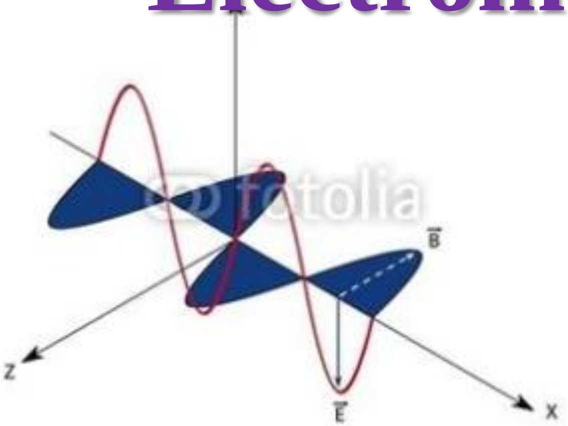
Electromagnetic Fields and Waves



Chapter 6
Electrostatic Field 1
-Electric Fields

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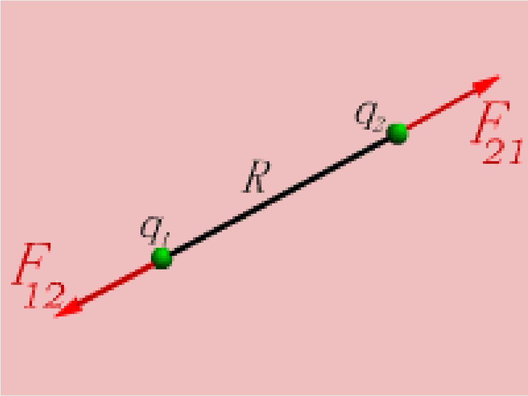
Coulomb’s Law

Coulomb’s law库伦定律 is an experimental law formulated in 1785 by the Charles Augustin de Coulomb. It deals with a point charge exerts on another point charge.



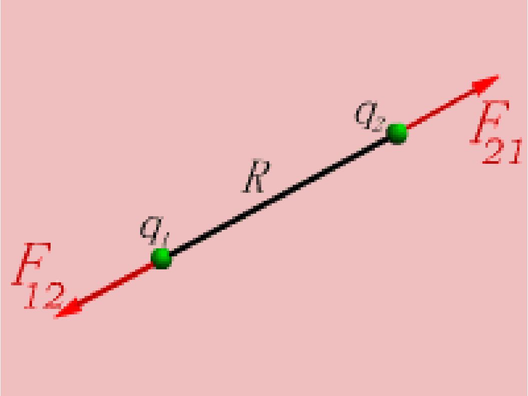
Coulomb torsion balance experiment

Coulomb's Law

The force F between two point charges q_1 and q_2 is:

1. Along the line joining them.
2. Directly proportional to the product $q_1 q_2$ of the charges.
3. Inversely proportional to the square of the distance R between them.

Expressed mathematically: $$F = \frac{k q_1 q_2}{R^2} \qquad (1)$$



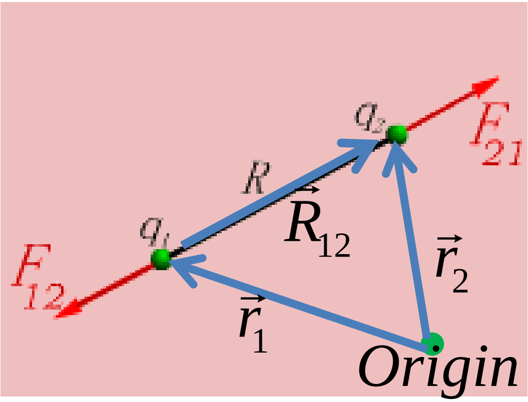
SI units: q_1 and q_2 are in Coulombs (C), R is in meters (m), and Force F is in Newtons (N), so that $k=1/(4\pi\varepsilon_0)$. The constant ε_0 is known as permittivity of free space and has the value

$$\varepsilon_0 = 8.854 \times 10^{-12} F/m$$

$$k = 9 \times 10^{9} m/F$$

Coulomb's Law

Thus, eq.(1) becomes: $$F = \frac{q_1 q_2}{4\pi\varepsilon_0 R^2} \qquad (2)$$

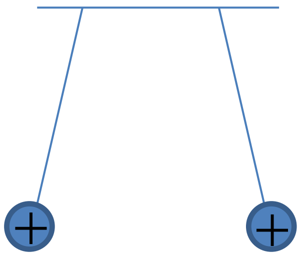


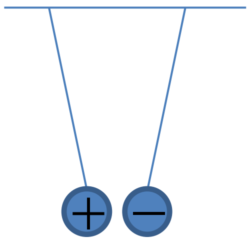
The charges are located at points have position vectors r_1 and r_2

