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Summary Sheet

Long May They Reign: How Majestic Dragons Can Survive In The New Realm

Summary

The dragon has always been imagined by people in both East and West . If a dragon lives in our world, it will be very meaningful and interesting to study its characteristics and interaction with ecosystem.

First, the basic physical characteristics of the dragon are studied. According to the growth characteristics of creatures, we establish **the Dragon's Weight Logistic Growth Model**. On the basis of this model, it is estimated that the weight of a mature dragon is between 10 tons and 20 tons. Then, by using the mechanics knowledge and analogy method , we analogize the trunk of the dragon to the elastic cylinder and estimate that the length of the trunk of the mature dragon is about 4 meters to 6 meters and that the length of the whole body of the mature dragon is about 20 meters to 35 meters. **Secondly**, we analyze the daily energy intake and expenditures of the dragon. Comparing the dragon with the creatures that have similar characteristics, we estimate the daily energy intake and diet of the dragon, and the rate of energy consumption during its flight. In addition, given the conditions of breathing fire, we establish **the Dragon's Fire Chemical Kinetics Model** so as to analyze the mechanism and energy consumption of breathing fire.**Thirdly**, on the basis of the results above, we consider the minimum requirement of the habitat for maintaining the dragon. In order to analyze the impact of the dragon on the ecosystem, we establish **the Dragon-Sheep-Habitat Model** based on the Logistic model. More importantly, we compare the difference of the resources for maintaining a dragon in arid, warm temperate and arctic regions, and analyze the living status of the dragon in these regions.

What's more, in the sensitivity analysis, we focus on three factors, namely the inherent growth rate of a dragon's weight, environmental factors and environmental temperature, in order to analyze their impacts on the weight of a mature dragon as they change. **Finally**, we write a letter to George R.R. Martin, the author of *A Song of Ice and Fire*, to provide some advice on maintaining the realistic ecological basis of the story. Although the dragon does not exist in the real world, the models established in this paper can also be applied to solve the problem about the growth of dinosaurs, the invasion of alien species and the design of new flamethrowers.

Note: To avoid copyright infringement, we draw and generate all the figures on our own.

Keywords: Physical Characteristics; Energy Expenditures and Caloric Intake; Dragon-Sheep-Habitat Model; Climate Impacts

Contents

1 Introduction

1.1 Problem Restatement

Dragon, the most legendary behemoths from antiquity appeared in the phenomenal TV series *Game of Thrones* adapted by George R.R. Martin's best-seller *Song of Ice and Fire*, which attracted eyeballs around the world. In the story, at the very beginning of their birth, they were small and thin, on the other hand, they were able to grow up at a rapid rate. Now, we assume that the three dragons in the series live in the real world and their growth meets basic ecological principles. We need to build mathematical models to solve the following problems:

1. Analyze the physical characteristics of a dragon and estimate its daily energy expenditures and the minimum of calorie intake.
2. Investigate the impact of the dragon's appearance on the ecological environment in the real world, and figure out the area that could sustain the dragons' life as well as the ecological conditions and the extent of the biomes that satisfy the necessary living conditions of them.
3. Resembling other animals, the dragon would migrate to various places as the local temperature changes. Therefore, we need evaluate the differences in the vital requirement of them living in an arid region, a warm temperate region, and the arctic region.

1.2 Problem Analysis

Since the dragon is a kind of beasts in fiction, for avoiding starting from scratch, we ought to consider them by analogy to animals existing in reality. We apply the related knowledge about ecology and biology etc. and the rule of thumb to help us clarify the ecological characteristics of the dragon. To help us conceive the image of the dragon better, we draw the conceptual graph, cf. Figure 1. n.b. To avoid disputes caused by copyright law, all figures in this paper are drawn, or generated by computer.

First of all, concerning the physical characteristics of the dragon, we focus on the trend between their weight and length change in accordance of time. Considering the above change is continuous, we set up a logistic growth differential equation, using the dragon's birth weight and the weight next year as two initial

Figure 1: The Hand Drawing Schematic Diagram of Dragon

values to obtain the weight function changing by time. Consequently, we could estimate the approximate range of the mature dragon's weight.

When analyzing the length change of the dragon, we assume that the dragon's limbs are on the ground when it is staying still, and regard it as an elastic cylinder. Based on mechanics, we can calculate the ratio of weight and length. Hence according to the initial value of weight and length of the dragon's birth, we can finally confirm the scale coefficient and the functional relationship of length to time. Considering that a dragon needs to maintain primary metabolism and life activities, the more weight it holds, the more energy it consumes. We estimate the daily energy consumption of a mature dragon according to an adult man. To maintain its long-term life activities, the dragon's daily energy intake must be approximately equal to its consumption.

Based on the above analysis, we study the habitat area of the three dragons and the conditions for the ecosystem and community size of the dragon. We assume that the three dragons will not compete with each other so that we only need to deal with the individual dragon entering a balanced ecosystem. Without loss of generality, we consider the dragon as the top predator of the ecosystem, regarding all animals as sheep and the area of the ecosystem as the habitat area of the dragon.

Then we let the number of sheep represent the supply of resources to the dragon in the habitat, and assume that this value is proportional to the ecosystem's area. Considering the factors affecting the number of sheep, we establish the Dragon-Sheep-Habitat Model based on differential equations and obtain a curve of the number of sheep over time.In order to ensure the sustainable supply of the ecosystem to the dragon, the initial number of sheep should have a minimum, otherwise the ecosystem will collapse due to the slow recovery.

To meet the minimum requirements for the dragon's activities, we consider this minimum as the community's capability to support the dragon in the habitat as the minimum standard for measuring the size of the ecosystem. Then, from this minimum value, we can refer the minimum habitat area for maintaining the dragon's survival. Hence, based on the above results, we discuss the impact of dragons on the ecosystem.

Considering the migration of dragons, the survival pressures of dragons in varied climate regions are different. We focus on the three different climate regions of the arid region, the warm temperate regions and the arctic region, to analyze the effect of different regions have on the survival resources of mature dragons. According to the above analysis of the daily energy intake of the dragon and the Dragon-Sheep-Habitat Model, we discuss the minimum requirement of resources for the dragon's long-term survival based on the different climatic conditions, the dragon's consumption and the ecosystem's supply difference, as well as the impact of the dragon on the local ecosystem.

2 Assumptions and Justifications

By adequate analysis of the problem, to simplify our model, we make the following well-justified assumptions.

1. The growth of dragons conforms to fundamental biological laws. If the dragon lives in the real world, as a species on the earth, its growth and maturation should be the same as other creatures, to meet the fundamental biological laws.

2. The dragon is a homothermal animal whose body temperature is not affected by the environment.

3. The dragon can fly and withstand huge wounds, and it will not easily die due to physical, chemical and biological attacks. According to the mythology and the TV series, we can consider that the dragon is with a capability to fly, and its flight satisfies the aerodynamics. In the meantime, the dragon is overgrowing, and its matured shape surpasses all the terrestrial creatures, and any species is unable to bear the *Dracarys* from the dragon.

4. The dragon's daily energy intake at least meets its daily energy consumption for maintaining its primary life activities and growth.

5. We suppose the birth length of a dragon is 30-40 cm. Since the dragon's Physiological information is lack of scientific basis, we acquire the above assumption by inferring from ancient reptile larvae and the TV series.

6. Once the dragon enters an ecosystem, it immediately becomes the top predator. Nevertheless, it will not cause catastrophic destruction to the biosphere. Besides, humankind can efficiently control the dragon.

3 Notations

For convenience, we use the following symbols in our models, cf. Table 1.

Table 1: Symbols and its Description

Symbols	Description
t	The time
$W(t)$	The dragon's weight function changing by time
W_{max}	The theoretical weight of the mature dragon
$L(t)$	The dragon's length function changing by time
E	The daily energy consumption of the dragon
T	The environmental temperature
r	The inherent growth rate of the dragon
σ	The environmental factor
A	The area of a balanced ecosystem
$N(t)$	The number of sheep function changing by time
K	The maximum carrying capacity of an ecosystem
b	The environmental carrying capacity coefficient

4 Dragon's Physical Characteristics and Diet

4.1 Description of Dragon's Weight and Length

To clearly describe the physiological characteristics of a dragon [1], we consider that the dragon lives in a warm temperate region with a suitable climate, assuming that the temperature of the region is maintaining at $25°C$. Therefore, the dragon can obtain sufficient food from the region during its growth and maturation. Also, based on morphology, the size of creatures cannot be increased indefinitely. Similarly, the dragon should be the same, which guarantees that when the dragon grows to maturation, its final weight and length will be limited. In the meantime, we define the dragon's body length as the straight distance from shoulder to hip.

4.1.1 Dragon's Weight Logistic Growth Model

Considering when the dragon is born, its weight $W(t)$ changes continuously by time. In the process of its growth to maturation, we regard that it is always healthy and will not die. Based on the above assumption, we denote that the theoretical weight of the mature dragon is $W_{max}\ kg$.

