函数，simplified called system function） is the z transform of $h[n]$:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

Where $z = e^{j\omega}$ ($H(\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$), ω is the (angular) frequency

- If $\mathbf{x}[n]$ is the input to a digital system, the resulting output is the digital convolution of $\mathbf{x}[n]$ with $h[n]$:

$$\mathbf{y}[n] = \sum_{k=-\infty}^{\infty} \mathbf{x}[n-k]h[k] = \mathbf{x}[n] * h[n]$$

$$\mathrm{Y}(z) = X(\mathrm{z}) \cdot H(\mathrm{z})$$

- Periodogram Method:
  - A method to estimate the power spectrum of stochastic signal samples to obtain the signal frequency domain characteristics.

Given a WSS process $\{x_n\}_{n=0}^{N-1}$, periodogram method estimate:

$$\hat{S}_X(\omega) = \frac{1}{N}|X(\omega)|^2 \qquad (11\text{-}1)$$

where $X(\omega)$ is the DTFT of $\{x_n\}_{n=0}^{N-1}$ and periodic with period 2π.

$$X(\omega) = \sum_{n=0}^{N-1} x_n e^{-jn\omega}$$

注：这里的(11-1)是离散信号的版本，theory ppt中的是连续信号的版本

Power spectrum

- Correlogram Method (Another method to estimate the power spectrum):
  - The function $\hat{S}(\omega)$ is periodic with period 2π, therefore the power spectrum of $\mathbf{x}[n]$ is:

$$\hat{S}_X(\omega) = \sum_{m=1-N}^{N-1} \hat{R}_X[m] e^{-jm\omega} \tag{11-2}$$

between 0 to 2π, or $-\pi$ to π, and

$$\hat{R}_X[m] = \frac{1}{\mathrm{N}} \sum_{\mathrm{n}=0}^{\mathrm{N}-1-|\mathrm{m}|} x(n)x(n+m)$$

注：这里的(11-2)是离散信号的版本，theory ppt中的是连续信号的版本

Power spectrum

- More about the Autocorrelation:

$$\hat{R}_X[m] = \alpha \sum_{\mathrm{n=0}}^{\mathrm{N-1-|m|}} x(n)x(n+m)$$

When $\alpha = \frac{1}{\mathrm{N}}$, the estimation is biased. Therefore we can use

$$\boldsymbol{E}\left(\hat{R}_X[m]\right) = \frac{\mathrm{N}-|m|}{\mathrm{N}} R_X[m] = w_N(n)R_X[m]$$

And $\boldsymbol{E}\left(\hat{S}_X(\omega)\right) = \sum_{\mathrm{m=1}-N}^{\mathrm{N-1}} w_N(m)R_X[m]e^{-jm\omega}$

Experiment – power spectrum

- Function periodogram in Matlab

pxx = periodogram(x)   % using the default rectangular window and returns the periodogram power spectral density (PSD) estimate

pxx = periodogram(x, window, nfft) %uses nfft points in the discrete Fourier transform (DFT).