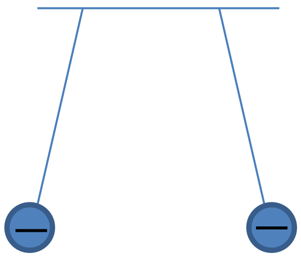
$$\vec{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0 R^2}\,\hat{a}_{R_{12}} \qquad (3)$$

$$\vec{R}_{12} = \vec{r}_2 - \vec{r}_1$$

$$\hat{a}_{R_{12}} = \frac{\vec{R}_{12}}{\left|\vec{R}_{12}\right|}$$



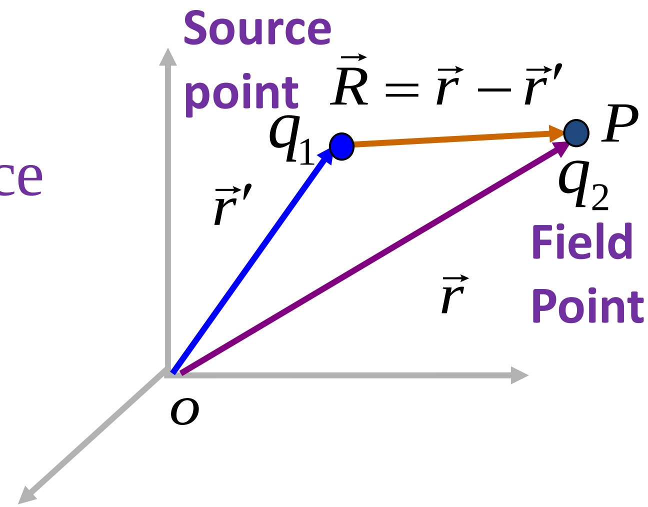




Coulomb's Law

Assume the electric strength of charge q_1 is $\vec{E}$

Another charge in electric field $\vec{E}$ has the force



$$\vec{F} = \vec{E} q_2$$

Compare with Coulomb's Law, we would know the force generated by the electric charge in free space is:

$$\vec{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0 R^2} \hat{a}_{R_{12}}$$

$$\vec{E} = \frac{q_1}{4\pi\varepsilon_0 R^2} \hat{a}_R = \frac{q_1}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}'\right|^2} \hat{a}_R = \frac{q_1(\vec{r} - \vec{r}')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}'\right|^3}$$

Note: $\vec{r}$ is the field point, $\vec{r}'$ is the source point.

Coulomb's Law

The relation between the electric field and charge is linear. Thus, Multiple charge satisfy the superposition叠加 principle原理.

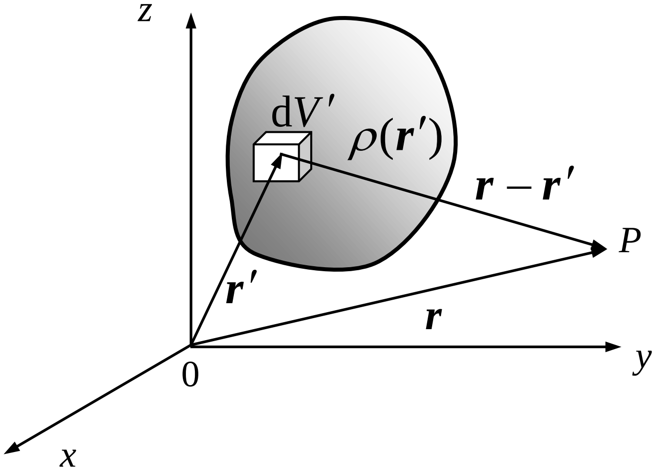
$$\vec{E} = \sum_{i=1}^{n} \frac{q_i(\vec{r} - \vec{r}_i')}{4\pi\varepsilon_0 \left| \vec{r} - \vec{r}_i' \right|^3}$$

Coulomb's law is only valid for point charge. Point charge is an ideal model of charge carrying object with zero volume. When the size of the object is much smaller than the distance, the object can be assume to be a point. Point charge model can be applied.

Coulomb's Law

The electric field of differential charge is:

$$d\vec{E} = \frac{(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}'\right|^3} dq(\vec{r}')$$



The total electric field of distributed charge:

1. Volume distributed electric charge: $\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho(\vec{r}')(\vec{r}-\vec{r}')}{\left|\vec{r}-\vec{r}'\right|^3} dV'$

2. Area distributed electric charge: $\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{S'} \frac{\rho_s(\vec{r}')(\vec{r}-\vec{r}')}{\left|\vec{r}-\vec{r}'\right|^3} dS'$

3. Line distributed electric charge: $\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{l'} \frac{\rho_l(\vec{r}')(\vec{r}-\vec{r}')}{\left|\vec{r}-\vec{r}'\right|^3} dl'$

