Electromagnetic Fields and Waves

Chapter 8

Electrostatic Field (3)

Boundary Condition

王世伟 (Sai-Wai Wong)

Email: wsw@szu.edu.cn

Boundary Condition of Electrostatic Field

The boundary conditions of electrostatic field

- Available equations of electrostatic field and potential meet:

$$\nabla \times \vec{E} = \vec{0} \quad \nabla \bullet \vec{D} = \rho$$

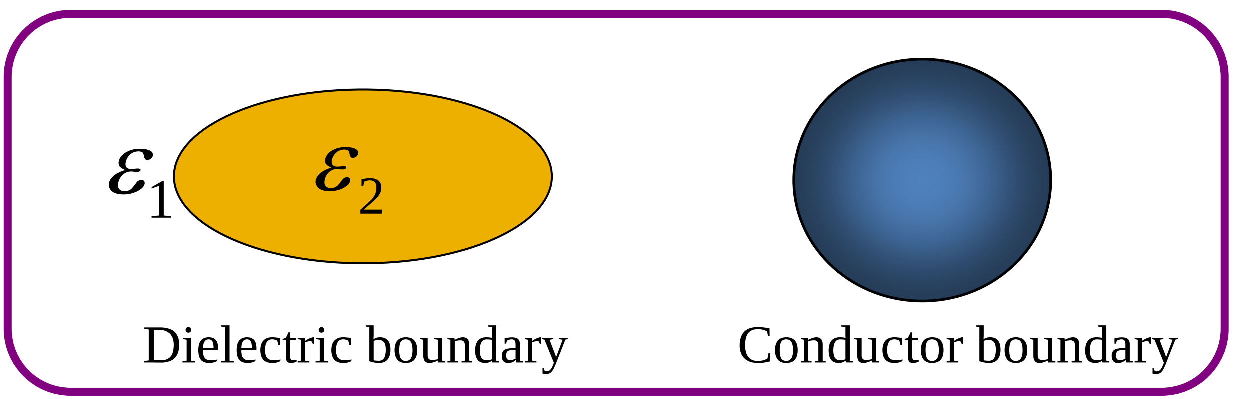
With source $\nabla^2 \phi = -\dfrac{\rho}{\varepsilon}$ Possion Equation

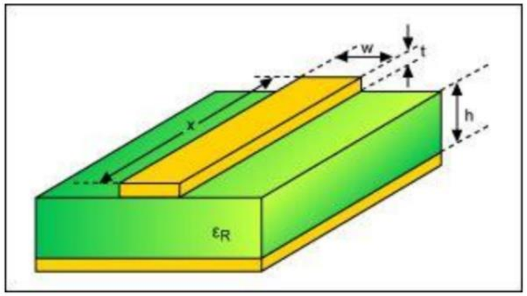
no source $\nabla^2 \phi = 0$ Laplace Equation

- These equations reflect the features of electrostatic field on a spatial空间 point. But they are differential equation and only fit continuous differentiable function.
- For the problem of mutational突变substrate, the field function is not continuous differentiable thus can not apply to the differential form of field equation.

The boundary conditions of electrostatic field

- Sometimes the problem is bounded and the differential form of field equation can not be used on the boundary.
- Thus, we need to find the fulfilled equation of **electrostatic field between the substrate and the boundary, namely, the boundary conditions of electrostatic field**.
- Generally, a common problem of boundary is the dielectric boundary and conductor boundary.
- The research method of boundary problem start from the integral form, because the equation with integral form is not restrict by the boundary.





