UniDB:
A Unified Diffusion Bridge Framework via Stochastic Optimal Control

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Arxiv: https://arxiv.org/pdf/2502.05749
Code: https://github.com/UniDB-SOC/UniDB

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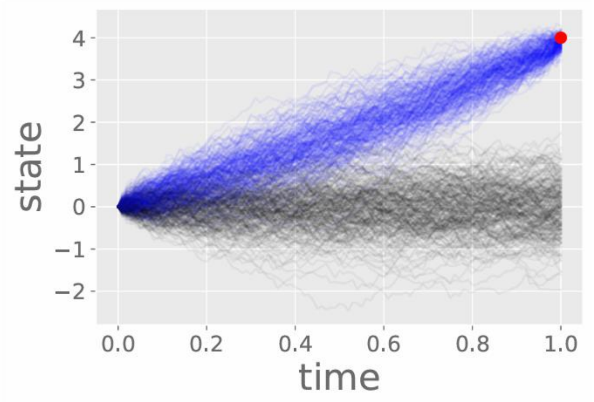
Doob's h-transform: one distribution to another

Doob's h-transform, a mathematical technique applied to stochastic processes, which rectifies the drift term of the forward diffusion process to pass through a preset terminal point $\mathbf{x_T} \in R^d$.

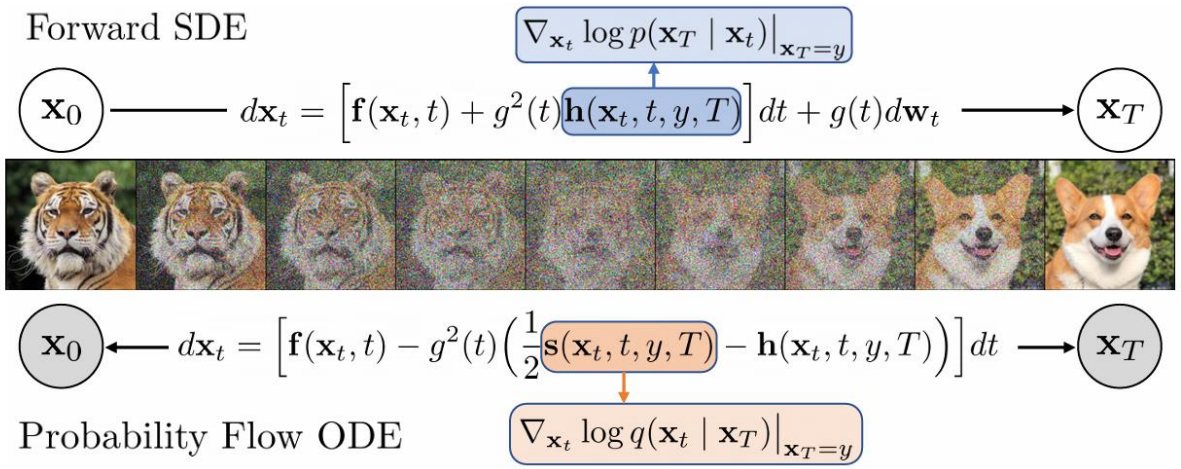
$$\mathbf{h}(x, t, y, T) = \nabla_{\mathbf{x}_t} \log p\left(\mathbf{x}_T \mid \mathbf{x}_t\right)\big|_{\mathbf{x}_t=x, \mathbf{x}_T=y}$$



$$d\mathbf{x}_t = \mathbf{f}\left(\mathbf{x}_t, t\right) dt + g^2(t) \boxed{\mathbf{h}\left(\mathbf{x}_t, t, y, T\right)} + g(t) d\mathbf{w}_t, \quad \mathbf{x}_0 \sim q_{\text{data}}(\mathbf{x}), \quad \mathbf{x}_T = y$$

$$d\mathbf{x}_t = \mathbf{f}\left(\mathbf{x}_t, t\right) dt + g(t) d\mathbf{w}_t$$

Diffusion Bridge with Doob's h-transform (DDBMs)



DDBMs proposed a diffusion bridge model (based on VE and VP process) using Doob's h-transform.

