Climate App instructions

June 11, 2021

1 What is the Climate App?

The Climate App is a web application to demonstrate how the surface temperature of a planet is affected by the sunlight, reflection and the greenhouse effect. The Climate App lets you explore the past, present and future of Earth's climate, as well as the climates of other worlds. The Climate App is a simple model of planetary climate, but one that captures the most important physics.

This application has a beginner version (Figure 1) and an advanced version (Figure 2). The two versions differ in the number of adjustable sliders. You can change versions using the switch below the title of the page. Different components in this application are described below. Some of them appear in both versions under different names, while others are used exclusively in the advanced version.

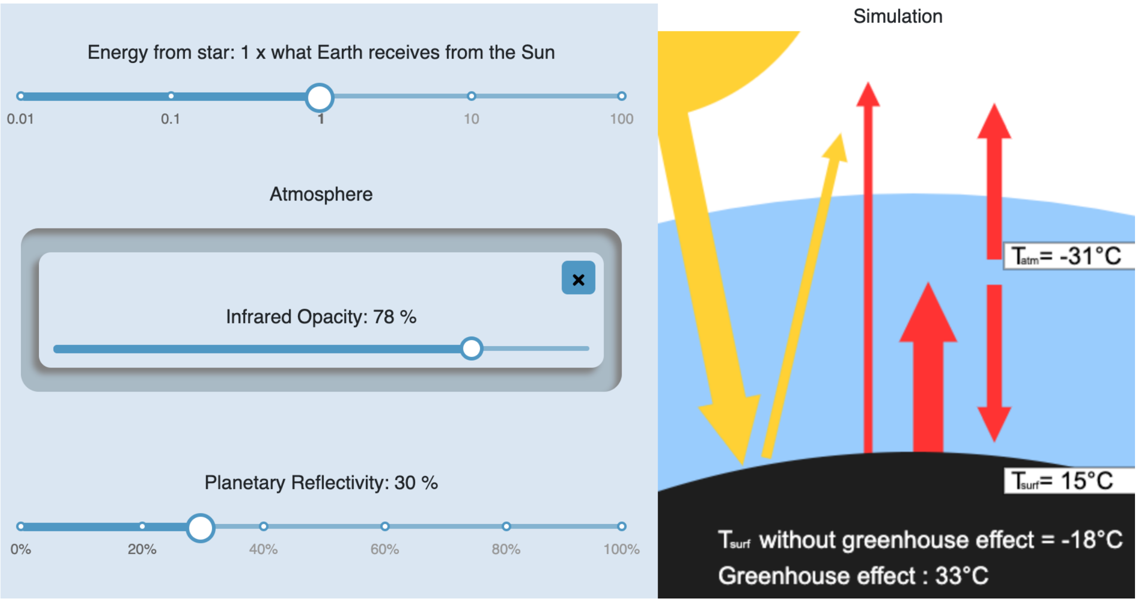

Figure 1: Screenshot of the beginner version of the Climate App.

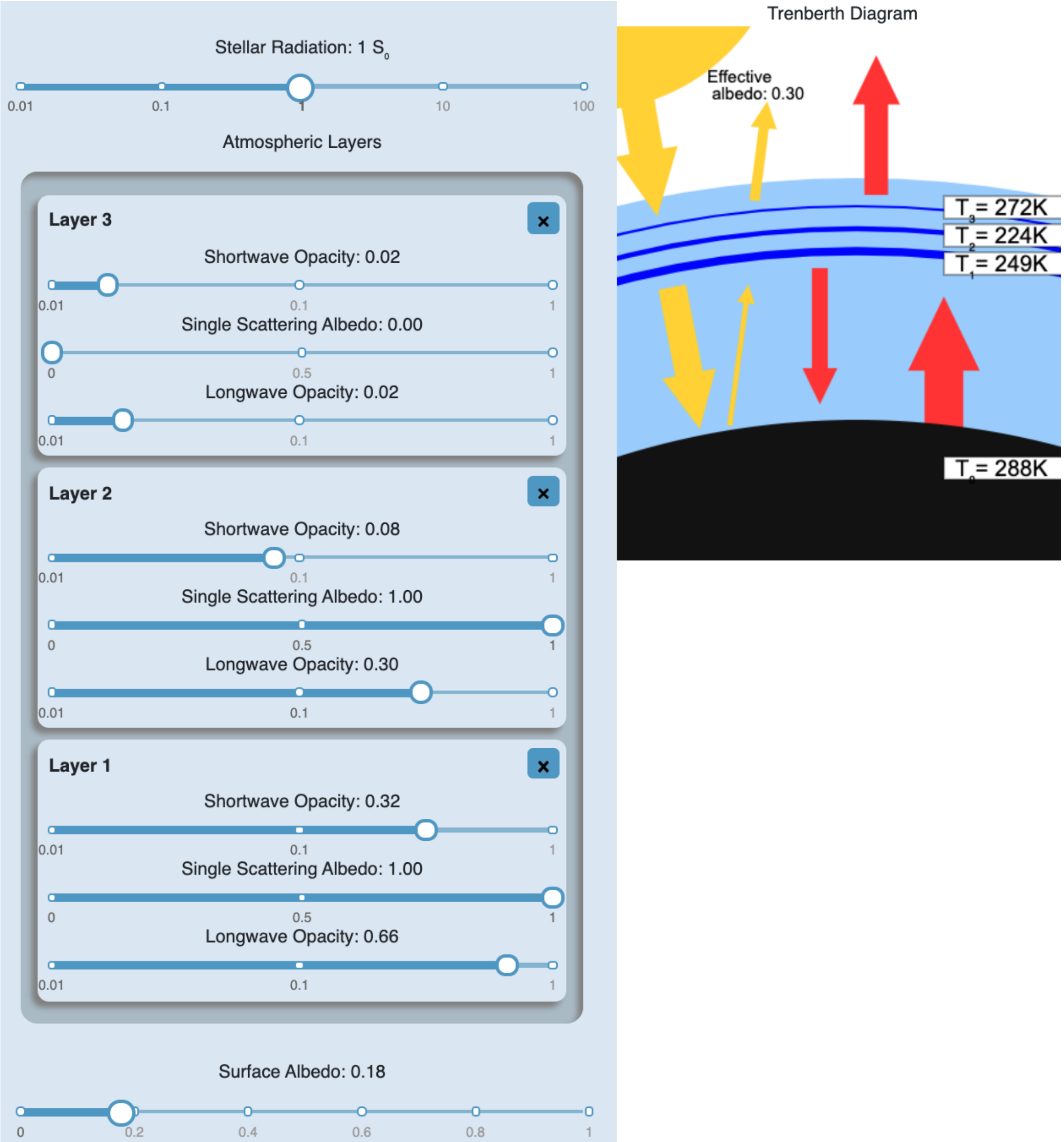

Figure 2: Screenshot of the advanced version of the Climate App.