Normal direction of electric field boundary condition

- Applied Gauss **theorem**, considering the infinitely small cylindrical surface， we have:

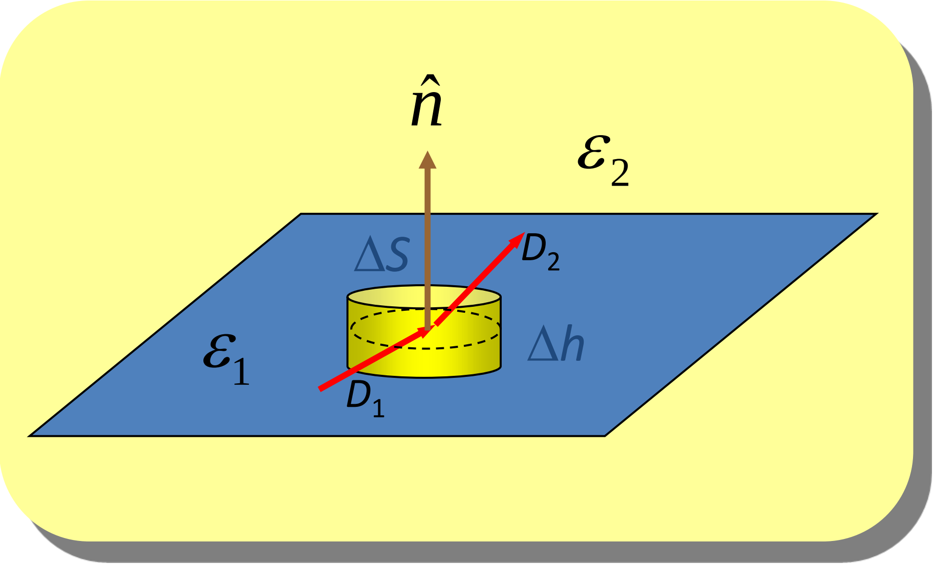
$$\oint_S \vec{D}\bullet d\vec{S} = Q$$

$$\int_S \vec{D}\bullet d\vec{S} = \vec{D}_1 \bullet \hat{n}\Delta S - \vec{D}_2 \bullet \hat{n}\Delta S$$

$$+\vec{D}_1 \bullet \hat{a}_R 4\pi R\Delta h/2$$

$$+\vec{D}_2 \bullet \hat{a}_R 4\pi R\Delta h/2$$

$$=(\vec{D}_1 - \vec{D}_2)\bullet \hat{n}\Delta S \qquad \Delta h \to 0$$

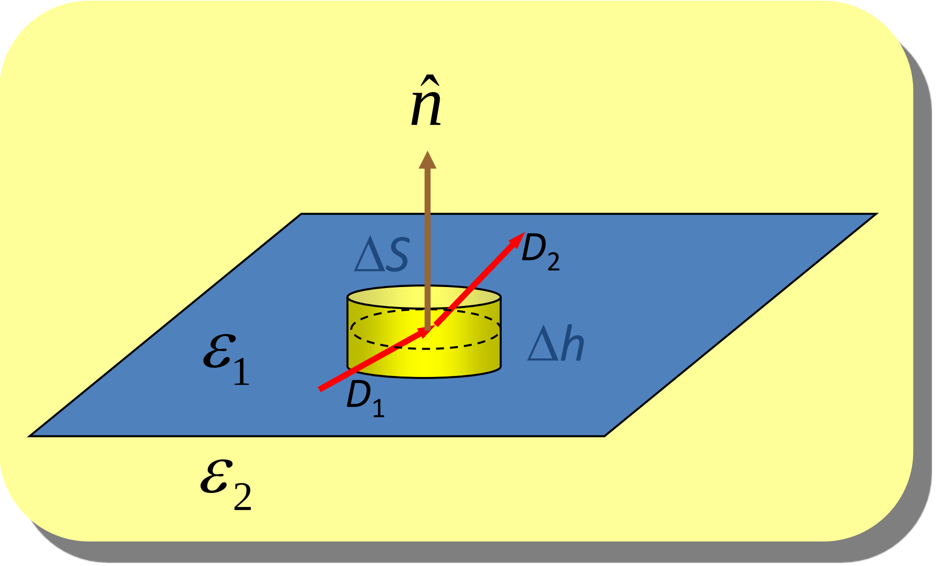


- At the same time

$$Q = \int_V \rho dv = \rho\Delta h\Delta S = \rho_s \Delta S \qquad \Delta h \to 0$$