By the Logistic retarded growth principle, we reckon that the effect of the weight growth rate should be embodied in the retardation force among the dragon's weight limitation and the environmental carrying capacity etc.

For now, we consider the most basic case: the growth without any limitation such as the environment and its maximum weight. Under the above situation, the dragon's weight should meet the equation (1).

$$\frac{dW(t)}{dt} = rW(t) \tag{1}$$

However, because of the morphological limitation of the dragon per se, the right side of the equation (1) should include the maximum weight retardation factor $(1 - W(t))/W_{max}$. Hence, we correct the equation (1) to the equation (2)

$$\frac{dW(t)}{dt} = rW(t)\left(1 - \frac{W(t)}{W_{max}}\right) \tag{2}$$

Besides, we consider the effect of the environment on the dragon's weight. Since for many organisms, their final size will decrease with increasing temperature [2]. In our model, we only consider the effect of temperature on the dragon's weight to substitute the environment effect.

We assume that the effect of temperature on the growth rate of dragons is linear, so we let σT refer to the inherent retardation of temperature on the growth rate. Thus correcting the retardation factor in equation (2) to $(1 - W(t)/W_{max} - \sigma T)$, we finally obtain the weight-time differential equation (3).

$$\frac{dW(t)}{dt} = rW(t)\left(1 - \frac{W(t)}{W_{max}} - \sigma T\right) \tag{3}$$

We choose the dragon's birth weight 10 kg as the initial value of the equation (3), i.e., $W(0) = 10$. We can possess the weight-time first-order nonlinear differential equation's Cauchy problem (4)

$$\begin{cases} \frac{dW(t)}{dt} = rW(t)\left(1 - \frac{W(t)}{W_{max}} - \sigma T\right) \\ W(0) = 30 \end{cases} \tag{4}$$

To solve the Cauchy problem (4), we denote that $\alpha = 1 - \sigma T$. Eventually, we find $W(t)$, the Logistic weight-time function (5).

$$W(t) = 10\alpha W_{max} \frac{e^{\alpha rt}}{\alpha K + 10e^{\alpha rt-1}} \tag{5}$$

Since the dragon's weight range from 30 to 40 kg next year, we assume that the growth rate of the dragon's weight is a constant. Based on the change in the dragon's weight during the year, we calculate the growth rate of the dragon's weight (6).

$$r = \frac{\Delta W(t)}{t} \tag{6}$$

We can know that the range of growth rate r is between 20 and 30 kg per year. Without loss of generality, we set the following parameters, cf. Table 2.

Table 2: The Parameter Values

$W(1)$ (kg)	r $(kg/year)$	T $(°C)$	σ
35	20	25	0.04

Hence, we can generate the figure of the equation (5), cf. Figure 2 .

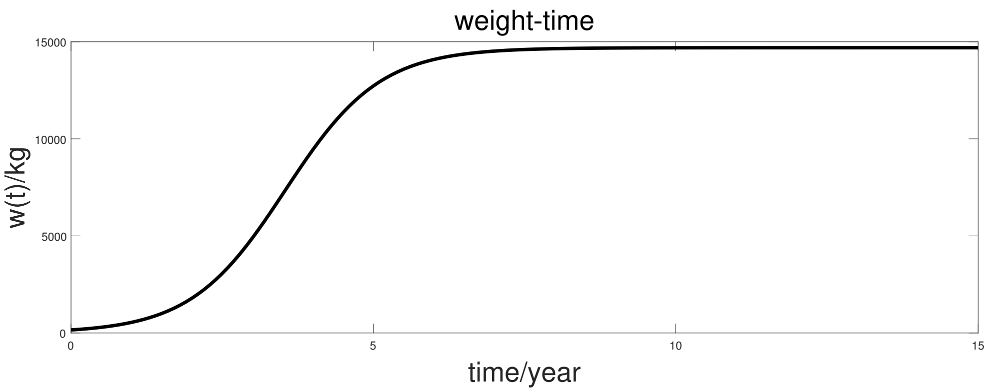

Figure 2: Dragon's Weight-Time Curve

4.1.2 Dragon's Length Analysis Based on Analogy

Consider the mature dragon is lying on the ground, with the limbs landing, staying still. Then we denote L m as the dragon's length, W kg as the dragon's weight. For simplifying the problem, we regard the dragons body as an elastic cylinder with length L m, diameter D m, base area S m^2, cf. Figure 3.

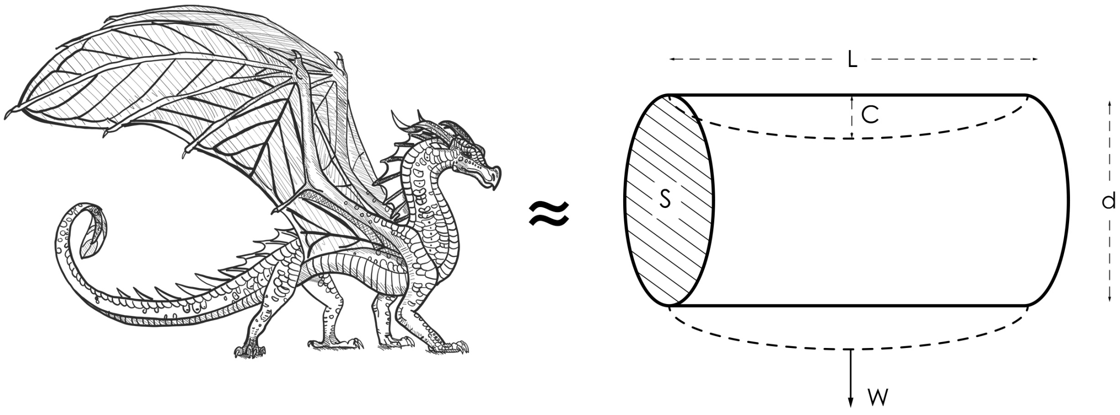

Figure 3: Regarding Dragon as an Elastic Cylinder by Analogy

Since the dragon's body sags under the influence of the gravity, we set the bending degree of the elastic cylinder to C m, which we can have the relationship (7) from mechanics [3].

$$\frac{C}{L} \propto \frac{WL^3}{Sd^2} \tag{7}$$

We know that during the evolution, C/L ought to be a constant [4]. Otherwise, creatures are unable to sustain their weight. Meanwhile, according to the relationship between mass and volume, base and diameter of the cylinder, we can obtain the relationship (8).

$$W \propto SL\ , \ \ S \propto d^2 \tag{8}$$

Via combining (7) and (8), we can finally obtain the equation (9), n.b. K is a scale coefficient.

$$W = kL^4 \tag{9}$$

Referring to the TV series, we assume that the infant dragon's length is from $0.7 \sim 0.9m$ [5]. Combined with the infant dragon's weight, we can calculate the scale coefficient K is around $15.24kg/m^4 \sim 41.65kg/m^4$. Since the mature dragon's weight is about $15ton$, according to the equation (9), we can know the mature dragon's length is around $4.35 \sim 5.63m$.

We choose $0.8m$ as the initial value of the dragon's length. Based on Figure 2, we generate the change graph between length and time, cf. Figure 4.

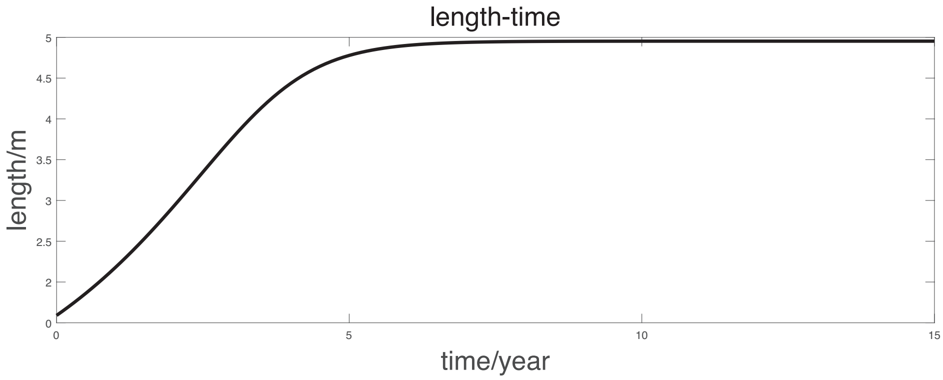

Figure 4: Dragon's Length-Time Curve

4.2 Dragon's Energy Expenditure and Caloric Intake

Inheriting the assumption of section 4.1, we consider that the dragon is a homeothermic animal [6]. The dragon's daily energy intake must guarantee its primary metabolism. At the same time, as a carnivore, the dragon can be analyzed by its daily meat intake. When examining its energy consumption, we believe that its most crucial energy consumption comes from the flight and fire breathing, thus establishing the primary models [7].

