

Stochastic Signal Processing

Lesson 9

Memoryless and Linear Systems II

Weize Sun

Memoryless and Linear Systems – Outline

- The average power – Time domain
- Differentiators (微分器)
- Power spectrum of the output of a linear system
 - The average power – Frequency domain
 - Differentiators
- Hilbert transforms

The average power – Time domain

We now give the average power $E\{|\mathbf{y}(t)|^2\}$ of the output of a system if its input is white noise.

- If the input to a linear system $h(t)$ is (complex) stationary white noise with autocorrelation $R_X(\tau) = q\delta(\tau)$, then:

$$\begin{aligned} E\{|\mathbf{y}(t)|^2\} &= R_Y(0) = R_X(0) * h(0) * h^*(0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_X(0 + \alpha - \beta) h(\alpha) h^*(\beta) d\alpha d\beta \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q\delta(\alpha - \beta) h(\alpha) h^*(\beta) d\alpha d\beta \quad (\text{for the } \delta(\alpha - \beta), \text{ now-zero only when } \alpha = \beta = t) \\ &= \int_{-\infty}^{\infty} qh(t)h^*(t)dt = qE \quad \text{where } E = \int_{-\infty}^{\infty} |h(t)|^2 dt \text{ is the energy of } h(t). \end{aligned}$$

Linear Systems – output autocorrelation

- In exam, we consider only the Steady state analysis (稳态分析) of stochastic processes, in this case

If input is stationary, output is stationary, for Steady state analysis (稳态分析) only

Properties when WSS ($\tau = t_1 - t_2$): (complex valued case)

$$R_{XY}(t_1, t_2) = R_X(t_1, t_2) * h^*(t_2) \quad \rightarrow \quad R_{XY}(\tau) = R_X(\tau) * h^*(-\tau)$$

$$R_{YX}(t_1, t_2) = R_X(t_1, t_2) * h(t_1) \quad \rightarrow \quad R_{YX}(\tau) = R_X(\tau) * h(\tau)$$

$$R_Y(t_1, t_2) = R_X(t_1, t_2) * h(t_1) * h^*(t_2) \quad \rightarrow \quad R_Y(\tau) = R_X(\tau) * h(\tau) * h^*(-\tau)$$

$$R_Y(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_X(t_1 - \alpha, t_2 - \beta) h(\alpha) h^*(\beta) d\alpha d\beta$$

$$R_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_X(\tau + \alpha - \beta) h(\alpha) h^*(\beta) d\alpha d\beta$$

Differentiators

- Recall

Linear Systems – output autocorrelation

- In exam, we consider only the Steady state analysis (稳态分析) of stochastic processes, in this case


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$$R_Y(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_X(t_1 - \alpha, t_2 - \beta) h(\alpha) h^*(\beta) d\alpha d\beta$$


$$R_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_X(\tau + \alpha - \beta) h(\alpha) h^*(\beta) d\alpha d\beta$$


- Now we discuss a widely used system called Differentiators

Differentiators

- A differentiator is a linear system whose output is the derivative of the input

$$L[\mathbf{x}(t)] = \frac{d\mathbf{x}(t)}{dt} = \mathbf{x}'(t)$$

We can, therefore, use the preceding results to find the mean of $\mathbf{x}'(t)$:

$$m_{\mathbf{x}'}(t) = L[m_{\mathbf{x}}(t)] = \frac{dm_{\mathbf{x}}(t)}{dt} = m'_{\mathbf{x}}(t)$$

Which is, to find the mean of $\mathbf{x}'(t)$, we can first take the mean of $\mathbf{x}(t)$ and thus get $m_{\mathbf{x}}(t)$, and then take the derivation $m_{\mathbf{x}}(t)$

- If $\mathbf{x}(t)$ is WSS, then $m_{\mathbf{x}}(t)$ is constant; hence $\frac{dm_{\mathbf{x}}(t)}{dt} = 0$, which is:

$$E\{\mathbf{x}'(t)\} = 0$$

Differentiators

- A differentiator is a linear system whose output is the derivative of the input $L[\mathbf{x}(t)] = \frac{d\mathbf{x}(t)}{dt} = \mathbf{x}'(t)$

We can, therefore, use the preceding results to find the autocorrelation of $\mathbf{x}'(t)$:

$$\left\{ \begin{array}{l} R_{XX'}(t_1, t_2) = L_2[R_{XX}(t_1, t_2)] = \frac{\partial R_{XX}(t_1, t_2)}{\partial t_2} \\ R_{X'X'}(t_1, t_2) = L_1[R_{XX'}(t_1, t_2)] = \frac{\partial R_{XX'}(t_1, t_2)}{\partial t_1} \end{array} \right. \longrightarrow R_{X'X'}(t_1, t_2) = \frac{\partial^2 R_{XX}(t_1, t_2)}{\partial t_1 \partial t_2}$$