Note: $\hat{n}$ is pointing from substrate 1 to substrate 2

Normal direction of electric field boundary condition



- Thus $(\vec{D}_1 - \vec{D}_2)\cdot\hat{n} = \rho_s$
- Written as scalar form

$$\boxed{D_{1n} - D_{2n} = \rho_s}$$

where $\rho_s = \lim_{\Delta S \to 0} \frac{Q}{\Delta S} = \lim_{\Delta h \to 0} \rho \Delta h$

is surface charge density on the boundary

- The difference of normal electric displacement on the substrate boundary is equal to surface free charge density on the boundary
- If no free charge on the dielectric boundary, the normal electric displacement on the substrate boundary is **continuous:**

$$D_{1n} = D_{2n}$$

Normal direction of electric field boundary condition

- As for isotropic dielectric, the boundary condition can be written as:

$$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$$

- If no free charge on the boundary:

$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

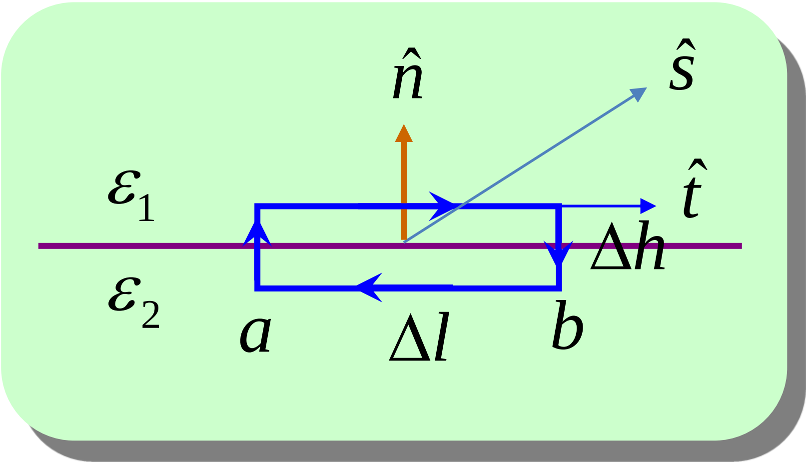
- This indicates the normal electric field intensity on the boundary is not continuous:
- If considering the bound charge density on the dielectric boundary, easily get:

$$E_{1n} - E_{2n} = (\rho_s + \rho_s') / \varepsilon_0$$

- Obviously, the distributed surface charge on the boundary causes the normal electric displacement and the electric field intensity discontinuous on the dielectric boundary

Tangential切向 boundary condition

- Applied electric field loop integral formula:



$$\oint_C \vec{E}\cdot d\vec{l} = \vec{E}_1\cdot\hat{t}\Delta l - \vec{E}_2\cdot\hat{t}\Delta l$$

$$+\vec{E}_1^a\cdot\hat{n}\Delta h/2 + \vec{E}_2^a\cdot\hat{n}\Delta h/2$$

$$-\vec{E}_1^b\cdot\hat{n}\Delta h/2 - \vec{E}_2^b\cdot\hat{n}\Delta h/2 = (\vec{E}_1 - \vec{E}_2)\cdot\hat{t}\Delta l \qquad \Delta h \to 0$$