Example 1

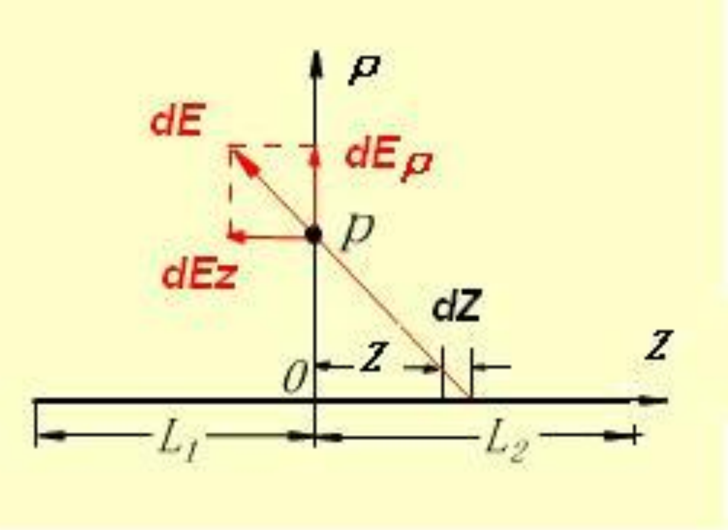
A line conductor of length L is uniformly distributed with charge density of ρ_l, try to find the electric field at point P.

Ans: Use the cylindrical coordinate system, locating the ρ-axis passing through the point *P* and the conductor overlaps with z-axis.

$$dE_\rho = \frac{\rho_l \rho dz'}{4\pi\varepsilon_o(\rho^2 + z'^2)^{3/2}}$$

$$dE_z = -\frac{\rho_l z' dz'}{4\pi\varepsilon_o(\rho^2 + z'^2)^{3/2}}$$

$$d\vec{E} = d\vec{E}_\rho + d\vec{E}_z = \frac{\rho_l(\rho\hat{a}_\rho - z'\hat{a}_z)dz'}{4\pi\varepsilon_o(\rho^2 + z'^2)^{3/2}}$$



In EM calculation, choose the right coordinate system is very importance to simplified the calculation

Example 1

A line conductor of length L is uniformly distributed with charge density of ρ_l, try to find the electric field at point P.

$$E_\rho = \frac{\rho_1 \rho}{4\pi\varepsilon_0}\int_{-L_1}^{L_2}\frac{dz'}{\left(\rho^2 + z'^2\right)^{3/2}}$$

$$\underline{\underline{z' = \rho\tan(t)}} \qquad \frac{\rho_1 \rho}{4\pi\varepsilon_0}\int_{arctan\frac{-L_1}{\rho}}^{arctan\frac{L_2}{\rho}}\frac{d\rho\tan(t)}{\left(\rho^2 + \rho^2 tan^2 t\right)^{3/2}}$$

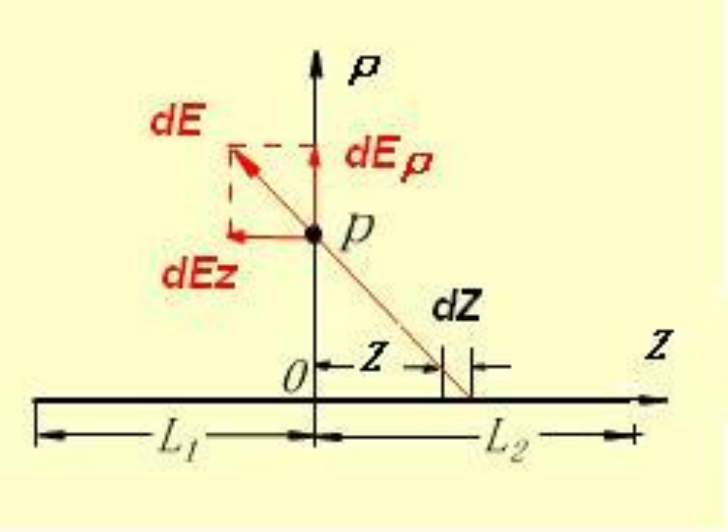
$$\underline{\underline{sec^2 t = 1 + tan^2}} \qquad \frac{\rho_1 \rho}{4\pi\varepsilon_0}\int_{arctan\frac{-L_1}{\rho}}^{arctan\frac{L_2}{\rho}}\frac{dtan(t)}{\rho^2 sec^3 t}$$

$$\underline{\underline{d\tan(t) = sec^2 tdt}} \qquad \frac{\rho_1 \rho}{4\pi\varepsilon_0}\int_{arctan\frac{-L_1}{\rho}}^{arctan\frac{L_2}{\rho}}\frac{dt}{\rho^2 sect}$$

$$= \frac{\rho_1 \rho}{4\pi\varepsilon_0}\int_{arctan\frac{-L_1}{\rho}}^{arctan\frac{L_2}{\rho}}\frac{\cos(t)dt}{\rho^2}$$

$$= \frac{\rho_1}{4\pi\varepsilon_0\rho}\left(\sin\left(arctan\frac{L_2}{\rho}\right) + \sin\left(arctan\frac{L_1}{\rho}\right)\right)$$

$$= \frac{\rho_1}{4\pi\varepsilon_0\rho}\left(\frac{L_2}{(\rho^2 + L_2{}^2)^{1/2}} + \frac{L_1}{(\rho^2 + L_1{}^2)^{1/2}}\right)$$