- Common window function in matlab

name	Matlab function
矩形窗	rectwin
三角窗	triang
Hamming	hamming

Experiment – power spectrum

- Practice 1

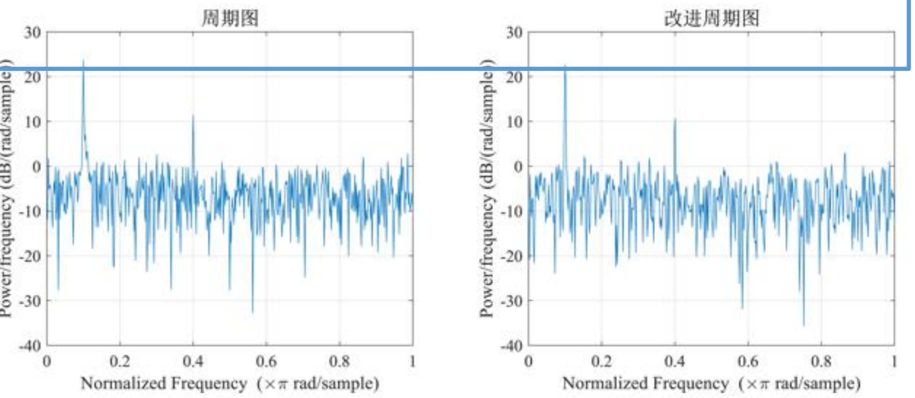
A random signal

$$X(t) = 1.8\cos(100\pi t) + 0.5cos(400\pi t) + N(t)$$

where $N(t)$ is White Gaussian Noise, and σ^2=1. Use rectangular window and Hamming window to estimate the power spectrum of $X(t)$

(Hint: use function periodogram)

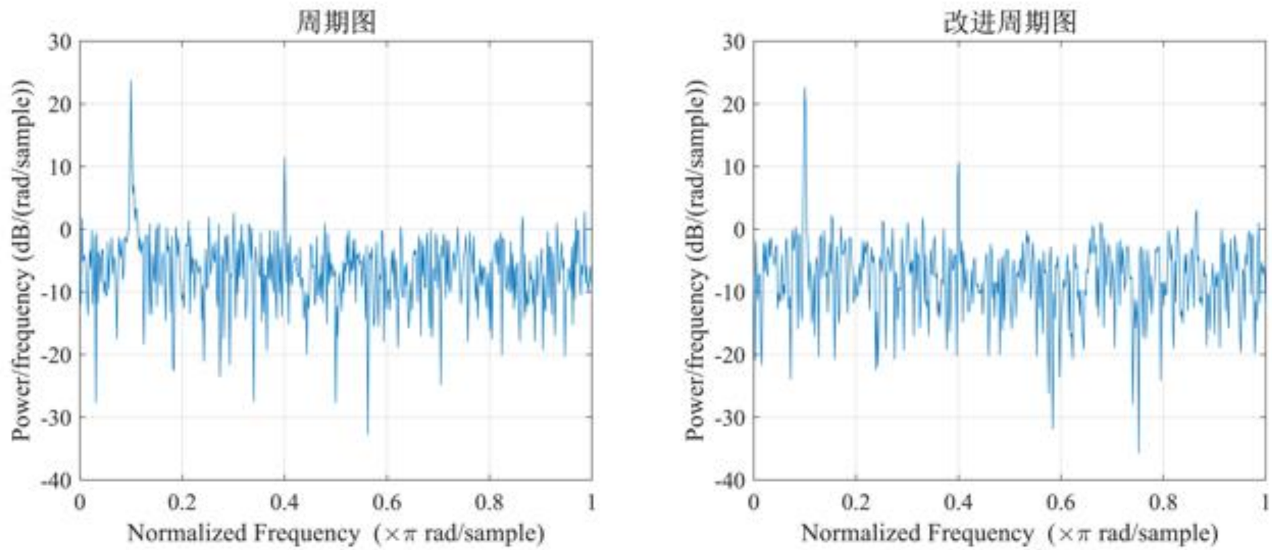
Experiment – power spectrum




```
fs = 1000; % sample rate
f1 = 50; %
f2 = 200; %
t = 0:1/fs:1-1/fs; %  sample point
x = 1.8*cos(2*pi*f1*t)+0.5*cos(2*pi*f2*t) + randn(size(t)); % length 1s
subplot(1,2,1)
periodogram(x,rectwin(length(x))); % Periodogram with rectangular Window
set(gca,'fontsize',12,'fontname','times');
title('\fontname{} Periodogram ','fontsize',14);
subplot(1,2,2)
periodogram(x,hamming(length(x))); % Periodogram with Hamming Window
set(gca,'fontsize',12,'fontname','times');
title('\fontname{} Modified Periodogram','fontsize',14);
set(gcf,'Units','centimeter','Position',[10 10 28 10]);
```