- Furthermore, for WSS process, $R_X(t_1, t_2) = R_X(\tau)$, $\tau = t_1 - t_2$, then

$$\left\{ \begin{array}{l} \frac{\partial R_X(t_1 - t_2)}{\partial t_2} = -\frac{dR_X(\tau)}{d\tau} \\ \frac{\partial^2 R_X(t_1 - t_2)}{\partial t_1 \partial t_2} = -\frac{d^2 R_X(\tau)}{d\tau^2} \end{array} \right. \longrightarrow \left\{ \begin{array}{l} R_{XX'}(\tau) = -R'_{XX}(\tau) \\ R_{X'X'}(\tau) = -R''_{XX}(\tau) \end{array} \right.$$

Power spectrum of the output of a linear system

- For the linear system and a stationary input $\mathbf{x}(t)$:

$$\mathbf{y}(t) = \mathbf{x}(t) * h(t) = \int_{-\infty}^{\infty} \mathbf{x}(t - \alpha) h(\alpha) d\alpha$$

- the autocorrelation/cross-correlation $R_Y(\tau)/R_{XY}(\tau)$ and power spectrum/cross power spectrum $S_Y(\omega)/S_{XY}(\omega)$ of the response are Fourier Transform pairs:

$$R_{XY}(\tau) = R_X(\tau) * h^*(-\tau) \quad \longleftrightarrow \quad S_{XY}(\omega) = S_X(\omega) H^*(\omega)$$

- Note that for real process, $R_{XY}(\tau) = R_X(\tau) * h(-\tau) \quad \longleftrightarrow \quad S_{XY}(\omega) = S_X(\omega) H^*(\omega)$, the conjugate in $H^*(\omega)$ refers to the Fourier Transform ' $h(-\tau)$ ', no matter it is in conjugate (complex valued) or not (real valued)

$$R_Y(\tau) = R_{XY}(\tau) * h(\tau) \quad \longleftrightarrow \quad S_Y(\omega) = S_{XY}(\omega) H(\omega)$$

Where $H(\omega)$ is the Fourier Transform of $h(t)$, also referred to as **Transfer function**.

Convolution in the time domain is multiplication in the frequency domain, and vice versa

(时域上的卷积等价于频域上的乘，反之亦然)

Power spectrum of the output of a linear system

$$R_{XY}(\tau) = R_X(\tau) * h^*(-\tau) \quad \longleftrightarrow \quad S_{XY}(\omega) = S_X(\omega)H^*(\omega)$$

$$R_Y(\tau) = R_{XY}(\tau) * h(\tau) \quad \longleftrightarrow \quad S_Y(\omega) = S_{XY}(\omega)H(\omega)$$

- Combining the above two equations we get:

$$R_Y(\tau) = R_X(\tau) * h(\tau) * h^*(-\tau) = R_X(\tau) * \rho(\tau)$$

\longleftrightarrow

$$S_Y(\omega) = S_X(\omega)H(\omega)H^*(\omega) = S_X(\omega)|H(\omega)|^2$$

where

$$\rho(\tau) = h(\tau) * h^*(-\tau) = \int_{-\infty}^{\infty} h(t + \tau)h^*(t)dt \leftrightarrow |H(\omega)|^2$$

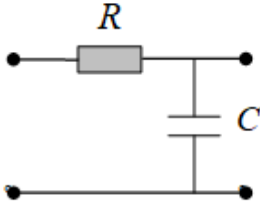
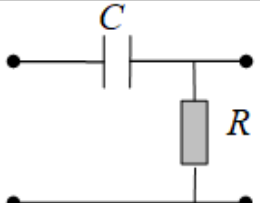
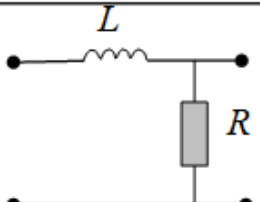
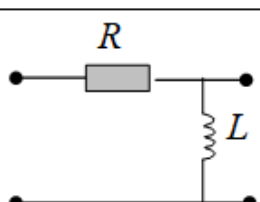
In particular, if $\mathbf{x}(t)$ is white noise with average power q , then

$$R_X(\tau) = q\delta(\tau) \quad \longleftrightarrow \quad S_X(\omega) = q$$

$$R_Y(\tau) = q\rho(\tau) \quad \longleftrightarrow \quad S_Y(\omega) = q|H(\omega)|^2$$