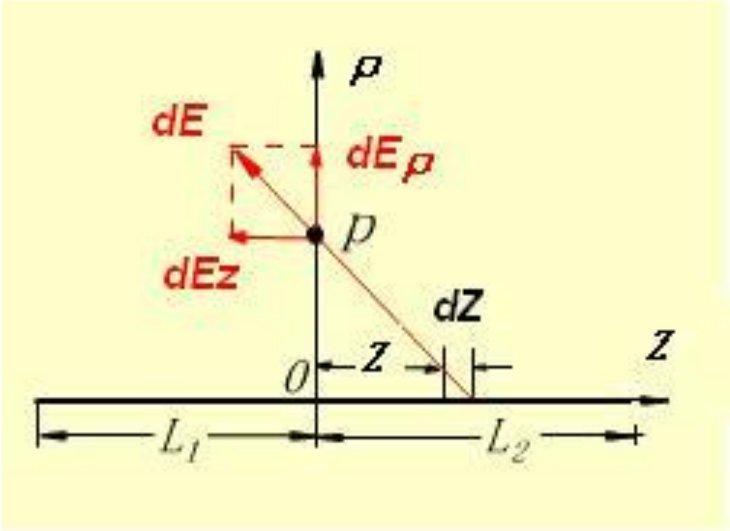


Example 1

A line conductor of length L is uniformly distributed with charge density of ρ_l, try to find the electric field at point P.

$$E_\rho = \frac{\rho_l \rho}{4\pi\varepsilon_o}\int_{-L_1}^{L_2}\frac{dz'}{(\rho^2+z'^2)^{3/2}}$$

$$= \frac{\rho_l}{4\pi\varepsilon_o\rho}\left[\frac{L_2}{(\rho^2+L_2^2)^{1/2}}+\frac{L_1}{(\rho^2+L_1^2)^{1/2}}\right]$$



$$E_z = -\frac{\rho_l}{4\pi\varepsilon_o}\int_{-L_1}^{L_2}\frac{z'dz'}{(\rho^2+z'^2)^{3/2}}$$

$$= \frac{\rho_l}{4\pi\varepsilon_o}\left[\frac{1}{(\rho^2+L_2^2)^{1/2}}-\frac{1}{(\rho^2+L_1^2)^{1/2}}\right]$$

$$\int x^a dx = \frac{x^{a+1}}{a+1}+c$$

If the conductor is infinitely long, then: $\vec{E} = E_\rho \hat{a}_\rho = \frac{\rho_l}{2\pi\varepsilon_o\rho}\hat{a}_\rho$

Gauss's Law高斯定理

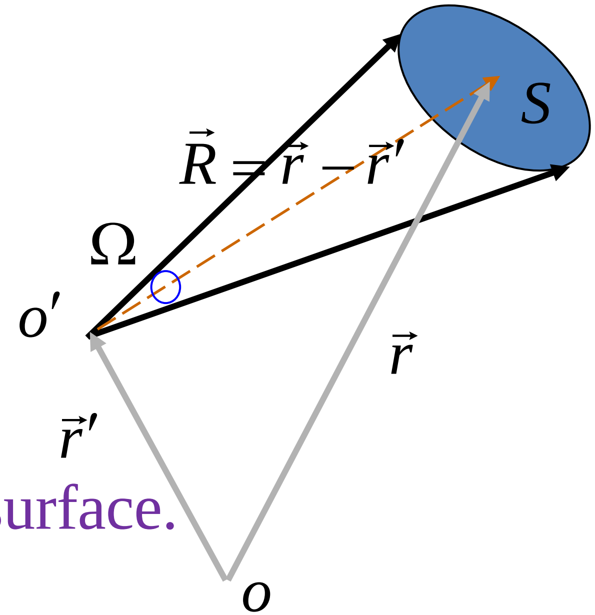
- Gauss's Law states that the total electric flux through any closed surface is equal to the total charge enclosed by that surface.

1. Gauss's law is an alternative statement of Coulomb's law, proper application of the divergence theorem to Coulomb's law results in Gauss's Law.

$$\oint_S \vec{E} \bullet d\vec{S} = \frac{Q}{\varepsilon_0}$$

2. Gauss's Law provides an easy means of finding $\vec{E}$ or $\vec{D}$ for symmetrical charge distributions such as a point charge, an infinite无限 line charge, an infinite cylindrical surface charge.

Solid Angle立体角



The angle of cone, its size is defined as the ratio of S to the R^2

$$\Omega = \frac{S}{R^2}$$

Obviously, the solid angle of a sphere is **4π.**

Solid angle can also calculate in arbitrary curve surface.

