1.1:

Substitution Method:

Hypothesis: $T(n) = O(n^2)$;

Show: $T(n) \le cn^2$ for some $c>0$;

Assume: $T(n/3) \le c(n/3)^2$;

$T(n) \le 2c(n/3)^2 + n^2$;

$= (2c/9)n^2 + n^2$;

$= (2c/9 + 1)n^2$;

To accomplish $T(n) \le cn^2$, we want $2c/9 + 1 \le c$;

Thus, $2c + 9 \le 9c \Rightarrow c \ge 9/7$

$T(n) \le cn^2$ for $c \ge 9/7$;

Therefore $T(n) = O(n^2)$.

1.2:

Recursion Method:

$T(n) = 2T(n/3) + n^2$;

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T (n)          n^2                                   => n^2
             /       \
           /           \
    (n/3)^2           (n/3)^2                        => 2/9 n^2
     /   \            /    \
(n/9)^2  (n/9)^2  (n/9)^2  (n/9)^2                   => (2/9)^2 n^2

...       ...       ...       ...

T(1) T(1) …             T(1) T(1)                    => Θ (n^(log_3 2))
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Total $\log_3 n$ layers.

$2^{\log_3 n} = n^{\log_3 2}$ T(1)s

Total: $O(n^2)$

2.(a):

$T(n) = 2T(n / 4) + n\sqrt{n}$;

$a = 2 > 0$; $b = 4 > 0$; $f(n) = n\sqrt{n}$;

$n^{\log_4 2} = n^{1/2}$;

$f(n) = n^{1.5} = \Omega(n^{\log_4 2+1})$; $\mathcal{E} = 1 > 0$;

$af(n/b) = 2(n/4)^{1.5} = 2 * 4^{-1.5} n^{1.5} = 0.25 n^{1.5} < cn^{1.5}$ for $c = 0.5 < 1$

Therefore: $T(n) = \Theta(n^{3/2})$

2.(b):

$T(n) = 2T(n/3) + 5^{\log_2 n}$

$a = 2 > 0$; $b = 3 > 0$; $f(n) = 5^{\log_2 n} = n^{\log_2 5} = n^{2.32}$;

$n^{\log_3 2} = n^{0.63}$;

So $f(n) = \Omega(n^{\log_3 2+\mathcal{E}})$; for $\mathcal{E} = \log_2 5 - \log_3 2 = 1.69 > 0$;

$af(n/b) = 2((n/3)^{\log_2 5}) = 2n^{\log_2 5} * (\frac{1}{3})^{\log_2 5} = 0.156\, n^{\log_2 5}$

$< cf(n)$ for $c = 0.3 < 1$;

Therefore: $T(n) = \Theta(n^{\log_2 5})$;

3.

Express the function $n^3/1000 - 100n^2 - 100n + 3$ in terms of Θ-notation.

$T(n) = n^3/1000 - 100n^2 - 100n + 3 = \Theta(n^3)$;

To prove that, there is a c1, c2 that:

$c1n^3 \le n^3/1000 - 100n^2 - 100n + 3 \le c2n^3$;

$c1 \le 1/1000 - 100/n - 100/n^2 + 3/n^3 \le c2$;

When n is very large:

$-100/n - 100/n^2 + 3/n^3 = 0$;

Therefore:

$c1 \le 1/1000 \le c2$;

We choose $n^0 =$ **10^6** (**in order to keep c1>0, n^0 must be large enough**), thus;

$c1(n^0)^3 \le 1000000^3 / 1000 - 100 * 1000000^2 - 100 * 1000000 + 3 \le c2(n^0)^3$;

$c1*10^{18} \le 10^{15} - 10^{14} - 10^8 + 3 \le c2*10^{18}$;

$c1 \le (10^{15} - 10^{14} - 10^8 + 3)/10^{18} \le c2$

Please note that:

$10^{15} - 10^{14} - 10^8 + 3 \approx 9 * 10^{14}$

Therefore: $(10^{15} - 10^{14} - 10^8 + 3)/10^{18} \approx 9 * 10^{-4}$

We combine these two inequalities together:

$0 < c1 \le 9 * 10^{-4} \le 1/1000 \le c2$;

So, when we choose **$n^0 = 10^6$; $c1 = 10^{-5}$; and $c2 = 10$;**

$c1n^3 \le n^3/1000 - 100n^2 - 100n + 3 \le c2n^3$ is always true when $n > n^0$;

Therefore, by definition: $T(n) = n^3/1000 - 100n^2 - 100n + 3 = \Theta(n^3)$;

Justification:

Let's choose $n' = 10^7 > n^0$;

$c1n'^3 = 10^{-5} * 10^{21} = 10^{16}$; $c2n'^3 = 10 * 10^{21} = 10^{22}$;

$n'^3/1000 - 100n'^2 – 100n' + 3 =$

$10^{21} * 10^{-3} – 100 * 10^{14} – 100 * 10^7 – 3 =$

$10^{18} – 10^{16} – 10^9 + 3 \approx$

$9.9 * 10^{17}$;

$c1n'^3 = 10^{16} < 9.9 * 10^{17} < c2n'^3 = 10^{22}$

4.

Substitution Method:

Hypothesis: T (n) = O(nlgn);

Show: T (n) ≤ cnlgn for some c>0;

Assume: T (n/3) ≤ c(n/3)lg(n/3);

T (2n/3) ≤ c(2n/3)lg(2n/3);

T(n) ≤ c(n/3)lg(n/3) + c(2n/3)lg(2n/3) + n

= cn/3(lgn - lg3) + 2cn/3(lgn - lg(3/2)) + n

= cn/3lgn - cn/3lg3 + 2cn/3lgn - 2cn/3lg(3/2) + n

= (cn/3lgn + 2cn/3lgn) - cn/3lg3 -2cn/3(lg3-lg2) + n

= cnlgn - cnlg3 + 2cn/3lg2 + n

= cnlgn + n (1 + 2c/3lg2 -clg3) ≤ cnlgn

To make this work, we need:

n(1 + 2c/3lg2 -clg3) ≤ 0

Because n is a positive integer, therefore:

1+2c/3lg2 - clg3 ≤ 0

c (2/3lg2 -lg3) ≤ -1

We can get 2/3lg2 - lg3 = - 0.92

So: -0.92c ≤ -1

c ≥ -1/-0.92

c ≥ 1.09

Now we find out, when c ≥ 1.09, T(n) ≤ cnlgn

Therefore T(n) = O (nlgn)