Powe spectrum of the output of a linear system

Some widely used Fourier Transform pairs

Circuit System	$H(\omega)$	$h(t)$
	$\frac{1}{1 + j\omega RC}$	$\frac{1}{RC} e^{-t/RC} U(t)$
	$\frac{j\omega RC}{1 + j\omega RC}$	$\delta(t) - \frac{1}{RC} e^{-t/RC} U(t)$
	$\frac{R}{R + j\omega L}$	$\frac{R}{L} e^{-Rt/L} U(t)$
	$\frac{j\omega L}{R + j\omega L}$	$\delta(t) - \frac{R}{L} e^{-Rt/L} U(t)$

The average power – Frequency domain

- As shown in page 3 of this ppt, the average power $E\{|\mathbf{y}(t)|^2\}$ of the output of a system is:

$$E\{|\mathbf{y}(t)|^2\} = R_Y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) |H(\omega)|^2 d\omega \geq 0$$

- This equation describes the filtering properties of a system when the input is a random process. It shows, for example, if $H(\omega) = 0$ for $|\omega| > \omega_0$ and $S_{xx}(\omega) = 0$ for $|\omega| < \omega_0$, then $E\{|\mathbf{y}(t)|^2\} = 0$.
- If $\mathbf{x}(t)$ is white noise with average power q (it means $R_X(\tau) = q\delta(\tau)$), and note that the Fourier Transform of $R_X(\tau) = q\delta(\tau)$ is $S_X(\omega) = q$, we get

$$E\{|\mathbf{y}(t)|^2\} = R_Y(0) = q \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega$$

- And $E = \int_{-\infty}^{\infty} |h(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega$ is the energy of $h(t)$. (时域和频域求积分都可以得到系统或信号的平均功率)

The average power – Time domain

We now give the average power $E\{|\mathbf{y}(t)|^2\}$ of the output of a system if its input is white noise.

- If the input to a linear system $h(t)$ is (complex) stationary white noise with autocorrelation $R_X(\tau) = q\delta(\tau)$, then:

$$E\{y^2(t)\} = R_Y(0) = R_X(0) * h(0) * h^*(0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_X(0 + \alpha - \beta) h(\alpha) h^*(\beta) d\alpha d\beta$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q\delta(\alpha - \beta) h(\alpha) h^*(\beta) d\alpha d\beta \quad (\text{for the } \delta(\alpha - \beta), \text{ now-zero only when } \alpha = \beta = t)$$

$$= \int_{-\infty}^{\infty} qh(t)h^*(t)dt = qE$$

where $E = \int_{-\infty}^{\infty} |h(t)|^2 dt$ is the energy of $h(t)$.

Linear Systems – output autocorrelation

• In exam, we consider only the Steady state analysis (稳态分析) of stochastic processes, in this case

If input is stationary, output is stationary, for Steady state analysis (稳态分析) only

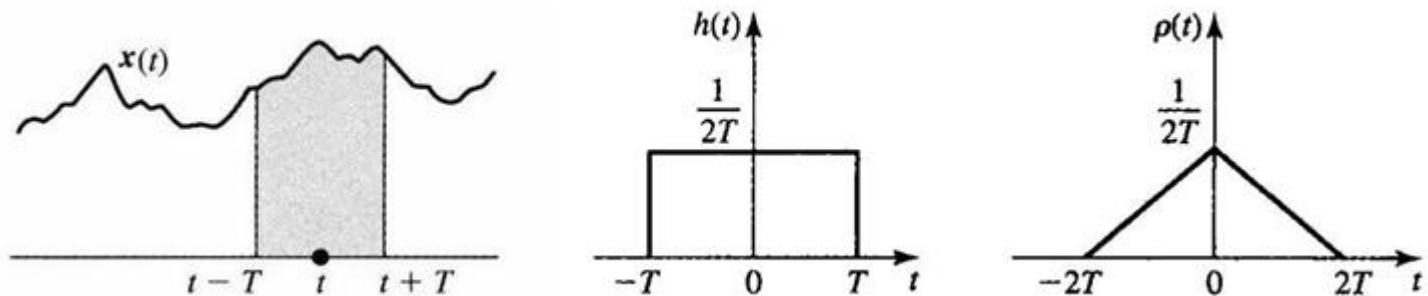
Properties when WSS ($\tau = t_1 - t_2$): (for complex process)