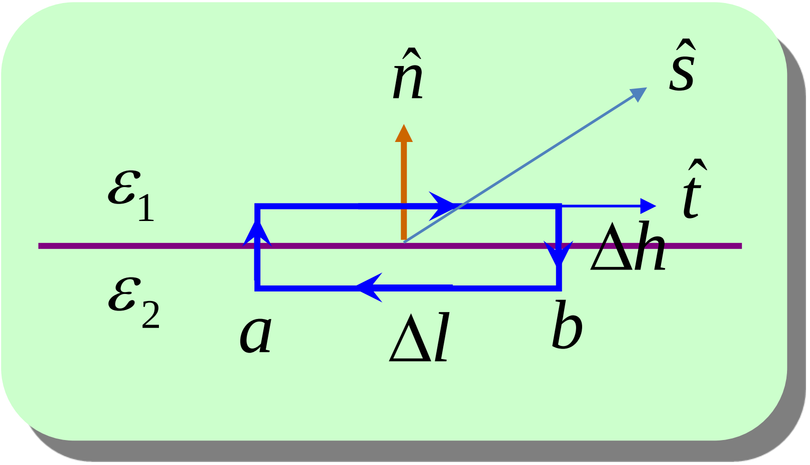
$$= 0$$

- Thus $(\vec{E}_1 - \vec{E}_2)\cdot\hat{t} = \vec{0}$
- Written as scalar form: $E_{1t} - E_{2t} = 0$
- Thus, the tangential切向 electric field is continuous on the boundary.

Tangential boundary condition

- Generally, we use the normal unit vector to represent boundary condition. Thus

$$\hat{t} = \hat{s} \times \hat{n}$$



- Then:

$$(\vec{E}_1 - \vec{E}_2) \bullet \hat{t} = (\vec{E}_1 - \vec{E}_2) \bullet (\hat{s} \times \hat{n}) = 0$$

$$= \hat{n} \times (\vec{E}_1 - \vec{E}_2) \bullet \hat{s} = \vec{0}$$

- Since $\hat{s}$ is arbitrary, so:

$$\hat{n} \times (\vec{E}_1 - \vec{E}_2) = \vec{0}$$

- The unit vector transformation discussed above is often used in electromagnetic field theory.

Perfect Dielectric Boundary

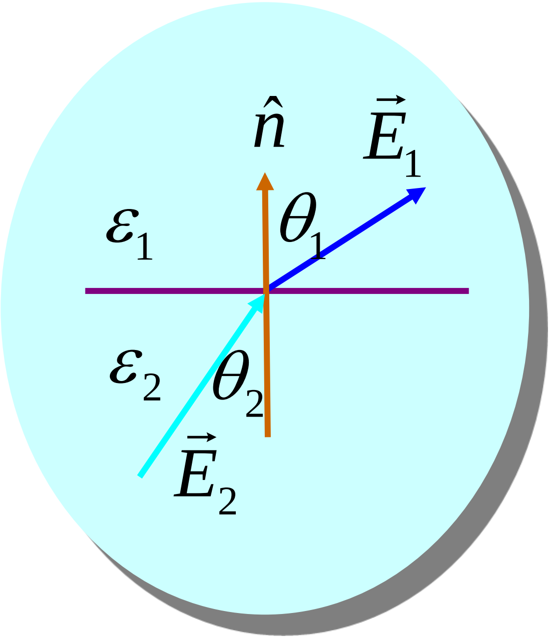
- Perfect dielectric boundary means that there is no free charge on the boundary.
- According to boundary condition:

$$\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2$$

$$E_1 \sin\theta_1 = E_2 \sin\theta_2$$

- Then

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$$



- From the two sides of the boundary, the electric field is not in the same direction on either side of boundary. Only when electric field and the boundary are perpendicular, the electric field has the same direction on either side of boundary.

