CSC304 Fall'22
Assignment 3

Due: December 5, 2022, by 11:59pm ET

Instructions:

1. Typed assignments are preferred (e.g., using LaTeX or Word), especially if your handwriting is possibly illegible or if you do not have access to a good quality scanner. Please submit a single PDF to MarkUs named "hwk3.pdf".
2. Remember our citation policy. You are free to read online material (though, if you find the exact homework problem, do not read the solution) and to discuss any problems with your peers. There are two rules you must follow: You must write the solution in your own words (it helps to not take any pictures or notes from your discussions), and you must cite any peers or online sources from where you obtained a significant hint.
3. Attempt to solve each problem, but if you have no clue how to approach a (sub)problem, remember the 20% rule: You can get 20% points for a (sub)problem by just stating "I don't know how to approach this question" (and 10% if you leave it blank but do not write such a statement).

Q1 [15 Points] Fun with Deferred Acceptance

Consider the Deferred Acceptance algorithm to find a stable matching between n men and n women where each participant has a strict ranking over participants of the opposite gender.

(a) [5 Points] Consider the following preferences for 4 men (M1 through M4) and 4 women (W1 through W4). Each row gives the preference of one individual, and the preference decreases from left (most preferred) to right (least preferred).

Men's Preferences

M1	W2	W4	W1	W3
M2	W3	W1	W4	W2
M3	W2	W3	W1	W4
M4	W4	W1	W3	W2

Women's Preferences

W1	M2	M1	M4	M3
W2	M4	M3	M1	M2
W3	M1	M4	M3	M2
W4	M2	M1	M4	M3

Run men-proposing deferred acceptance (MPDA) and women-proposing deferred acceptance (WPDA) on this instance. For each algorithm, describe each iteration: who proposes to whom in that iteration, and who is engaged to whom at the end of the iteration.

(b) [5 Points] Suppose there are k "good" men and k "good" women such that in the preference ranking of each woman (resp. man), the top k men (resp. women) are precisely the k good men (resp. women) in some order. That is, every participant prefers the k good participants of the opposite gender to the other participants of the opposite gender. Show that in any stable matching, the k good men must be matched to the k good women.

(c) [5 Points] Show that if men-proposing deferred acceptance (MPDA) and women-proposing deferred acceptance (WPDA) return the same stable matching, then it is the unique stable matching in that instance. [Hint: Use the optimality and pessimality results.]

Q2 [15 Points] Do voting rules think alike?

A group of 100 students voted to elect their monitor. They submitted ranked votes over 4 candidates: Alex, Charlie, Pat, and Kim. The votes are as follows:

- 32 students voted Alex $\succ$ Kim $\succ$ Charlie $\succ$ Pat
- 27 students voted Charlie $\succ$ Pat $\succ$ Kim $\succ$ Alex
- 25 students voted Pat $\succ$ Charlie $\succ$ Kim $\succ$ Alex
- 16 students voted Kim $\succ$ Pat $\succ$ Alex $\succ$ Charlie

Calculate the winner of the election according to plurality, Borda, STV, Copeland, and maximin. Show your work.

Q3 [15 Points] Cake Cutting

Imagine a cake consisting of the interval $[0, 1]$. The cake has three homogeneous sections: $[0, 5/12]$ is strawberry flavoured, $[5/12, 7/12]$ is vanilla flavoured, and $[7/12, 1]$ is chocolate flavoured. Here, homogeneous means that within each flavour, different parts of the cake are indistinguishable from each other. There are three agents with the following preferences.

- Agent 1 has no preference over flavours, so she has value 1 spread out uniformly across the entire cake $[0, 1]$.
- Agent 2 only likes the strawberry flavour, so she has value 1 spread out uniformly across $[0, 5/12]$ and has zero value for the remaining cake.
- Agent 3 only likes the chocolate flavour, so she has value 1 spread out uniformly across $[7/12, 1]$ and has zero value for the remaining cake.

(a) [5 Points] Suppose we cut each of the three sections ($[0, 5/12]$, $[5/12, 7/12]$, and $[7/12, 1]$) into 3 equal parts, and give each agent one part from each section. Is this proportional? Is it envy-free? Is it Pareto optimal? Justify your answer.

(b) [10 Points] Recall that the social welfare of an allocation A is simply the sum of the values derived by the three agents, i.e., $sw(A) = \sum_{i=1}^{3} V_i(A_i)$. Answer the following questions, and justify your answers.

- Which allocation has the highest social welfare?
- Which envy-free allocation has the highest social welfare among all envy-free allocations?
- Which Pareto optimal allocation has the highest social welfare among all Pareto optimal allocations?

