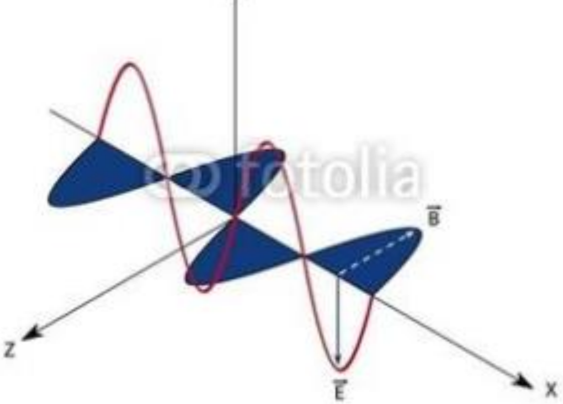
Electromagnetic Fields and Waves



Chapter 9
Electrostatic Field (4)-
Capacitor

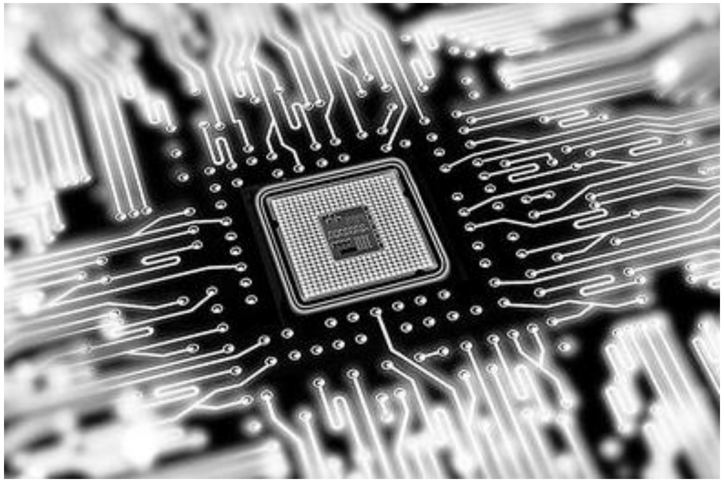
王世伟 Sai-Wai Wong

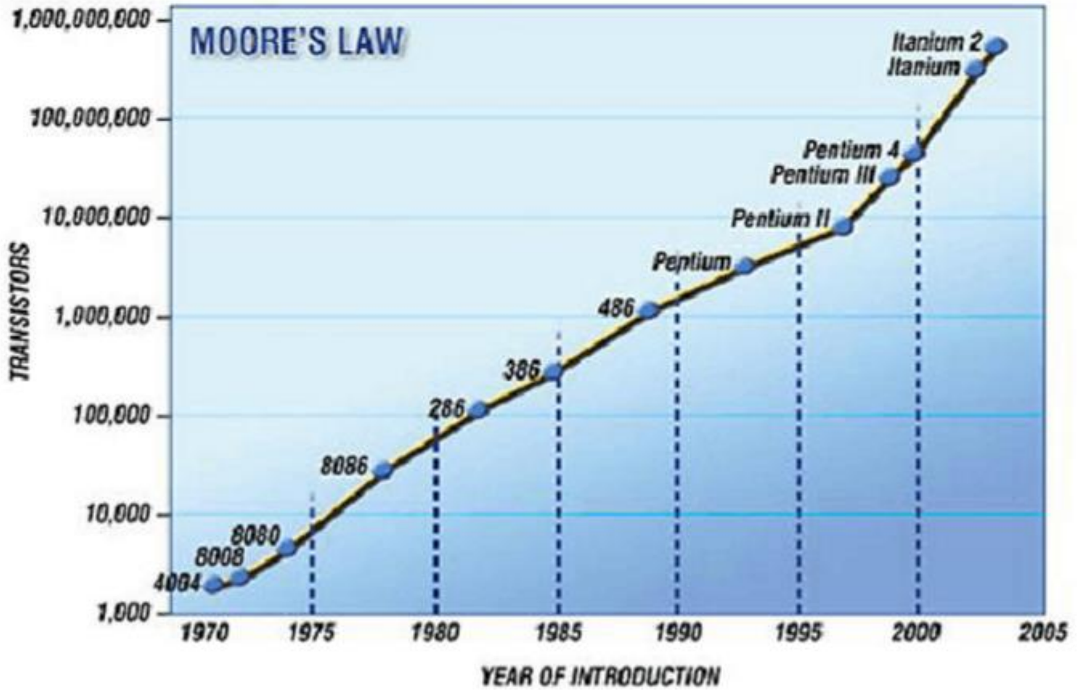
Email:wsw@szu.edu.cn

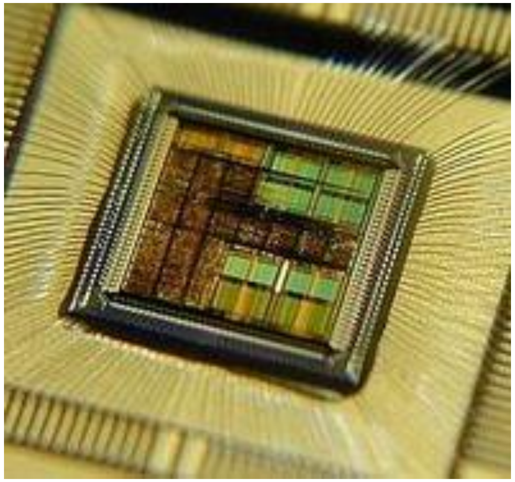
Capacitor

- **Dual-conductor Capacitance**
- **Capacitance in Multi-conductor Systems**

Capacitance







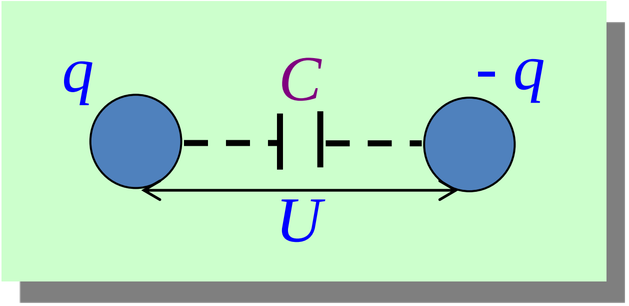
Year 2000 Pentium 4, 180nm, 42 million transistors
Year 2010 Inter-core i7-980X, 32nm,1.17 billion transistors
Year 2013 Inter-core i7-4960X, 22nm,1.86 billion transistors

Capacitance

➢ Dual-conductor Capacitance

✓ Electricity of the two conductors are ***q*** and ***–q***, respectively, potential difference between them is ***U***, then the capacitance (***C***) is defined by:

$$C(F) = \frac{q(C)}{U(V)}$$