Differential area element dS to point o', its solid angle is:

$$d\Omega = \frac{\hat{a}_R \bullet d\vec{S}}{R^2} = \frac{(\vec{r} - \vec{r}') \bullet d\vec{S}}{|\vec{r} - \vec{r}'|^3}$$

The solid angle of whole curve surface S to point o' is

$$\Omega = \int_S \frac{(\vec{r} - \vec{r}') \bullet d\vec{S}}{|\vec{r} - \vec{r}'|^3}$$

Gauss's Law

If S is the closed curve, then

$$\Omega = \oint_S \frac{(\vec{r}-\vec{r}')\cdot d\vec{S}}{|\vec{r}-\vec{r}'|^3} = \begin{cases} 4\pi & \vec{r}' \text{ inside } S \\ 0 & \vec{r}' \text{ outside } S \end{cases}$$

The prove is leave out.

Gauss's Law: $\oint_S \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0}$

Q is the net charge enclosed by the area S

C.F Gauss

Mathematician

（1777-1855）

Gauss's Law

Gauss's Law: $\oint_S \vec{E}\bullet d\vec{S} = \frac{Q}{\varepsilon_0}$

Prove: according to the formula of electric field

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}dq'$$

Since dS and dq' can be interchanged in the integral:

$$\oint_S \vec{E}\bullet d\vec{S} = \frac{1}{4\pi\varepsilon_0}\int\left[\oint_S \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}dS\right]dq'$$

Applied the formula: $\Omega = \oint_S \frac{(\vec{r}-\vec{r}\,')\bullet d\vec{S}}{|\vec{r}-\vec{r}\,'|^3} = \begin{cases} 4\pi \\ 0 \end{cases}$

We have: $\oint_S \vec{E}\bullet d\vec{S} = \frac{1}{\varepsilon_0}\begin{cases} \int dq' & \vec{r}\,' \text{ inside } S \\ 0 & \vec{r}\,' \text{ outside } S \end{cases} = \frac{Q}{\varepsilon_0}$ Done!

Gauss's Law

Electric flux density: $\vec{D} = \varepsilon_0 \vec{E}$ also called electric displacement vector电位移矢量

The Gauss's Law can be expressed as: $\oint_S \vec{E} \bullet d\vec{S} = \dfrac{Q}{\varepsilon_0}$ ⇨ $\oint_S \vec{D} \bullet d\vec{S} = Q$

This is the integral form of Gauss's Law.

According to the definition of divergence, we have:

$$\nabla \bullet \vec{E} = \lim_{\Delta V \to 0} \frac{\oint_S \vec{E} \bullet d\vec{S}}{\Delta V} = \frac{1}{\varepsilon_0} \lim_{\Delta V \to 0} \frac{\int dq'}{\Delta V}$$

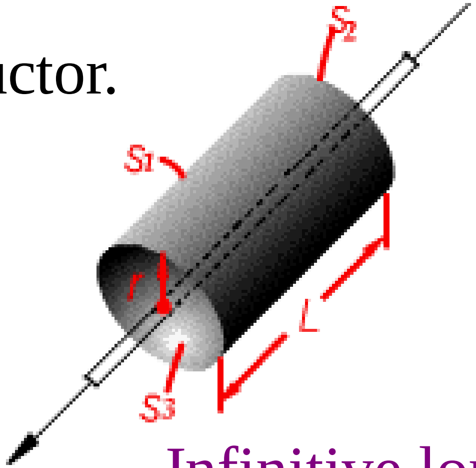
For volume distributed charge, electric charge density is ρ, we have

$$\nabla \bullet \vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{or} \quad \nabla \bullet \vec{D} = \rho$$

This is the differential form of Gauss's Law.

Example 2

An infinitive long line conductor has line charge density of ρ_l. Try to find the electric field in the space near the conductor.



Infinitive long conductor

Ans: $\vec{D}$ is perpendicular to the conductor and in the shape of radiating. When the radius r values are the same, the magnitudes of $\vec{D}$ is the same.

Define an enclosed surface with length of L and radius r be the Gaussian surface.

$$\oint_S \vec{D}\cdot d\vec{S} = \oint_{S_1} \vec{D}_1 \cdot d\vec{S}_1 + \oint_{S_2} \underbrace{\vec{D}_2 \cdot d\vec{S}_2}_{0} + \oint_{S_3} \underbrace{\vec{D}_3 \cdot d\vec{S}_3}_{0} = \rho_l L$$

$$D_1 2\pi r L = \rho_l L$$

$$\vec{D}_1 = \frac{\rho_l}{2\pi r}\vec{a}_r \qquad \vec{E}_1 = \frac{\vec{D}_1}{\varepsilon_0} = \frac{\rho_l}{2\pi\varepsilon_0 r}\vec{a}_r$$

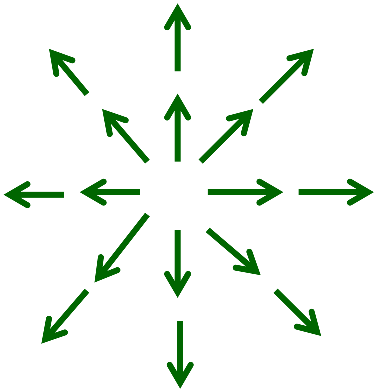
Curl of Electric Charge

We know the expression of electric field of charge, we can calculate its curl. However, we can use some techniques to simplified the calculation.

