1.1:

Substitution Method:

Hypothesis: $T(n) = O(n^2)$;

Show: $T(n) \leq cn^2$ for some $c>0$;

Assume: $T(n/3) \leq c(n/3)^2$;

$T(n) \leq 2c(n/3)^2 + n^2$;

$= (2c/9)n^2 + n^2$;

$= (2c/9 + 1)n^2$;

To accomplish $T(n) \leq cn^2$, we want $2c/9 + 1 \leq c$;

Thus, $2c + 9 \leq 9c \Rightarrow c \geq 9/7$

$T(n) \leq cn^2$ for $c \geq 9/7$;

Therefore $T(n) = O(n^2)$.

1.2:

Recursion Method:

$T(n) = 2T(n/3) + n^2$;

$T(n)$ n^2 $\Rightarrow n^2$

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$(n/3)^2$ $(n/3)^2$ $\Rightarrow 2/9\ n^2$

/ \ / \

$(n/9)^2$ $(n/9)^2$ $(n/9)^2$ $(n/9)^2$ $\Rightarrow (2/9)^2\ n^2$

… … … …

$T(1)$ $T(1)$ … $T(1)$ $T(1)$ $\Rightarrow \Theta(n^{\log_3 2})$

Total $\log_3 n$ layers.

$2^{\log_3 n} = n^{\log_3 2}$ $T(1)$s Total: $O(n^2)$

2.(a):

$T(n) = 2T(n / 4) + n\sqrt{n}$;

$a = 2 > 0$; $b = 4 > 0$; $f(n) = n\sqrt{n}$;

$n^{\log_4 2} = n^{1/2}$;

$f(n) = n^{1.5} = \Omega(n^{\log_4 2+1})$; $\varepsilon = 1 > 0$;

$af(n/b) = 2 (n/4)^{1.5} = 2 * 4^{-1.5} n^{1.5} = 0.25 n^{1.5} < cn^{1.5}$ for $c = 0.5 < 1$

Therefore: $T(n) = \Theta(n^{3/2})$

2.(b):

$T(n) = 2T(n/3) + 5^{\log_2 n}$

$a = 2 > 0$; $b = 3 > 0$; $f(n) = 5^{\log_2 n} = n^{\log_2 5} = n^{2.32}$;

$n^{\log_3 2} = n^{0.63}$;

So $f(n) = \Omega(n^{\log_3 2+\varepsilon})$; for $\varepsilon = \log_2 5 - \log_3 2 = 1.69 > 0$;

$af(n/b) = 2((n/3)^{\log_2 5}) = 2n^{\log_2 5} * (\frac{1}{3})^{\log_2 5} = 0.156\, n^{\log_2 5}$

$< cf(n)$ for $c = 0.3 < 1$;

Therefore: $T(n) = \Theta(n^{\log_2 5})$;

3.

Exercise 2.2-1

Express the function $n^3/1000 - 100n^2 - 100n + 3$ in terms of Θ-notation.

$T(n) = n^3/1000 - 100n^2 - 100n + 3 = \Theta(n^3)$;

To prove that, there is a c1, c2 that:

$c1n^3 \le n^3/1000 - 100n^2 - 100n + 3 \le c2n^3$;

$c1 \le 1/1000 - 100/n - 100/n^2 + 3/n^3 \le c2$;

When n is very large:

$-100/n - 100/n^2 + 3/n^3 = 0$;

Therefore:

$c1 \le 1/1000 \le c2$;

We choose $n^0 =$ **10^6** (in order to keep c1>0, n^0 must be large enough), thus;

$c1 \le 1000000^3 / 1000 - 100 * 1000000^2 - 100 * 1000000 + 3 \le c2$;

$c1 \le 10^{15} - 10^{14} - 10^8 + 3 \le c2$;

We combine these two inequalities together:

$c1 \le 1/1000 \le 10^{15} - 10^{14} - 10^8 + 3 \le c2$;

So, when we choose **$n^0 = 10^6$; $c1 = 10^{-5}$; and $c2 = 10^{15}$:**

$c1n^3 \le n^3/1000 - 100n^2 - 100n + 3 \le c2n^3$ is always true;

Therefore, by definition: $T(n) = n^3/1000 - 100n^2 - 100n + 3 = \Theta(n^3)$;

4.

Substitution Method:

Hypothesis: $T(n) = O(n\lg n)$;

Show: $T(n) \le cn\lg n$ for some $c>0$;

Assume: $T(n/3) \le c(n/3)\lg(n/3)$;

$T(2n/3) \le c(2n/3)\lg(2n/3)$;

$$
\begin{aligned}
T(n) &\le c(n/3)\lg(n/3) + c(2n/3)\lg(2n/3) + n \\
&= cn/3(\lg n - \lg 3) + 2cn/3(\lg n - \lg(3/2)) + n \\
&= cn/3\lg n - cn/3\lg 3 + 2cn/3\lg n - 2cn/3\lg(3/2) + n \\
&= (cn/3\lg n + 2cn/3\lg n) - cn/3\lg 3 - 2cn/3(\lg 3 - \lg 2) + n \\
&= cn\lg n - cn\lg 3 + 2cn/3\lg 2 + n \\
&= cn\lg n + n(1 + 2c/3\lg 2 - c\lg 3) \le cn\lg n
\end{aligned}
$$

To make this work, we need:

$$n(1 + 2c/3\lg 2 - c\lg 3) \le 0$$

Because n is a positive integer, therefore:

$$1+2c/3\lg 2 - c\lg 3 \le 0$$

$$c(2/3\lg 2 - \lg 3) \le -1$$

We can get $2/3\lg 2 - \lg 3 = -0.92$

So: $-0.92c \le -1$

$c \ge -1/-0.92$

$c \ge 1.09$

Now we find out, when $c \ge 1.09$, $T(n) \le cn\lg n$

Therefore $T(n) = O(n\lg n)$