4.2.1 Dragon's Diet and Energy Consumption during Flight

To simplify the problem, we assume that the dragon is unable to breathe fire in this model. Hence, we can just investigate the energy consumption and intake in flight-only case. We compare the mature dragon with a Northman, a direwolf and a sparrow in different aspects [8].

Based on the daily energy intake of an adult Northman (c. $2,000cal/day$), we estimate the daily energy intake of the mature dragon by calculating the ratio of the weight of an adult Northman to the mature dragon, and it is about 400,000 calories [9].

Next, based on the daily meat intake of a direwolf (c. $12kg/day$), by calculating the ratio of the direwolf's weight to the mature dragon. We estimate that the mature dragon's daily meat consumption is roughly $900kg$, if the sheep's weight is $100kg$, then the dragon needs to consume about sheep $9pcs/day$.

Finally, based on the fat consumption of a sparrow per hour under normal flight, it is about $0.08g/hour$. By calculating the ratio of the sparrow's weight to the mature dragon, similarly, we estimate that the mature dragon will consume around $80kg/hour$. For a more intuitive comparison among the different creatures and dragon, we draw the schematic diagram cf. Figure 5.

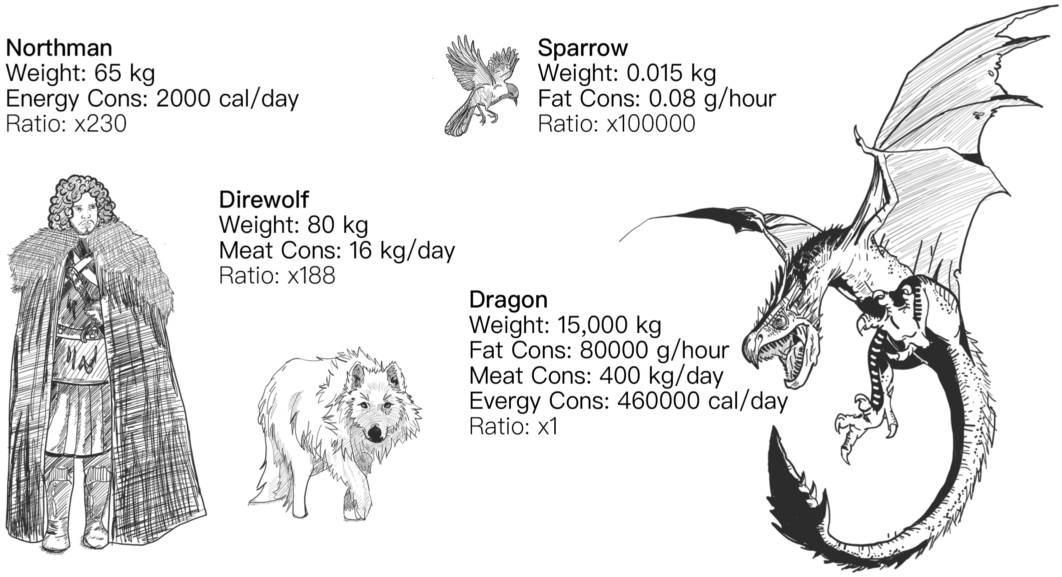

Figure 5: Consumption and Intake of Dragon vs. Other Creatures

4.2.2 Dragon's Fire Chemical Kinetics Model

Estimation of the Temperature of Dragon's Fire We can estimate the temperature by the status of burning objects. In the TV series, the Queen was fighting against the enemy troops while she was riding on the dragon. We suppose that the equipment of soldiers is made of iron. After the dragon's fire attack, all material is melt and vapored incompletely. The melting point of iron is $1538°C$, and the boiling point is $2862°C$ [10]. As a result, it can be certain that the temperature of the dragon's fire is around $2862°C$.

Chemical Component of Fire's Fuel According to the approximate temperature of $2862°C$, we list the burning temperature of common organic fuels [11], cf. Table 3.

Table 3: Burning Temperature List

Fuel	Temp.($°C$)	Fuel	Temp.($°C$)	Fuel	Temp. ($°C$)
Methanol	1100	Crude	1100	Gas	2020
Ethanol	1180	Gasoline	1200	LPG	2120
Acetone	1000	Acetylene	2127	Methane	1800
Ether	**2861**	Hydrogen	700	Ethane	1895

From Table 3, the burning temperature of ether is most approximate to the dragon's fire. Hence, we could regard ether as the fire's fuel.

Chemical Kinetics Mechanism for Producing Ether According to the Ockham's Razor Principle [12], the mechanism of chemicals produced by an organism should be the simplest. The simplest preparation method of ether is the dehydration of ethanol to ether under the catalysis of high temperature and concentrated sulfuric acid. We assume that the dragon can produce special catalysts, which can liberalize the strict reaction condition, as shown by the following chemical equation.

$$2\,CH_3CH_2OH\,(C_2H_6O) \xrightarrow[Concentrated\ H_2SO_4]{Special\ Catalyst} CH_3CH_2OCH_2CH_3\,(C_4H_{10}O)\uparrow + H_2O\uparrow$$

Thus, we can construct the structure of the dragon's fire-breathing organ consisting of gland A providing concentrated sulfuric acid, gland B providing anhydrous ethanol, and gland C as a reaction vessel. The high pressure inside the gland C and the catalyst can promote the reaction and keep the ether produced in a liquid state, which is easy to store and enhance the ignition point.

The sphincter is a valve controlling the excretion of ether. When the sphincter is tightened, the ether is locked in the gland C. When the sphincter relaxes, with the air entering, the pressure of the gland C rapidly decreases. Finally, the ether evaporates and ejects from the valve. The gaseous ether is mixed with the special catalyst, the ignition point is lowered to below the environmental temperature when reacted with oxygen in the air to generate carbon dioxide and water. Then a large amount of heat is released. The schematic diagram of the gland structure of the dragon and the chemical equation of the ether combustion is shown in Figure 6.

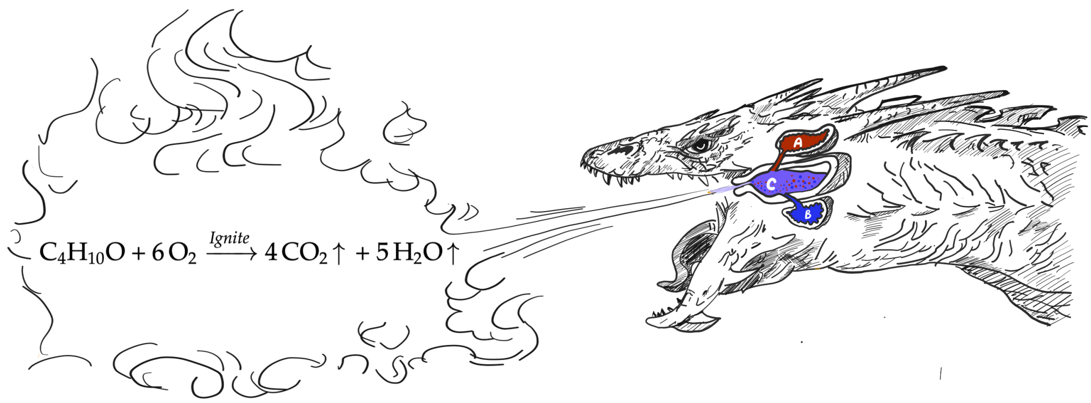

Figure 6: The Schematic Diagram of the Gland Structure

The sulfuric acid in the gland is produced by the stomach acid and sulfur-containing amino acids (methionine, cystine, cysteine) under the action of a catalyst, wherein the methionine needs to be obtained from meat. It also explains why the dragon needs to consume a large amount of meat for maintaining fuel production. Ethanol is an intermediate product in the decomposition of carbohydrates.

Estimation of Energy Consumption by Breathing Fire Through combustion heat of ether and the storage volume of gland C, we can estimate the energy consumption when the dragon is breathing fire. According to the TV series, we assume that the storage volume of gland C is roughly equivalent to the human's full stomach, denoting the gland storage volume V, the burning heat of ether Q at $25°C, 100kPa$. Suppose the pressure inside the gland C is the critical pressure of ether $3637.6kPa$. Based on that, we define the density of liquid ether as ρ, the molar mass of ether as M, the amount of substance as N, and the mass as m. Suppose the dragon consumes one gland of ether in a day, we finally define the energy consumption as Σ, cf. Table 4.

Table 4: Parameters and Their Values

V (m^3)	Q (kJ/mol)	ρ (kg/m^3)	m (kg)	M (kg/mol)	N (mol)
0.002	2752.9	265	0.74	$\rho V =$ 0.53	m/M =7.16

We can calculate the energy consumption: $\Sigma = Q\rho V/M = 19716.72\ kJ$. As a result, provided the dragon breathes fire, the energy consumption in a day will increase by $19716.72\ kJ$.

n.b. Since there are no fire-breathing creatures in the real world, in order to ensure the stcapability and capability of our models, in the subsequent analysis, we will not consider the consumption of energy by the dragon's fire.

