CSC304 Fall'22
Assignment 2

Due: November 11, 2022, by 11:59pm ET

Instructions:

1. Typed assignments are preferred (e.g., using LaTeX or Word), especially if your handwriting is possibly illegible or if you do not have access to a good quality scanner. Please submit a single PDF to MarkUs named "hwk2.pdf".

2. Remember our citation policy. You are free to read online material (though, if you find the exact homework problem, do not read the solution) and to discuss any problems with your peers. There are two rules you must follow: You must write the solution in your own words (it helps to not take any pictures or notes from your discussions), and you must cite any peers or online sources from where you obtained a significant hint.

3. Attempt to solve each problem, but if you have no clue how to approach a (sub)problem, remember the 20% rule: You can get 20% points for a (sub)problem by just stating "I don't know how to approach this question" (and 10% if you leave it blank but do not write such a statement).

Q1 [20 Points] VCG in a Continuous Space

The government decides to help out three companies by building a single warehouse that they can all share. The warehouse is to be built at a location $x \in [0, 10]$. Each company $i \in \{1, 2, 3\}$ is located at y_i, where $y_1 = 1$, $y_2 = 5$, and $y_3 = 9$. A warehouse at location x provides value $100 - (y_i - x)^2$ to each company i simultaneously.

Under the VCG mechanism, decide where the warehouse should be built, and how much payment each company should be charged. Show your calculation.

Q2 [20 Points] Sponsored Search

Consider the following sponsored search auction with three slots s_1, s_2, s_3, and four bidders A, B, C, D. The *clickthrough rate* of each slot and the *value per click* for each bidder are as follows.

Recall that in sponsored search, a bidder with value per click v that is assigned a slot with click-through rate c and charged payment *per click* p has expected value $v \cdot c$, expected payment $p \cdot c$, and expected utility $(v - p) \cdot c$.

Values per Click	Bidders	Slots	Clickthrough Rates
4	A	s_1	0.3
8	B	s_2	0.5
5	C	s_3	0.2
2	D		

(a) [5 Points] Under VCG, determine the allocation of slots to bidders and the payments charged to bidders *per click*. What is the total expected revenue under this mechanism? Show your calculation for full credit.

(b) [5 Points] Under GSP, determine the allocation of slots to bidders and the payments charged to bidders *per click*, assuming bidders bid truthfully. What is the total expected revenue under this mechanism? Show your calculation for full credit.

(c) [10 Points] Note that if bidders bid truthfully, GSP obtains at least as much expected revenue as VCG in this example. Show that this is true generally for all sponsored search auctions, not just the one drawn above.

Q3 [20 Points] Bayes-Nash Equilibria

Consider an auction between a buyer and a seller. Seller s wants to sell a good for which his own value is v_s. Buyer b has value v_b for the good. Both v_s and v_b are drawn independently from the uniform distribution $U[0,1]$. Each player knows his own value, but only knows the distribution of the other player's value.

Here's how the auction goes down. The seller and the buyer independently write down some prices (say p_s and p_b, respectively) on different pieces of paper. Then, they compare their prices. If it turns out that $p_s \leq p_b$, then the item is sold at price p_s, the seller gets utility $p_s - v_s$, and the buyer gets utility $v_b - p_s$. On the other hand, if $p_s > p_b$, the item remains unsold and both players get zero utility.

Is there a Bayes-Nash equilibrium in which the buyer honestly writes down her value (i.e. $p_b = v_b$)? If so, what is the seller's price p_s in this equilibrium? Show your calculation.

[Hint: Assume $p_b = v_b$, and derive the optimal price p_s for the seller to write. Then, check whether given this strategy of the seller, $p_b = v_b$ is indeed the optimal price for the buyer to write.]

Q4 [20 Points] Second-Price Auction with a Twist

In class, we proved that the second-price auction is strategyproof. Consider using the second-price auction for selling a single item, but with a twist. After all the bids are submitted, the auctioneer is supposed to sell the item to the highest bidder at price equal to the second-highest bid. The auctioneer usually does that, but with some probability p, makes an error and sells the item to the *second-highest bidder* at price equal to the second-highest bid (i.e., the winner's own bid).

You show up to such an auction and see that there is only one bidder other than yourself. Your value for the item is v. You know that this other bidder has a value v' drawn from the uniform distribution $U[0,1]$ and that the other bidder, unaware of the fact that the auctioneer has this tendency to make an error, is going to truthfully bid their value v'. What is the optimal bid that you should submit? Your answer can depend on your own value v and the probability of error p.