✓ As electric charge is proportional to potential, electric charge and potential have no effect on the capacitance, capacitance is only related to the shape, position of the conductor and the substrate around the conductor, which is just a proportional constant, reflecting the interrelation of the conductors.

Capacitance

➢ Isolated-conductor Capacitance

✓ The capacitance of the isolated-conductor is consider as capacitance between the conductor and the referenced potential (ground or infinity), if the conductor has the charge $\boldsymbol{q}$ with the potential $\boldsymbol{\Phi}$, the capacitance can be obtained by:

$$C = \frac{q}{\Phi}$$

➢ Capacitance Calculation

Set(q) ➡ **Solve (E)** ➡ **Calculate(U)** ➡ **Calculate(C)**

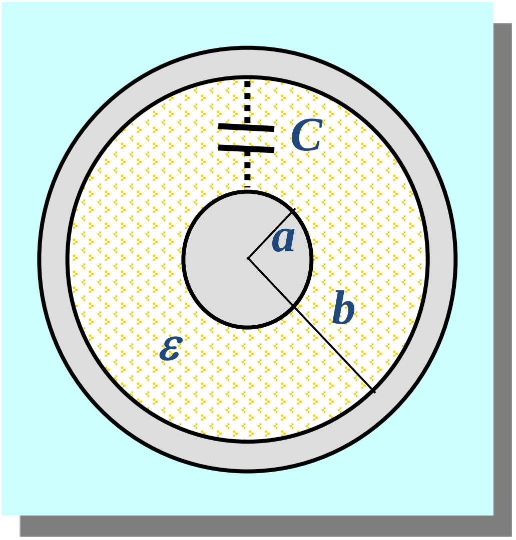
✓ For complex problems, the series and parallel capacitance are introduced:

Series: $$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

Parallel: $$C = C_1 + C_2$$

Example 1

- ***Capacitance for Spherical capacitor***



✓ The charge of this two conductors are ***q*** and ***–q***, respectively, electric field between the conductors is spherically symmetric due to the spherical symmetry of the spherical capacitor and the distribution of the electric charge, so:

$$\vec{E} = E_r \hat{a}_r$$

✓ This informs us to use the Gauss' law:

$$\oint_S \vec{E} \bullet d\vec{S} = E_r 4\pi r^2 = \frac{q}{\varepsilon} \quad \Rightarrow \quad E_r = \frac{q}{4\pi r^2 \varepsilon} \qquad a < r < b$$

Example 1

Potential difference between the inner and outer conductors , called $\boldsymbol{U_{ab}}$:

$$U_{ab} = \int_a^b E_r dr = \frac{q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$E_r = \frac{q}{4\pi r^2 \varepsilon} \qquad a < r < b$$

Capacity of the spherical capacitor, called $\boldsymbol{C}$:

$$C = \frac{q}{U_{ab}} = \frac{4\pi\varepsilon ab}{b-a} = \varepsilon\frac{S}{d}$$

$d = b - a$ distance between the two sphere surfaces

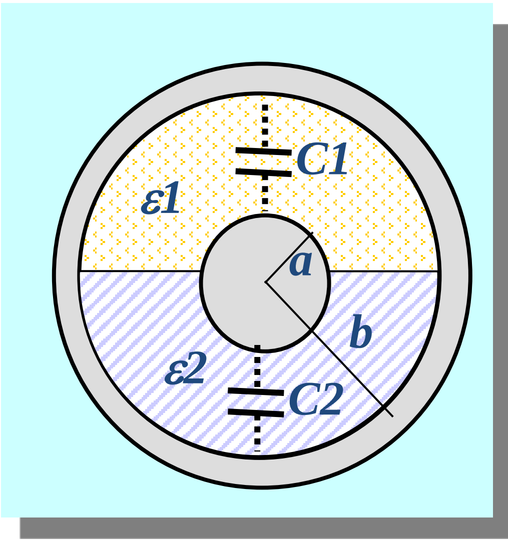
$S = 4\pi ab$ geometric mean 几何平均数of the surface area of the spheres

Example 2

➢ ***Capacitance of Spherical capacitor with filling two kinds of substrates both in hemisphere contents.***

Solution 1: Strength of tangential electric field on the substrate interface is continuous, thus:

$$\vec{E}_1 = \vec{E}_2 = E_r \hat{a}_r$$



This informs us to use the Gauss' law:

$$\oint_S \vec{D} \cdot d\vec{S} = 2\pi r^2 \varepsilon_1 E_1 + 2\pi r^2 \varepsilon_2 E_2 = q$$

$$E_r = \frac{q}{2\pi r^2 (\varepsilon_1 + \varepsilon_2)} \qquad a < r < b$$

Example 2

Potential difference between the inner and outer conductors , called $\boldsymbol{U_{ab}}$:

$$U_{ab} = \frac{q}{2\pi(\varepsilon_1 + \varepsilon_2)}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Capacity of the spherical capacitor, called $\boldsymbol{C}$:

$$C = \frac{q}{U_{ab}} = \frac{2\pi(\varepsilon_1 + \varepsilon_2)ab}{b-a}$$

Example 2

Solution 2: Using the property of the parallel capacitors from the *Example.1*, we get that:

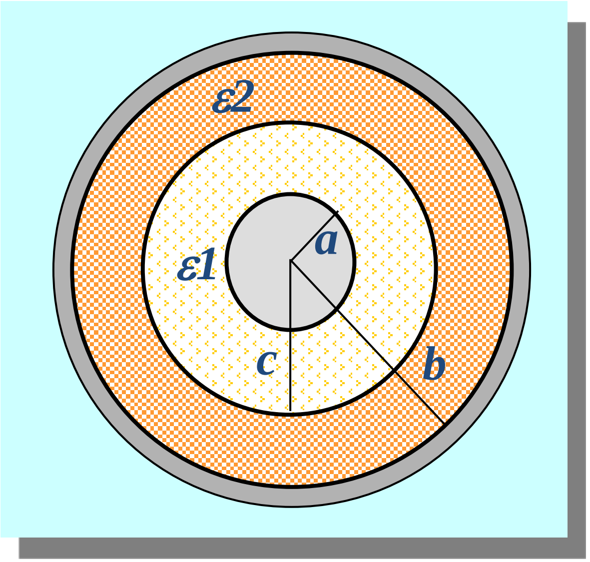
$$C_1 = \frac{1}{2}\frac{4\pi\varepsilon_1 ab}{b-a} \qquad C_2 = \frac{1}{2}\frac{4\pi\varepsilon_2 ab}{b-a}$$

Two capacitors are in parallel, so:

$$C = C_1 + C_2 = \frac{2\pi(\varepsilon_1+\varepsilon_2)ab}{b-a}$$

Thinking

How to obtain the capacitance of the spherical capacitor filling with different and spherically symmetric substrates?

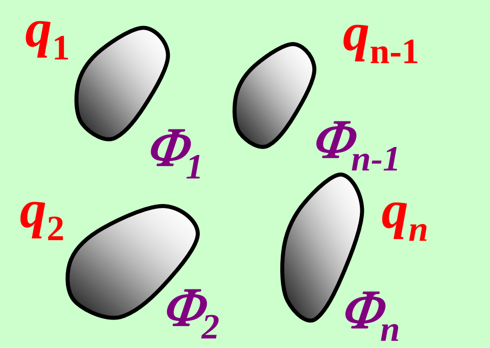


Capacitance in Multi-conductor Systems

➢ Potential Coefficient Matrix

✓ For the systems with ***n*** charged conductors, the potential of any points in space are generated by the electric charges on the conductor surfaces, which fulfill *Superposition叠加 principle*, so the potential of the conductors are expressed by :

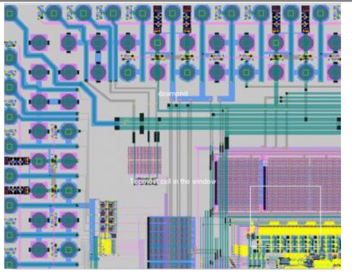
$$\begin{cases} \phi_1 = p_{11}q_1 + p_{12}q_2 + \cdots + p_{1n}q_n \\ \phi_2 = p_{21}q_1 + p_{22}q_2 + \cdots + p_{2n}q_n \\ \cdots \\ \phi_n = p_{n1}q_1 + p_{n2}q_2 + \cdots + p_{nn}q_n \end{cases}$$



Multi-conductor system

✓ The Matrix form is:

$$[\phi] = [p][q]$$



Capacitance in Multi-conductor Systems

✓ [$\boldsymbol{p}$] is called the Potential Coefficient Matrix, which is only related to the shape, position of the conductor, the substrate properties and the selection of the referenced potential.