Bonus Question

Q4 [10 Points] Voting on a Line

Suppose n voters and m candidates are located on the real line. The location of voter i is x_i, and the location of candidate j is y_j. Assume that all locations are distinct, voters and candidates are sorted (so $x_1 \leq \ldots \leq x_n$ and $y_1 \leq \ldots \leq y_m$), and no voter is ever equidistant to two candidates (for all i, j, j': $|x_i - y_j| \neq |x_i - y_{j'}|$).

Suppose each voter i ranks the candidates in the increasing order of their distance to her, i.e., $j \succ_i j'$ if and only if $|x_i - y_j| < |x_i - y_{j'}|$. Prove that in the resulting preference profile, there must exist a *weak* Condorcet winner (who is preferred to every other candidate by at least $n/2$ voters).

[Hint: For each candidate j, consider the interval $B_j = \left[\frac{y_{j-1}+y_j}{2}, \frac{y_j+y_{j+1}}{2}\right]$, where $y_0 = -\infty$ and $y_{m+1} = \infty$. Now, consider the smallest j such that the number of voters in $\cup_{k=1}^{j} B_k$ is at least $n/2$.]

Q1

- Men-Proposing Deferred Acceptance (MPDA):

1. Initially, no one has proposed, no one is engaged, and no one is matched.
2. While some man m is unengaged:
   - $w \leftarrow m$'s most preferred woman to whom m has not proposed yet
   - m proposes to w
   - If w is unengaged:
     - m and w are engaged
   - Else if w prefers m to her current engaged partner m'
     - m and w are engaged, m' becomes unengaged
   - Else: w rejects m
3. Match all engaged pairs.

MPDA:

Iterations	Proposal	Engagement
1	M1 → W2	M1 ⇔ W2
2	M2 → W3	M1 ⇔ W2 , M2 ⇔ W3
3	M3 → W2 (accepted !)	M3 ⇔ W2 , M2 ⇔ W3
4	M1 → W4	M3 ⇔ W2 , M2 ⇔ W3 M1 ⇔ W4
5	M4 → W4 (rejected)	M3 ⇔ W2 , M2 ⇔ W3 M1 ⇔ W4
6	M4 → W1	M3 ⇔ W2 , M2 ⇔ W3 M1 ⇔ W4 , M4 ⇔ W1

WPDA :

Iterations	Proposal	Engagement
1	W1 → M2	W1 ⟺ M2
2	W2 → M4	W1 ⟺ M2 , W2 ⟺ M4
3	W3 → M1	W1 ⟺ M2 , W2 ⟺ M4 , W3 ⟺ M1
4	W4 → M2 (rejected)	W1 ⟺ M2 , W2 ⟺ M4 , W3 ⟺ M1
5	W4 → M1 (accepted !)	W1 ⟺ M2 , W2 ⟺ M4 , W4 ⟺ M1
6	W3 → M4 (accepted)	W1 ⟺ M2 , W3 ⟺ M4 , W4 ⟺ M1
7	W2 → M3 (accepted)	W1 ⟺ M2 , W3 ⟺ M4 , W4 ⟺ M1 , W2 ⟺ M3

b) Go from women's perspective:

(since relationship is symmetric, we only have to prove one gender)

Assume $\exists$ stable matching s.t. $\exists$ man $x \notin$ "good" men, and x is matched w/ some "good" woman y'

$\Rightarrow$ $\exists$ "good" man x' who is not matched w/ a "good" woman

$\Rightarrow$ x' is unmatched, or matched w/ woman $y \notin$ "good" women.

Note that $\forall$ woman, ranking of $x' > x$

$\therefore$ y' prefers x' over her current partner x

Similarly, $\forall$ man, ranking of $y' > y$, and $\underline{y' > \text{unmatched}}$

since y' appears in the preference ranking of every man

$\therefore$ x' prefers y' over his status (single, or with y)

$\Rightarrow$ (x', y') is a blocking pair

$\Rightarrow$ Our current matching is unstable

$\rightarrow\leftarrow$ Contradic'n, so every man matched to a "good" woman must be a "good" man

$\square$

c) Assume the matching returned by MPDA & WPDA is NOT unique.