Q1. Maximize welfare:

$$\underbrace{\text{total welfare}}_{\text{let } = \omega_t} = 100-(1-x)^2 + 100-(5-x)^2 + 100-(9-x)^2$$

$$= 300-(1-x)^2-(5-x)^2-(9-x)^2$$

$$\therefore \frac{\partial \omega_t}{\partial x} = -2(1-x)(-1) - 2(5-x)(-1) - 2(9-x)(-1)$$

$$= 2[(1-x)+(5-x)+(9-x)]$$

$$= 2[15-3x]$$

$\hookrightarrow$ let $= 0$

$\Rightarrow x = 5$

For company 1, or C_1:

if C_1 not present, then:

$$\frac{\partial \omega_t}{\partial x} = 2(5-x)+2(9-x)$$

$$= 2[14-2x]$$

$\hookrightarrow x = 7$

$$\text{welfare of } C_1 + C_2 = 200-(5-7)^2-(9-7)^2$$

$$= 200-4-4$$

$$= 192$$

if C_1 present, then:

$$\text{welfare of } C_1 + C_2 = 200-(5-5)^2-(9-5)^2$$

$$= 200-4^2 = 184$$

$\therefore$ payment for $C_1 = 192 - 184 = 8$

For C_2:

if C_2 not present:

$$W_t = 200 - (1-x)^2 - (9-x)^2$$

$$\frac{\partial W_t}{\partial x} = 2(1-x) + 2(9-x) = 20 - 4x = 0$$

$$\Rightarrow x = 5$$

same as if C_2 is present

$\therefore$ C_2 has no influence on welfare of C_1 and C_3

$\therefore$ C_2 pays 0

For C_3:

if C_3 not present:

$$W_t = 200 - (1-x)^2 - (5-x)^2$$

$$\frac{\partial W_t}{\partial x} = 2(1-x) - 2(5-x)$$

$$= 12 - 4x = 0 \qquad \therefore x = 3$$

$$\text{welfare of } C_1 + C_2 = 200 - (1-3)^2 - (5-3)^2$$

$$= 200 - 4 - 4$$

$$= 192$$

if C_3 present:

$$\text{welfare of } C_1 + C_2 = 200 - (1-5)^2 - (5-5)^2$$

$$= 200 - 16$$

$$= 184$$

$\therefore$ C_3 payment $= 192 - 184$

$$= 8$$

Q2. a) Under VCG, bidders bid their true value

Bidders w/ higher bid get better slots, so:

$B \to S_2$, $C \to S_1$, $A \to S_3$, D gets nothing

	expected payment	payment per click
B	$5(0.5-0.3) + 4(0.3-0.2) + 2(0.2-0) = 1.8$	$1.8/0.5 = \frac{18}{5}$
C	$4(0.3-0.2) + 2(0.2-0) = 0.8$	$0.8/0.3 = \frac{8}{3}$
A	$2(0.2-0) = 0.4$	$0.4/0.2 = 2$
D	0	0

total expected revenue to the auctioneer:

$= 1.8 + 0.8 + 0.4$

$= 3.0$

b) Since bidders bid truthfully, we have same as before:

$B \to S_2$, $C \to S_1$, $A \to S_3$, D gets nothing

under GSP, the payment per click of kth highest bidder = value per click of (k+1)th highest bid

	payment per click	expected payment
B	5	5×0.5
C	4	4×0.3
A	2	2×0.2
D	0	0

total expected revenue $= 2.5 + 1.2 + 0.4$

$= 4.1$

c) let be an auction of n bidders and m slots, where $n > m$. Assume bidders bid truthfully, then:

Under VCG, the expected payment to the kth highest bidder is $\sum_{i=k+1}^{n} V_i (C_{i-1} - C_i)$, where V_i = value per click of ith highest bidder

$\underline{C_i}$ = ith highest clickthrough rate

↳ (all of $C_{m+1}, \dots, C_n = 0$)

Under GSP, the expected payment of kth highest bidder is $C_k \cdot V_{k+1}$

We will prove something stronger, $\forall k$, the kth highest bidder pays more under GSP:

WTS $\sum_{i=k+1}^{n} V_i \cdot (C_{i-1} - C_i) \leq C_k \cdot V_{k+1}$:

$$\sum_{i=k+1}^{n} V_i \cdot (C_{i-1} - C_i) = V_{k+1} \cdot (C_k - C_{k+1}) + V_{k+2} \cdot (C_{k+1} - C_{k+2}) + \dots + V_n \cdot (C_{n-1} - C_n)$$

$$\leq V_{k+1} \cdot (C_k - C_{k+1}) + V_{k+1} \cdot (C_{k+1} - C_{k+2}) + \dots + V_{k+1} \cdot (C_{n-1} - C_n)$$