We know: $$\nabla \frac{1}{|\vec{r}-\vec{r}'|} = -\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

Thus, the electric field can be expressed as

$$\vec{E} = -\frac{q}{4\pi\varepsilon_0} \nabla \frac{1}{|\vec{r}-\vec{r}'|}$$



We also know the equality $\nabla \times (\nabla u) = \vec{0}$

Thus, $$\nabla \times \vec{E} = \vec{0}$$

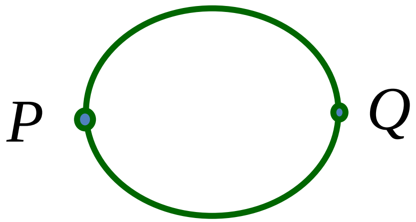
Electric field of charge is a field of zero curl

Curl of Electric Charge

In integral form

$$\oint_C \vec{E} \bullet d\vec{l} = \int_S \nabla \times \vec{E} \bullet d\vec{S} = 0$$

The integration between two arbitrary points of electric field in zero curl loop is independent of path.



Electric Potential电势

- The curl of an electrostatic field is zero:

$$\nabla \times \vec{E} = \vec{0}$$

- The electric field can be expressed using a scalar ϕ , Thus, we have:

$$\vec{E} = -\nabla \phi$$

- The ϕ is called electric potential. “-” sign means the gradient of the potential is opposite direction of E-field.

- Obviously, electric field $\vec{E}$ has three variable in three dimensional space, however electric potential ϕ has only scalar variables.
- Using electric potential to calculate the curl of electric field is much easier.

Electric Potential

From the electric equation, we have

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 R^2}\hat{a}_R = \frac{q}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}'\right|^2}\hat{a}_R = \frac{q(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}'\right|^3}$$

Using the equation $\vec{E} = -\nabla\phi$, we have

$$\phi = \frac{q}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}'\right|} + C(\vec{r})$$

Function $C(\vec{r})$ needs to satisfy the condition of $\nabla C = 0$

From $\nabla \times \vec{E} = 0$, the potential variable ϕ can not be uniquely determined. We need extra information, e.g. reference potential point

Electric Voltage电压

- Voltage is the line integral of two points in an electric field

$$\int_P^Q \vec{E}\bullet d\vec{l} = -\int_P^Q \nabla\phi\bullet d\vec{l} = -\int_P^Q \frac{\partial\phi}{\partial l}\,dl$$

$$= -\int_P^Q d\phi = \phi(P) - \phi(Q)$$

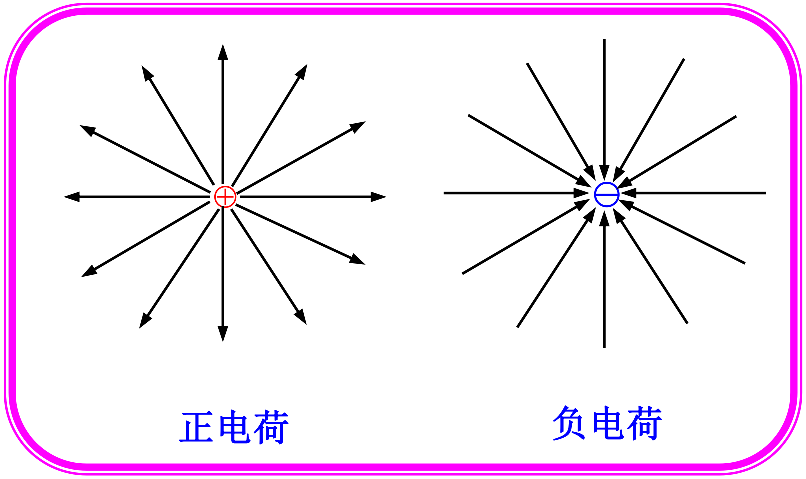
- Voltage between two points is the electric potential difference between two points, which is independence of the integral path.

- If zero potential (ground) is chosen as a reference point Q, then the voltage is equal to the potential of point P.

$$\int_P^Q \vec{E}\bullet d\vec{l} = \phi(P) - \phi(Q) = \phi(P)$$

Properties of electric lines

- The choice of reference point is not unique. In application, we choose infinitive large ground (zero potential) as the reference.
- Electric lines properties:
- Electric lines equation $\vec{E} \times d\vec{l} = \vec{0}$
- Start at +ve charge, ended at –ve charge.