Known:

$\forall$ man m, MPDA returns $\underline{best(m)}$ and WPDA returns $\underline{worst(m)}$

most preferred valid partner of m

least preferred valid partner of m

MPDA & WPDA results the same, so:

let $w = best(m) = worst(m) \in$ m's valid partners set V_m

The above holds iff $|V_m| = 1$

$\Rightarrow$ m has exactly one valid partner

So every man has exactly one valid partner, similarly, every woman has exactly one valid partner

However, we assumed the matching returned by MPDA & WPDA not unique, so $\exists$ man or woman who has more than one valid partner

$\rightarrow\leftarrow$ Contradic'n. so the matching must be unique

$\square$

Q2.

Plurality :

each voter awards 1 pt to his top alternative

Alex : 32 pts , Charlie : 27 pts , Pat : 25 pts ,
Kim : 16 pts

$\Rightarrow$ Alex wins

Borda :

3 2 1 0

- 32 students voted Alex $\succ$ Kim $\succ$ Charlie $\succ$ Pat
- 27 students voted Charlie $\succ$ Pat $\succ$ Kim $\succ$ Alex
- 25 students voted Pat $\succ$ Charlie $\succ$ Kim $\succ$ Alex
- 16 students voted Kim $\succ$ Pat $\succ$ Alex $\succ$ Charlie

Alex : $3 \times 32 + 0 \times 27 + 0 \times 25 + 1 \times 16 = 112$

Charlie : $1 \times 32 + 3 \times 27 + 2 \times 25 + 0 \times 16 = 163$

Pat : $0 \times 32 + 2 \times 27 + 3 \times 25 + 2 \times 16 = 161$

Kim : $2 \times 32 + 1 \times 27 + 1 \times 25 + 3 \times 16 = 164$

$\Rightarrow$ Kim wins

STV:

eliminate alternative w/ least plurality votes

denote people by their initials (e.g. Alex → A):

# of students voted this ranking

32	27	25	16
A	C	P	K
K	P	C	P
C	K	K	A
P	A	A	C

plurality votes:

K: 16

P: 25

C: 27

A: 32

↓ eliminate K

32	27	25	16
A	C	P	P
C	P	C	A
P	A	A	C

plurality votes:

P: 41

C: 27

A: 32

↓ eliminate C

32	27	25	16
A	P	P	P
P	A	A	A

plurality votes :

P: 68

A : 32

$\downarrow$ eliminate A

Pat wins

Copeland :

score(x) = # of alternatives x beats in pairwise elections

score(Alex) :

Alex vs. Kim , then 32 people vote Alex
27+25+16 = 68 people vote Kim
$\Rightarrow$ Alex loses

Alex vs. Charlie , then 32+16 = 48 people vote A
27+25 = 52 people vote C
$\Rightarrow$ Alex loses

Alex vs. Pat , then 32 people vote A

68 people vote P

$\Rightarrow$ Alex loses

so score(Alex) = 0

(since Alex beats 0 others in a pairwise election)

Score(Kim):

Kim vs. Alex, Kim wins

Kim vs. Charlie, 32 + 16 = 48 people vote K

27 + 25 = 52 people vote C

$\Rightarrow$ Kim loses

Kim vs. Pat, 32+16 = 48 people vote K

52 people vote P

$\Rightarrow$ Kim loses

so score(Kim) = 1

Score (Charlie):

Charlie vs. Alex: Charlie wins

vs. Kim: Charlie wins

vs. Pat: 32 + 27 = 59 people vote C

25 + 16 = 41 people vote P

$\Rightarrow$ Charlie wins

so score(Charlie) = 3

Since Charlie beats everyone in a pairwise election, he must have the max score

(alternatively, score(Pat) = 2 < 3)

$\Rightarrow$ Charlie wins

Maximin:

$\text{score}(x) = \min_y n_{x \succ y}$

score(Alex) = $\min\{32, 48, 32\} = 32$

(refer to work in Copeland)

score(Charlie) = $\min\{52, 52, 59\} = 52$

score(Pat) = $\min\{68, 52, 41\} = 41$

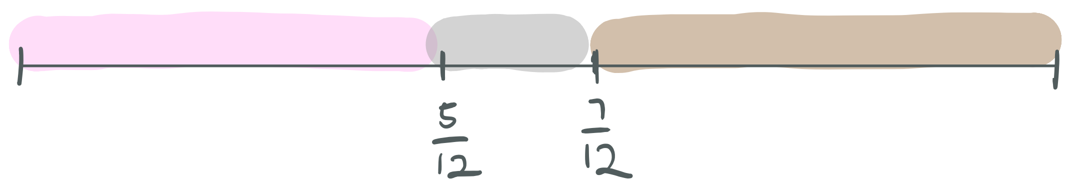
score(Kim) = $\min\{68, 48, 48\} = 48$

Charlie has max score

$\Rightarrow$ Charlie wins

Q3.

a)



- Prop: $\forall i \in N: V_i(A_i) \geq 1/n$
- EF: $\forall i, j \in N: V_i(A_i) \geq V_i(A_j)$

- Definition: We say that A is Pareto optimal if for any other allocation B, it cannot be that $V_i(B_i) \geq V_i(A_i)$ for all i and $V_i(B_i) > V_i(A_i)$ for some i.

