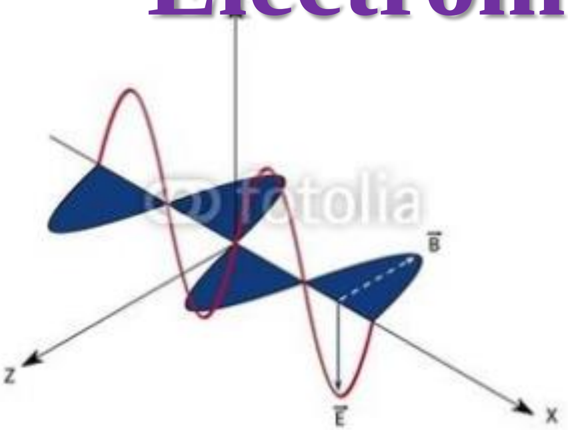
Electromagnetic Fields and Waves



Chapter 10

Electrostatic Field (5)

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Energy

- **Energy of charges system**
- **Energy of Electrostatic Field**
- **Electrostatic force**

Introduction

- Energy is the basic attribute of substance物质, electrostatic field naturally posses energy as a kind of substance.
- Generally, energy is divided into kinetic动能 energy and potential位能 energy. Electrostatic field is generated by static charge (static balance system), so there is no kinetic energy, only potential energy.
- Energy conservation principle is the rule to judge whether electrostatic field posses energy. As we all know, the charge will be affected by force in the electrostatic field. If no other forces are present, the charge moves. This is due to the work done做功 by electrostatic field of the charge. According to the energy conservation principle, we can conclude that the work done by the electrostatic field is converted from the energy of electrostatic field.

- Likewise, if put a charge in the electrostatic field, the work done by the external force will convert into energy of electrostatic field and stored in the system.
- The work done 做功 by E to the charge q is:

$$W=\int_C \vec{f}\cdot d\vec{l}=\int_C q\vec{E}\cdot d\vec{l}=q\int_C \vec{E}\cdot d\vec{l}=q\phi \qquad \phi=\int_C \vec{E}\cdot d\vec{l}$$

- Since the electrostatic field is conservative field, has nothing to do with the integral path. Thus, the work and the energy have nothing to do with the work path. This also indicates: **Electrostatic field energy is potential energy, it only relates to the charge distribution in the system, nevertheless, has nothing to do with the formation process of this distribution.**

Isolated charged body energy

- Supposing the quantity of electric charged body is Q and it is moved gradually from infinity. There is no electric field at first, so the external force does not have work done when moving the first charge dq. When the second dq is moving in, supposing the potential is Φ， the external force have work done: Φdq. When the electric charge quantity increases to the final value Q， the all work done by external forces is :

$$W_e = \int_0^Q \phi(q)\,\mathrm{d}q$$

- The potential of isolated charged body is : $\phi(q) = \dfrac{q}{C}$

- Thus, the work done by the external forces, namely the energy of electrostatic field is:

$$W_{\mathrm{e}} = \frac{1}{2}\phi Q = \frac{Q^2}{2C}$$

Energy of distributed charge

- Supposing in a linear and isotropic medium, a charged system with electric charge density ρ has set up , and the potential is Φ. Due to the electrostatic field energy is only related to the final charge distribution, while has nothing to do with charging process of ρ, so we can select a simple charging process:
  - Supposing the changes of ρ in the charging process :

$$\rho(x) = \rho x \qquad 0 \leq x \leq 1$$

when x=0， start charging；

when x=1， charging is over.

Energy of distributed charge

- Because the potential Φ is proportional to the charge distribution ρ, the potential changes in the process of charging is:

$$\phi(x) = \phi x \qquad 0 \le x \le 1$$

- So, in the entire charging process, the total energy of electric charge system is:

$$W_e = \int_V \int_0^\rho \phi(x) d\rho dv = \int_V \int_0^1 \rho\phi x dx dv = \frac{1}{2}\int_V \rho\phi dv$$