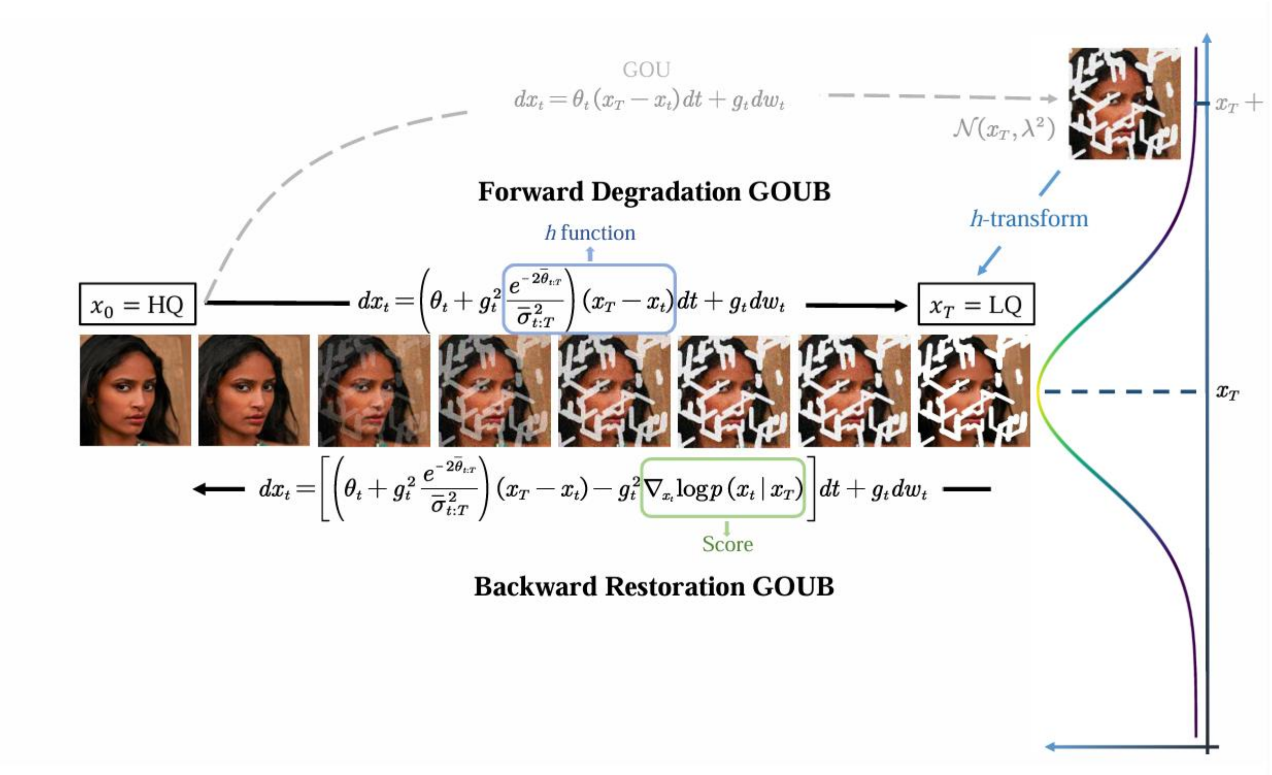
$$\text{VE}: \mathrm{d}\mathbf{x}_t = g_t\, \mathrm{d}\mathbf{w}_t$$

$$\text{VP}: \mathrm{d}\mathbf{x}_t = -\frac{1}{2}g_t^2\mathbf{x}_t\, \mathrm{d}t + g_t\, \mathrm{d}\mathbf{w}_t$$

- ✓ Be designed to establish fixed endpoints between two distinct distributions.
- ✓ Solve the SDE based on these learned scores to transition from one endpoint distribution to another.
- ✗ Lack the mean information of the terminal distribution x_T, which restricts the quality of the generated images, particularly in image restoration tasks.

Diffusion Bridge with Doob's h-transform (GOUB)

$$\text{GOU}: \mathrm{d}\mathbf{x}_t = \theta_t\,(\boxed{\boldsymbol{\mu}} - \mathbf{x}_t)\mathrm{d}t + g_t\,\mathrm{d}\mathbf{w}_t$$



- Introduce mean information of the terminal distribution $\boxed{\mu = x_T}$ with Doob's h-transform
- Both VE and VP processes are special cases of GOU process

Problems & Limitations

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- The theoretical mechanisms by which Doob's h-transform governs the bridging process remain poorly understood, lacking a rigorous framework to unify its empirical success.
- Existing methods frequently degrade high-frequency details—such as sharp edges and fine textures—resulting in outputs with blurred or oversmoothed artifacts that compromise perceptual fidelity. These limitations underscore the need for both theoretical grounding and enhanced detail preservation in diffusion bridges.

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Stochastic Optimal Control

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The SOC problem aims to design the controller $u_{t,\gamma}$ to drive the dynamic system from x_0 to x_T with minimum cost.

$$\min_{\mathbf{u}_{t,\gamma}\in\mathcal{U}} \int_0^T \frac{1}{2}\|\mathbf{u}_{t,\gamma}\|_2^2 dt + \frac{\gamma}{2}\|\mathbf{x}_T^u - x_T\|_2^2$$
$$\text{s.t. } \mathbf{dx}_t = (f(\mathbf{x}_t, t) + g_t\mathbf{u}_{t,\gamma})\mathrm{d}t + g_t\,\mathbf{dw}_t, \mathbf{x}_0^u = x_0$$

$||u_{t,\gamma}||_2^2$: instantaneous cost

$\frac{\gamma}{2}||x_T^u - x_T||_2^2$: terminal cost with its penalty coefficient

- ✓ RB-Modulation operationalized SOC via a simplified SDE structure for training-free style transfer using pre-trained diffusion models.
- ✓ DBFS leveraged SOC to construct diffusion bridges in infinite-dimensional function spaces and propose that there has an equivalence between SOC and Doob's h-transform.
- ✗ Do not address the intrinsic limitations of Doob's h -transform and how can equivalence be established.