1.1 Energy Flows, a.k.a. the Trenberth Diagram

The panel to the right of the sliders is a visual representation of the planet's climate (it may appear at the bottom of the page on a smart phone). Yellow arrows represent visible light from the star (not at all to scale at the top left of the diagram): in the basic app it is either absorbed or reflected at the planet's surface (shown in black/gray at the bottom). Red arrows represent thermal (infrared) light emitted by the planet's surface and atmosphere (the pale blue swath above the planet's surface, also not to scale). The yellow arrows track the shortwave radiation, and the red arrows track the longwave thermal emission. The width of each arrow is proportional to the importance of that energy flow as compared to the amount of starlight that reaches the planet. This sort of schematic showing energy flows in an planet's atmosphere is called a Trenberth Diagram.

Moving the sliders changes the planetary parameters and hence the way that energy flows in and out of the atmosphere, which is ultimately what governs the surface temperature. Temperatures are reported in Celsius for the basic version and Kelvin for the advanced version. The beginner version of the app also reports the magnitude of the greenhouse effect in degrees Celsius; this is the difference between the surface temperature and the value it would take if the infrared opacity of the atmosphere were zero. The default values of the sliders when you open the Climate App, both the beginner and advanced versions, are approximately the average values for modern-day Earth.

Features in the Advanced Climate App

The effective albedo gives the ratio between the shortwave energy that emerges from the top of the atmosphere and the incoming stellar radiation. It may be different from the user-defined surface albedo because the advanced app allows for scattering of light by the atmospheric layers.

In the advanced app there would be a lot of arrows between each layer of the atmosphere, so the app only shows the net flows of energy out the top and bottom of the atmosphere. We can still see each layer inside, and their thickness is proportional to their longwave emissivity.

1.2 The Climate App Sliders

The panel on the left lets you vary the planet's properties and hence its climate (the panel may appear at the top of the page on a smart phone).

Energy from Star (Stellar Radiation in Advanced App)

Also known as "instellation", this is the amount of starlight that reaches the top of the planet's atmosphere; for planets in the Solar System, the star in question is none other than the Sun! Incident starlight is the most important factor determining a planet's climate. For convenience, the Climate App expresses the instellation in terms of the the solar constant, the average flux that the Earth receives from the Sun (341 W/m^2). For example, a value of 10 on this slider would mean that the planet receives ten times the amount of stellar radiation that we receive on Earth. The slider is a logarithmic scale, which allows for values a hundred times smaller or greater than the solar constant.

Planetary Reflectivity (Surface Albedo in Advanced App)

This value determines how reflective the planetary surface is. A value of 0 means the planet is perfectly black and absorbs all incoming stellar radiation; there are many planets close to this limit. No planet, on the other hand, reflects all incoming radiation, so the Climate App restricts the albedo to values less than 99%. The default value for the planetary reflectivity in the beginner app is 30%, while in the advanced app the surface albedo is 18% and the effective albedo is 30%; the Earth's albedo is approximately 30%. While moving this slider, you can observe the planet in the simulation canvas change shade.

Albedo and instellation determine how much energy the planet absorbs every second and hence how much energy it must radiate away every second in order to remain in equilibrium. Although this energy balance is only approximate from one second to another, it must be maintained on timescales of years.

In the advanced app, starlight can also be reflected back to space by the atmospheric layer(s) so the top-of-atmosphere effective albedo is reported in the Trenberth diagram. The effective albedo can be greater than or less than the underlying surface albedo, depending on the shortwave properties of the atmosphere.

Atmosphere (Atmospheric Layers in Advanced App)

In this box, you will find settings for the atmosphere. In the beginner version the atmosphere is treated as a single layer (Figure 1). You may remove the atmosphere by clicking the $\times$ sign and add it back by clicking on a + box which only appears when your system does not have an atmosphere.

For the advanced version, you may have up to three atmospheric layers, each having different properties. This simulates the vertical changes in atmospheric properties. For example, Earth's atmosphere has more water vapour in the lower, warmer layers compared to the higher and colder layers. In this case, lower layers would have greater infrared opacity comparing to higher layers. You can click on the + box to add a new layer, and the $\times$ on a layer to remove it.

- **Infrared Opacity (Longwave Emissivity in Advanced App)**: This slider represents the greenhouse gases in the atmosphere. Each atmospheric layer has an adjustable longwave emissivity, its effectiveness in absorbing and emitting thermal radiation. A value of 1 absorbs all infrared radiation that hits it and re-radiates that energy equally up and down.
- **Shortwave Opacity (only in Advanced app)**: As shortwave (stellar) radiation encounters an atmospheric layer, the photons may pass through without interaction, be absorbed or be scattered by molecules and aerosols in the atmosphere. This slider determines the fraction of interacting shortwave radiation at the corresponding layer (absorbed + scattered). A value of 1 means that all of the shortwave photons are either absorbed or scattered.
- **Single Scattering Albedo (only in Advanced app)**: Single scattering albedo refers to the fraction of interacting shortwave photons that are scattered rather than absorbed. For example, a value of 0.1 indicates that out of all the shortwave photons having interacted with the atmosphere, 10% were scattered (and the remaining 90% were absorbed). The Climate App assumes isotropic scattering, so half of the scattered photons keep going in their original direction of travel, while the other half are back-scattered from whence they came.

2 How does the Climate App work?

The Climate App is based on a few key concepts.

2.1 Shortwave and Longwave Radiation

All objects emit radiation, whether you can see it or not. Hot objects like the Sun emit most of their light at visible wavelengths (the range of light that human eye can see, not coincidentally). Somewhat cooler objects like planets and people tend to emit most of their radiation at longer infrared wavelengths (beyond the red edge of the rainbow). Since visible and thermal right interact with the planetary surface and atmosphere in qualitatively different ways, it is customary in climate science to distinguish between the shortwave radiation received from a star and longwave radiation emitted by the planet (among experts, this is known as the two band, or double grey approximation). In the Climate App visualizations, shortwave light is denoted by yellow arrows, while thermal radiation is denoted by red arrows.