- If the charge distributed as surface charge ρ_s :

$$W_e = \frac{1}{2}\int_S \rho_s \phi ds$$

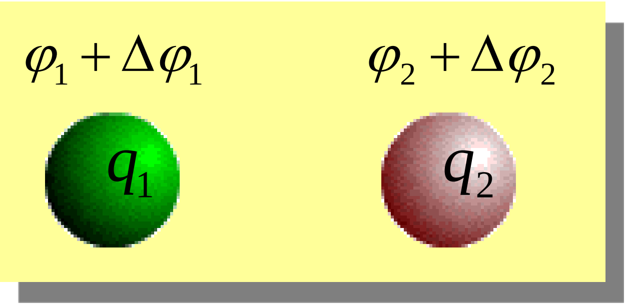
- If the charge distributed as linear charge ρ_l :

$$W_e = \frac{1}{2}\int_C \rho_l \phi dl$$

The Energy of N charge bodies

Set the charge quantity and the potential of each charged body are q_i and Φ_i in the system, and similarly, we can obtain the energy of N charged bodies system:

$$W_e = \frac{1}{2}\sum_{i}^{n} q_i \phi_i$$



Considering a system with two charges:

$$W_e = \frac{1}{2}\sum_{i=1}^{2} \phi_i q_i = \frac{1}{2}\left[q_1(\varphi_1 + \Delta\varphi_1) + q_2(\varphi_2 + \Delta\varphi_2)\right]$$

$$= \overset{\text{Self-energy}}{\frac{1}{2}(q_1\varphi_1 + q_2\varphi_2)} + \overset{\text{Mutual-energy}}{\frac{1}{2}(q_1\Delta\varphi_1 + q_2\Delta\varphi_2)}$$

The energy of a system is constitute with self-energy and mutual-energy. The self-energy is the work done by “compressing” many charges to q, and the mutual-energy is generated by the interaction between many charged bodies.

The Energy of The Point charge

- *N* charged bodies energy formula also can be used for *N* point charges system, namely:

$$W_e = \frac{1}{2}\sum_{i}^{n} q_i \phi_i$$

- But due to the potential generated by the point charge in its place is infinitive, so the energy term $q_i \Phi_{ii}$ is also infinitive.

- Therefore when the above formula is used in point charge system. The Φ_i is the potential, which generated by other charges at the position of charge i. In other words, the energy formula of point charge, only considering the mutual-energy.

The Energy of Electrostatic Field

- The energy formulas discussed above are about the energy of charge system , and it is electrostatic energy either. But the direct relationship with electrostatic field is not given.
- Substituting $\rho = \nabla \cdot \vec{D}$ into above energy formulas, and using:

$$\nabla \cdot \left(\phi \vec{D}\right) = \nabla \phi \cdot \vec{D} + \phi \nabla \cdot \vec{D}$$

$$\boxed{W_e = \frac{1}{2} \int_V \rho \phi dv}$$

- we can obtain:

$$W_e = \frac{1}{2} \int_V \left(\nabla \cdot \vec{D}\right) \phi dv = \frac{1}{2} \int_V \left(\nabla \cdot \left(\phi \vec{D}\right) - \nabla \phi \cdot \vec{D}\right) dv$$

$$= \frac{1}{2} \int_S \phi \vec{D} ds + \frac{1}{2} \int_V \vec{E} \cdot \vec{D} dv$$

- Setting:

$$w_e = \frac{1}{2} \vec{E} \cdot \vec{D} \, , \, \vec{D} \cdot \hat{n} = \rho_{se}$$

The Energy of Electrostatic Field

- Therefore: $$W_e = \frac{1}{2}\int_S \phi\rho_{se} ds + \int_V w_e dv$$
  - This equation indicates: w_e is the energy density of electric field , ρ_{se} represents the equivalent surface charge density
  - The electrostatic energy in the bounded region V is equal to the sum of the energy of electrostatic field and equivalent surface charge on the boundary
- If *V* is the whole region, *S* represents the sphere at infinity. Setting the potential at infinity is zero, so surface integral item of above equation is zero, then:

$$W_e = \int_V w_e dv$$

The energy does not meet the linear superposition principle

- For isotropic medium , $\vec{D} = \varepsilon \vec{E}$ so:

$$W_e = \frac{1}{2}\int_V \varepsilon E^2 dv$$

$$\boxed{w_e = \frac{1}{2}\vec{E}\cdot\vec{D}}$$

Discussion of two kinds of formulas

- So far, we have obtained two sorts of methods for calculating the energy of charge system.
- The energy formulas calculated by using charge potential seemingly indicate that the charge energy is concentrated in charges and the charge is the carrier of energy. The space would be no energy when no charges exist. This is the distance function point of view. This is the view point of action at a distance超距作用 in the early years.
- The energy formulas expressed by electric field told us: As long as the electric field is existed, the energy will exist, even though the region without charge. This is the point of view from field.