Then agent 1 receives $\frac{1}{3}$ of total cake

$\Rightarrow V_1(A_1) = \frac{1}{3}$

Agent 2 only values strawberry, and she receives $\frac{1}{3}$ of strawberry

$\Rightarrow V_2(A_2) = \frac{1}{3}$

Agent 3 only values chocolate, and she has $\frac{1}{3}$ chocolate

$\Rightarrow V_3(A_3) = \frac{1}{3}$

- this is proportional, since $\forall i \in N$, $V_i(A_i) = \frac{1}{3}$
- this is EF, since everyone receives identical piece of cake, so $\forall i, j \in N$, $V_i(A_i) = V_i(A_j)$
- this is not PO, we can give strawberry from agent 3 to agent 2 (let this allocation be A'), then:
  - $V_1(A_1') = V_1(A_1)$
  - $V_3(A_3') = V_3(A_3)$
  - $V_2(A_2') > V_2(A_2)$

b) • Suppose we cut at x and y

Agent 2 gets $[0, x]$ for $x \leq \frac{5}{12}$

(There is no point to cut beyond $\frac{5}{12}$, since Agent 2 has 0 value for chocolate or vanilla)

$$V_2([0,x]) = \frac{x}{5/12} = \frac{12}{5}x$$

Agent 3 gets $[y,1]$ for $y \geq \frac{7}{12}$

(similar argument)

$$V_3([y,1]) = \frac{1-y}{5/12} = \frac{12}{5}(1-y)$$

Agent 1 gets $[x,y]$

$$V_1([x,y]) = \frac{y-x}{1} = y-x$$

$$\underbrace{\text{Total welfare}}_{\text{let } = w} = \frac{12}{5}x + \frac{12}{5}(1-y) + (y-x)$$

$$= \frac{7}{5}x + \frac{12}{5} - \frac{7}{5}y$$

maximized w occurs at max x and min y

$$\Rightarrow x = \frac{5}{12}, \ y = \frac{7}{12}$$

so give $[0, \frac{5}{12}]$ to agent 2,

$[\frac{5}{12}, \frac{7}{12}]$ to agent 1,

$[\frac{7}{12}, 1]$ to agent 3

- Want EF:

1. since $y \geq \frac{7}{12}$, agent 2 can only envy agent 1,
   (since agent 2 doesn't value anything beyond $\frac{7}{12}$, she can't envy agent 3)

2. similarly, since $x \leq \frac{5}{12}$, agent 3 can only envy agent 1

3. agent 1 can envy both agents 2 and 3, since she values the entire cake

Maximize w under constraints:

$\Rightarrow$ In agent 1's perspective:

$$y - x \geq \underbrace{1-y}_{V_1(A_3)} \quad \text{AND} \quad y - x \geq \underbrace{x}_{V_1(A_2)}$$

$$\text{so} \quad y \geq \frac{1+x}{2} \quad \text{AND} \quad y \geq 2x$$

$\Rightarrow$ In agent 2's perspective:

$$\frac{x}{5/12} \geq \underbrace{\frac{\frac{5}{12} - x}{5/12}}_{V_2(A_1)}$$

so $x \geq \frac{5}{12} - x$

so $x \geq \frac{5}{24}$ $\quad\quad \therefore x \in [\frac{5}{24}, \frac{5}{12}]$

$\Rightarrow$ In agent 3's perspective :

$$\frac{1-y}{5/12} \geq \frac{y - 7/12}{5/12} \quad V_3(A_1)$$

so $1-y \geq y - 7/12$

so $\frac{19}{12} \geq 2y$

so $y \leq \frac{19}{24}$ $\quad\quad \therefore y \in [\frac{7}{12}, \frac{19}{24}]$

Maximize $w = \frac{7}{5}(x-y) + \frac{12}{5}$ under the constraints $0 \geq \frac{x+1}{2} - y$ and $0 \geq 2x - y$

By Wolfram, $x = \frac{1}{3}$ and $y = \frac{2}{3}$

So give $[0, \frac{1}{3}]$ to agent 2,

$[\frac{1}{3}, \frac{2}{3}]$ to agent 1,

$[\frac{2}{3}, 1]$ to agent 3

- Want PO:

WTS the first allocation we had in this question is PO.

i.e. $[0, \frac{5}{12}]$ to agent 1,
$[\frac{5}{12}, \frac{7}{12}]$ to agent 2,
$[\frac{7}{12}, 1]$ to agent 3

let this allocation be A

pf/ Assume A is not PO, so $\exists B'$ s.t.
$\forall i \in N, V_i(B_i') \geq V_i(A_i)$ and $\exists i \in N, V_i(B_i') > V_i(A_i)$

But, since $V_2(A_2) = V_3(A_3) = 1$,
for all other allocations B, $V_2(B_2) \leq V_2(A_2)$
$V_3(B_3) \leq V_3(A_3)$

so $V_2(B_2') = V_2(A_2)$ and $V_3(B_3') = V_3(A_3)$,

↳ so B_2' still gives entire strawberry to 2 and entire chocolate to 3, otherwise agents 2 & 3 won't be as happy.

$\Rightarrow$ for B' to be PO, $V_1(B_1') > V_1(A_1)$

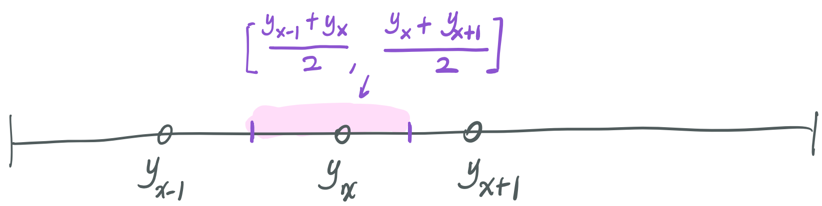
But we can't give more cake to agent 1 w/out sacrificing some cake from agents 2 & 3

$\rightarrow\leftarrow$ Contradic'n, so A is PO

$\square$

Since A is the allocation w/ absolute highest social welfare, it is also the PO allocation w/ highest social welfare.

Q4



For candidate j, every voter in interval $\left[\frac{y_{j-1}+y_j}{2}, \frac{y_j+y_{j+1}}{2}\right]$ would vote j over every other candidate

For every voter on the left of j, they would vote j over every candidate on the right of j

For every voter on the right of j, they would vote j over every candidate on left of j

i.e. 1. voter on $\left(-\infty, \frac{y_j+y_{j+1}}{2}\right]$ would vote on j over candidates $y_{j+1}, \dots, y_m$

2. voter on $\left[\frac{y_{j-1}+y_j}{2}, \infty\right)$ would vote on j over candidates $y_0, \dots, y_{j-1}$

$\{y_0, \dots, y_{j-1}\} \cup \{y_{j+1}, \dots, y_m\} = \{y_0, \dots, y_m\} \setminus \{y_j\}$

Consider smallest j s.t. # voters in $\bigcup_{k=1}^{j} B_k \geq \frac{n}{2}$

WTS there's at least $\frac{n}{2}$ voters $\in \left(-\infty, \frac{y_j + y_{j+1}}{2}\right]$ ①

and at least $\frac{n}{2}$ voters $\in \left[\frac{y_{j-1} + y_j}{2}, \infty\right)$ ②

①: holds by the j we picked

②: we know that # voters $\in \bigcup_{k=1}^{j-1} B_k < \frac{n}{2}$

(otherwise we would choose $j-1$ over j)

$\therefore$ # voters in $\bigcup_{k=j}^{m} B_k > \frac{n}{2}$

so $> \frac{n}{2}$ voters in $\left[\frac{y_{j-1} + y_j}{2}, \infty\right)$

so $\forall k \in \{0, \dots, m\} \setminus \{j\}$, at least $\frac{n}{2}$ voters would vote y_j over y_k.

so y_j is the weak Condorcet winner

Does y_j always exist ?

j is the smallest element $\in \underbrace{\{ j : \#\text{ voters in } \bigcup_{k=1}^{j} B_k \geq \frac{n}{2} \}}_{\text{let } = S}$

S is non-empty, since :

$$\bigcup_{k=1}^{m} B_k = (-\infty, \infty) \text{ , so } \#\text{ voters in } \bigcup_{k=1}^{m} B_k = n \geq \frac{n}{2}$$

$$\text{so } m \in S$$

S is a set of positive integers, since :

$\bigcup_{k=1}^{x} B_k$ is undefined for every non-positive integer x

$$\text{so } x \notin S$$

$\therefore$ By well-ordering principle, j always exists

$\Rightarrow$ y_j always exists

$\Rightarrow$ weak Condorcet winner always exists

$\square$