5 Dragon-Sheep-Habitat Model

5.1 Analysis of Dragon's Habitat Demand

We consider the ecosystem that has reached an ecological balance, with an area of S. At some point, the mature dragon enters the ecosystem, denoting this moment as the initial time. We assume that the dragon can prey the rest of the animals. Without loss of generality, we can regard all the animals as sheep, the current ecosystem as a grassland ecosystem.

We reckon that the number of sheep $N(t)$ can be regarded as the supply of resources to the dragon at the time t. We regard the area of the ecosystem as the habitat area of the dragon.The sheep in the ecosystem embodies an indicator to measure the capability of the community to support the dragon in the habitat, and it can also be regarded as an indicator of the size of the community.

Denote the maximum carrying capacity of the ecosystem to the sheep is K, which is proportional to the area S. Since the ecosystem is balanced before the initial time ($t = 0$), we can obtain the initial condition (10) that the ecosystem should satisfy.

$$N(0) = K \tag{10}$$

Consider the limitation of environmental resources on the number of sheep and the growth of sheep per se. According to the retardation growth principle, in the absence of dragons, we can obtain the differential equation (11) that the number of sheep should satisfy [13]. n.b. The parameter p is the inherent growth rate of the sheep.

$$\frac{dN(t)}{dt} = pN(t)(1 - \frac{N(t)}{K}) \tag{11}$$

Since the dragon's entry will influent the ecosystem and affect the number of sheep, we modify the equation (11) to the following form (12). n.b. The parameter a is the theoretical predation of a mature dragon within a day, and its unit is pcs/day.

$$\frac{dN(t)}{dt} = pN(t)(1 - \frac{N(t)}{K}) - a \tag{12}$$

We combine the equation (10) and (12) to obtain the number of sheep change one-order non-linear equation's Cauchy problem (13) when the dragon enter the ecosystem.

$$\begin{cases} \frac{dN(t)}{dt} = pN(t)(1 - \frac{N(t)}{K}) - a \\ N(0) = K \end{cases} \tag{13}$$

Through our analysis of the daily energy intake of the mature dragon, we can know that the mature dragon needs to consume about $900kg$ meat per day (sheep's weight is about $100kg$), that is about 9 sheep. Without loss of generality, we can set that $a = 9, p = 0.01$. Furthermore, we select the different values of the ecosystem carrying capacity of sheep, and then we can solve the equation (13) to the graph about the number of sheep changing by time. e.g. Taking $K = 5000$, we obtain the graph as shown in Figure 7.

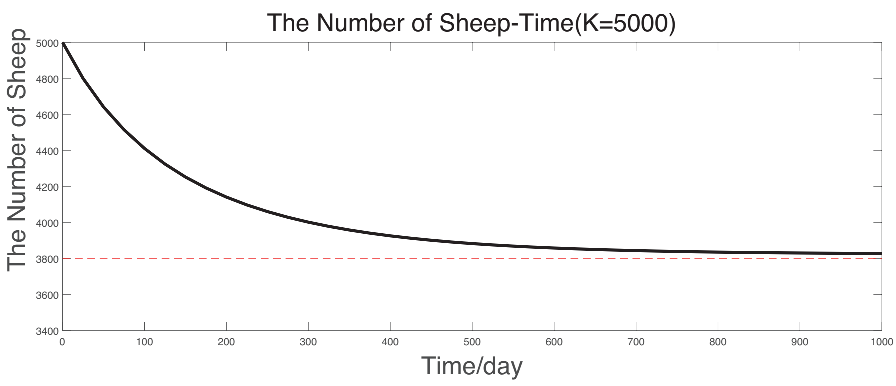

Figure 7: The Number of Sheep Changing by Time when K=5000

It can be seen from Figure 7, as the dragon enters the ecosystem, the number of sheep continues to decline, but the rate of decline gradually slows down. At about $t = 700$ (around the second year), the ecosystem is almost balanced. In fact, the value of K cannot be arbitrary, in order to satisfy the long-term survival of the mature dragon, when the K value is too small, the rate at the dragon preying the sheep is far greater than the recovery rate of the number of sheep. The number of sheep will soon reach zero, which will cause the ecosystem to collapse. e.g. Taking $K = 3000$, we obtain the graph as shown in Figure 8.

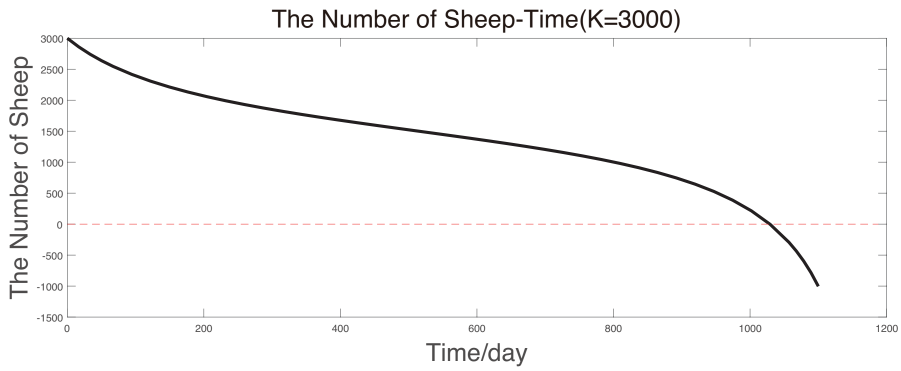

Figure 8: The Number of Sheep Changing by Time when K=3000

Figure 8 shows that $N(t)$ reaches zero at about $t = 1000$ (around the third year). Therefore, we conclude that in order to meet the basic living needs of the dragon, and for maintaining the ecosystem's sustainable development, the value of K should have critical value. We choose different K values and finally obtain the forms of the corresponding ecosystem, cf. Table 5.

Table 5: K Value and Corresponding Ecosystem

K Value	Ecosystem Development Trend
100	Ecosystem Collapses in 10 Days
500	Ecosystem Collapses in 2 Months
1000	Ecosystem Collapses in 5 Months
2000	Ecosystem Collapses in 1 Year
3000	Ecosystem Collapses in 3 Years
3500	Ecosystem Collapses in 10 Years
4000	Ecosystem Maintains Long-Term Balance with 2600 Sheep
5000	Ecosystem Maintains Long-Term Balance with 3800 Sheep

From Table 5, the critical value of K is about 4000. We assume that the ecosystem's area A is proportional to its maximum environmental carrying capacity. Then we have the relationship (14). n.b. The parameter b is the coefficient of the environmental carrying capacity, indicating the number of sheep that can be carried per square meter of the ecosystem's area.

$$K = bA \tag{14}$$

In the case that the pasture will not degrade, about 30 mu of grassland can raise one sheep, taking into account the active area of the sheep, and about $150mu$ of land carries one sheep. From this ratio, if you want to carry $4,000$ sheep, about $600,000mu$ of land (i.e., $400km^2$). Therefore, we estimate that the habitat of dragons can be at least $400km^2$. Considering that the area of Yellowstone National Park is $900,000$ hectares [14], so the Park can feed about 22 dragons.

Based on the above analysis, we can obtain the minimum requirements for the dragon's habitat and the minimum standards for the community's capability to support the dragon in the habitat. Here are our conclusions:

- The habitat for an individual dragon is at least $400km^2$, around 120 Central Parks in New York [10], and we draw a schematic graph cf. Figure 9.

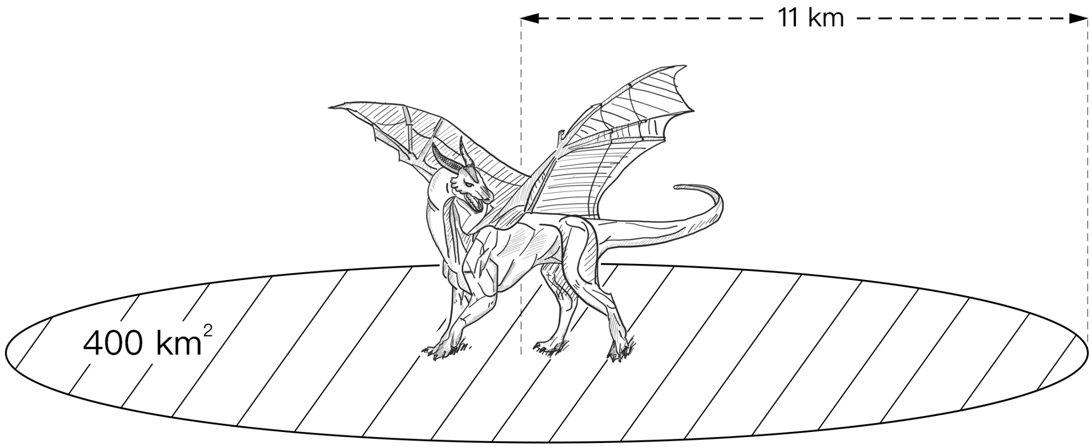

Figure 9: The Minimum Area of Dragon's Habitat

- At least 4,000 sheep in the habitat could guarantee the long-term support for the dragon. i.e., The community of the ecosystem needs at least 4,000 herbivores and carnivores. From biology, the energy transfer efficiency between food chains is about 20%, so the number of herbivores in the community is about 3,000, and carnivores is about 1,000. We draw the trophic structure of the community which dragons participated, cf. Figure 10.