Power spectrum of the output of a linear system

- Quick example – Moving average: The integral $y(t) = \frac{1}{2T} \int_{t-T}^{t+T} x(\alpha) d\alpha$ is the average of the process $x(t)$ in the interval $(t-T, t+T)$. Clearly, $y(t)$ is the output of a system with input $x(t)$ and impulse response a rectangular pulse as in the below Fig. The corresponding $\rho(\tau)$ is a triangle. In this case,

$$H(\omega) = \frac{1}{2T} \int_{-T}^T 1 \times e^{-j\omega\tau} d\tau = \frac{\sin T\omega}{T\omega} \quad S_Y(\omega) = S_X(\omega) \frac{\sin^2 T\omega}{T^2\omega^2}$$

- Thus $H(\omega)$ takes significant values only in the interval $[-T, T]$ centered at the origin. Hence the moving average suppresses the high-frequency components of the input
- It is a simple low-pass filter.



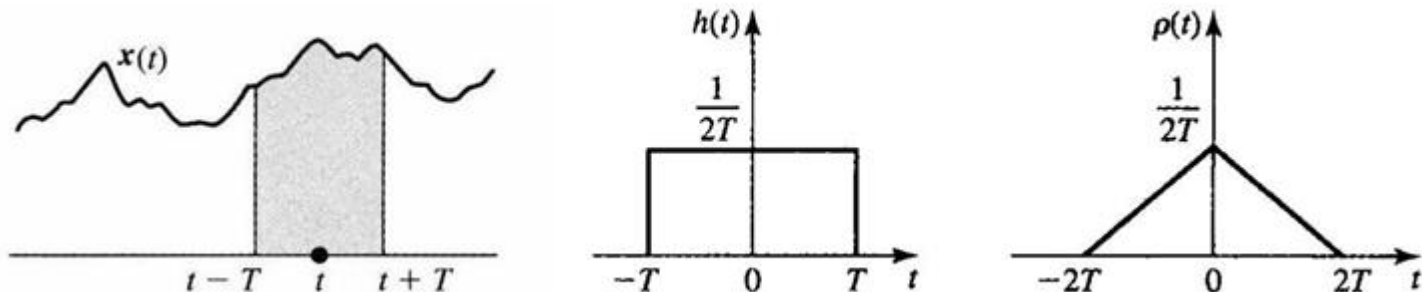
Powe spectrum of the output of a linear system

- Since $\rho(\tau) = h(\tau) * h^*(-\tau)$ is a triangle, it follows from $R_Y(\tau) = R_X(\tau) * h(\tau) * h^*(-\tau) = R_X(\tau) * \rho(\tau)$ that

$$R_Y(\tau) = \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|\alpha|}{2T}\right) R_X(\tau - \alpha) d\alpha$$

- Note that we can define the **time average** of the process $\mathbf{x}(t)$ as:

$$\mathbf{m}_T = \frac{1}{2T} \int_{-T}^T \mathbf{x}(t) dt$$



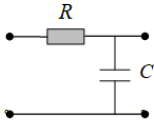
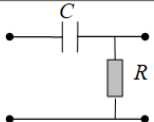
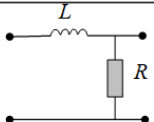
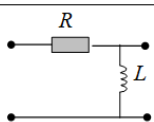
Powe spectrum of the output of a linear system

- Example 1: A **stationary process** $\mathbf{x}(t)$ with autocorrelation $R_X(\tau) = \sigma_X^2 e^{-\beta|\tau|}$ is applied to a linear system with

$$h(t) = \begin{cases} \alpha e^{-\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Please calculate the autocorrelation of the output, or says, calculate $R_Y(\tau)$ where $\tau = t_1 - t_2$.

(This is the example 2 from last lesson, this time, we use spectrum domain kind of method to solve this problem)

Circuit System	$H(\omega)$	$h(t)$
	$\frac{1}{1 + j\omega RC}$	$\frac{1}{RC} e^{-t/RC} U(t)$
	$\frac{j\omega RC}{1 + j\omega RC}$	$\delta(t) - \frac{1}{RC} e^{-t/RC} U(t)$
	$\frac{R}{R + j\omega L}$	$\frac{R}{L} e^{-Rt/L} U(t)$
	$\frac{j\omega L}{R + j\omega L}$	$\delta(t) - \frac{R}{L} e^{-Rt/L} U(t)$

Powe spectrum of the output of a linear system

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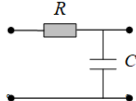
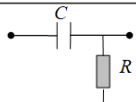
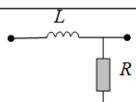
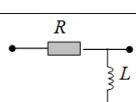
Solution:

- $h(t) = \begin{cases} \alpha e^{-\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$ is the RC circuit $h(t) = \alpha e^{-\alpha t} U(t)$, $\alpha = \frac{1}{RC}$

$$\rightarrow S_X(\omega) = 2\sigma_X^2 \frac{\beta}{\beta^2 + \omega^2}$$

$$H(\omega) = \frac{\alpha}{j\omega + \alpha}$$

$$\rightarrow S_Y(\omega) = S_X(\omega) |H(\omega)|^2 = 2\sigma_X^2 \frac{\beta}{\beta^2 + \omega^2} \frac{\alpha^2}{\omega^2 + \alpha^2}$$

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	$\frac{1}{1 + j\omega RC}$	$\frac{1}{RC} e^{-t/RC} U(t)$
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Powe spectrum of the output of a linear system

- Example 1: A **stationary process** $\mathbf{x}(t)$ with autocorrelation $R_X(\tau) = \sigma_X^2 e^{-\beta|\tau|}$ is applied to a linear system with

$$h(t) = \begin{cases} \alpha e^{-\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Please calculate the autocorrelation of the output, or says, calculate $R_Y(\tau)$ where $\tau = t_1 - t_2$.

Solution:

$$S_Y(\omega) = 2\sigma_X^2 \frac{\beta}{\beta^2 + \omega^2} \frac{\alpha^2}{\omega^2 + \alpha^2} = \frac{\alpha\sigma_X^2}{\alpha^2 - \beta^2} \left(\frac{2\alpha\beta}{\omega^2 + \beta^2} - \frac{2\alpha\beta}{\omega^2 + \alpha^2} \right)$$

Thus

$$R_Y(\tau) = \frac{\alpha\sigma_X^2}{\alpha^2 - \beta^2} (\alpha e^{-\beta|\tau|} - \beta e^{-\alpha|\tau|})$$

It is much easier than the convolutional (time domain) based method introduced last week!

Powe spectrum of the output of a linear system

- Example 2: For the two linear system $h_Y(t)$ and $h_Z(t)$ whose Transfer functions are $H_Y(\omega) = \frac{\alpha}{2\alpha + j\omega}$ and $H_Z(\omega) = \frac{\alpha + j\omega}{2\alpha + j\omega}$, and the power spectrum of the stationary input $X(t)$ is $S_X(\omega) = \frac{4\lambda}{4\lambda^2 + \omega^2}$, calculate $S_Y(\omega)$, $S_Z(\omega)$ and $S_{ZY}(\omega)$

Solution:

Power spectrum of the output of a linear system

- Example 2: For the two linear systems $h_Y(t)$ and $h_Z(t)$ whose transfer functions are $H_Y(\omega) = \frac{\alpha}{2\alpha + j\omega}$ and $H_Z(\omega) = \frac{\alpha + j\omega}{2\alpha + j\omega}$, and the power spectrum of the stationary input $X(t)$ is $S_X(\omega) = \frac{4\lambda}{4\lambda^2 + \omega^2}$, calculate $S_Y(\omega)$, $S_Z(\omega)$ and $S_{ZY}(\omega)$

Solution:

$$S_Y(\omega) = S_X(\omega) |H_Y(\omega)|^2 = \frac{4\lambda}{4\lambda^2 + \omega^2} \frac{\alpha^2}{4\alpha^2 + \omega^2}$$

$$S_Z(\omega) = S_X(\omega) |H_Z(\omega)|^2 = \frac{4\lambda}{4\lambda^2 + \omega^2} \frac{\alpha^2 + \omega^2}{4\alpha^2 + \omega^2}$$

Powe spectrum of the output of a linear system

- Example 2: For the two linear system $h_Y(t)$ and $h_Z(t)$ whose Transfer functions are $H_Y(\omega) = \frac{\alpha}{2\alpha + j\omega}$ and $H_Z(\omega) = \frac{\alpha + j\omega}{2\alpha + j\omega}$, and the power spectrum of the stationary input $X(t)$ is $S_X(\omega) = \frac{4\lambda}{4\lambda^2 + \omega^2}$, calculate $S_Y(\omega)$, $S_Z(\omega)$ and $S_{ZY}(\omega)$