Experiment – power spectrum

Practice 1



Formula: ω = 2 π f / f_s

Maximum values appear in ω=0.1π and ω=0.4π, corresponding to f =50Hz and f =200Hz,

Hamming window is able to reduce side lobe, especially in ω=0.1π

Experimental Report 3

- Basic 1 (40 points)： A random signal

$$X(t) = \sin(\omega_1 t) + 2cos(\omega_2 t) + N(t)$$

where ω_1=100π ,ω_2=150π, $N(t)$ is zero mean White Gaussian Noise.

1) Under and σ^2=0.1, generate the signal for 2s (signal length), and use appropriate sample rate to do the sampling. Plot the periodogram with different window, and compare the results, describe the differences.

2) Analyze the effects of sampling rate, signal length and the value of σ^2 (now you can change the value of σ^2) on the estimation of the power spectrum using the periodogram (use rectangular Window only).

Experimental Report 3

- Basic 1 (40 points)： A random signal

$$X(t) = \sin(\omega_1 t) + 2cos(\omega_2 t) + N(t)$$

where ω_1=100π ,ω_2=150π, $N(t)$ is zero mean White Gaussian Noise.

3) According to (11-1) and (11-2), design your own Periodogram and Correlogram function (write the function in Matlab yourself), and compare the difference of two methods. Replace these two functions with the default Periodogram function in 2), and plot the figures again, and show the comparison between your own Periodogram and Correlogram function and the default Periodogram function used in 2)

$$\hat{S}_X(\omega) = \frac{1}{\mathrm{N}} |X(\omega)|^2 \tag{11-1}$$

$$\hat{S}_X(\omega) = \sum_{m=1-N}^{N-1} \hat{R}_X[m] e^{-jm\omega} \tag{11-2}$$

Note: you should submit your codes that can generate the figures in 3). The codes should be runnable!

Experimental Report 3

- Knowledge explanation: Now the signal becomes

$$X(t) = \sin(\omega_1 t) \; + 2cos(\omega_2 t) + 4cos(\omega_I t) + N(t)$$

where ω_1=100π ,ω_2=150π, $4cos(\omega_I t)$ is the interference, and ω_I uniformly distributed in $[50\pi, 80\pi]$, $N(t)$ is zero mean White Gaussian Noise with σ^2.

Note that in one independent run, as ω_I uniformly distributed in $[50\pi, 80\pi]$, and we assume that it is $\omega_I = 70\pi$ in that independent run, if we use the Periodogram or Correlogram function as in (11-1) and (11-2), the resulting figure will give us a large peak at 70π, this is why we call it interference.

In this case, can we find out the peaks ω_1=100π ,ω_2=150π?

Experimental Report 3

- Knowledge explanation: Now the signal becomes

$$X(t) = \sin(\omega_1 t) + 2cos(\omega_2 t) + 4cos(\omega_I t) + N(t)$$

In this case, can we find out the peaks ω_1=100π ,ω_2=150π?