➢ Physical significance of the potential parameter

$$p_{ii} = \left.\frac{\phi_i}{q_i}\right|_{q_j=0,\, j=1,2,\cdots n,\, j\neq i} \qquad p_{ji} = \left.\frac{\phi_j}{q_i}\right|_{q_j=0,\, j=1,2,\cdots n, i\neq j}$$

✓ The former expresses potential of conductor i when only conductor i has electric charges

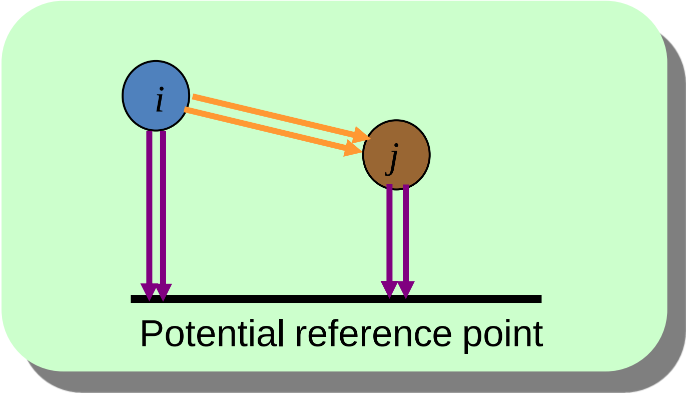
✓ The latter expresses interactive potential between conductor i *and* j when only conductor i has electric charges :

Capacitance in Multi-conductor Systems

✓ If conductor *i* has positive charges, other conductors have no charge, then p_{ii} is the potential of the conductor *i*, i.e. Φ_i , p_{ji} is the potential of conductor *j*, i.e. Φ_j.

➢ Properties of the potential parameters

✓ When conductor *i* has positive charges, other conductors have no charge, the electric power line emits from conductor *i*, part of it terminates at the referenced potential point, rest of it terminates at other conductors *j*.



Capacitance in Multi-conductor Systems

✓ Because conductor j has no charge, the electric power line terminates at j is equal to that emits from it, besides, the electric power line emit from it will terminates at referenced potential point or other conductors. The potential declines along the direction of the electric power line, so:

$$p_{ii} > p_{ij} \geq 0 \qquad (i \neq j)$$

$$p_{ji} = p_{ij} \qquad [p] = [p]^T$$

✓ If the substrate of the systems is reciprocity media, the potential coefficient also fulfill the reciprocity characteristics, the potential matrix is a symmetric matrix:

$$\phi_i > \phi_j \geq 0 \qquad (i \neq j)$$

Capacitance in Multi-conductor Systems

➢ Capacitance Coefficient Matrix

$$\begin{cases} q_1 = \beta_{11}\phi_1 + \beta_{12}\phi_2 + \cdots + \beta_{1n}\phi_n \\ q_2 = \beta_{21}\phi_1 + \beta_{22}\phi_2 + \cdots + \beta_{2n}\phi_n \\ \cdots \\ q_n = \beta_{n1}\phi_1 + \beta_{n2}\phi_2 + \cdots + \beta_{nn}\phi_n \end{cases}$$

✓ The Matrix form is: $[q] = [\beta][\phi]$

✓ **$[\beta]$** is called the Capacitance Coefficient Matrix, it's obviously that: $[\beta] = [p]^{-1}$

✓ If all conductors is grounded except conductor i, then $\boldsymbol{\beta_{ii}}$ is the capacitance coefficient of the conductor i, $\boldsymbol{\beta_{ji}}$ is the interactive capacitance coefficient between conductor i *and* j.

Capacitance in Multi-conductor Systems

✓ If all conductors is grounded except conductor i, and conductor i has a potential of +1V, then β_{ii} is the electric charge of the conductor i, β_{ji} is the induction electric charge of conductor j.

✓The conductor i has a positive potential, thus, $q_i > 0$. According to characteristics of the charge induction and the principle of conservation of charge, $q_j < 0$, besides, the absolute value of the charge of other $(n\text{-}1)$ conductors does not exceed the charge of the conductor i, i.e.:

$$-\sum_{\substack{j=1 \\ j \neq i}}^{n} q_j \leq q_i$$

Capacitance in Multi-conductor Systems

- ✓ Then,

$$\beta_{ii} > 0, \quad \beta_{ij} \le 0 \quad (i \ne j), \quad \sum_{j=1}^{n} \beta_{ij} \ge 0$$