2.2 The Stefan-Boltzmann Law

The Stefan-Boltzmann law quantifies the relation between the temperature and total emitted flux (at all wavelengths) for a perfectly absorbing and emitting object, a so-called blackbody: $F_{\rm bb} = \sigma T^4$, where $F_{\rm bb}$ is the emitted flux (units of W/m^2), T is the temperature of the object in Kelvin, and σ is the Stefan-Boltzmann constant: $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}.

Real materials are not perfect blackbodies: they do not perfectly absorb and radiate. Since the surface and atmosphere of a planet emit at thermal wavelengths, we quantify the ability of an object to radiate by its longwave emissivity, $\epsilon_{\rm LW}$. The flux emitted by an element of the climate system is therefore given by $F = \epsilon_{\rm LW}\sigma T^4$. The surface emits this amount of radiation upwards, while an atmospheric layer emits this amount of radiation upwards and the same amount downwards.

2.3 Kirchhoff's Law of Thermal Radiation

Kirchhoff's law of thermal radiation dictates that a material's ability to emit radiation at some wavelength is equal to its ability to absorb radiation at that same wavelength. For example, gases that readily absorb infrared light—so-called greenhouse gases—must also emit well at those same wavelengths.

Solid and liquid surfaces clearly do not absorb all of the light that hits them (otherwise they would appear perfectly black!). Indeed, the surface reflectance (or albedo) slider in the Climate App controls the fraction of visible light that the planetary surface reflects. But these same materials are very close to being perfect blackbodies in the thermal infrared, they all have $\epsilon_{\rm LW} \approx 1$. To very good approximation, we therefore adopt $\epsilon_{\rm LW} \equiv 1$ for the planetary surface in the Climate App.

A planet's atmosphere, on the other hand, can have an infrared emissivity anywhere between zero to one ($0 \leq \epsilon_{\rm LW} \leq 1$), depending on the presence of greenhouse gases. It is precisely what the longwave emissivity slider controls (thermal opacity in the beginner version). In the beginner version the atmosphere cannot absorb shortwave radiation, so if the longwave emissivity is set to zero then the atmosphere simply cannot absorb any energy and its temperature is ill-defined and we remove the atmospheric temperature label.

In the beginner version of the app we assume that shortwave radiation does not interact with the atmosphere ($\epsilon_{\rm SW} \equiv 0$). For the advanced version, however, we allow for the absorption or scattering of shortwave radiation ($0 < \epsilon_{\rm SW} \leq 1$).

2.4 Radiative Equilibrium

A planet must radiate as much energy to space as it absorbs from its star, at least on average. If radiation is the only way to move energy around in a climate system, then the total absorbed and emitted flux must be equal for each component of the climate system (atmospheric layer or planetary surface). To understand why this equilibrium arises, imagine a patch of ground that absorbs more energy than it emits: it will warm up and hence will radiate away more heat, finding a new equilibrium.

Radiation is not the only way to move energy around. Heat conduction is rarely important for planetary climate[1] so we completely neglect it in the Climate App. Convection, on the other hand, operates in many planetary atmosphere, including the Earth's. Since convection just another means of moving heat from the planetary surface up to the atmosphere, it can roughly be thought of as contributing to the red arrow pointing from the surface up to the atmosphere.

The concept of radiative equilibrium is illustrated in the right panel of the Climate App: the total width of all arrows from the planetary surface equal the total width of all arrows reaching the surface. The same situation applies for each atmospheric layer, and indeed even at the top of the atmosphere: the same amount of energy enters the atmosphere from above as exits to space.

2.5 The Two Stream Approximation

Photons can go in any direction, but it is expedient for one-dimensional climate calculations such as these to assume that light is either travelling straight up or straight down (this simplification is known to aficionados as the two stream approximation). Conceptually, it is easy to see how the messy reality of multi-directional photons might boil down to the two-stream approximation: any ray of light must be travelling somewhat towards or away from the planet, but typical rays must pass through a bit more atmosphere because they are not entering or exiting the atmosphere perfectly vertically.

3 Deriving the Climate App Equations

All of the key atmospheric physics described above have been known since the late 19$^{\text{th}}$ century. In fact, a climate model comparable to the Beginner Climate App was first developed by the scientist Svante Arrhenius in 1896. We present here a derivation along the lines of what one would find in any introductory book on climate, as well as a more involved derivation for the Advanced Climate App.

3.1 Beginner Version

In this version we consider the simplest greenhouse model, where the incoming solar radiation passes through the atmosphere without incident. (The equations are identical if light is reflected by the atmosphere rather than the surface.)

[1] A notable exception is understanding the seasonal cycles under ground, useful for planting gardens and building foundations.

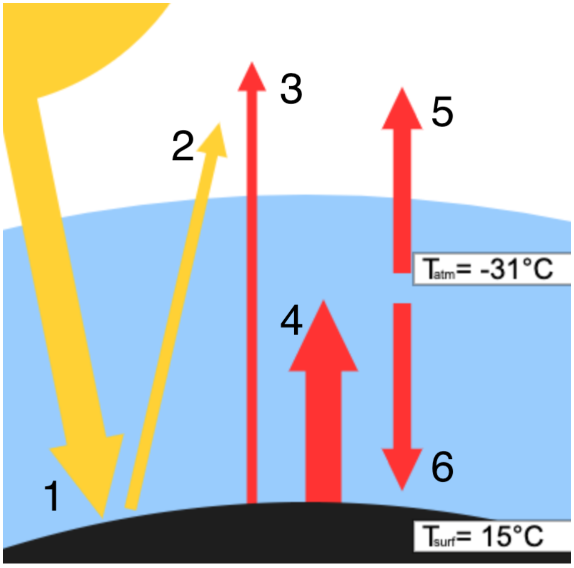

Figure 3: Simulation used in the beginner version with numbered arrows.

