Zhu Li's Test Flight Problem Set

For the Stanford MOOC "Introduction to Mathematical Thinking" course.

Disclaimer:

0 is not considered a natural number in this problem set.

For each question, you must enter your answer into the appropriate entry field in the Test Flight module (TEX entry is possible), or you may upload a file (JPEG, scanned PDF of handwritten solution, PDF from a Word file, etc.) Your answers will be peer graded according to the course rubric

YOU ARE EXPECTED TO WORK ALONE ON THIS PROBLEM SET.

1: Say whether the following is true or false and support your answer by a proof.

$$(\exists m \in \mathcal{N})(\exists n \in \mathcal{N})(3m + 5n = 12)$$

Proof:

$\because$ m and n are natural numbers.
$\therefore m \geq 1$ and $n \geq 1$ and $m, n \in \mathcal{Z}$.
$\therefore \forall m \in \mathcal{N}, 3m + 5n \geq 3m + 5, \forall n \in \mathcal{N}, 3m + 5n \geq 3 + 5n$.
$\therefore$ For $3m + 5n \leq 12, m \leq \lfloor \frac{12-5}{3} \rfloor, n \leq \lfloor \frac{12-3}{5} \rfloor$
$\therefore 1 \leq m \leq 2, 1 \leq n \leq 1$
$\therefore$ There're finite pairs of (m,n) s.t. $3m + 5n \leq 12$.

All possible enumerations are:
$3 \times 1 + 5 \times 1 = 8$
$3 \times 2 + 5 \times 1 = 11$

$\therefore$ None of the possible enumerations sums up to 12.
$\therefore$ There **doesn't exist** natural numbers m and n, s.t. $3m + 5n = 12$.
Q.E.D.

2: Say whether the following is true or false and support your answer by a proof:
The sum of any five consecutive integers is divisible by 5 (without remainder).

Proof:

Let n be an integer, thus $n \in \mathcal{N}$.

$\therefore$

$$\begin{aligned} S &= \sum_{i=0}^{4}(n+i) \\ &= 5n + (0+1+2+3+4) \\ &= 5n + 10 \\ &= 5(n+2) \end{aligned}$$

$\because n+2 \in \mathcal{N}$.
$\therefore$ 5 divides S.
$\therefore$ **The sum of any five consecutive integers is divisible by 5.**
Q.E.D.

3: Say whether the following is true or false and support your answer by a proof:
For any integer n, the number $n^2 + n + 1$ is odd.

Proof:

Let n be an integer, thus $n = 2k$ or $n = 2k + 1$, where $k \in \mathcal{Z}$.

When $n = 2k$,

$$\begin{aligned} n^2 + n + 1 &= 4k^2 + 2k + 1 \\ &= 2(2k^2 + k) + 1 \end{aligned}$$

$\therefore n^2 + n + 1$ is odd when n is even.

When $n = 2k + 1$,

$$\begin{aligned} n^2 + n + 1 &= (2k+1)^2 + (2k+1) + 1 \\ &= 4k^2 + 6k + 3 \\ &= 2(2k^2 + 3k + 1) + 1 \end{aligned}$$

$\therefore n^2 + n + 1$ is odd when n is odd.

$\therefore \forall n \in \mathcal{Z}, n^2 + n + 1$ **is odd.**
Q.E.D.

4: Prove that every odd natural number is of one of the forms 4n + 1 or 4n + 3, where n is an integer.

Proof:

Let x be an odd natural number.
$\therefore x = 2y + 1, y \in \mathcal{Z}$

When y is an odd natural number, $y = 2n + 1, n \in \mathcal{Z}$
$\therefore x = 2y + 1 = 2(2n + 1) + 1 = 4n + 3$

When y is an even natural number, $y = 2n, n \in \mathcal{Z}$
$\therefore x = 2y + 1 = 2(2n) + 1 = 4n + 1$

$\therefore \exists n \in \mathcal{N}, s.t.\ x = 4n + 1$ or $x = 4n + 3$
Q.E.D.

5: Prove that for any integer n, at least one of the integers n, n + 2, n + 4 is divisible by 3.

Proof:

$\because n + 4 \equiv n + 1 \mod 3$
$\therefore$ The problem is equivalent for the group $(n, n + 1, n + 2)$.

$\forall n \in \mathcal{Z}, \exists k \in \mathcal{Z}, \exists r \in \{0, 1, 2\}, s.t.\ n = 3k + r.$
$\therefore$ The problem is equivalent for the group $(r, r + 1, r + 2)$.

Suppose **none of $(r, r + 1, r + 2)$ is divisible by 3**.
$\therefore$ The 3 remainders of $(r, r + 1, r + 2)$ can take at most 2 values from $\{1, 2\}$.
$\therefore \exists x, y \in \{r, r + 1, r + 2\}, x < y, s.t.\ x \equiv y \mod 3$
$\because y - x \in \{1, 2\}$
$\therefore x \not\equiv y \mod 3$
Thus the assumption is contradictory.

$\therefore$ At least one of $\{r, r + 1, r + 2\}$ is divisible by 3.
$\therefore$ At least one of $\{n, n + 2, n + 4\}$ is divisible by 3.
Q.E.D.