Perfect Dielectric Boundary

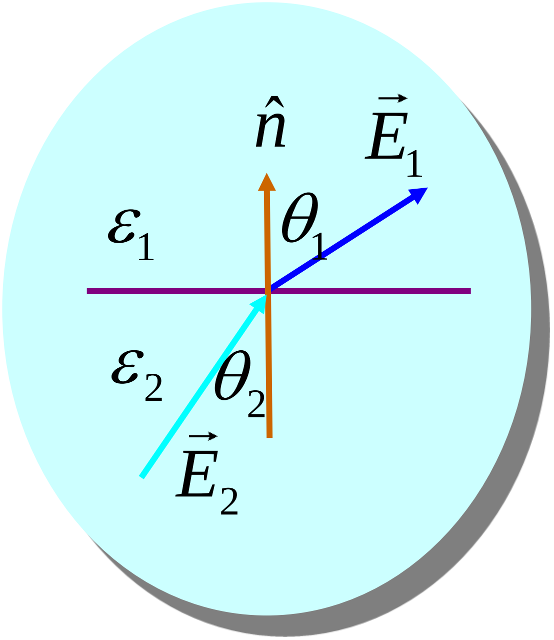
- Apply the relationship $\vec{E} = -\nabla\phi$, we can get:

$$\hat{n}\cdot\vec{E} = -\hat{n}\cdot\nabla\phi = -\frac{\partial\phi}{\partial n}$$

n is the variable along normal direction

$$\hat{t}\cdot\vec{E} = -\hat{t}\cdot\nabla\phi = -\frac{\partial\phi}{\partial t}$$

t is the variable along tangential direction



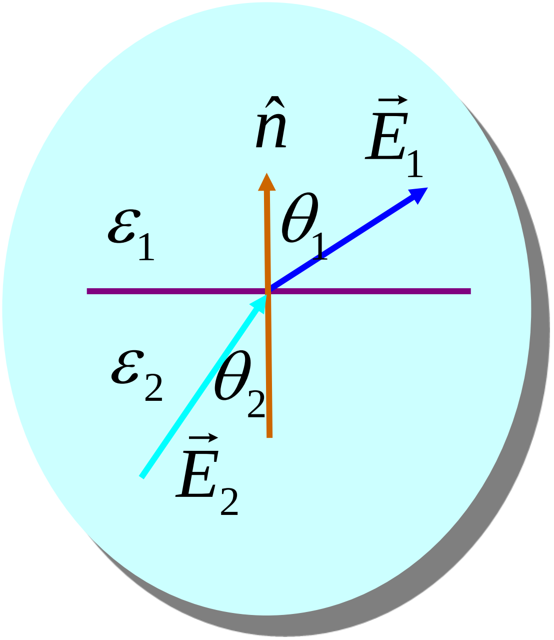
- Substitute above equations into electric field boundary conditions, we get:

$$\varepsilon_1\frac{\partial\phi_1}{\partial n} - \varepsilon_2\frac{\partial\phi_2}{\partial n} = -\rho_s$$

$$\frac{\partial\phi_1}{\partial t} - \frac{\partial\phi_2}{\partial t} = 0 \quad \Rightarrow \quad \phi_1 - \phi_2 = const.$$

Perfect Dielectric Boundary

- If the potential reference is not given, const can be an arbitrary constant. However, if the electric potential reference is given, const is an fixed number. Because the boundary condition satisfy any point on the boundary, thus, const can only be zero.



- Finally, the boundary condition of potential at the boundary of two dielectrics are:

$$\phi_1 = \phi_2 \qquad\qquad \varepsilon_1 \frac{\partial \phi_1}{\partial n} - \varepsilon_2 \frac{\partial \phi_2}{\partial n} = -\rho_s$$

- The physical meaning of these two equations are: the electric potential is continue at the boundary. The partial derivative of electric potential at normal direction of the boundary is discontinue.