- ✓ Similarly, if the substrate of the systems is reciprocity media, the capacitance coefficient matrix also fulfills the reciprocity characteristics:

$$\beta_{ji} = \beta_{ij} \qquad [\beta] = [\beta]^T$$

Capacitance in Multi-conductor Systems

➢ Partial Capacitance

✓ Let the electric charges of all the conductors expressed by the voltage among the conductors, which we have the concept of Partial Capacitance.

$$\begin{cases} q_1 = (\beta_{11} + \beta_{12} + \cdots + \beta_{1n})\phi_1 + \beta_{12}(\phi_2 - \phi_1) + \cdots + \beta_{1n}(\phi_n - \phi_1) \\ q_2 = \beta_{21}(\phi_1 - \phi_2) + (\beta_{21} + \beta_{22} + \cdots + \beta_{2n})\phi_2 + \cdots + \beta_{2n}(\phi_n - \phi_2) \\ \cdots \\ q_n = \beta_{n1}(\phi_1 - \phi_n) + \beta_{n2}(\phi_2 - \phi_n) + \cdots + (\beta_{n1} + \beta_{n2} + \cdots + \beta_{nn})\phi_n \end{cases}$$

✓ Let:

$$C_{ii} = \sum_{j}^{n} \beta_{ij} \qquad C_{ij} = -\beta_{ij} \quad (i \neq j)$$

$$U_{ii} = \phi_i \qquad U_{ij} = \phi_i - \phi_j \quad (i \neq j)$$

Capacitance in Multi-conductor Systems

✓ Then:

$$\begin{cases} q_1 = C_{11}U_{11} + C_{12}U_{12} + \cdots + C_{1n}U_{1n} \\ q_2 = C_{21}U_{21} + C_{22}U_{22} + \cdots + C_{2n}U_{2n} \\ \cdots \\ q_n = C_{n1}U_{n1} + C_{n2}U_{22} + \cdots + C_{nn}U_{nn} \end{cases}$$

✓ $\boldsymbol{C_{ii}}$ is the self-partial capacitance, which is the partial capacitance between the referenced point and conductor i of the system with n conductors.

✓ $\boldsymbol{C_{ji}}$ is the partial capacitance between conductor i and j.

✓ Obviously, $C_{ij} \geq 0$

✓ For the multi-conductors systems using reciprocity media

$$C_{ji} = C_{ij}$$

Capacitance in Multi-conductor Systems

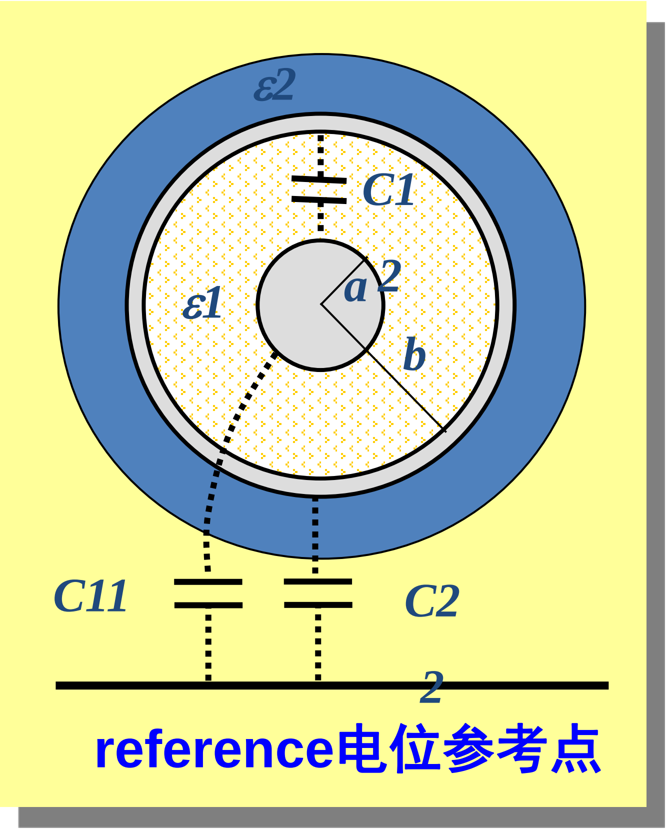
- ✓ The formulas of the partial capacitance express that the capacitance network exist in the multi-conductors systems, which reflects the interaction between the conductors (induction or coupling耦合), so getting the capacitance network means getting the properties of the multi-conductors system in the electrostatic field.