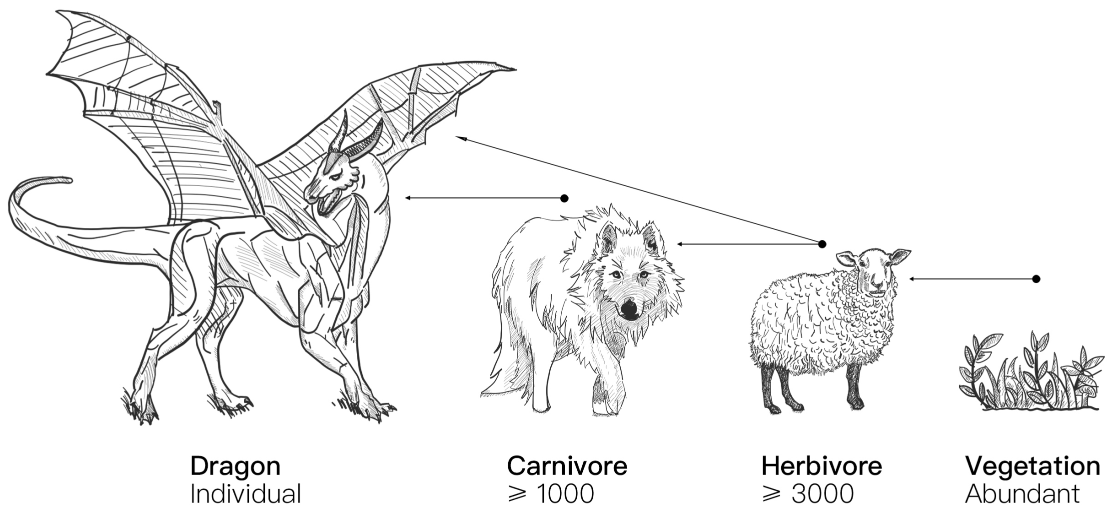

Figure 10: Food Chain with Dragon's Participation

5.2 Dragon's Impact on The Ecosystem: Balance or Collapse?

We still consider the ecosystem reached an ecological balance, which can sustain itself in the long run without the dragon. Assuming that the dragon can prey all creatures in the ecosystem, we regard the food chain of the ecosystem as a dragon-sheep-grass model. Based on the above analysis of the habitat and K value of the dragon, we can draw the following conclusions:

- When the K value is very small (e.g. less than 1000), the ecosystem will be destructed in a short term due to the dragon's entry. i.e., the sheep in the food chain are extinct, then the dragon has no food source and will eventually become extinct. i.e., the ecosystem cannot self-recover in time in this situation.
- Since the K value has the critical value, when the K value is greater than or equal to the critical value, the number of sheep in the food chain tends to be stable. The ecosystem can provide long-term survival needs for the dragon, i.e., the entire ecosystem will return to balance.

6 Impact of Three Regions on Dragon's Survival

6.1 Difference in Daily Energy Intake of Dragons in Three Regions

We still assume that the dragon is a homothermal animal, i.e., it needs to maintain its body temperature relatively stable in the changes of environmental temperature. Consider three different regions in climate: an arid region, a warm temperate region, and the arctic region. The three regions have different temperatures, especially in the arctic region, where the dragon consumes more energy per day to maintain its activities and metabolism.

Combined with the metabolic rate of the human body at different temperatures, we speculate that the dragon's daily energy intake in the arid region and the arctic region should be 0.9 times and 1.3 times, respectively, in the warm temperate region. According to our analysis of the dragon's daily energy intake, we can know that the dragon needs to consume about $460,000$ calories per day (i.e., 9 sheep) in the warm temperate region, thus we calculate the energy consumption in the arid region is $414,000$ calories and $598,000$ calories in the arctic region, which is equivalent to consuming 8 and 12 sheep per day.

6.2 Three Regions' Capability about Self-Recovery and Supporting Dragons

There are huge differences in vegetation and animal resources in the three regions. The more the number and variety of animals and plants, the higher the species richness of the ecosystem and the self-recovery capability in the regions (i.e., return to the original state after destructed). It is clear that the species richness in the warm temperate region is much greater than the arid and arctic regions. Therefore the self-recovery capability of the warm temperate region is far greater than that in the arid and arctic regions.

Based on the Dragon-Sheep-Habitat Model, we use the parameter p in the model as an indicator of the self-recovery capability of an ecosystem. We assume that in the arid region and the arctic region $p = 0.005$, and in the warm temperate region $p = 0.01$. Furthermore, under different climatic conditions, the distribution density of creatures in an ecosystem is different. In the warm temperate region, the distribution density of creatures is much greater than that in the arid and arctic regions. We set the parameter b in the model as an indicator of the distribution density of the area. We assume that b is $10pcs/hm^2$ in the warm temperate region and $13pcs/hm^2$ in the arid and arctic regions.

6.3 Minimum Resource and Habitat Demand for Dragon's Survival in Three Regions

Combining the analysis of 6.1 and 6.2, we substitute the parameters p (i.e., the inherent growth rate of the number of sheep) and b (i.e., the environmental carrying capacity coefficient) into the equation (13) to calculate the critical value of K (i.e., maximum carrying capacity in three regions). cf. (15).

$$K_{arid} = 6500,\ K_{warm} = 4000,\ K_{arctic} = 10000 \tag{15}$$

According to the environmental carrying capacity coefficient b in different regions, we can calculate the minimum habitat area under three regions. cf. (16).

$$A_{arid} = 187,000hm^2,\ A_{warm} = 40,000hm^2,\ A_{arctic} = 133,000hm^2 \tag{16}$$

In addition, we draw a diagram of the minimum demand of dragon's survival in the three regions as follows. cf. Figure 11.

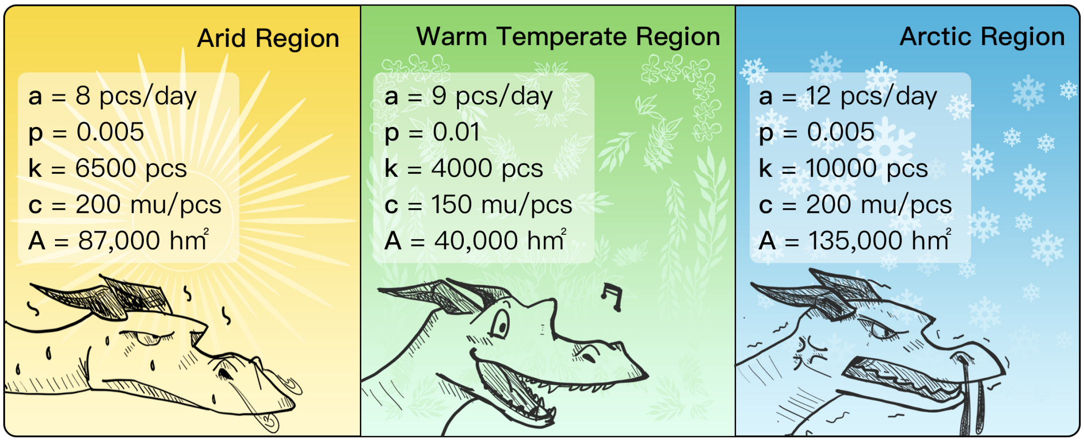

Figure 11: Minimum Demand of Dragon's Survival in Three Regions

7 Sensitivity Analysis

In the Dragon's Weight Logistic Growth Model, the dragon's weight after one year, the inherent growth rate of the dragon's weight, the environmental factor, and the environmental temperature all have impacts on the model's result. Since the dragon's weight after one year is closely related to the inherent growth rate, we only consider the effects of changes in inherent growth rate, environmental factor and environmental temperature on the model.

7.1 Impact of Inherent Growth Rate r on Dragon's Weight

According to the equation (6), we can calculate the inherent growth rate r of the dragon's weight from $20kg/year$ to $30kg/year$. We fix the environmental temperature T as $25°C$, the environmental factor σ as 0.04. Then we take a linear increase of r by step 2, and plot the function $W(t)$ as shown in Figure 12.

As we can see in Figure 12, the weight W(t) of the mature dragon varies from $1.2tons$ to $4.5tons$ at a rate of $2kg/year$, which is weakly relative to its total weight. Therefore $W(t)$ is not sensitive to the change of r, which also shows that the model per se is non-sensitive to r.