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UniDB: A Unified Diffusion Bridge Framework via Stochastic Optimal Control

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- Generalize existing diffusion bridge models based on Stochastic optimal control like DDBMs and GOUB, offering a comprehensive understanding and extension of Doob's h-transform by incorporating general forward SDE forms.
- Derive closed-form solutions for the SOC problem, demonstrating that Doob's h -transform is merely a special case within UniDB, revealing inherent limitations in the existing diffusion bridge approaches.
- Require minimal code modification, ensuring easy implementation.
- Achieve state-of-the-art results in various image restoration tasks.

Construct Diffusion Bridge via Stochastic Optimal Control

UniDB

$$\min_{\mathbf{u}_{t,\gamma}\in\mathcal{U}} \int_0^T \frac{1}{2}\|\mathbf{u}_{t,\gamma}\|_2^2 dt + \frac{\gamma}{2}\|\mathbf{x}_T^u - x_T\|_2^2$$

$$\text{s.t. } \mathrm{d}\mathbf{x}_t = (f_t\mathbf{x}_t + h_t\mathbf{m} + g_t\mathbf{u}_{t,\gamma})\mathrm{d}t + g_t\,\mathrm{d}\mathbf{w}_t, \mathbf{x}_0^u = x_0$$

$$\mathbf{u}_{t,\gamma}^* = g_t e^{\bar{f}_{t:T}} \frac{x_T - e^{\bar{f}_{t:T}}\mathbf{x}_t - \mathbf{m}e^{\bar{f}_T}\bar{h}_{t:T}}{d_{t,\gamma}}$$

$$\mathbf{x}_t = e^{\bar{f}_t}\left(\frac{d_{t,\gamma}}{d_{0,\gamma}}x_0 + \frac{e^{\bar{f}_T}\bar{g}_t^2}{d_{0,\gamma}}x_T + \left(\bar{h}_t - \frac{e^{2\bar{f}_T}\bar{h}_T\bar{g}_t^2}{d_{0,\gamma}}\right)\mathbf{m}\right)$$

$$\mathbf{f}(\mathbf{x}_t, t) = f_t\mathbf{x}_t + h_t\mathbf{m}$$

Combined a given state vector term $\mathbf{m}$ with the same dimension as $\mathbf{x_t}$ and its related coefficient h_t to improve the generality of the linear SDE form.

As for the backward process, the backward reverse SDE and Mean-ODE are respectively formulated as:

$$\mathrm{d}\mathbf{x}_t = \left[f_t\mathbf{x}_t + h_t\mathbf{m} + g_t\mathbf{u}_{t,\gamma}^* - g_t^2\nabla_{\mathbf{x}_t}p\left(\mathbf{x}_t \mid x_T\right)\right]\mathrm{d}t + g_t\,\mathrm{d}\tilde{\mathbf{w}}_t$$

$$\mathrm{d}\mathbf{x}_t = \left[f_t\mathbf{x}_t + h_t\mathbf{m} + g_t\mathbf{u}_{t,\gamma}^* - g_t^2\nabla_{\mathbf{x}_t}p\left(\mathbf{x}_t \mid x_T\right)\right]\mathrm{d}t$$

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Theorem: *For the SOC problem ,when $\gamma \to \infty$, the optimal controller becomes $u_{t,\infty}^{*} = g_t \nabla_{x_t} \log p\,(x_T \mid x_t)$, which means that Doob's h-transform is a special case of UniDB.*

Existing diffusion bridge models using Doob's h-transform are merely special instances of UniDB, which offers a unified approach to diffusion bridges through SOC.

Proposition: Consider the SOC problem, denote $\mathcal{J}(\mathbf{u}_{t,\gamma}, \gamma) \triangleq \int_0^T \frac{1}{2} ||\mathbf{u}_{t,\gamma}||_2^2 dt + \frac{\gamma}{2} ||\mathbf{x}_T^{\mathbf{u}} - x_T||_2^2$ as the overall cost of the system, $u_{t,\gamma}^{*}$ as the optimal controller:

$$\mathcal{J}\left(\mathbf{u}_{t,\gamma}^{*}, \gamma\right) \leq \mathcal{J}\left(\mathbf{u}_{t,\infty}^{*}, \infty\right)$$

The overall cost with a finite γ is more favorable than when $\gamma \to \infty$. Existing diffusion bridge models ($\gamma \to \infty$) result in suboptimal performance with blurred or overly smoothed image details. Therefore, maintaining the penalty coefficient γ as a hyper-parameter is a more effective approach.