Discussion of two kinds of formulas

- On the topic of electrostatic field, the view point of action at a distance and the view point of field were hard to prove which one is correct. Later on, the electromagnetic wave was found when researching the time-varying electromagnetic field, and the view point of field was prevailing占优.
- The energy formulas calculated by using charge potential can only calculate the energy of the whole space, while The energy formulas expressed by electric field can calculate the energy of the local region. The two formulas are equivalent when considering the whole space.

Example 1

Calculating the energy of a conducting sphere with radius *a* and electric charge quantity *Q*. The dielectric constant of medium which surrounded conducting sphere is ε.

- Solution 1： Using the potential of charge. For a conducting sphere with radius *a* and electric quantity *Q*, its potential is :

$$\phi = \frac{Q}{4\pi\varepsilon a}$$

- Applied the energy formula using charge potential， we can obtain:

$$W_{\mathrm{e}} = \frac{1}{2}\phi Q = \frac{Q^2}{8\pi\varepsilon\, a}$$

- Solution 2： Using the energy formula with electric field. The electric field intensity of a conducting sphere with electric charge quantity *Q* is:

$$E = \frac{Q}{4\pi\varepsilon r^2}$$

- The energy density is :

$$w_e = \frac{1}{2}ED = \frac{Q^2}{32\pi^2\varepsilon r^4}$$

- Then calculating the volume integral of whole place outside of the ball:

$$W_e = \int_V w_e dv = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_a^{\infty} w_e r^2 \sin\theta \, dr$$

$$= \frac{Q^2}{8\pi\varepsilon\, a}$$

$$\boxed{dV = R^2 \sin\theta dR d\theta d\varphi}$$

Electrostatic Force

- The Coulomb's law has told us that the force of electrostatic field to the charge is: $\vec{F} = q\vec{E}$
- Sometimes energy conversion can be used to simplify the calculation of force--virtual displacement
  - When arbitrary one charged body in the charged body system has a small displacement, there will be a relationship between the work done and the energy.
  - Using the transformation between work done and energy to calculate the force on the charged body.
- Setting a system with (n+1) charged bodies, only Red charged body has a displacement dg, f_g is electric field force, then the charge of the potential of a charged body in the system will change, and the relationship between the work and the energy is :

$$dW = dW_e + f_g dg = \sum \phi_k dq_k$$

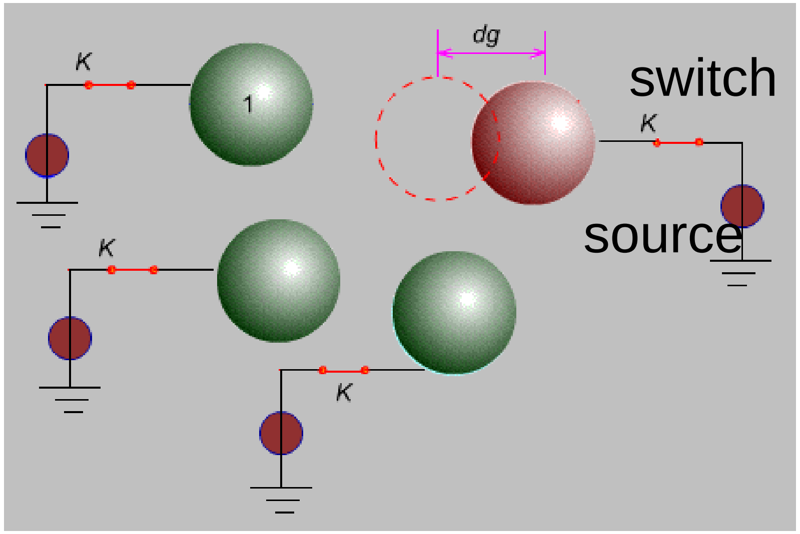
Energy from external source = Increase of electrostatic energy + Work done of E-field