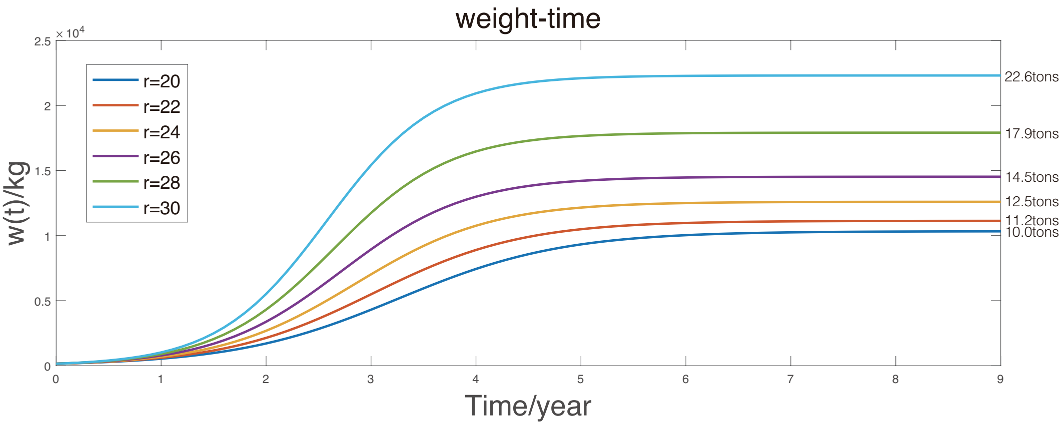

Figure 12: Sensitivity Analysis on Inherent Growth Rate r

7.2 Impact of Environmental factor σ on Dragon's Weight

In our model, the environmental factor measures the retardation effect of the environment on the dragon's weight. We fix the environmental temperature T as $25°C$, the inherent growth rate r of the dragon's weight as $20kg/year$. We select different σ values, and plot the function $W(t)$ as shown in Figure 13.

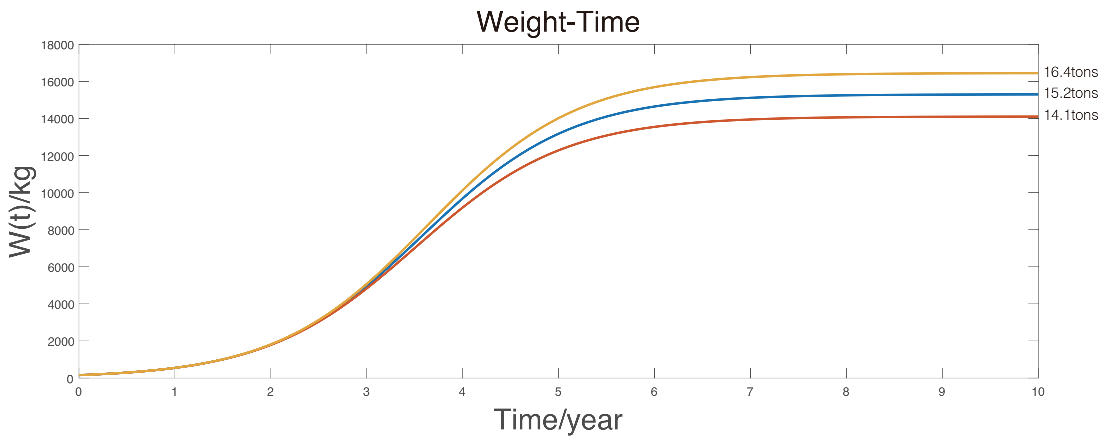

Figure 13: Sensitivity Analysis on Environmental Factor σ

It can be seen from Figure 13 that as the σ increases, the stable value of W(t) is decreasing. It indicates that the environment has effects on the dragon's weight while the temperature remains constant.

7.3 Impact of Environmental Temperature T on Dragon's Weight

In the end, we consider the effect of environmental temperature on the dragon's weight. We fix r as $20kg/year$ and σ as 0.04. Select different T values and plot the function $W(t)$ as shown in Figure 14.

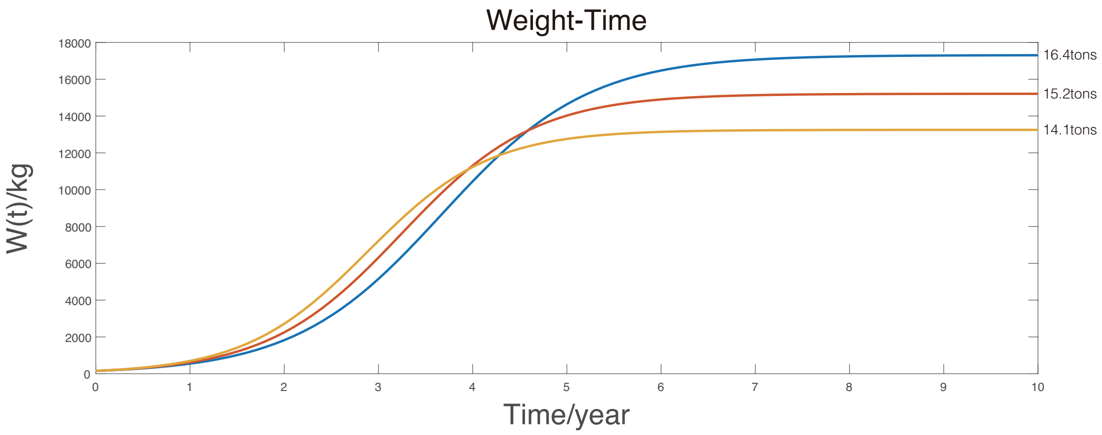

Figure 14: Sensitivity Analysis on environmental Temperature T

From Figure 14, we can see the difference in temperature has effects on the weight $W(t)$. Due to the correlation between the dragon's length and weight, these parameters reflect the size of the dragon. i.e., the larger their values, the larger the dragon's size. Therefore, according to Figure 14, we can obtain that the size of the mature dragon is smaller in hot regions, on the contrary, the size is larger in cold regions, which satisfies the ecological laws. And also it illustrates the rationality of our model.

8 Model Evaluation and Further Discussion

8.1 Evaluation of Models

Like the creature in fiction, the dragon is mysterious and enchanting. In order to explore the characteristics of dragons, we combine the characteristics of the real creatures, physics, chemistry, biology, ecology, and other knowledge to establish series of models, the Dragon's Weight Logistic Growth Model, the Dragon's Fire Chemical Kinetics Model, and the Dragon-Sheep-Habitat Model, etc. We answer the following questions precisely by analogy, such as the analysis of the dragon's basic characteristics, the dragon's daily energy intake and consumption, the minimum requirement for the habitat to support the dragons, the minimum area of the habitat, and the impact on local ecosystems with the dragon's entry.

Throughout our modeling, we set reasonable assumptions to provide convenience to build and solve models. We also take the sensitivity analysis which verified the stability of the models.

We innovatively analyze the dragon's fire-breathing mechanism and explore the possibilities and operations of its implementation. Under the assumption that the dragon can breathe fire, we established the Chemical Kinetics Model of the dragon's fire to investigate the physiological and energy characteristics of the fire-breathing mechanism.

However, our models still have the following weaknesses:

- Our model contains many parameters. The values of these parameters largely depend on biological knowledge. Due to the diversity and complexity of the creatures, the values will be very different, which will affect the results of the models.

- To simplify the model, our Dragon-Sheep-Habitat Model is aimed at the mature dragon, which does not account changes in energy intake during the dragon's growth, hence the model has limitations to some extent.

8.2 Further Discussion

In our Dragon-Sheep-Habitat Model, we explore the problems about the mature dragons. De facto, if we add the related variables and functions, based on the original model, we can also investigate the dragon from infancy to maturation, such as its patterns of the energy intake changes, its impact on the ecosystem, and its habitat requirements etc.

Our models are combined with the relevant knowledge, generally satisfy mathematical principles and natural laws. Although the dragon does not exist in the real world, the Dragon-Sheep-Habitat Model can also try to solve the invasion problem of huge alien species, the Dragon's Weight Logistic Growth Model can try to solve the dinosaurs' growth problem, the Dragon's Fire Chemical Kinetics Model can assist the design of a new type flamethrower.

9 A Letter to Martin

(See next two pages)

A Letter to George R.R. Martin

Team # 1910246

Team # 1910246
MCM/ICM Contest
Problem A
Game of Ecology

January 29, 2019

Mr.
George R.R. Martin
The Creator of
Song of Ice and Fire

Dear Martin,

We hope everything is fine with you.

Thank you very much for creating the epic series and the outstanding novels *A Song of Ice and Fire*, which we are immensely addicted, as well as the TV series *Game of Thrones* adapted from your books. The dragons appearing in the TV series are fantastic and enchanting. The figure of the dragon has struck a chord with us, which aroused our curiosity and interest in researching dragons.

According to your descriptions of dragons in the novel, we start to consider that if a dragon lives in the real world, how can its essential physical characteristics be? What is its living habits like? How many resources should we provide for it to maintain its survival and growth? What impacts would it make on the local ecosystem? Based on these problems, we combine mathematics, physics, chemistry, biology, ecology and other related subjects to try to establish various models to analyze the characteristics of dragons and to make scientific answers to these questions.