UniDB unifies diffusion bridge models

DDBMs (VE) corresponds to UniDB with hyper-parameter $\mathcal{H}_{VE}(f_t = 0, h_t = 0, \gamma \to \infty)$

DDBMs (VP) corresponds to UniDB with hyper-parameter $\mathcal{H}_{VP}\left(f_t = -\frac{1}{2} g_t^2, h_t = 0, \gamma \to \infty\right)$

GOUB corresponds to UniDB with hyper-parameter $\mathcal{H}_{GOU}(f_t = \theta_t, h_t = -\theta_t, \mathbf{m} = \boldsymbol{\mu}, \gamma \to \infty)$

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An Example: UniDB-GOU

Algorithm 1 UniDB Training

repeat

Take a pair of images $\mathbf{x}_0 = x_0$, $\mathbf{x}_T = x_T$

$t \sim \text{Uniform}(\{1, ..., T\})$

$\sigma_{t-1,\theta} = g_t$

$a_{t,\gamma} = e^{-\bar{\theta}_t} \frac{\bar{\sigma}^2_{t:T}}{\bar{\sigma}^2_T} \quad \leftarrow \quad$ GOUB

$a_{t,\gamma} = e^{-\bar{\theta}_t} \frac{\gamma^{-1} + \bar{\sigma}^2_{t:T}}{\gamma^{-1} + \bar{\sigma}^2_T} \quad \leftarrow \quad$ UniDB-GOU

$\mathbf{x}_t = a_{t,\gamma} x_0 + (1 - a_{t,\gamma}) x_T + \bar{\sigma}'_t \epsilon$

$\bar{\boldsymbol{\mu}}_{t,\gamma} = a_{t,\gamma} x_0 + (1 - a_{t,\gamma}) x_T$

$\boldsymbol{\mu}_{t-1,\theta} = \mathbf{x}_t - \left(\theta_t + g_t^2 \frac{e^{-2\bar{\theta}_{t:T}}}{\bar{\sigma}^2_{t:T}} \right) (x_T - \mathbf{x}_t) + \frac{g_t^2}{\bar{\sigma}'^2_t} \boldsymbol{\epsilon}_\theta(\mathbf{x}_t, x_T, t) \quad \leftarrow \quad$ GOUB

$\boldsymbol{\mu}_{t-1,\theta} = \mathbf{x}_t - \left(\theta_t + g_t^2 \frac{e^{-2\bar{\theta}_{t:T}}}{\gamma^{-1} + \bar{\sigma}^2_{t:T}} \right) (x_T - \mathbf{x}_t) + \frac{g_t^2}{\bar{\sigma}'^2_t} \boldsymbol{\epsilon}_\theta(\mathbf{x}_t, x_T, t) \quad \leftarrow \quad$ UniDB + GOU

$\boldsymbol{\mu}_{t-1,\gamma} = \bar{\boldsymbol{\mu}}_{t-1,\gamma} + \frac{\bar{\sigma}'_{t-1} a_{t,\gamma}}{\bar{\sigma}'^2_t a_{t-1,\gamma}} (\mathbf{x}_t - \bar{\boldsymbol{\mu}}_{t,\gamma})$

Take gradient descent step on $\nabla_\theta \left(\mathcal{L}_\theta = \mathbb{E}_{t,\mathbf{x}_0,\mathbf{x}_t,\mathbf{x}_T} \left[\frac{1}{2\sigma^2_{t-1,\theta}} \|\boldsymbol{\mu}_{t-1,\theta} - \boldsymbol{\mu}_{t-1,\gamma}\| \right] \right)$

until converged

Algorithm 2 UniDB Sampling

Input: LQ images $\mathbf{x}_T = x_T$.

for $t = T$ **to** 1 **do**

$z \sim N(0, I)$ if $t > 1$, else $z = 0$

$\mathbf{x}_{t-1} = \mathbf{x}_t - \left(\theta_t + g_t^2 \frac{e^{-2\bar{\theta}_{t:T}}}{\gamma^{-1} + \bar{\sigma}^2_{t:T}} \right) (x_T - \mathbf{x}_t) + \frac{g_t^2}{\bar{\sigma}'^2_t} \boldsymbol{\epsilon}_\theta(\mathbf{x}_t, x_T, t) - g_t z$

end for

Return HQ images $\mathbf{x}_0$

Minor code modifications!!!!