Constant Charge System

❖ The system is not connected to external power (K is open), when a virtual displacement occurs, the charge q_k on the charged body stays the same. Therefore, the external power fails to provide energy to the system:

$$dW = dW_e + f_g dg$$

$$= \sum \phi_k dq_k = 0$$



$$f = -\frac{\partial W_e}{\partial g}\bigg|_{q_k = const.}$$

- When the external power is not connect, the electric field force work done will lead to reduce the electrostatic field energy

Constant Potential System

- The system connected to external power (K is closed), when a virtual displacement occurs, the capacitance of the system will change. In order to keep potential ϕ_k on the charged body remains the same, the external source must supply extra charge to the system. Thus the system gain energy.

$$\boxed{dW = dW_e + f_g dg}$$

$$dW = \sum \phi_k dq_k \qquad dW = 2dW_e \qquad dW_e = d\left(\frac{1}{2}\sum \phi_k q_k\right) = \frac{1}{2}\sum \phi_k dq_k$$

$$\sum \phi_k dq_k = \frac{1}{2}\sum \phi_k dq_k + fdg$$

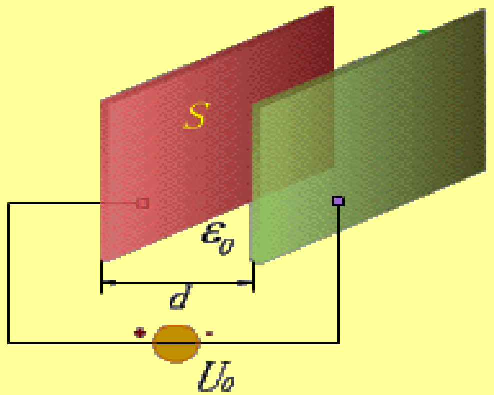
$$\Downarrow$$

$$2dW_e = dW_e + fdg \Longrightarrow f = +\frac{\partial W_e}{\partial g}\bigg|_{\phi_k = const.}$$

- Half of the energy which provided by external energy is used to increment of the electrostatic field energy, the other half is the work done by the electric field force.

Example 2

Calculating the force between two conducting plates of charged plate capacitor .



Parallel plate capacitor

- Solution 1: Constant potential system:

$$W_e = \frac{1}{2} C U^2 , C = \frac{\varepsilon_0 S}{d}$$

$$f = + \frac{\partial W_e}{\partial g} \Big|_{\phi_k = const} = \frac{U^2}{2} \cdot \frac{\partial C}{\partial d} = - \frac{U^2 \varepsilon_0 S}{2d^2} < 0$$

- Solution 2: Constant charge system:

$$W_e = \frac{1}{2} \cdot \frac{q \bullet q}{C}$$

$$f = - \frac{\partial W_e}{\partial g} \Big|_{q_k = const} = - \frac{q^2}{2C^2} \cdot \frac{\partial C}{\partial d} = - \frac{U^2}{2} \cdot \frac{\varepsilon_0 S}{d^2} < 0$$

Example 3

Calculating the force per unit area applied on the surface of a charged soap bubble

- Solution： setting the soap bubble with electric charge quantity q and radius r， thus the potential of the charged soap bubble is :

$$\phi = \frac{q}{4\pi\varepsilon_0 r}$$

- The energy carried by the soap bubble is :