From the model we established and the analyze the dragon's daily caloric intake and energy expenditure. We can know that from its infancy to maturation, the food intake of the dragon is increasing. The growth rate of the food intake is about one sheep per year. When the dragon becomes mature, it should at least consume nine sheep per day to maintain its primary activities. Thus, this food intake is quite large, that is to say, the dragon may have a significant impact on the local ecosystem.

To ensure the scientificalness and rationality of the dragons in the books, we have paid attention to the impacts of the dragon on the real ecosystem, especially the migration of dragons. Therefore, we have considered that the difference of the living conditions of the dragons and the impacts they make on the ecosystem when the dragon lives in an arid region, a warm temperate region or the Arctic region. In consequence, we suggest that to maintain the realistic ecological basis to support your story, and you may well pay attention to the following points:

- The best living area for dragons is the warm temperate region, where the climate is humid, with high species richness. Therefore, the capacity for maintaining the dragons is enormous enough.
- The dragon should live far away from the arid region and the Arctic region. The dragon itself will not like these kinds of regions with lousy living conditions and climates.
- We try to analyze the mechanism of the dragon breathing fire. We have found that the energy expenditure for breathing fire are very huge. Therefore, we hope that the times for the dragon to breath fire should not be too many per day, because this behavior costs a lot for its body to burden.
- According to our models' conclusions, the weight of the mature dragon may range from 10 ton to 20 ton. Therefore, the energy consumption is enormous when it is flying over a long distance. Be careful about the limitation of the flight distance and time.
- In the foraging activities, a dragon should seek for food in a wide range of areas. Considering the compatibility of the dragon with the ecosystem, to maintain the balance of it, you ought to let the dragons avoid causing catastrophic damage to the ecosystem.

We hope that our suggestions can help you with your subsequent creations, and it will be greatly honored for us if our recommendations are adopted! *A Song of Ice and Fire* really makes us enchanted and satisfied. We admire your imagination, and your capacity is shown in arranging such a large story setting. By the way, your mania fan in our team draw a picture as a gift for you.

We wish you every success in the coming year. If you want more information, please feel free to write to us. Looking forward to your new books again!

Yours Sincerely,

Team # 1910246

References

[1] May, R.M. The Ecology of dragons[J]. Nature Publishing Group, 1976, 264: 16

[2] Begon, M. & C. R. Townsend. Ecology From Individuals to Ecosytems [M]. Beijing: Higher education publishing house, 2016: 34-35.

[3] Cristescu, N.D. & E.M. Craciun et al. Mechanics of Elastic Composites [M]. Florida: CRC Press, 2003: 45-70.

[4] Wikipedia Evolution https://en.wikipedia.org/wiki/Evolution. [Online; accessed 26-February-2019]

[5] Youtube. Game Of Thrones All dragon scenes seasons 1-7. https://www.youtube.com/watch?v=DKKPDlwRMFk. [Online; accessed 26-February-2019]

[6] Wikipedia. Warm-blooded. https://en.wikipedia.org/wiki/Warm-blooded. [Online; accessed 26-February-2019]

[7] Ramsey, J. J. & K. Hagopian. Energy Expenditure and Restriction of Energy Intake: Could Energy Restriction Alter Energy Expenditure in Companion Animals? [J]. The Journal of Nutrition, 2006, 136(7): 1958-1966.

[8] Green J. A. & L. G. Halsey et al. Estimating energy expenditure of animals using the accelerometry technique: activity, inactivity and comparison with the heart-rate technique [J]. The Journal of Experimental Biology, 2009, 212: 471-482.

[9] Wikipedia. Calorie. https://en.wikipedia.org/wiki/Calorie. [Online; accessed 27-February-2019].

[10] Wikipedia. iron. https://en.wikipedia.org/wiki/Iron. [Online; accessed 27-February-2019].

[11] LIU G.Q. Chemical and chemical data sheet [M]. Amoy: Xiamen University PressïijŇ2001:42-43.

[12] Wikipedia. Occam’s razor. https://en.wikipedia.org/wiki/Occam%27s_razor. [Online; accessed 28-February-2019].

[13] Jiang Q. Y. & J. X. Xie & J. Ye. Mathematical model [M]. Beijing: Higher education publishing house, 2011: 236-230.

[14] Wikipedia. Yellowstone National Park. https://en.wikipedia.org/wiki/Yellowstone_National_Park. [Online; accessed 28-February-2019].

[15] Wikipedia. Central Park. https://en.wikipedia.org/wiki/Central_Park. [Online; accessed 28-February-2019].

Appendices

Appendix A Plots

the curve of the function N(t), on the condition of $a = 9$, $p = 0.01$ with different K values.

