

1.1:

Substitution Method:

Hypothesis: $T(n) = O(n^2)$;

Show: $T(n) \leq cn^2$ for some $c > 0$;

Assume: $T(n/3) \leq c(n/3)^2$;

$$T(n) \leq 2c(n/3)^2 + n^2;$$

$$= (2c/9)n^2 + n^2;$$

$$= (2c/9 + 1)n^2;$$

To accomplish $T(n) \leq cn^2$, we want $2c/9 + 1 \leq c$;

Thus, $2c + 9 \leq 9c \Rightarrow c \geq 9/7$

$T(n) \leq cn^2$ for $c \geq 9/7$;

Therefore $T(n) = O(n^2)$.

1.2:

Recursion Method:

$$T(n) = 2T(n/3) + n^2;$$

$$T(n) \quad n^2 \quad \Rightarrow n^2$$

$$\begin{array}{ccc} & / & \backslash \\ & / & \backslash \\ (n/3)^2 & & (n/3)^2 \end{array} \quad \Rightarrow 2/9 n^2$$

$$\begin{array}{cccc} / & \backslash & / & \backslash \\ (n/9)^2 & (n/9)^2 & (n/9)^2 & (n/9)^2 \end{array} \quad \Rightarrow (2/9)^2 n^2$$

...

$$T(1) \ T(1) \ \dots \quad T(1) \ T(1) \quad \Rightarrow \Theta(n^{\log_3 2})$$

Total $\log_3 n$ layers.

$$2^{\log_3 n} = n^{\log_3 2} T(1)s$$

Total: $O(n^2)$

2.(a):

$$T(n) = 2T(n/4) + n\sqrt{n};$$

$$a = 2 > 0; \quad b = 4 > 0; \quad f(n) = n\sqrt{n};$$

$$n^{\log_4 2} = n^{1/2};$$

$$f(n) = n^{1.5} = \Omega(n^{\log_4 2 + 1}); \quad \epsilon = 1 > 0;$$

$$af(n/b) = 2(n/4)^{1.5} = 2 * 4^{-1.5} n^{1.5} = 0.25 n^{1.5} < cn^{1.5} \text{ for } c = 0.5 < 1$$

$$\text{Therefore: } T(n) = \Theta(n^{3/2})$$

2.(b):

$$T(n) = 2T(n/3) + 5^{\log_2 n}$$

$$a = 2 > 0; \quad b = 3 > 0; \quad f(n) = 5^{\log_2 n} = n^{\log_2 5} = n^{2.32};$$

$$n^{\log_3 2} = n^{0.63};$$

$$\text{So } f(n) = \Omega(n^{\log_3 2 + \epsilon}); \text{ for } \epsilon = \log_2 5 - \log_3 2 = 1.69 > 0;$$

$$af(n/b) = 2((n/3)^{\log_2 5}) = 2n^{\log_2 5} * (\frac{1}{3})^{\log_2 5} = 0.156 n^{\log_2 5} < cf(n) \text{ for } c = 0.3 < 1;$$

$$\text{Therefore: } T(n) = \Theta(n^{\log_2 5});$$

3.

Exercise 2.2-1

Express the function $n^3/1000 - 100n^2 - 100n + 3$ in terms of Θ -notation.

$$T(n) = n^3/1000 - 100n^2 - 100n + 3 = \Theta(n^3);$$

To prove that, there is a c_1, c_2 that:

$$c_1 n^3 \leq n^3/1000 - 100n^2 - 100n + 3 \leq c_2 n^3;$$

$$c_1 \leq 1/1000 - 100/n - 100/n^2 + 3/n^3 \leq c_2;$$

When n is very large:

$$-100/n - 100/n^2 + 3/n^3 = 0;$$

Therefore:

$$c_1 \leq 1/1000 \leq c_2;$$

We choose $n^0 = 10^6$ (in order to keep $c_1 > 0$, n^0 must be large enough), thus;

$$c_1 \leq 1000000^3 / 1000 - 100 * 1000000^2 - 100 * 1000000 + 3 \leq c_2;$$

$$c_1 \leq 10^{15} - 10^{14} - 10^8 + 3 \leq c_2;$$

We combine these two inequalities together:

$$c_1 \leq 1/1000 \leq 10^{15} - 10^{14} - 10^8 + 3 \leq c_2;$$

So, when we choose $n^0 = 10^6$; $c_1 = 10^{-5}$; and $c_2 = 10^{15}$:

$$c_1 n^3 \leq n^3/1000 - 100n^2 - 100n + 3 \leq c_2 n^3 \text{ is always true;}$$

Therefore, by definition: $T(n) = n^3/1000 - 100n^2 - 100n + 3 = \Theta(n^3)$;

4.

Substitution Method:

Hypothesis: $T(n) = O(n \lg n)$;

Show: $T(n) \leq cn \lg n$ for some $c > 0$;

Assume: $T(n/3) \leq c(n/3) \lg(n/3)$;

$T(2n/3) \leq c(2n/3) \lg(2n/3)$;

$$\begin{aligned} T(n) &\leq c(n/3) \lg(n/3) + c(2n/3) \lg(2n/3) + n \\ &= cn/3(\lg n - \lg 3) + 2cn/3(\lg n - \lg(3/2)) + n \\ &= cn/3 \lg n - cn/3 \lg 3 + 2cn/3 \lg n - 2cn/3 \lg(3/2) + n \\ &= (cn/3 \lg n + 2cn/3 \lg n) - cn/3 \lg 3 - 2cn/3(\lg 3 - \lg 2) + n \\ &= cn \lg n - cn \lg 3 + 2cn/3 \lg 2 + n \\ &= cn \lg n + n(1 + 2c/3 \lg 2 - c \lg 3) \leq cn \lg n \end{aligned}$$

To make this work, we need:

$$n(1 + 2c/3 \lg 2 - c \lg 3) \leq 0$$

Because n is a positive integer, therefore:

$$1 + 2c/3 \lg 2 - c \lg 3 \leq 0$$

$$c(2/3 \lg 2 - \lg 3) \leq -1$$

We can get $2/3 \lg 2 - \lg 3 = -0.92$

So: $-0.92c \leq -1$

$$c \leq -1/-0.92$$

$$c \leq 1.09$$

Now we find out, when $c \leq 1.09$, $T(n) \leq cn \lg n$

Therefore $T(n) = O(n \lg n)$