Solution:

$$\begin{aligned} R_{ZY}(\tau) &= E[Z(t_1)Y(t_2)] = E[Z(t + \tau)Y(t)] \\ &= E\left[\int_{-\infty}^{+\infty} Z(t + \tau)X(t - u)h_Y(u)du\right] \\ &= \int_{-\infty}^{+\infty} E[Z(t + \tau)X(t - u)]h_Y(u) du \\ &= \int_{-\infty}^{+\infty} R_{ZX}(\tau + u)h_Y(u) du \end{aligned}$$

$$\begin{aligned} R_{ZY}(\tau) &= R_{ZX}(\tau) * h_Y(-\tau) \\ &= R_{XX}(\tau) * h_Z(\tau) * h_Y(-\tau) \end{aligned}$$

It means

Which is $R_{ZY}(\tau) = R_{XY}(\tau) * h_Z(\tau) = R_{XX}(\tau) * h_Y(-\tau) * h_Z(\tau)$

Thus

$$S_{ZY}(\omega) = S_X(\omega)H_Y^*(\omega)H_Z(\omega)$$

Fourier
Transform



$$S_{ZY}(\omega) = \frac{4\lambda}{4\lambda^2 + \omega^2} \frac{\alpha}{2\alpha - j\omega} \frac{\alpha + j\omega}{2\alpha + j\omega} = \frac{4\lambda}{4\lambda^2 + \omega^2} \frac{\alpha^2 + j\alpha\omega}{4\alpha + \omega^2}$$

Power spectrum of the output of a linear system

- Example 2: For the two linear systems $h_Y(t)$ and $h_Z(t)$ whose Transfer functions are $H_Y(\omega) = \frac{\alpha}{2\alpha + j\omega}$ and $H_Z(\omega) = \frac{\alpha + j\omega}{2\alpha + j\omega}$, and the power spectrum of the stationary input $X(t)$ is $S_X(\omega) = \frac{4\lambda}{4\lambda^2 + \omega^2}$, calculate $S_Y(\omega)$, $S_Z(\omega)$ and $S_{YZ}(\omega)$

Quick question:



$$S_{ZY}(\omega) = \frac{4\lambda}{4\lambda^2 + \omega^2} \frac{\alpha}{2\alpha - j\omega} \frac{\alpha + j\omega}{2\alpha + j\omega} = \frac{4\lambda}{4\lambda^2 + \omega^2} \frac{\alpha^2 + j\alpha\omega}{4\alpha + \omega^2}$$

Then $S_{YZ}(\omega) = ?$

Power spectrum of the output of a linear system

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Quick question:



$$S_{ZY}(\omega) = \frac{4\lambda}{4\lambda^2 + \omega^2} \frac{\alpha}{2\alpha - j\omega} \frac{\alpha + j\omega}{2\alpha + j\omega} = \frac{4\lambda}{4\lambda^2 + \omega^2} \frac{\alpha^2 + j\alpha\omega}{4\alpha + \omega^2}$$

Then $S_{YZ}(\omega) = ?$

Note that $S_{YZ}(\omega) = S_{ZY}^*(\omega)$, thus

$$S_{YZ}(\omega) = \frac{4\lambda}{4\lambda^2 + \omega^2} \frac{\alpha^2 - j\alpha\omega}{4\alpha + \omega^2}$$

Powe spectrum of the output of a linear system - Differentiators

- The derivative $\mathbf{x}'(t)$ of a process $\mathbf{x}(t)$ can be considered as the output of a linear system with input $\mathbf{x}(t)$ and system function $j\omega$. Which is:

$$R_{XX'}(\tau) = -\frac{dR_X(\tau)}{d\tau} \quad \longleftrightarrow \quad S_{XX'}(\omega) = -j\omega S_X(\omega)$$

$$R_{X'X}(\tau) = \frac{dR_X(\tau)}{d\tau} \quad \longleftrightarrow \quad S_{X'X}(\omega) = j\omega S_X(\omega)$$

$$R_{X'X'}(\tau) = -\frac{d^2 R_{XX}(\tau)}{d\tau^2} \quad \longleftrightarrow \quad S_{X'X'}(\omega) = \omega^2 S_X(\omega)$$

- The n -th derivative $\mathbf{y}(t) = \mathbf{x}^{(n)}(t)$ of $\mathbf{x}(t)$ is the output of a system with input $x(t)$ and system function $(j\omega)^n$, thus:

$$S_Y(\omega) = |j\omega|^{2n} S_X(\omega); \quad R_Y(\tau) = (-1)^n R_X^{(2n)}(\tau)$$

Powe spectrum of the output of a linear system - Differentiators

- Quick example – The differential equation

$$\mathbf{y}'(t) + c\mathbf{y}(t) = \mathbf{x}(t)$$

specifies a linear system with input $\mathbf{x}(t)$, output $\mathbf{y}(t)$, and system function $1/(j\omega + c)$.