GOUB and UniDB-GOU

$$e^{-\bar{\theta}_t} \frac{\bar{\sigma}^2_{t:T}}{\bar{\sigma}^2_T} \Rightarrow e^{-\bar{\theta}_t} \frac{\gamma^{-1} + \bar{\sigma}^2_{t:T}}{\gamma^{-1} + \bar{\sigma}^2_T}$$

$$\underbrace{g_t \, \mathrm{h} = \frac{g_t e^{-2\bar{\theta}_{t:T}} (x_T - \mathbf{x}_t)}{\bar{\sigma}^2_{t:T}}}_{\text{GOUB}} \Rightarrow \underbrace{\mathbf{u}^*_{t,\gamma} = \frac{g_t e^{-2\bar{\theta}_{t:T}} (x_T - \mathbf{x}_t)}{\gamma^{-1} + \bar{\sigma}^2_{t:T}}}_{\text{UniDB-GOU}}$$

Experiment setup

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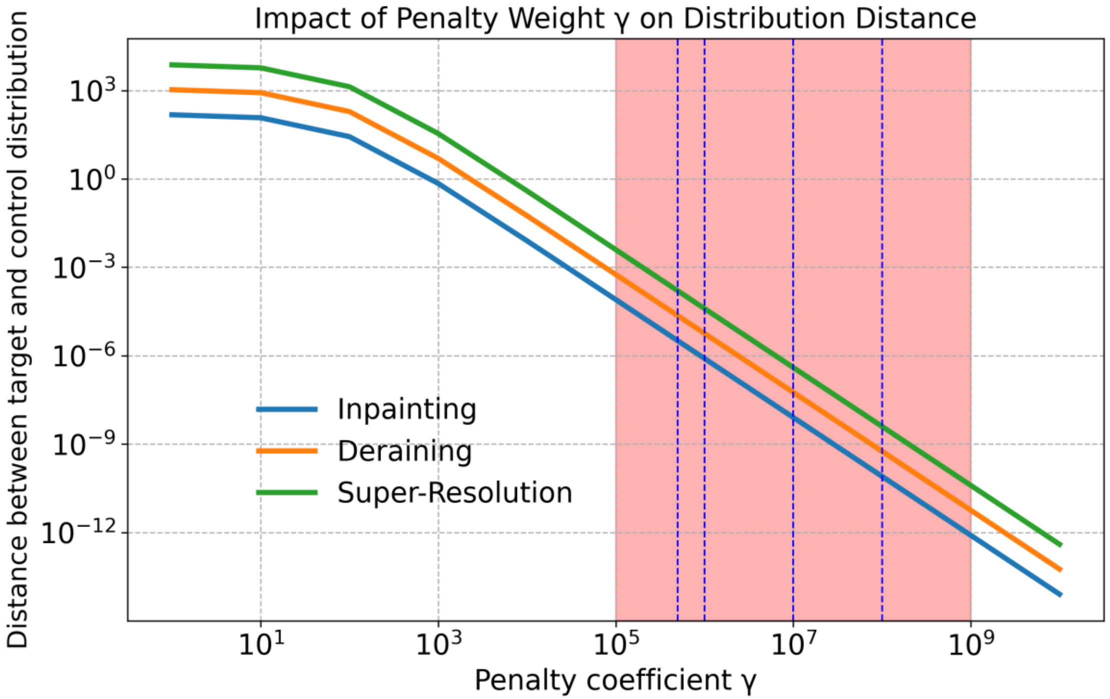
$$\|\mathbf{x}_T^u - x_T\|_2^2 = \frac{e^{-2\bar{\theta}_T}}{\left(1+\gamma\lambda^2\left(1-e^{-2\bar{\theta}_T}\right)\right)^2}\|x_T - x_0\|_2^2$$

Analyze the l_2-norm distances
between the two terminal distributions

Better choices of γ, balance the control cost
$\int_0^T \frac{1}{2}||\mathbf{u}_{t,\gamma}||_2^2 dt$ and the terminal cost $\frac{\gamma}{2}||\mathbf{x}_\mathbf{T}^\mathbf{u} - x_T||_2^2$.



***Notations**

- x_0: the initial state distribution
- $\mathbf{x}_T^u$: the terminal distribution by the controller
- x_T: the pre-defined terminal distribution

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Comparison with state-of-the-art in 4×Super-resolution