Figure 3 illustrates this simple climate model. Incoming solar radiation (1) is partially absorbed by the surface and partially reflected back into space (2). The surface then heats up and radiates away energy in form of longwave radiation (red arrows). Some of this energy is makes it to space (3) but some is absorbed by the atmosphere (4). The atmosphere must radiate away the energy it absorbs from the surface in order to remain in radiative equilibrium. Unlike the surface, the atmosphere radiates both upwards (5) and downwards (6). Arrows 4 and 6 illustrate the two-way energy exchange between the planetary surface and the atmosphere.

Since the temperatures of the surface and the atmosphere are interdependent, we can establish a system of equations to solve for both temperatures. In fact, a more complicated model with three or more layers follows the same principle, where the system of equations may become much more complicated as all layers are interdependent, as shown in Section 3.2.

The guiding principle is radiative equilibrium: each component of the climate system must emit as much energy as it absorbs. There are two unknowns: the surface temperature, T_{surf}, and the atmospheric temperature, T_{atm}. We therefore need two equations in order to solve the problem. In each case we put the absorbed power on the left hand side and the emitted power on the right. The atmospheric balance is

$$\epsilon_{\mathrm{LW}} \sigma T_{\mathrm{surf}}^4 = 2\epsilon_{\mathrm{LW}} \sigma T_{\mathrm{atm}}^4, \tag{1}$$

where ϵ_{LW} is the longwave emissivity (infrared opacity in the Beginner version). The left hand side is the flux emitted by the surface that is absorbed by the atmosphere (arrow 4) and the right hand side is the sum of the upward and downward fluxes emitted by the atmosphere (arrows 5 and 6). Inspection of Equation 1 reveals many of the same symbols on both sides of the equality, which begs for simplification, yielding $T_{\mathrm{surf}}^4 = 2T_{\mathrm{atm}}^4$.

The flux balance at the surface is

$$F_* - AF_* + \epsilon_{\mathrm{LW}} \sigma T_{\mathrm{atm}}^4 = \sigma T_{\mathrm{surf}}^4, \tag{2}$$

where F_* is the stellar flux reaching the planet and A is the planetary albedo (planetary reflectivity in the Beginner App). The first term on the left hand side is the incident stellar flux (arrow 1), the second term is the reflected light (arrow 2) and the third term is the downward atmospheric emission (arrow 6); on the right hand side there is the emission from the planetary surface.

Substituting our expression for atmospheric radiative equilibrium, $T_{\rm surf}^4 = 2T_{\rm atm}^4$, into Equation 1, we obtain

$$(1 - A)F_* + \epsilon_{\rm LW}\sigma T_{\rm atm}^4 = \sigma(2T_{\rm atm}^4), \tag{3}$$

which can be solved for the atmospheric temperature:

$$T_{\rm atm} = \frac{F_*}{\sigma}\left(\frac{1 - A}{2 - \epsilon_{\rm LW}}\right)^{1/4} \tag{4}$$

and from thence the surface temperature:

$$T_{\rm surf} = \frac{F_*}{\sigma}\left(\frac{1 - A}{1 - \epsilon_{\rm LW}/2}\right)^{1/4}. \tag{5}$$

For the Climate App's default values of $A = 0.30$ and $\epsilon_{\rm LW} = 0.78$, we obtain $T_{\rm atm} = 242$ K $= -31$°C and $T_{\rm surf} = 288$ K $= 15$°C, as shown in Figure 1 and in good agreement with average values on Earth today.

Lastly, the Beginner version of the Climate App reports the magnitude of the greenhouse effect. This is calculated by comparing the surface temperature with what it would have been in the absence of the greenhouse effect, i.e., $\epsilon_{\rm LW} = 0$. For the default parameters, the greenhouse effect is 33°C, again in good agreement with modern-day Earth.

3.2 Advanced Version

Using the same reasoning as the simple greenhouse model, we can derive the system of equations that allow us to solve for the temperatures of the planetary surface and three atmospheric layers (see Appendix A). However, the model used in the advanced version is more complicated (Figure 4). In addition to the extra atmospheric layers, the advanced Climate App includes two new parameters (sliders) for each atmospheric layer to control how they interact with shortwave radiation (light from the star). These parameters are the shortwave opacity, $\epsilon_{\rm SW}$, and single-scattering albedo, α. When it arrives at an atmospheric layer, shortwave radiation can do three things: transmission, $1 - \epsilon_{\rm SW}$, scattering $\epsilon_{\rm SW}\alpha$, or absorption, $\epsilon_{\rm SW}(1 - \alpha)$.

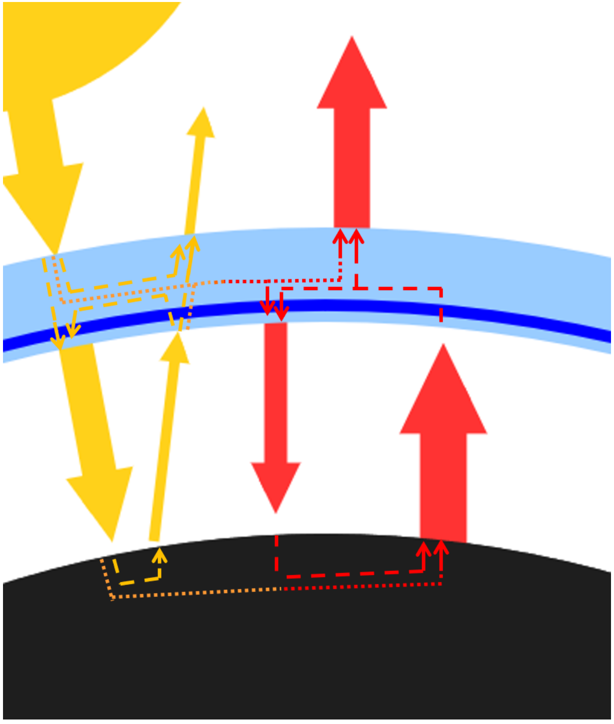

Figure 4: Schematic showing how the various fluxes (arrows in Climate App) are connected to each other for advanced version of the Climate App.