6: A classic unsolved problem in number theory asks if there are infinitely many pairs of 'twin primes', pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

Proof:

Suppose there exists a triple $(n, n+2, n+4), s.t.$ they're all primes.
By the proof I've done in **question 5**, at least one of $(n, n+2, n+4)$ is divisible by 3.
$\therefore$ When a prime is divisible by 3, the prime itself must be 3.
$\therefore$ One of $(n, n+2, n+4)$ must be 3.
$\because$ n is prime, we have $n \geq 2, n+2 \geq 4, n+4 \geq 6$.
$\therefore$ n must be 3.
$\therefore$ The triple must be $(3, 5, 7)$.

$\therefore$ The only prime triple is 3, 5, 7.
Q.E.D.

7: Prove that for any natural number n,

$$2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$$

Proof:

Let $S = \sum_{i=1}^{n} 2^i$, thus $2S = \sum_{i=1}^{n} 2^{i+1}$
$\therefore$

$$\begin{aligned} 2S - S &= 2^{n+1} + (\sum_{i=2}^{n} 2^i) - (\sum_{i=2}^{n} 2^i) - 2 \\ &= 2^{n+1} - 2 \end{aligned}$$

$\therefore S = 2^{n+1} - 2$
$\therefore 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$
Q.E.D.

8: Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, then for any fixed number $M > 0$, the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.

Proof:

$\because$ The sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$.
$\therefore \forall \epsilon > 0, \exists N_0 \in \mathcal{N}, \forall n > N_0, s.t. |a_n - L| < \epsilon.$
$\therefore \forall \epsilon > 0, \exists N_0 \in \mathcal{N}, \forall n > N_0, s.t. M|a_n - L| < M\epsilon.$
$\because M > 0$
$\therefore$ Let $\epsilon' = M\epsilon$, we have $\epsilon' > 0$.
$\therefore \forall \epsilon' > 0, \exists N_0 \in \mathcal{N}, \forall n > N_0, s.t. |(Ma_n) - (ML)| < \epsilon'.$
$\therefore \forall M > 0$, we have $\lim_{n \to \infty} \{Ma_n\} = ML$.
Q.E.D.

9: Given an infinite collection $A_n, n = 1, 2, \ldots$ of intervals of the real line, their intersection is defined to be.

$$\bigcap_{n=1}^{\infty} A_n = \{x | (\forall n)(x \in A_n)\}$$

Give an example of a family of intervals $A_n, n = 1, 2, \ldots$, such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n = \phi$. Prove that your example has the stated property.

Proof:

Let there be a sequence $\{a_n\}$, where $a_n = \frac{1}{n}, n \in \mathcal{N}$.
$\therefore \forall \epsilon > 0$, let $N_0 = \lceil \frac{1}{\epsilon} \rceil$.
$\therefore N_0 \geq \frac{1}{\epsilon}, \frac{1}{N_0} \leq \epsilon$.
$\therefore \forall n > N_0$, we have. $\therefore$

$$\begin{aligned} |a_n - 0| &= |a_n| \\ &= |\frac{1}{n}| \\ &= \frac{1}{n} \\ &< \frac{1}{N_0} \\ &\leq \epsilon \end{aligned}$$

$\therefore \forall \epsilon > 0, \exists N_0 \in \mathcal{N}, \forall n > N_0, |a_n - 0| < \epsilon.$
$\therefore \lim\limits_{n \to \infty} a_n = 0$

Let $A_n = (0, a_n)$.
$\therefore \forall n \in \mathcal{N}, a_{n+1} < a_n$.
$\therefore \forall n \in \mathcal{N}, A_{n+1} \subset A_n$.

$\because \lim\limits_{n \to \infty} a_n = 0$
$\therefore \lim\limits_{n \to \infty} A_n = (0, \lim\limits_{n \to \infty} a_n) = (0, 0) = \phi$

$\because A_1 \subset A_2 \subset \ldots A_n$.
$\therefore$

$$\begin{aligned} \bigcap_{n=1}^{\infty} A_n &= \lim_{n \to \infty} \bigcap_{i=1}^{n} A_i \\ &= \lim_{n \to \infty} A_n \\ &= \phi \end{aligned}$$

$\therefore$ The sequence of intervals $A_n = (0, \frac{1}{n})$ satifies the stated property.
Q.E.D.

10: Give an example of a family of intervals $A_n, n = 1, 2, \ldots$, such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number. Prove that your example has the stated property.

Proof:

Let there be a sequence $\{a_n\}$, where $a_n = \frac{1}{n}, n \in \mathcal{N}$.
By the proof of **question 9**, we have $\lim\limits_{n \to \infty} a_n = 0$

Let $A_n = [0, a_n]$.
The proof of property $A_{n+1} \subset A_n$ is the same as in **question 9**.

As for the limit of A_n

$$\begin{aligned} \lim_{n \to \infty} A_n &= [0, \lim_{n \to \infty} a_n] \\ &= [0, 0] \\ &= \{0\} \end{aligned}$$

$\because A_1 \subset A_2 \subset \ldots A_n$.
$\therefore$

$$\begin{aligned} \bigcap_{n=1}^{\infty} A_n &= \lim_{n \to \infty} \bigcap_{i=1}^{n} A_i \\ &= \lim_{n \to \infty} A_n \\ &= \{0\} \end{aligned}$$

$\therefore$ The sequence of intervals $A_n = [0, \frac{1}{n}]$ satifies the stated property.
Q.E.D.