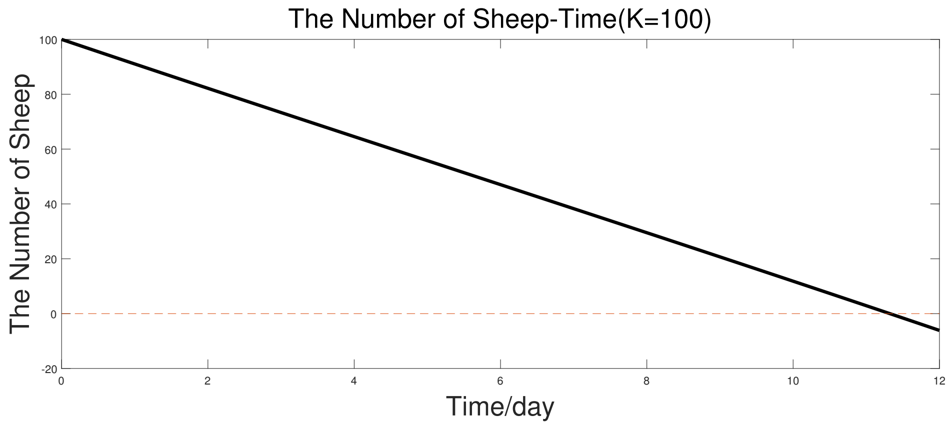

Figure 15: K=100, Ecosystem Collapses in 10 Days

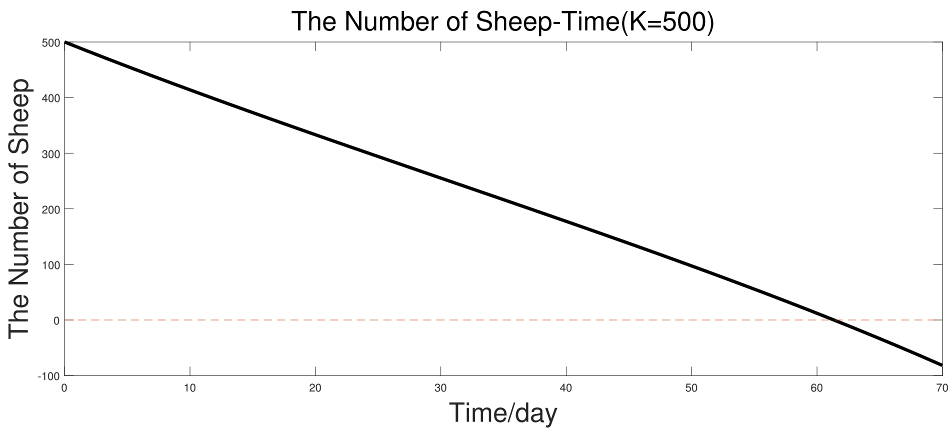

Figure 16: K=500, Ecosystem Collapses in 2 Months

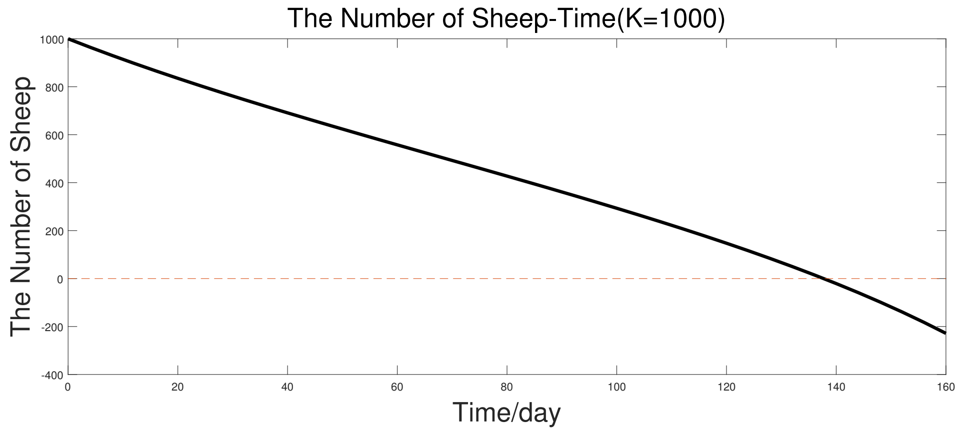

Figure 17: K=1000, Ecosystem Collapses in 5 Months

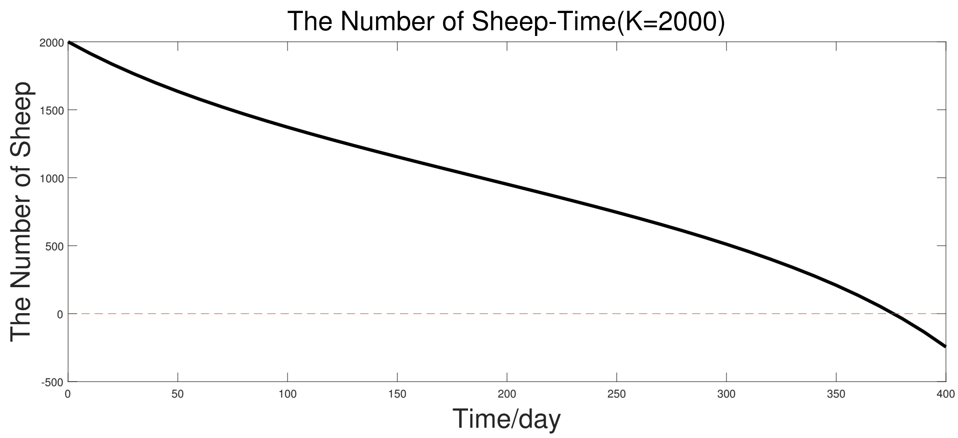

Figure 18: K=2000, Ecosystem Collapses in 1 year

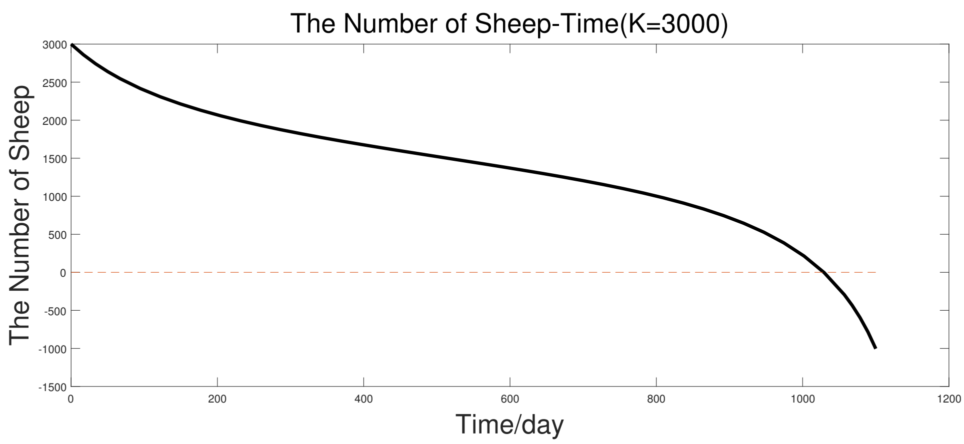

Figure 19: K=3000, Ecosystem Collapses in 3 years

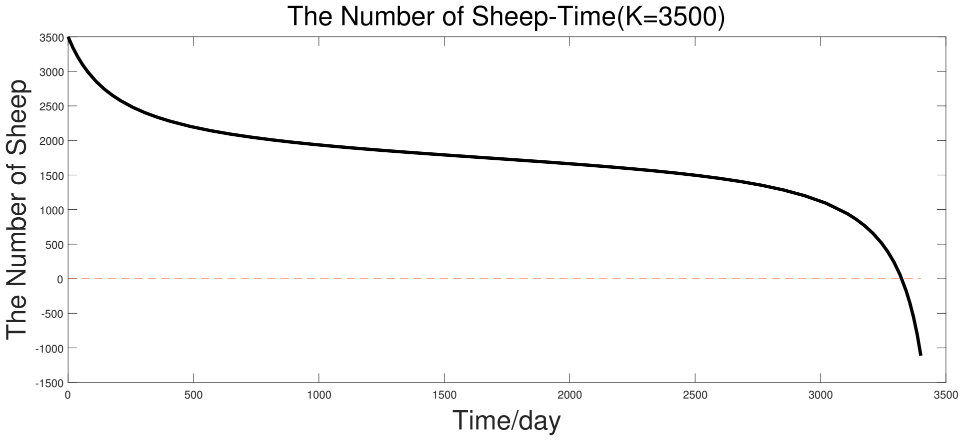

Figure 20: K=3500, Ecosystem Collapses in 10 years

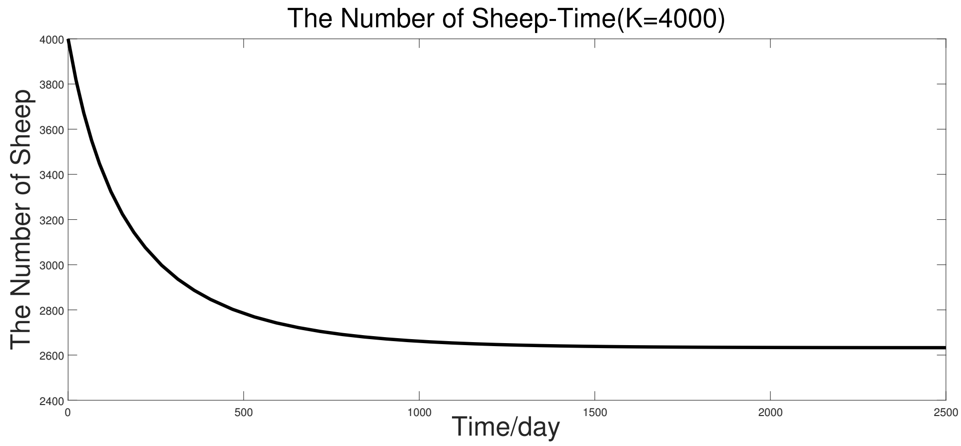

Figure 21: K=4000, Ecosystem Maintains Long-Term Balance

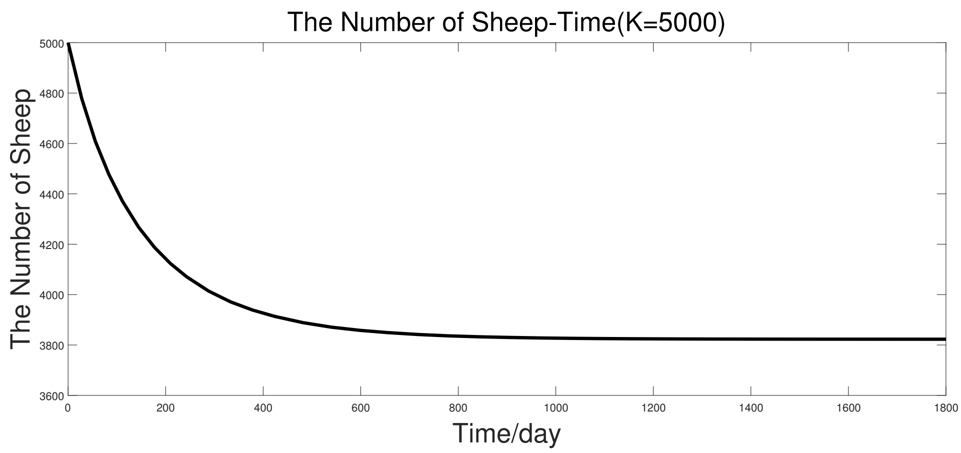

Figure 22: K=5000, Ecosystem Maintains Long-Term Balance

Appendix B Codes

MATLAB Source Code: Plotting the Curve of the Function W(t)

```
%%%weight - time curve
p0=10;%%%Initial weight value
k=10694;%%%Theoretical upper limit
r=20;%%%Growth rate
a=0.064;%%%Environmental impact factor
t=0:0.1:15;%%%timeline

p_u=a*k*p0*exp(a*r*t);%%%weight fraction molecule
p_l=a*k+p0*(exp(a*r*t)-1);%%%Weight fraction denominator
p=p_u./p_l/a;%%%Weight

%%%Draw a curve of body weight over time
plot(t,p)
xlabel('time/year')
ylabel('w(t)/kg')
title('weight-time')
```

MATLAB Source Code: Plotting the Curve of the Function L(t)

```
%Draw the  image of body length changes over time

l0=0.8;%Initial value of body length
l=l0*(p/10).^(1/4);%Relationship between body length and body weight

%Draw an image of body length over time
plot(t,l)
hold on
xlabel('t/year')
ylabel('length/m')
title('length-t')
```

MATLAB Source Code: Dragon-Sheep-Habitat Model

```
function function3(a,r,k)
%%%Dragon Sheep Habitat Model Code
x0=k;

s=@(t,x) r*x.*(1-x/k)-a;

%Numerical Solution
[t x]=ode45(s,[0 1000],x0);

%Draw horizontal line y=0
plot(t,x)
y=zeros(1,length(x));
hold on
plot(t,y)
hold off

end
```