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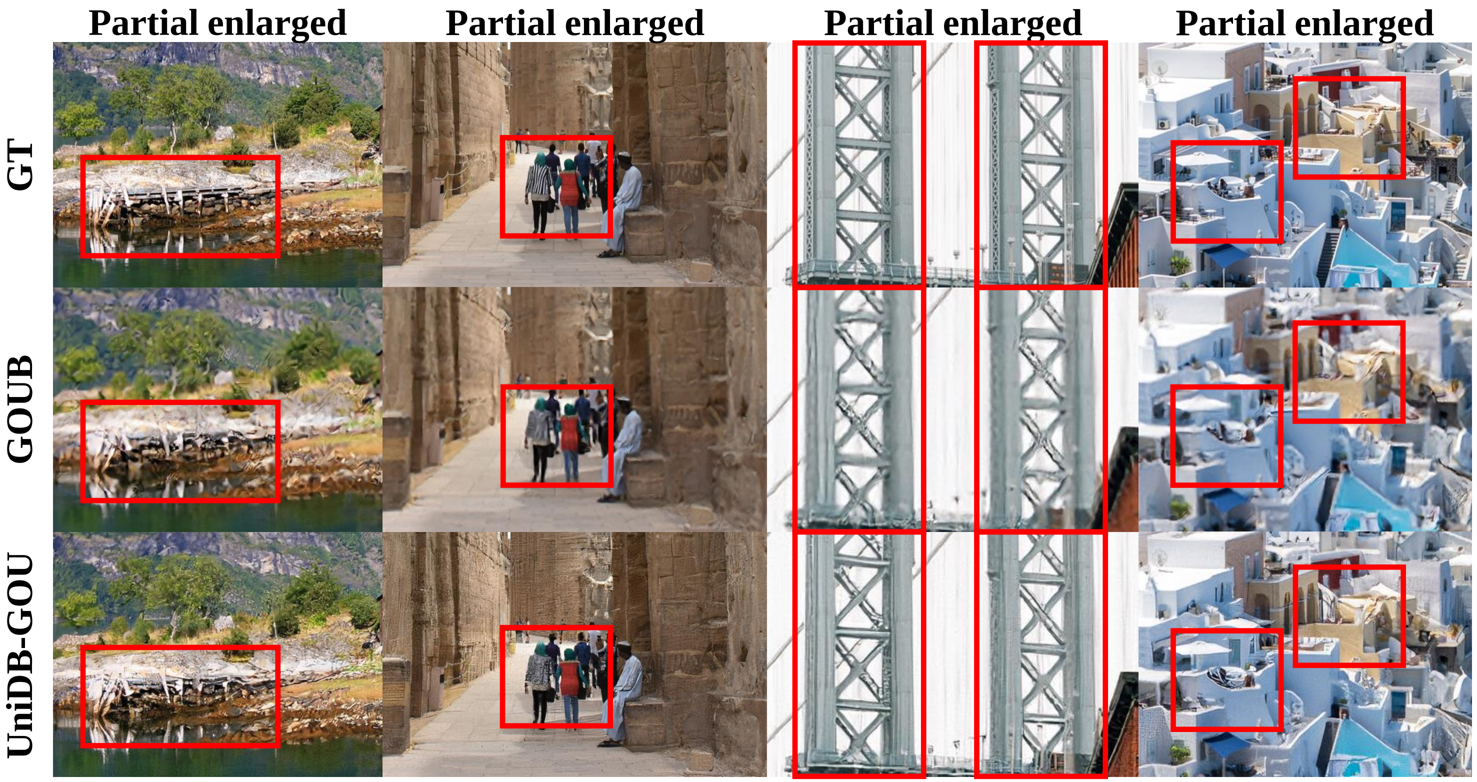

Table 1. **Image 4×Super-Resolution.** Qualitative evaluation of the DIV2K datasets with baselines.

METHOD	PSNR↑	SSIM↑	LPIPS↓	FID↓
DDRM	24.35	0.5927	0.364	78.71
IR-SDE	25.90	0.6570	0.231	45.36
GOUB (SDE)	26.89	0.7478	0.220	20.85
GOUB (ODE)	28.50	0.8070	0.328	22.14
UniDB (SDE)	25.46	0.6856	**0.179**	**16.21**
UniDB (ODE)	**28.64**	**0.8072**	0.323	22.32

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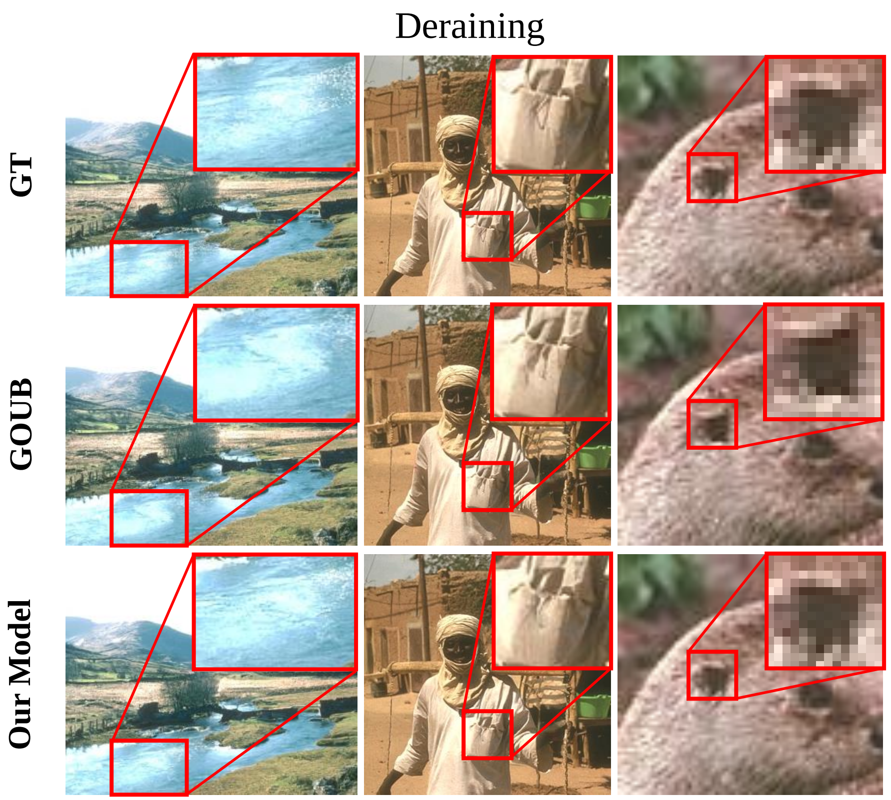
Comparison with state-of-the-art in Deraining and Inpainting