- ✓The capacitance network of the multi-conductors systems has a wide application, such as multi-core power transmission line, multi-conductor transmission line, LSI interconnect lines.

- ✓ To obtain the partial capacitance of the multi-conductor systems, we can use the formulas, definition of the partial capacitance and the characteristic of the series and parallel capacitors.

Example 3

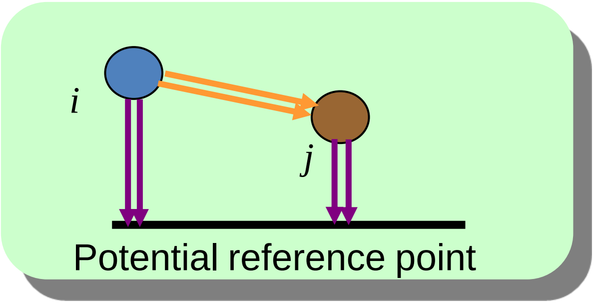
Find the partial capacitance of spherical capacitor in dual-conductor system.



Solution 1: Selecting the infinity as the referenced point and using the definition of the partial capacitance.

✓ C_{11} is the self-partial capacitance between the inner conductor and the referenced point. As the inner conductor is shielded by the outer conductor, the electric power line emit from the inner conductor cannot reach to the referenced point directly, so:

$$C_{11} = 0$$



Capacitance in Multi-conductor Systems

✓ $\boldsymbol{C_{22}}$ is the partial capacitance between the outer conductor and the referenced point, i.e. the capacitance of isolated outer conductor. If it has an electric charge of q, using the Gauss theorem, we obtain the electric field strength beyond the outer conductor:

$$\vec{E} = \frac{q}{4\pi r^2 \varepsilon_2} \hat{a}_r \qquad r > b$$

✓ Thus, the potential on the outer conductor:

$$\phi_2 = \int_b^\infty \vec{E} \cdot \hat{a}_r dr = \frac{q}{4\pi\varepsilon_2 b} \Rightarrow C_{22} = \frac{q}{\phi_2} = 4\pi\varepsilon_2 b$$

✓ $\boldsymbol{C_{12}}$= $\boldsymbol{C_{21}}$ is the partial capacitance between the inner and outer conductor, according to Example 1:

$$C_{12} = C_{21} = \frac{4\pi\varepsilon_1 ab}{b-a}$$

Capacitance in Multi-conductor Systems

Solution 2: Using the formulas.

✓ The charge of the inner and outer conductor are $\boldsymbol{q_1}$ and $\boldsymbol{q_2}$, respectively, using Gauss theorem, we obtain that:

$$\vec{E}_1 = \frac{q_1}{4\pi\varepsilon_1 r^2}\hat{a}_r \ \left(a < r < b\right) \qquad \vec{E}_2 = \frac{q_1 + q_2}{4\pi\varepsilon_2 r^2}\hat{a}_r \ \left(r > b\right)$$

$$\phi_1 = \int_a^b \vec{E}_1 \cdot \hat{a}_r dr + \int_b^\infty \vec{E}_2 \cdot \hat{a}_r dr$$

$$= \frac{q_1}{4\pi\varepsilon_1}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{1}{4\pi\varepsilon_2 b}(q_1 + q_2)$$

$$= \frac{1}{4\pi}\left(\frac{1}{\varepsilon_1 a} - \frac{1}{\varepsilon_1 b} + \frac{1}{\varepsilon_2 b}\right)q_1 + \frac{1}{4\pi\varepsilon_2 b}q_2$$

$$\phi_2 = \int_b^\infty \vec{E}_2 \cdot \hat{a}_r dr = \frac{1}{4\pi\varepsilon_2 b}(q_1 + q_2)$$