- Never intersect with each other.
- Perpendicular to the equal potential surface.
- The density of the lines corresponding to the density of the electric field.

Electric Potential

- Electric potential in a volume.

$$\phi = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(r')}{R}\,dV'$$

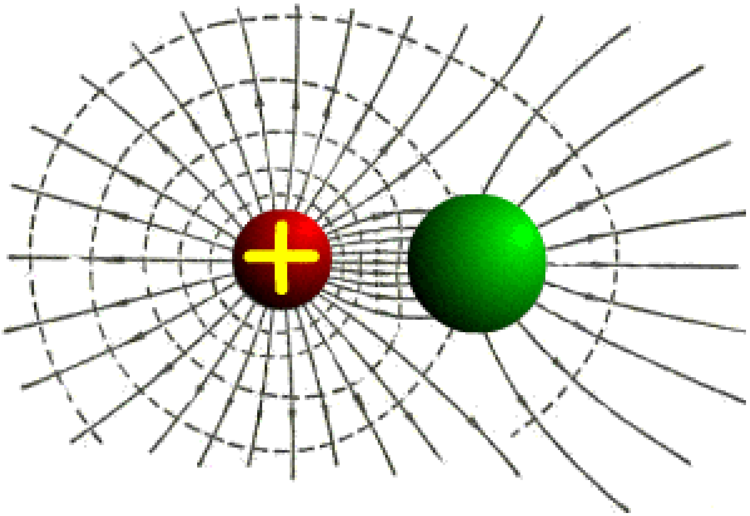
- Electric potential in an area.

$$\phi = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_S(r')}{R}\,dS'$$

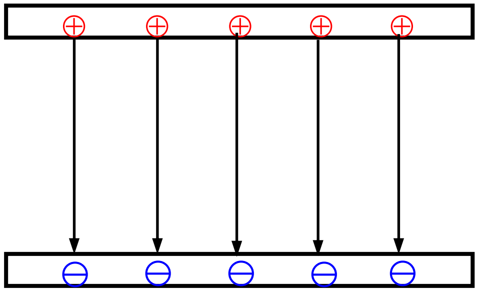
- Electric potential in a line.

$$\phi = \frac{1}{4\pi\varepsilon_0}\int_l \frac{\rho_l(r')}{R}\,dl'$$

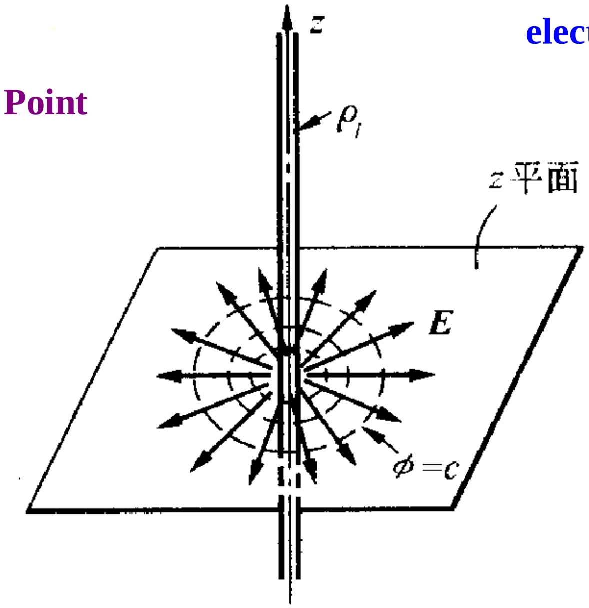
Electric field distribution



The electric field distribution of the Point charge and conductor without ground



Parallel plate carrying electric charge

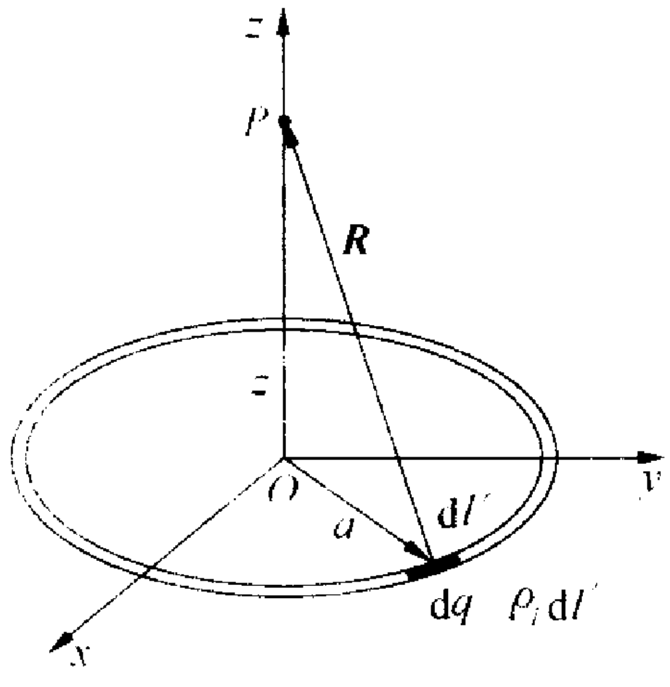


无限长直线电荷电场

Electric field of a infinitive long charge carrying conductor rod

Example 3

A circular-shape line electric charge with the radius *a* locates in the xoy-plane in the vacuum, and the line charge density is ρ_l , please calculate the electric potential and the intensity of the electric field at the point *P(0,0,z)*.