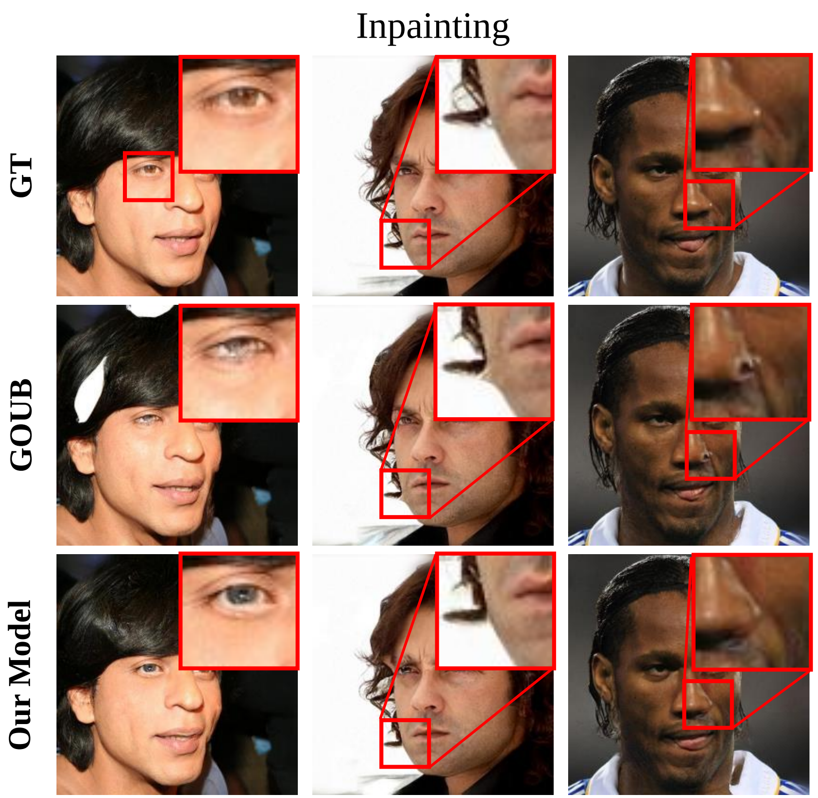
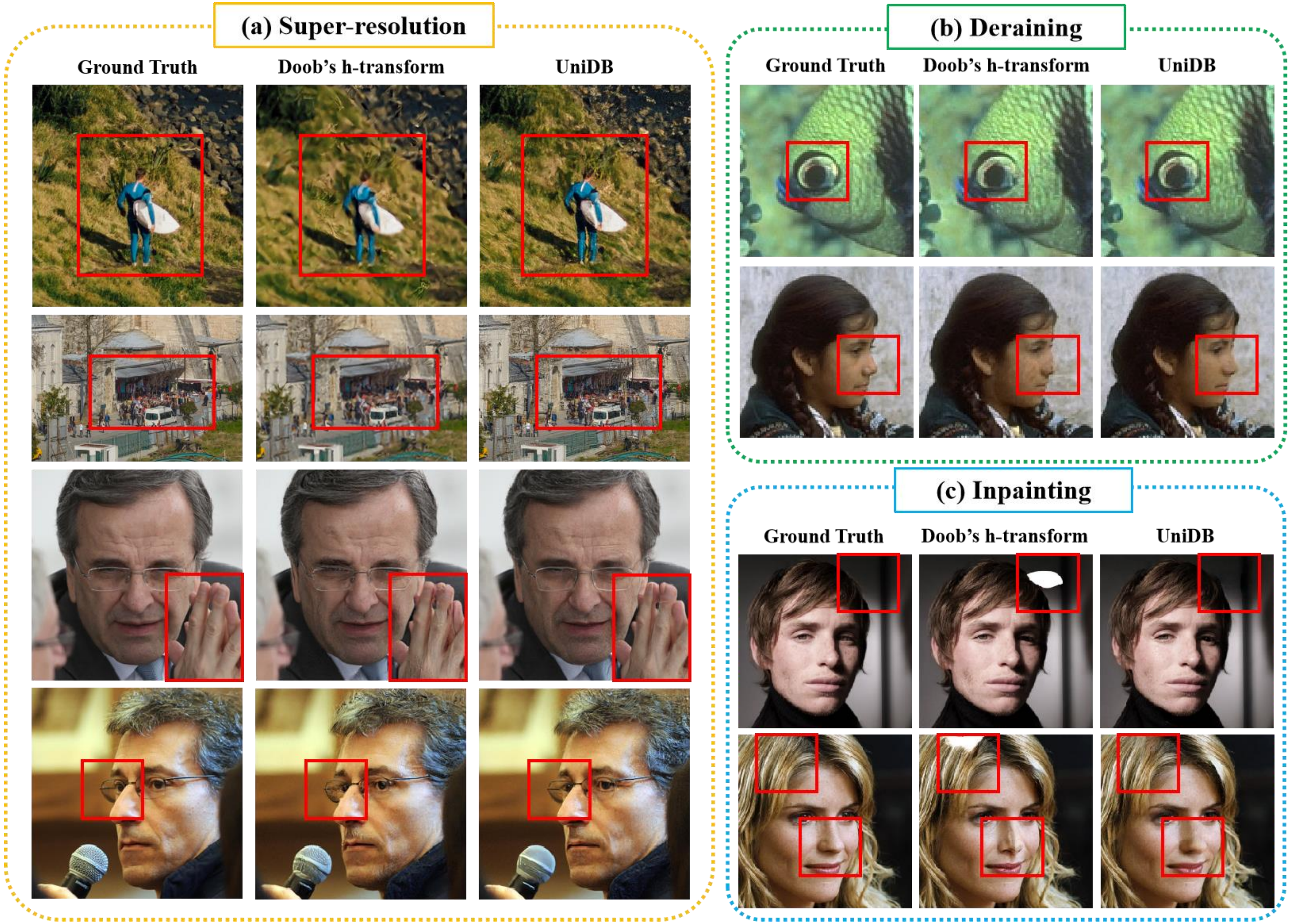