Capacitance in Multi-conductor Systems

✓ Using the formulas of the potential coefficient:

$$p_{ii}=\left.\frac{\phi_i}{q_i}\right|_{q_j=0,\, j=1,2,\cdots n,\, j\neq i} \qquad p_{ji}=\left.\frac{\phi_j}{q_i}\right|_{q_j=0,\, j=1,2,\cdots n, i\neq j}$$

$$\boxed{\phi_1=\frac{1}{4\pi}\left(\frac{1}{\varepsilon_1 a}-\frac{1}{\varepsilon_1 b}+\frac{1}{\varepsilon_2 b}\right)q_1+\frac{1}{4\pi\varepsilon_2 b}q_2} \qquad \boxed{\phi_2=\frac{1}{4\pi\varepsilon_2 b}(q_1+q_2)}$$

$$p_{11}=\left.\frac{\phi_1}{q_1}\right|_{q_2=0}=\frac{1}{4\pi}\left(\frac{1}{\varepsilon_1 a}-\frac{1}{\varepsilon_1 b}+\frac{1}{\varepsilon_2 b}\right) \qquad p_{12}=\left.\frac{\phi_1}{q_2}\right|_{q_1=0}=\frac{1}{4\pi\varepsilon_2 b}$$

$$p_{22}=\left.\frac{\phi_2}{q_2}\right|_{q_1=0}=\frac{1}{4\pi\varepsilon_2 b} \qquad p_{21}=\left.\frac{\phi_2}{q_1}\right|_{q_2=0}=\frac{1}{4\pi\varepsilon_2 b}$$

Capacitance in Multi-conductor Systems

✓ According to the capacitance coefficient matrix:

$$p_{11} = \frac{1}{4\pi}\left(\frac{1}{\varepsilon_1 a} - \frac{1}{\varepsilon_1 b} + \frac{1}{\varepsilon_2 b}\right) \qquad p_{12} = \frac{1}{4\pi\varepsilon_2 b}$$

$$p_{22} = \frac{1}{4\pi\varepsilon_2 b}$$

$$p_{21} = \frac{1}{4\pi\varepsilon_2 b}$$

$$[\beta] = [p]^{-1} = \frac{1}{\det[p]}\begin{bmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{bmatrix}$$

$$\det[p] = p_{11}p_{22} - p_{12}p_{21} = \frac{b-a}{(4\pi b)^2 a\varepsilon_1\varepsilon_2}$$

Capacitance in Multi-conductor Systems

- ✓ Then:

$$\beta_{11} = \frac{p_{22}}{\det[p]} = \frac{4\pi\varepsilon_1 ab}{b-a}$$

$$\beta_{12} = \beta_{21} = \frac{-p_{12}}{\det[p]} = -\frac{4\pi\varepsilon_1 ab}{b-a}$$

$$\beta_{22} = \frac{p_{11}}{\det[p]} = 4\pi\varepsilon_2 b + \frac{4\pi\varepsilon_1 ab}{b-a}$$

- ✓ The partial capacitance is:

$$C_{11} = \beta_{11} + \beta_{12} = 0$$

$$C_{12} = C_{21} = -\beta_{12} = \frac{4\pi\varepsilon_1 ab}{b-a}$$

$$C_{22} = \beta_{12} + \beta_{22} = 4\pi\varepsilon_2 b$$

Conclusion

- **Capacitance in Isolated conductor**

- **Capacitance in Dual-conductor**

- **Capacitance in Multi-conductor Systems**

Assignment8

Conducting spherical shells with radius a=10cm and b=30cm are maintained at a potential of 100V such that $V(r=b)=0$ and $V(r=a)=100V$. Determine V and $\vec{E}$ in the region between the shells. If $\varepsilon_r=2.5$ in the region, determine the total charge induced on the shells and the capacitance of the capacitor.