For now we sweep all of the shortwave complications under the rug by using F_3, F_2, F_1 and F_0 to denote the shortwave flux absorbed at layer 3, layer 2, layer 1 of the atmosphere and at the surface, respectively. This yields a fairly compact form for the four coupled equations, starting with the top of the atmosphere (layer 3):

$$
\begin{aligned}
2\epsilon_{\mathrm{LW3}}T_3^4 &= \frac{F_3}{\sigma} + T_0^4\epsilon_{\mathrm{LW3}}(1-\epsilon_{\mathrm{LW1}})(1-\epsilon_{\mathrm{LW2}}) + T_1^4\epsilon_{\mathrm{LW1}}(1-\epsilon_{\mathrm{LW2}})\epsilon_{\mathrm{LW3}} + T_2^4\epsilon_{\mathrm{LW2}}\epsilon_{\mathrm{LW3}}\\
2\epsilon_{\mathrm{LW2}}T_2^4 &= \frac{F_2}{\sigma} + T_0^4\epsilon_{\mathrm{LW2}}(1-\epsilon_{\mathrm{LW1}}) + T_1^4\epsilon_{\mathrm{LW1}}\epsilon_{\mathrm{LW2}} + T_3^4\epsilon_{\mathrm{LW2}}\epsilon_{\mathrm{LW3}}\\
2\epsilon_{\mathrm{LW1}}T_1^4 &= \frac{F_1}{\sigma} + T_0^4\epsilon_{\mathrm{LW1}} + T_2^4\epsilon_{\mathrm{LW1}}\epsilon_{\mathrm{LW2}} + T_3^4\epsilon_{\mathrm{LW1}}(1-\epsilon_{\mathrm{LW2}})\epsilon_{\mathrm{LW3}}\\
T_0^4 &= \frac{F_0}{\sigma} + T_1^4\epsilon_{\mathrm{LW1}} + T_2^4\epsilon_{\mathrm{LW2}}(1-\epsilon_{\mathrm{LW1}}) + T_3^4\epsilon_{\mathrm{LW3}}(1-\epsilon_{\mathrm{LW2}})(1-\epsilon_{\mathrm{LW1}}),
\end{aligned}
$$

where ϵ_{LW1}, ϵ_{LW2} and ϵ_{LW3} are the longwave emissivity of the three atmopsheric layers (recall that we assume the surface has a longwave emissivity of 1). Solving this system of equation gives:

$$T_3^4 = \frac{\epsilon_{\mathrm{LW3}} F_1 + \epsilon_{\mathrm{LW3}} F_2 + \epsilon_{\mathrm{LW3}} F_0 + F_3}{\sigma \epsilon_{\mathrm{LW3}} (2 - \epsilon_{\mathrm{LW3}})}$$

$$T_2^4 = \frac{\begin{pmatrix} -\epsilon_{\mathrm{LW2}} 2\epsilon_{\mathrm{LW3}} F_1 - \epsilon_{\mathrm{LW2}} 2\epsilon_{\mathrm{LW3}} F_2 - \epsilon_{\mathrm{LW2}} 2\epsilon_{\mathrm{LW3}} F_0 - \epsilon_{\mathrm{LW2}} 2F_3 \\ +\epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} F_1 + 2\epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} F_2 + \epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} F_0 + 2\epsilon_{\mathrm{LW2}} F_1 \\ + 2\epsilon_{\mathrm{LW2}} F_3 + 2\epsilon_{\mathrm{LW2}} F_0 - \epsilon_{\mathrm{LW3}} F_2 + 2F_2 \end{pmatrix}}{\sigma \epsilon_{\mathrm{LW2}} (\epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} - 2\epsilon_{\mathrm{LW2}} - 2\epsilon_{\mathrm{LW3}} + 4)}$$

$$T_1^4 = \frac{\begin{pmatrix} -2\epsilon_{\mathrm{LW1}} 2\epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} F_1 - \epsilon_{\mathrm{LW1}} 2\epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} l_2 - 2\epsilon_{\mathrm{LW1}} 2\epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} F_0 \\ +2\epsilon_{\mathrm{LW1}} 2\epsilon_{\mathrm{LW2}} F_1 - \epsilon_{\mathrm{LW1}} 2\epsilon_{\mathrm{LW2}} F_3 + 2\epsilon_{\mathrm{LW1}} 2\epsilon_{\mathrm{LW2}} F_0 + 2\epsilon_{\mathrm{LW1}} 2\epsilon_{\mathrm{LW3}} F_1 \\ +\epsilon_{\mathrm{LW1}} 2\epsilon_{\mathrm{LW3}} F_2 + 2\epsilon_{\mathrm{LW1}} 2\epsilon_{\mathrm{LW3}} F_0 + 2\epsilon_{\mathrm{LW1}} 2F_2 + 2\epsilon_{\mathrm{LW1}} 2F_3 \\ +4\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} F_1 + 2\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} F_2 + 3\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} F_0 \\ -4\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW2}} F_1 + 2\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW2}} F_3 - 2\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW2}} F_0 - 4\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW3}} F_1 \\ -2\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW3}} F_2 - 2\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW3}} F_0 - 4\epsilon_{\mathrm{LW1}} F_2 - 4\epsilon_{\mathrm{LW1}} F_3 \\ - 4\epsilon_{\mathrm{LW1}} F_0 - \epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} F_1 + 2\epsilon_{\mathrm{LW2}} F_1 + 2\epsilon_{\mathrm{LW3}} F_1 - 4F_1 \end{pmatrix}}{\sigma \epsilon_{\mathrm{LW1}} (\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} - 2\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW2}} - 2\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW3}} + 4\epsilon_{\mathrm{LW1}} - 2\epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} + 4\epsilon_{\mathrm{LW2}} + 4\epsilon_{\mathrm{LW3}} - 8)}$$

$$T_0^4 = \frac{\begin{pmatrix} -2\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} F_1 - \epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} F_2 - 2\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} F_0 + 2\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW2}} F_1 \\ -\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW2}} F_3 + 2\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW2}} F_0 + 2\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW3}} F_1 + \epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW3}} F_2 + 2\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW3}} F_0 \\ +2\epsilon_{\mathrm{LW1}} F_2 + 2\epsilon_{\mathrm{LW1}} F_3 + 3\epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} F_1 + 2\epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} F_2 + 2\epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} F_0 - 2\epsilon_{\mathrm{LW2}} F_1 \\ + 2\epsilon_{\mathrm{LW2}} F_3 - 2\epsilon_{\mathrm{LW3}} F_1 - 2\epsilon_{\mathrm{LW3}} F_2 - 4F_1 - 4F_2 - 4F_3 - 8F_0 \end{pmatrix}}{\sigma (\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} - 2\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW2}} - 2\epsilon_{\mathrm{LW1}} \epsilon_{\mathrm{LW3}} + 4\epsilon_{\mathrm{LW1}} - 2\epsilon_{\mathrm{LW2}} \epsilon_{\mathrm{LW3}} + 4\epsilon_{\mathrm{LW2}} + 4\epsilon_{\mathrm{LW3}} - 8)}$$