带均匀线电荷的圆环

- Select a line element dl' in the ring, and the electric charge in the line element is:

$$dq = \rho_l dl'$$

Example 3

- Distance between source point and field point:

$$R = \sqrt{z^2 + a^2}$$

- Using cylindrical coordinates, the electric potential at the point P is obtained by :

$$\phi = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi a}\frac{\rho_l}{R}dl' = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi a}\frac{\rho_l}{\sqrt{z^2+a^2}}dl' = \frac{a\rho_l}{2\varepsilon_0\sqrt{z^2+a^2}}$$

- Using the property that the symmetrical distribution of the electric field, the electric field at the point P only has the z-axis component, that is:

$$\vec{E} = \hat{a}_z E_z = -\nabla\phi = -\hat{a}_z\frac{\partial\phi}{\partial z} = \hat{a}_z\frac{a\rho_l z}{2\varepsilon_0\left(z^2+a^2\right)^{3/2}}$$

Example 4

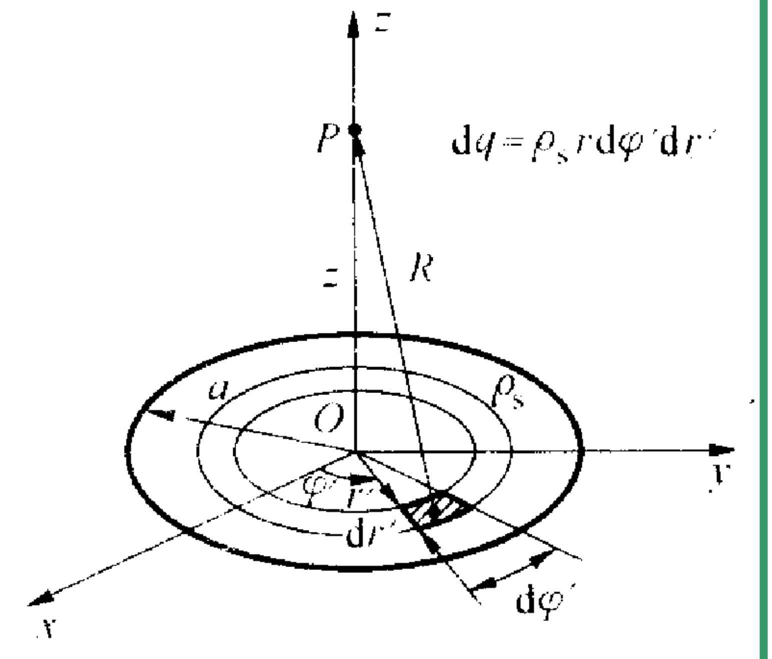
A circular-shape charged disc with the radius a ,and the electric charge area density ρ_S , what is the electric field intensity along the z-axis of the disc .

- A circular ring with radius r and width dr', when dr' is small enough, the distance between source point and field point is :

$$R=\sqrt{z^2+r'^2}$$

- The charge in the circular ring can be obtained as follow:

$$dq=\rho_S r' d\varphi' dr'$$
$$=\rho_S 2\pi r' dr'$$



带均匀面电荷的圆盘

Example 4

- Electric potential at the point *P(0,0,z)* in the z-axis:

$$d\phi = \frac{2\pi r' \rho_S dr'}{4\pi\varepsilon_0 \sqrt{z^2 + r'^2}} = \frac{r' \rho_S dr'}{2\varepsilon_0 \sqrt{z^2 + r'^2}}$$

$$\phi = \int_0^a \frac{r' \rho_S dr'}{2\varepsilon_0 \sqrt{z^2 + r'^2}} = \frac{\rho_S}{2\varepsilon_0}\left(\sqrt{z^2 + a^2} - z\right)$$

- Electric field intensity:

$$\vec{E} = \hat{a}_z E_z = -\nabla\phi = -\hat{a}_z \frac{\partial\phi}{\partial z} = \hat{a}_z \frac{\rho_S}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + a^2}}\right)$$

Assignment5

The finite sheet $0 \leq x \leq 1, 0 \leq y \leq 1$ on the $z=0$ plane has a charge density $\rho_s = xy(x^2+y^2+25)^{3/2}$ nC/m^2. Find

(a) The total charge on the sheet

(b) The electric field at (0,0,5)

(c) The force experienced by a -1 mC charge located at (0,0,5)