$$W_e = \frac{1}{2}\phi q = \frac{q^2}{8\pi\varepsilon_0 r}$$

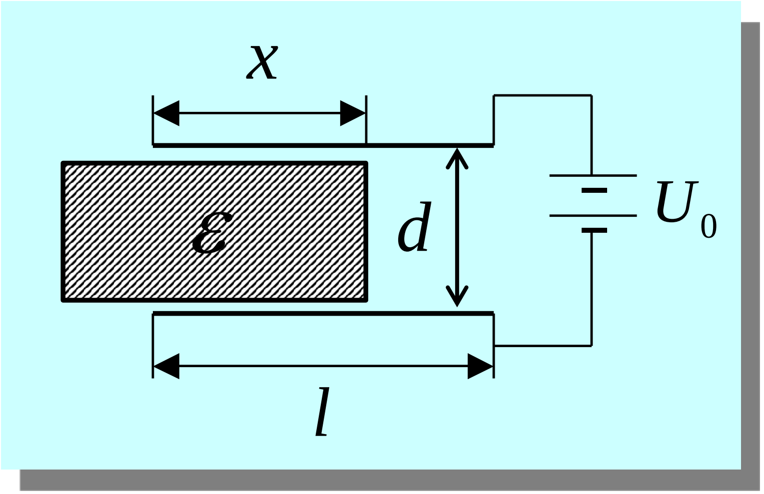
- Using constant charge system, we can obtain:

$$F = -\frac{\partial W_e}{\partial r}\bigg|_{q=const} = \frac{q^2}{8\pi\ \varepsilon_0 r^2}(\mathrm{N/m^2})$$

Example 4

Calculating the force apply on the dielectric block in the parallel plate capacitor along the *x* direction.

- Solution 1： constant potential system. Ignoring fringe effect, in the capacitor:



$$E = \frac{U_0}{d} \qquad W_e = \frac{1}{2} ED$$

- The energy is :

$$W_e = \frac{1}{2}\int_V \varepsilon E^2 dV = \frac{1}{2}\varepsilon\left(\frac{U_0}{d}\right)^2 xwd + \frac{1}{2}\varepsilon_0\left(\frac{U_0}{d}\right)^2 (l-x)wd$$

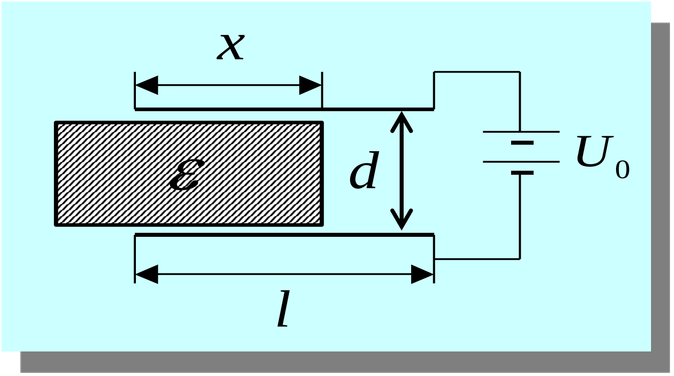
- Thus:

$$F_x = \frac{\partial W_e}{\partial x}\Big|_{\phi=const} = \frac{1}{2}(\varepsilon-\varepsilon_0)\left(\frac{U_0}{d}\right)^2 wd = \frac{1}{2}(\varepsilon-\varepsilon_0)E^2 wd$$

❖ Solution 2： constant charge system. First calculating the electric field expressed by q:

$$E_1 = E_2 = E, \quad \rho_{s1} = D_1 = \varepsilon E, \quad \rho_{s2} = D_2 = \varepsilon_0 E$$

$$\rho_{s1} xw + \rho_{s2} \left(l - x\right) w = q$$



$$E = \frac{q}{\varepsilon xw + \varepsilon_0 \left(l - x\right) w}$$

❖ The energy is :

$$W_e = \frac{1}{2} \varepsilon E_1^2 xwd + \frac{1}{2} \varepsilon_0 E_2^2 \left(l - x\right) wd = \frac{q^2 d}{2\left[\varepsilon xw + \varepsilon_0 \left(l - x\right) w\right]}$$

❖ Thus:

$$f_x = -\frac{\partial W_e}{\partial x}\Big|_{q=const} = \frac{q^2 dw\left(\varepsilon - \varepsilon_0\right)}{2\left[\varepsilon xw + \varepsilon_0 \left(l - x\right) w\right]^2} = \frac{1}{2} E^2 \left(\varepsilon - \varepsilon_0\right) wd$$

Conclusion

- The energy of charge system
- The energy of electrostatic field
- Electrostatic force
- Virtual displacement

Assignment9

A metal sphere of radius 10cm has surface charge density of $10nC/m^2$. Calculate the electric energy stored in the system. (71.06nJ)