The values of the shortwave flux terms F_3, F_2, F_1 and F_0 were determined as follows. In the one-dimensional scattering model used in the Climate App, we assume an asymmetry factor of 0 at each atmospheric layer, i.e., isotropic scattering. This means that half of the photons scattered at a given layer are scattered back in the direction they came from, while the other half are scattered forward. We treat the forward scattering case as equivalent to transmission (since we use an 1-D model, we ignore the scattering angle). We the isotropic scattering assumption, we can solve the following system of equation to obtain the shortwave terms (see appendix B for the solutions):

$$\begin{aligned}
F_3 &= (I + u_2)\epsilon_{\mathrm{SW3}}(1 - \alpha_3) \\
F_2 &= (d_3 + u_1)\epsilon_{\mathrm{SW2}}(1 - \alpha_2) \\
F_1 &= (d_2 + u_0)\epsilon_{\mathrm{SW1}}(1 - \alpha_1) \\
F_0 &= d_1(1 - A) \\
u_0 &= d_1 A \\
d_1 &= 0.5u_0\epsilon_{\mathrm{SW1}}\alpha_1 + d_2(1 - \epsilon_{\mathrm{SW1}}) + 0.5d_2\epsilon_{\mathrm{SW1}}\alpha_1 \\
u_1 &= u_0(1 - \epsilon_{\mathrm{SW1}}) + 0.5d_2\epsilon_{\mathrm{SW1}}\alpha_1 + 0.5u_0\epsilon_{\mathrm{SW1}}\alpha_1 \\
d_2 &= 0.5u_1\epsilon_{\mathrm{SW2}}\alpha_2 + d_3(1 - \epsilon_{\mathrm{SW2}}) + 0.5d_3\epsilon_{\mathrm{SW2}}\alpha_2 \\
u_2 &= 0.5d_3\epsilon_{\mathrm{SW2}}\alpha_2 + u_1(1 - \epsilon_{\mathrm{SW2}}) + 0.5u_1\epsilon_{\mathrm{SW2}}\alpha_2 \\
d_3 &= 0.5u_2\epsilon_{\mathrm{SW3}}\alpha_3 + I(1 - \epsilon_{\mathrm{SW3}}) + 0.5I\epsilon_{\mathrm{SW3}}\alpha_3 \\
u_3 &= u_2(1 - \epsilon_{\mathrm{SW3}}) + 0.5I\epsilon_{\mathrm{SW3}}\alpha_3 + 0.5u_2\epsilon_{\mathrm{SW3}}\alpha_3
\end{aligned}$$

where A is the surface albedo, I is the incident stellar flux, ϵ_{SW1}, ϵ_{SW2} and ϵ_{SW3} are the shortwave opacities of each atmospheric layer and α_1, α_2 and α_3 are their single scattering albedo.

The total shortwave radiation heading upward from layer i is denoted by u_i. Likewise, the total shortwave radiation heading downward from layer i is denoted by d_i. For example, in figure 5, the black arrow represents shortwave radiation transmitted upward from layer 1 (u_1). It is the sum of three components indicated by red arrows (from left to right): a fraction of shortwave radiation reflected upward from the surface (u_0) which goes through layer 1 without any interaction ($u_0(1-\epsilon_{\mathrm{SW1}})$), a fraction of radiation transmitted downward from layer 2 (d_2) which scatters back as it reaches layer 1 ($\frac{1}{2}d_2\epsilon_{\mathrm{SW1}}\alpha_1$), and another fraction of radiation reflected upward from the surface (u_0) which scatters forward as it reaches layer 1 ($\frac{1}{2}u_0\epsilon_{\mathrm{SW1}}\alpha_1$). The factor of $\frac{1}{2}$ comes from the assumption of isotropic scattering.

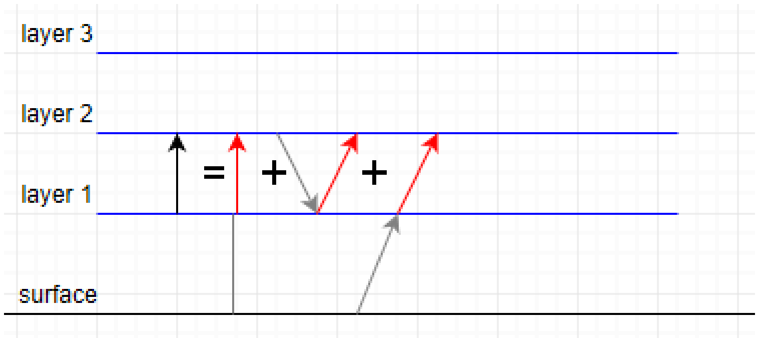

Figure 5: Diagram showing how the expression of u1 (red arrow) was obtained.

4 Who Made the Climate App?

The Climate App was envisioned and developed by Prof. Nicolas B. Cowan and his trainees at McGill University. The first version of the Climate App was developed by Dr. Joel C. Schwartz as an off-line application. The current online version of the Climate App was developed by Anthony Courchesne. The advanced version and other improvements have been made by Lan Xi Zhu. If you find any bugs or errors in the Climate App or this documentation, please email nicolas.cowan@mcgill.ca.

A Three-layer model without shortwave opacity

For three atmospheric layers but no shortwave absorption, the system of equations to solve is:

$$2\epsilon_3 T_3^4 = \epsilon_3 T_0^4(1-\epsilon_1)(1-\epsilon_2) + \epsilon_3\epsilon_1 T_1^4(1-\epsilon_2) + \epsilon_3\epsilon_2 T_2^4$$

$$2\epsilon_2 T_2^4 = \epsilon_2 T_0^4(1-\epsilon_1) + \epsilon_2\epsilon_1 T_1^4 + \epsilon_2\epsilon_3 T_3^4$$

$$2\epsilon_1 T_1^4 = \epsilon_1 T_0^4 + \epsilon_1\epsilon_2 T_2^4 + \epsilon_1\epsilon_3 T_3^4(1-\epsilon_2)$$

$$T_0^4 = \frac{I(1-A)}{\sigma} + \epsilon_1 T_1^4 + \epsilon_2 T_2^4(1-\epsilon_1) + \epsilon_3 T_3^4(1-\epsilon_2)(1-\epsilon_1),$$

where ϵ_i refers to longwave emissivity at the ith layer. As one would expect, one can also obtain these equations from the full advanced app equations and neglecting shortwave emissivity ($\epsilon_{\mathrm{SW1}} = \epsilon_{\mathrm{SW2}} = \epsilon_{\mathrm{SW3}} = 0$).