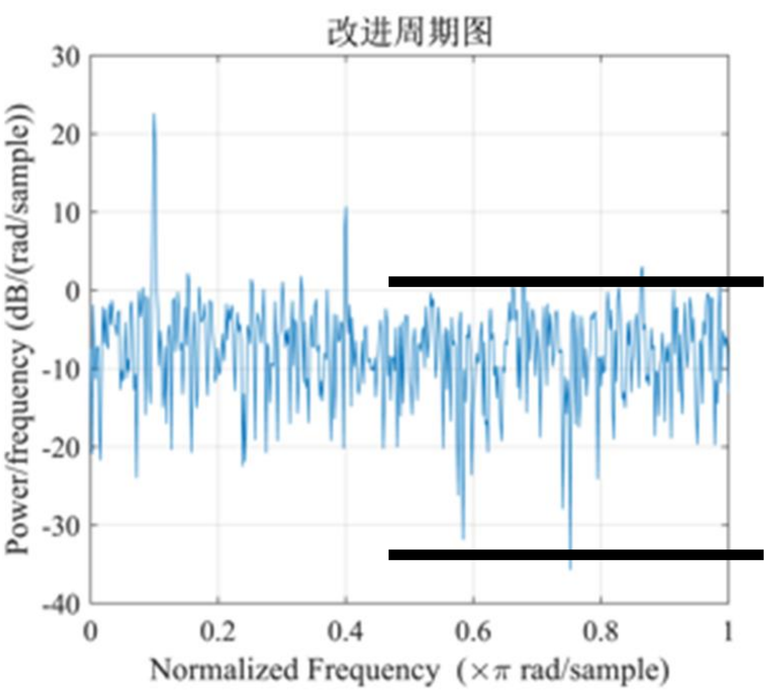
Note that, $\sin(\omega_1 t) + 2cos(\omega_2 t)$ is a deterministic signal (if in each time, we start the sampling from $t = 0$), while $4cos(\omega_I t)$ and $N(t)$ are stochastic processes. In fact, for $4cos(\omega_I t)$ and $N(t)$, if we want to find their power spectrum, we cannot sample it and plot its Periodogram or Correlogram once only! We should sample them as many as possible times and take the average!

Experimental Report 3

- Knowledge explanation: Now the signal becomes

$$X(t) = \sin(\omega_1 t) \ + 2cos(\omega_2 t) + 4cos(\omega_I t) + N(t)$$

For example, if we sample $4cos(\omega_I t)$ and $N(t)$ ∞ times and take an average. Then $4cos(\omega_I t)$ might give a 'rectangular peak' in $[50\pi, 80\pi]$, and $N(t)$ will give a 'flat line' in all frequencies but not a 'wave' like this.



However, sampling $\sin(\omega_1 t) \ + 2cos(\omega_2 t)$ ∞ times will always give two peaks in ω_1 and ω_2!

In this case, we will get the real power spectrum: two peaks, one 'rectangular peak' in $[50\pi, 80\pi]$, and a 'flat line'

Experimental Report 3

- Basic 2 (20 points): Now the signal becomes

$$X(t) = \sin(\omega_1 t) + 2cos(\omega_2 t) + 4cos(\omega_I t) + N(t)$$

where ω_1=100π ,ω_2=150π, $4cos(\omega_I t)$ is the interference, and ω_I uniformly distributed in $[50\pi, 80\pi]$, $N(t)$ is zero mean White Gaussian Noise with σ^2.

1) Set σ^2=0.1 and use appropriate sample rate to do the sampling. For $M = 100$ runs, in each run, generate the signal for 2s, compute the periodogram (11-1) (use this only, do not use the default periodogram of Matlab), and take the average of these M runs to get the power spectrum. Plot the periodogram of the 1st, 50nd, 100nd run and the power spectrum. (there are totally four figures)

Experimental Report 3

- Basic 2 (20 points): Now the signal becomes

$$X(t) = \sin(\omega_1 t) + 2cos(\omega_2 t) + 4cos(\omega_I t) + N(t)$$

where ω_1=100π ,ω_2=150π, $4cos(\omega_I t)$ is the interference, and ω_I uniformly distributed in $[50\pi, 80\pi]$, $N(t)$ is zero mean White Gaussian Noise with σ^2.

2) Plot the power spectrum for different σ^2 and provide analysis. (at least select 4 values of σ^2, at least one should be small so that will not effect the observation of $\sin(\omega_1 t) + 2cos(\omega_2 t) + 4cos(\omega_I t)$, and at least one should be large so that it is hard to observe $\sin(\omega_1 t) + 2cos(\omega_2 t) + 4cos(\omega_I t)$)

Note: you should submit your codes that can generate the figures in 1). The codes should be runnable!