Table 2. **Image Deraining.** Qualitative evaluation of the Rain100H datasets with baselines.

METHOD	PSNR↑	SSIM↑	LPIPS↓	FID↓
MAXIM	30.81	0.9027	0.133	58.72
MHNet	31.08	0.8990	0.126	57.93
IR-SDE	31.65	0.9041	0.047	18.64
GOUB (SDE)	31.96	0.9028	0.046	18.14
GOUB (ODE)	34.56	0.9414	0.077	32.83
UniDB (SDE)	32.05	0.9036	**0.045**	**17.65**
UniDB (ODE)	**34.68**	**0.9426**	0.074	31.16

Table 4. **Image Inpainting.** Qualitative evaluation of the CelebA-HQ 256×256 datasets with baselines.

METHOD	PSNR↑	SSIM↑	LPIPS↓	FID↓
PromptIR	30.22	0.9180	0.068	32.69
DDRM	27.16	0.8993	0.089	37.02
IR-SDE	28.37	0.9166	0.046	25.13
GOUB (SDE)	28.98	0.9067	0.037	4.30
GOUB (ODE)	31.39	0.9392	0.052	12.24
UniDB (SDE)	29.20	0.9077	**0.036**	**4.08**
UniDB (ODE)	**31.67**	**0.9395**	0.052	11.98


(a) Super-resolution
Ground Truth
Doob's h-transform
UniDB
(b) Deraining
Ground Truth
Doob's h-transform
UniDB
(c) Inpainting
Ground Truth
Doob's h-transform
UniDB

Ablation Study on different penalty coefficients γ

Table 2. Quantitative evaluation results for DIV2K, CelebA-HQ and Rain100H of UniDB-GOU with different penalty coefficients γ.

TASKS	METRICS	Different γ				
		5×10^5	1×10^6	1×10^7	1×10^8	∞
Image 4×Super-Resolution	**PSNR↑**	24.94	24.72	25.46	25.06	**26.89**
	SSIM↑	0.6419	0.6587	0.6856	0.6393	**0.7478**
	LPIPS↓	0.234	0.199	**0.179**	0.289	0.220
	FID↓	20.33	18.37	**16.21**	23.76	20.85
Image Inpainting	**PSNR↑**	28.73	29.15	**29.20**	28.65	28.98
	SSIM↑	0.9065	0.9068	**0.9077**	0.9062	0.9067
	LPIPS↓	0.038	0.036	**0.036**	0.039	0.037
	FID↓	4.49	4.12	**4.08**	4.64	4.30
Image Deraining	**PSNR↑**	29.44	31.96	32.00	**32.05**	31.96
	SSIM↑	0.8715	0.9018	0.9029	**0.9036**	0.9028
	LPIPS↓	0.058	0.045	0.046	**0.045**	0.046
	FID↓	24.96	18.37	17.87	**17.65**	18.14