The solutions for the temperatures of the three atmospheric layers, T_3, T_2 and T_1 and the surface temperature, T_0 are:

$$T_3^4 = \frac{I(1-A)}{\sigma(2-\epsilon_3)}$$

$$T_2^4 = \frac{I(1-A)(2+\epsilon_3-\epsilon_2\epsilon_3)}{\sigma(2-\epsilon_2)(2-\epsilon_3)}$$

$$T_1^4 = \frac{I(1-A)(4+2\epsilon_2-2\epsilon_1\epsilon_2+2\epsilon_3-2\epsilon_1\epsilon_3-3\epsilon_2\epsilon_3+2\epsilon_1\epsilon_2\epsilon_3)}{\sigma(2-\epsilon_1)(2-\epsilon_2)(2-\epsilon_3)}$$

$$T_0^4 = \frac{2I(1-A)(4-\epsilon_1\epsilon_2-\epsilon_1\epsilon_3-\epsilon_2\epsilon_3+\epsilon_1\epsilon_2\epsilon_3)}{\sigma(2-\epsilon_1)(2-\epsilon_2)(2-\epsilon_3)}.$$

B Shortwave absorption terms

The solutions of the shortwave absorption terms, F_3, F_2, F_1 and F_0, are:

$$F_3 = \frac{\left(\begin{array}{l} I \cdot \epsilon_{\mathrm{SW3}} \cdot (1-\alpha_3)\Big(\big(A\alpha_1^2\alpha_2^2\alpha_3\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW2}}^2\epsilon_{\mathrm{SW3}} - 4A\alpha_1^2\alpha_2\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW2}} \\ -A\alpha_1^2\alpha_3\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW3}}(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2)^2 - 4A\alpha_1\alpha_2\alpha_3\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}}\epsilon_{\mathrm{SW3}} + 16A\alpha_1\epsilon_{\mathrm{SW1}} \\ -A\alpha_2^2\alpha_3\epsilon_{\mathrm{SW2}}^2\epsilon_{\mathrm{SW3}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2 + 4A\alpha_2\epsilon_{\mathrm{SW2}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2 \\ +A\alpha_3\epsilon_{\mathrm{SW3}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2)^2 - 2\alpha_1\alpha_2^2\alpha_3\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}}^2\epsilon_{\mathrm{SW3}} \\ +8\alpha_1\alpha_2\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}} + 2\alpha_1\alpha_3\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW3}}(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2)^2 + 8\alpha_2\alpha_3\epsilon_{\mathrm{SW2}}\epsilon_{\mathrm{SW3}} - 32\big) \\ -(\alpha_3\epsilon_{\mathrm{SW3}} - 2\epsilon_{\mathrm{SW3}} + 2)\big(A\alpha_1^2\alpha_2^2\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW2}}^2 - A\alpha_1^2\alpha_2\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW2}}(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2) \\ -2A\alpha_1^2\alpha_2\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW2}} + 2A\alpha_1^2\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW2}}(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2) + 4A\alpha_1^2\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW2}} \\ -4A\alpha_1^2\epsilon_{\mathrm{SW1}}^2 - 4A\alpha_1\alpha_2\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}} + 4A\alpha_1\epsilon_{\mathrm{SW1}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2) + 8A\alpha_1\epsilon_{\mathrm{SW1}} \\ -A\alpha_2^2\epsilon_{\mathrm{SW2}}^2(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2 + A\alpha_2\epsilon_{\mathrm{SW2}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2) \\ +2A\alpha_2\epsilon_{\mathrm{SW2}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2 - 8A\epsilon_{\mathrm{SW1}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2) - 16A\epsilon_{\mathrm{SW1}} \\ -2A\epsilon_{\mathrm{SW2}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2) - 4A\epsilon_{\mathrm{SW2}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2 \\ +16A - 2\alpha_1\alpha_2^2\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}}^2 + 2\alpha_1\alpha_2\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}}(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2) \\ +4\alpha_1\alpha_2\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}} - 4\alpha_1\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}}(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2) \\ - 8\alpha_1\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}} + 8\alpha_1\epsilon_{\mathrm{SW1}} + 8\alpha_2\epsilon_{\mathrm{SW2}}\big)\Big) \end{array}\right)}{\left(\begin{array}{l} A\alpha_1^2\alpha_2^2\alpha_3\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW2}}^2\epsilon_{\mathrm{SW3}} - 4A\alpha_1^2\alpha_2\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW2}} - A\alpha_1^2\alpha_3\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW3}}(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2)^2 \\ -4A\alpha_1\alpha_2\alpha_3\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}}\epsilon_{\mathrm{SW3}} + 16A\alpha_1\epsilon_{\mathrm{SW1}} - A\alpha_2^2\alpha_3\epsilon_{\mathrm{SW2}}^2\epsilon_{\mathrm{SW3}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2 \\ +4A\alpha_2\epsilon_{\mathrm{SW2}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2 + A\alpha_3\epsilon_{\mathrm{SW3}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2 \\ \cdot(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2)^2 - 2\alpha_1\alpha_2^2\alpha_3\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}}^2\epsilon_{\mathrm{SW3}} + 8\alpha_1\alpha_2\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}} \\ + 2\alpha_1\alpha_3\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW3}}(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2)^2 + 8\alpha_2\alpha_3\epsilon_{\mathrm{SW2}}\epsilon_{\mathrm{SW3}} - 32 \end{array}\right)}$$