This is because: $\mathbf{y}'(t) + c\mathbf{y}(t) = \mathbf{x}(t) \rightarrow j\omega Y(\omega) + cY(\omega) = X(\omega)$, then

$$H(\omega) = Y(\omega)/X(\omega) = 1/(j\omega + c)$$

We assume that $\mathbf{x}(t)$ is white noise with $R_X(\tau) = q\delta(\tau)$. Then we obtain

$$S_Y(\omega) = S_X(\omega) \frac{1}{\omega^2 + c^2} = \frac{q}{\omega^2 + c^2}; \quad R_Y(\tau) = \frac{q}{2c} e^{-c|\tau|}$$

- Note that $E\{\mathbf{y}^2(t)\} = R_{yy}(0) = q/2c$.
- Similarly, if $\mathbf{y}''(t) + b\mathbf{y}'(t) + c\mathbf{y}(t) = \mathbf{x}(t)$, and $S_X(\omega) = q$, then we have:

$$H(\omega) = \frac{1}{-\omega^2 + jb\omega + c} \quad S_Y(\omega) = \frac{q}{(c - \omega^2)^2 + b^2\omega^2}$$

$$R_{XX'}(\tau) = -\frac{dR_X(\tau)}{d\tau} \longleftrightarrow S_{XX'}(\omega) = -j\omega S_X(\omega)$$

$$R_{X'X}(\tau) = \frac{dR_X(\tau)}{d\tau} \longleftrightarrow S_{X'X}(\omega) = j\omega S_X(\omega)$$

$$R_{X'X'}(\tau) = -\frac{d^2 R_{XX}(\tau)}{d\tau^2} \longleftrightarrow S_{X'X'}(\omega) = \omega^2 S_X(\omega)$$

Powe spectrum of the output of a linear system - Differentiators

- Example 3: given a linear time-invariant system with input $x(t)$ and output $y(t)$ satisfying

$$\frac{d^2y(t)}{dt^2} - 4\frac{dy(t)}{dt} + 2y(t) = 3\frac{dx(t)}{dt} + x(t)$$

Find $H(\omega)$ and $|H(\omega)|^2$

Solution:

Powe spectrum of the output of a linear system - Differentiators

- Example 3: given a linear time-invariant system with input $x(t)$ and output $y(t)$ satisfying

$$\frac{d^2y(t)}{dt^2} - 4\frac{dy(t)}{dt} + 2y(t) = 3\frac{dx(t)}{dt} + x(t)$$

Find $H(\omega)$ and $|H(\omega)|^2$

Solution:

$$(j\omega)^2Y(\omega) - 4j\omega Y(\omega) + 2Y(\omega) = 3j\omega X(\omega) + X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{3j\omega + 1}{(j\omega)^2 - 4j\omega + 2} = \frac{1 + 3j\omega}{(2 - \omega^2) - 4j\omega}$$

$$|H(\omega)|^2 = \frac{1 + (3\omega)^2}{(2 - \omega^2)^2 + (4\omega)^2} = \frac{1 + 9\omega^2}{\omega^4 + 12\omega^2 + 4}$$

Hilbert transforms

- A system with system function

$$H(\omega) = -j\text{sgn}\omega = \begin{cases} -j & \omega > 0 \\ j & \omega < 0 \end{cases}, \text{ where } \text{sgn}\omega = \begin{cases} 1 & \omega > 0 \\ -1 & \omega < 0 \end{cases}$$

is called a **quadrature filter** (正交滤波器). The corresponding impulse response is $1/\pi t$.

- $H(\omega)$ is all-pass with -90° phase shift; hence its response to $\cos\omega t$ is $\cos(\omega t - 90^\circ) = \sin\omega t$ and its response to $\sin\omega t$ is $\sin(\omega t - 90^\circ) = -\cos\omega t$.
- The response of a quadrature filter to a **real process** $\mathbf{x}(t)$ is denoted by $\hat{\mathbf{x}}(t)$ and it is called the **Hilbert transform** of $\mathbf{x}(t)$. Thus