Perfect Electric Conductor: PEC

- The perfect conductor is lossless conductor. The surface of perfect conductor is called ***Perfect Electric Wall***. ***Perfect Electric Conductor(PEC)*** is the fulfilled equations of electric field on perfect electric wall.
- As we all know， the electrostatic field is always zero in the conductor, and potential is a constant (equipotential). If we treat the conductor as substrate 2, we can easily obtained the PEC of perfect electric wall based on the boundary condition of substrate interface:

$$\hat{n}\cdot\vec{D}=\rho_s \qquad \hat{n}\times\vec{E}=0$$

$$\frac{\partial\phi}{\partial n}=-\frac{\rho_s}{\varepsilon} \qquad \phi=const$$

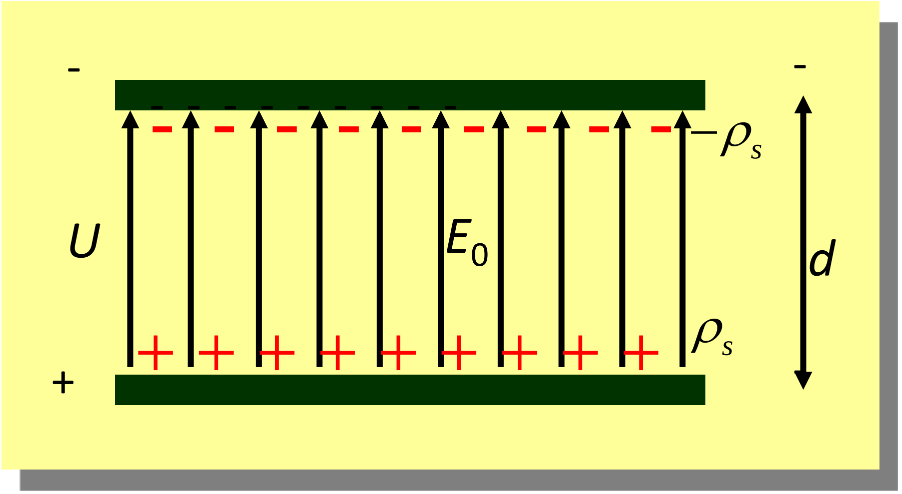
$$\hat{n}\times(\vec{E}_1-\vec{E}_2)=\vec{0}$$
$$\varepsilon_1 E_{1n}-\varepsilon_2 E_{2n}=\rho_s$$
$$\varepsilon_1\frac{\partial\phi_1}{\partial n}-\varepsilon_2\frac{\partial\phi_2}{\partial n}=-\rho_s$$
$$\phi_1=\phi_2$$

- Thus, the tangential electric field

on the perfect electric wall is zero.

Example 1

Two infinite parallel conductive plates , the gap between two plates *d* is quite small compare to the length and width of the parallel conductive plates. The plates are disconnected after fully charged by connecting a D.C. voltage (U), then insert a homogeneous dielectric plate $\varepsilon_r=9$ into the gap between two plates. Assuming the thickness of the dielectric plate is slightly smaller than *d*. Solve the electric intensities before and after inserting a dielectric plate into the parallel conductive plates.



- Ans： without dielectric plate, the electric intensity of the parallel conductive plates is:

$$E_0 = \frac{U}{d}$$

- The direction of E_0 is from positive to negative shown in right.

- On the boundary between the lower plate and the air, according to the boundary condition $D_{1n} - D_{2n} = \rho_s$ and $D_{1n} = D_1 = D_0$, $D_2 = 0$ (the E is zero inside the conductor) on the boundary, we can get:

$$\rho_s = D_0 = \varepsilon_0 E_0 = \varepsilon_0 \frac{U}{d}$$

- And apply the same condition to the upper plate and air, we can obtain the free charge surface density of the upper plate (lower surface), that is $-\rho_s$ 。

- Due to first outage and then insert the dielectric plate, moreover the dielectric plate is homogeneous, thus the total free charge and distribution(ρ_s) on the conductor surface stay the same after putting the dielectric, i.e.,