$$F_2 = \frac{\left(\begin{array}{l} 4I \cdot \epsilon_{\mathrm{SW2}}(\alpha_3\epsilon_{\mathrm{SW3}} - 2\epsilon_{\mathrm{SW3}} + 2)[A\alpha_1^2\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW2}}(\alpha_2 - 1) \\ -A\alpha_1^2\epsilon_{\mathrm{SW1}}^2(\alpha_2 - 1) + A\alpha_1\epsilon_{\mathrm{SW1}}(\alpha_2 - 1)(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2) \\ -2A\epsilon_{\mathrm{SW1}}(\alpha_2 - 1)(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2) - 4A\epsilon_{\mathrm{SW1}}(\alpha_2 - 1) \\ -A\epsilon_{\mathrm{SW2}}(\alpha_2 - 1)(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2 + 4A(\alpha_2 - 1) \\ - 2\alpha_1\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}}(\alpha_2 - 1) + 2\alpha_1\epsilon_{\mathrm{SW1}}(\alpha_2 - 1) + 4\alpha_2 - 4] \end{array}\right)}{\left(\begin{array}{l} A\alpha_1^2\alpha_2^2\alpha_3\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW2}}^2\epsilon_{\mathrm{SW3}} - 4A\alpha_1^2\alpha_2\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW2}} - A\alpha_1^2\alpha_3\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW3}}(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2)^2 \\ -4A\alpha_1\alpha_2\alpha_3\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}}\epsilon_{\mathrm{SW3}} + 16A\alpha_1\epsilon_{\mathrm{SW1}} - A\alpha_2^2\alpha_3\epsilon_{\mathrm{SW2}}^2\epsilon_{\mathrm{SW3}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2 \\ +4A\alpha_2\epsilon_{\mathrm{SW2}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2 + A\alpha_3\epsilon_{\mathrm{SW3}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2 \\ \cdot(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2)^2 - 2\alpha_1\alpha_2^2\alpha_3\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}}^2\epsilon_{\mathrm{SW3}} + 8\alpha_1\alpha_2\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}} \\ + 2\alpha_1\alpha_3\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW3}}(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2)^2 + 8\alpha_2\alpha_3\epsilon_{\mathrm{SW2}}\epsilon_{\mathrm{SW3}} - 32 \end{array}\right)}$$

$$F_1 = \frac{\begin{pmatrix} 4I \cdot \epsilon_{\mathrm{SW1}}(\alpha_3\epsilon_{\mathrm{SW3}} - 2\epsilon_{\mathrm{SW3}} + 2)[-A\alpha_1\alpha_2\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}}(\alpha_1 - 1) \\ +2A\alpha_1\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}}(\alpha_1 - 1) + A\alpha_2\epsilon_{\mathrm{SW2}}(\alpha_1 - 1)(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2) \\ -4A\epsilon_{\mathrm{SW1}}(\alpha_1 - 1) - 2A\epsilon_{\mathrm{SW2}}(\alpha_1 - 1)(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2) \\ + 4A(\alpha_1 - 1) + 4\alpha_1 + 2\alpha_2\epsilon_{\mathrm{SW2}}(\alpha_1 - 1) - 4\epsilon_{\mathrm{SW2}}(\alpha_1 - 1) - 4] \end{pmatrix}}{\begin{pmatrix} A\alpha_1^2\alpha_2^2\alpha_3\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW2}}^2\epsilon_{\mathrm{SW3}} - 4A\alpha_1^2\alpha_2\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW2}} - A\alpha_1^2\alpha_3\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW3}}(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2)^2 \\ -4A\alpha_1\alpha_2\alpha_3\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}}\epsilon_{\mathrm{SW3}} + 16A\alpha_1\epsilon_{\mathrm{SW1}} - A\alpha_2^2\alpha_3\epsilon_{\mathrm{SW2}}^2\epsilon_{\mathrm{SW3}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2 \\ +4A\alpha_2\epsilon_{\mathrm{SW2}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2 + A\alpha_3\epsilon_{\mathrm{SW3}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2 \\ \cdot(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2)^2 - 2\alpha_1\alpha_2^2\alpha_3\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}}^2\epsilon_{\mathrm{SW3}} + 8\alpha_1\alpha_2\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}} \\ + 2\alpha_1\alpha_3\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW3}}(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2)^2 + 8\alpha_2\alpha_3\epsilon_{\mathrm{SW2}}\epsilon_{\mathrm{SW3}} - 32 \end{pmatrix}}$$

$$F_0 = \frac{\begin{pmatrix} 4I \cdot (\alpha_3\epsilon_{\mathrm{SW3}} - 2\epsilon_{\mathrm{SW3}} + 2)[4A + 2\alpha_1\epsilon_{\mathrm{SW1}}(A - 1) + \alpha_2\epsilon_{\mathrm{SW2}}(A - 1)(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2) \\ - 4\epsilon_{\mathrm{SW1}}(A - 1) - 2\epsilon_{\mathrm{SW2}}(A - 1)(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2) - 4] \end{pmatrix}}{\begin{pmatrix} A\alpha_1^2\alpha_2^2\alpha_3\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW2}}^2\epsilon_{\mathrm{SW3}} - 4A\alpha_1^2\alpha_2\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW2}} - A\alpha_1^2\alpha_3\epsilon_{\mathrm{SW1}}^2\epsilon_{\mathrm{SW3}}(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2)^2 \\ -4A\alpha_1\alpha_2\alpha_3\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}}\epsilon_{\mathrm{SW3}} + 16A\alpha_1\epsilon_{\mathrm{SW1}} - A\alpha_2^2\alpha_3\epsilon_{\mathrm{SW2}}^2\epsilon_{\mathrm{SW3}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2 \\ +4A\alpha_2\epsilon_{\mathrm{SW2}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2 + A\alpha_3\epsilon_{\mathrm{SW3}}(\alpha_1\epsilon_{\mathrm{SW1}} - 2\epsilon_{\mathrm{SW1}} + 2)^2 \\ \cdot(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2)^2 - 2\alpha_1\alpha_2^2\alpha_3\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}}^2\epsilon_{\mathrm{SW3}} + 8\alpha_1\alpha_2\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW2}} \\ + 2\alpha_1\alpha_3\epsilon_{\mathrm{SW1}}\epsilon_{\mathrm{SW3}}(\alpha_2\epsilon_{\mathrm{SW2}} - 2\epsilon_{\mathrm{SW2}} + 2)^2 + 8\alpha_2\alpha_3\epsilon_{\mathrm{SW2}}\epsilon_{\mathrm{SW3}} - 32 \end{pmatrix}}$$