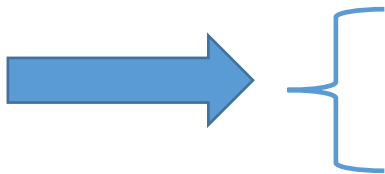
$$\hat{\mathbf{x}}(t) = \mathbf{x}(t) * \frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathbf{x}(\alpha)}{t - \alpha} d\alpha$$

where $*$ is the convolution operator

Hilbert transforms

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$$\hat{\mathbf{x}}(t) = \mathbf{x}(t) * \frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathbf{x}(\alpha)}{t - \alpha} d\alpha$$



$$S_{X\hat{X}}(\omega) = jS_{XX}(\omega)\text{sgn}\omega = -S_{\hat{X}X}(\omega)$$

$$S_{\hat{X}\hat{X}}(\omega) = S_{XX}(\omega)$$

Hilbert transforms

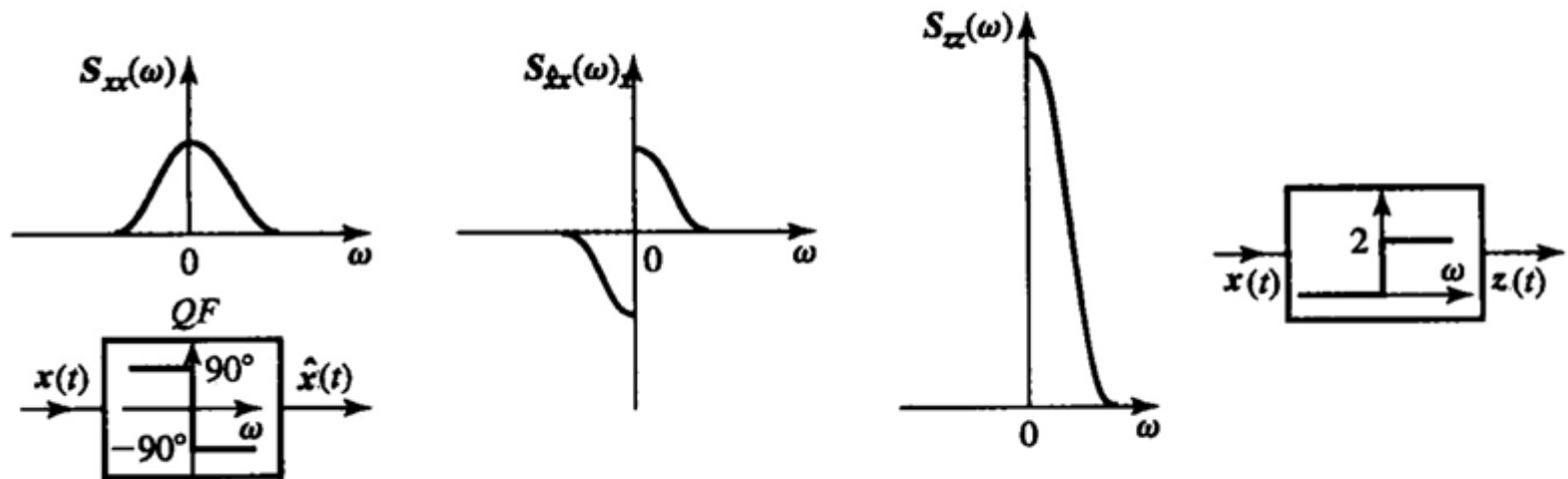
- The complex process $\mathbf{z}(t) = \mathbf{x}(t) + j\hat{\mathbf{x}}(t)$ is called the analytic signal associated with $\mathbf{x}(t)$. Clearly, $\mathbf{z}(t)$ is the response of the system

$$H(\omega) = 1 + j(-j\text{sgn } \omega) = 2U(\omega)$$

- with input $\mathbf{x}(t)$. we have:

$$S_{ZZ}(\omega) = 4S_{XX}(\omega)U(\omega) = 2S_{XX}(\omega) + 2jS_{\hat{X}X}(\omega)$$

$$R_{ZZ}(\tau) = 2R_{XX}(\tau) + 2jR_{\hat{X}X}(\tau)$$



More examples

1: find the mean and autocorrelation of $Y(t)$ if it is an Poisson impact process (泊松冲击过程) inputted to the Differentiator.

Hint: for Poisson impact process $X(t)$:

$$E[X(t)] = \lambda t$$

$$R_X(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$

2: given a process $X(t) = A \cos(\omega_0 t + \varphi)$ where A and ω_0 are constants, φ uniformly distributed in $(0, 2\pi)$. Input the $X(t)$ to an differentiator and get $X'(t)$, calculate the mean, autocorrelation and power spectrum of $X'(t)$.

- This week:

- Text book: 7.3 (part of 'power spectrum together with systems')
- Red book: 3.2.2, 3.2.3

- Next week:

- Text book: 7.4
- Red book: 3.3 (限带过程) 注：限带过程的讲解将以此书为主导

Experiment

- Go on with Experiment 2