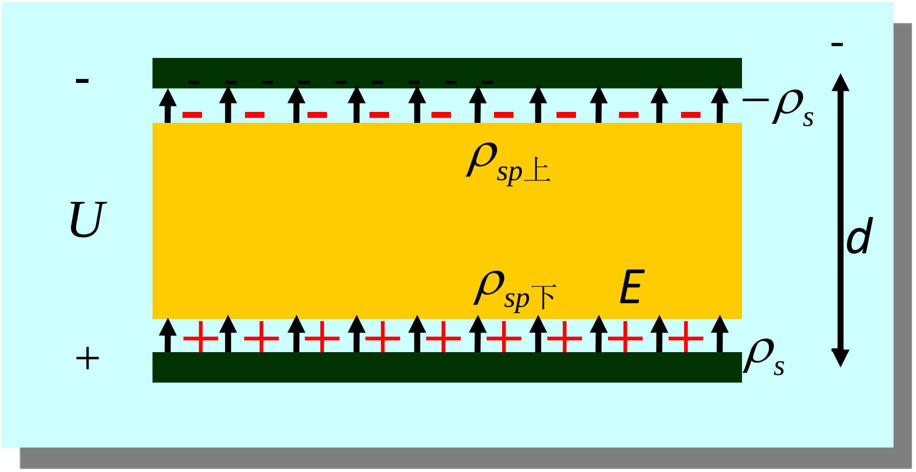
$$D_0 = \rho_s = \varepsilon_0 \frac{U}{d}$$

- There is net no free surface charge on the dielectric, and the dielectric is parallel to the conductor and the dielectric plate. When applying the boundary condition of the electric displacement normal vector is continuous on this boundary, we can get the electric displacement D in the dielectric:

$$D = D_0 = \varepsilon_0 \frac{U}{d}$$

- The electric field intensity E in the dielectric is:

$$E = \frac{D}{\varepsilon} = \frac{D}{9\varepsilon_0} = \frac{E_0}{9}$$



- The $\vec{E}$ on the boundary of dielectric and air is discontinuous. Because the bound charge appear on the dielectric surface, and the additional field intensity which generate by bound charge weaken the original field intensity.

❖ The intensity of polarization in the dielectric $\vec{P}$ is:

$$\vec{P} = \chi_e \varepsilon_0 \vec{E} = \left(\varepsilon_r - 1\right) \varepsilon_0 \frac{E_0}{9} \hat{a}_y = \frac{8}{9} \rho_s \hat{a}_y$$

❖ According to : $\rho_{sp} = \vec{P} \bullet \hat{n}$

❖ We can get: $\rho_{sp上} = \vec{P} \bullet \hat{a}_y = \frac{8}{9} \rho_s$

$$\rho_{sp下} = \vec{P} \bullet \left(-\hat{a}_y\right) = -\frac{8}{9} \rho_s$$

Example 2

Consider the boundary of polystyrene ($\varepsilon = 2.6\varepsilon_0$) and air. At the side of polystyrene, the electric field intensity is 2500 V/m and have an angle with the boundary normal direction of 20 degree: (1) Determine the angle from normal direction at boundary, (2) Calculate the electric field *E* and electric displacement *D* of the air.

- Solution： (1)Let the polystyrene and air as medium 1 and 2, respectively. We have

$$\tan\theta_2 = \frac{\varepsilon_2}{\varepsilon_1}\tan\theta_1 = 0.14 \quad \Rightarrow \quad \theta_2 = 8^\circ$$

- (2)Since $E_{1t} = E_{2t}$ at the boundary $E_1 \sin\theta_1 = E_2 \sin\theta_2$
- so $E_2 = E_1 \dfrac{\sin\theta_1}{\sin\theta_2} = 6144(V/m)$
- Therefore

$$D_2 = \varepsilon_2 E_2 = 5.44\times10^{-8}(C/m^2)$$

Assignment7，作业档案名请用格式：姓名+作业7

Region $y \le 0$ consists of a perfect conductor while region $0 \le y$ is a dielectric medium ($\varepsilon_r=2$) as show in the following figure. If there is a surface charge of 2nC/m^2 on the conductor, determine $\vec{E}$ and $\vec{D}$ at

(a) A(3, -2, 2)

(b) B(-4, 1, 5)