(k+1'th highest bidder bids more than k+2'th, ..., n'th highest bidder)

$$= V_{k+1} [(C_k - \underline{C_{k+1}}) + (\underline{C_{k+1}} - C_{k+2}) + \dots + (C_{n-1} - C_n)]$$

cancel out

$$= V_{k+1} [C_k - C_n]$$

$$\leq V_{k+1} \cdot C_k \quad \text{(since } C_n \geq 0\text{)}$$

□

Q3. Assume $p_b = V_b$, then for seller:

$$E[\text{seller utility}] = p(\text{seller wins}) \cdot E[\text{seller utility} \mid \text{win}]$$
$$+$$
$$p(\text{seller loses}) \cdot E[\text{seller utility} \mid \text{lose}]$$

when seller wins, i.e. when seller bids higher, utility (seller) = 0

" loses, i.e. $\underbrace{\text{when seller bids lower}}$, utility (seller) $= p_s - V_s$

⇒ buyer would be charged w/ this price p_s

- $p(\text{seller win}) = p(p_s > p_b)$ ← Assume $p_b = V_b$

$= p(p_s > V_b)$ ↖ drawn from $U[0,1]$

$= p(V_b < p_s)$

$= p_s$

- $p(\text{seller loses}) = 1 - p(\text{seller wins})$

$= 1 - p_s$

$\therefore \underbrace{E[\text{seller utility}]}_{\text{let } = E_s} = (1-p_s)(p_s - V_s)$

$= p_s - V_s - p_s^2 + p_s V_s$

$\frac{\partial E_s}{\partial p_s} = 1 - 2p_s + V_s = 0$ ↪ let $=0$ to find max

$\Rightarrow 2p_s = 1 + V_s$

$p_s = \frac{1+V_s}{2}$

Next, given $p_s = \frac{1+V_s}{2}$:

$$E[\text{buyer util.}] = p(\text{buyer wins}) \cdot E[\text{buyer util} \mid \text{win}]$$
$$+$$
$$p(\text{buyer loses}) \cdot E[\text{buyer util} \mid \text{lose}]$$

- $p(\text{buyer wins}) = p(p_b \geq p_s)$

$$= p\left(p_b \geq \frac{1+V_s}{2}\right)$$
$$= p(2p_b - 1 \geq V_s)$$
$$= p(V_s \leq 2p_b - 1)$$
$$= 2p_b - 1$$

since $V_s \sim U[0,1]$

- $E[\text{buyer util} \mid \text{win}] = V_b - E[p_s]$ where $p_s \leq p_b$

$$\Rightarrow \frac{1+V_s}{2} \leq p_b$$
$$\Rightarrow 1+V_s \leq 2p_b$$
$$\Rightarrow V_s \leq 2p_b - 1$$

$$= V_b - E\left[\frac{1+V_s}{2}\right]$$
$$= V_b - \left(\frac{1}{2} + \frac{1}{2}E[V_s]\right)$$

$2p_b - 1 \leq 1$, and so $V_s \sim U[0, 2p_b - 1]$

$$= V_b - \frac{1}{2} - \frac{1}{2}\left[\frac{2p_b - 1 - 0}{2}\right]$$
$$= V_b - \frac{1}{2} - \frac{1}{2}\left[p_b - \frac{1}{2}\right]$$
$$= V_b - \frac{1}{4} - \frac{p_b}{2}$$

- $E[\text{buyer util} \mid \text{lose}] = 0$

$\therefore \underbrace{E[\text{buyer util.}]}_{\text{let } = E_b} = p(\text{buyer wins}) \cdot E[\text{buyer util} \mid \text{win}] + 0$

$= (2p_b - 1)(V_b - \frac{1}{4} - \frac{p_b}{2})$

$= -p_b^2 + 2p_bV_b - V_b + \frac{1}{4}$

then $\frac{\partial E_b}{\partial p_b} = 2V_b - 2p_b$ (let this $= 0$)

$\Rightarrow p_b = V_b$

$\therefore$ Such BNE where $p_b = V_b$ exists, and $p_s = \frac{1+V_s}{2}$

Q4. Let myself be denoted by a_1, the other agent by a_2

- a_1's bid is b_1
- a_2's bid is b_2

$$E[\text{utility to } a_1] = P(a_1 \text{ winning}) \cdot E[a_1 \text{ utility} \mid \text{winning}]$$
$$+$$
$$p(a_1 \text{ losing}) \cdot E[a_1 \text{ utility} \mid \text{losing}]$$

- If I win the auction, then I either:
  1. get nothing since auctioneer makes a mistake w.p. p
  2. get utility $\underbrace{(v - b_2)}_{\text{value - payment}}$ w.p. $(1-p)$
- If I lose the auction, then either:
  1. get utility $v - b_1$ by mistake w.p. p
  2. get nothing w.p. $(1-p)$

since a_2 thought this was a VCG auction

- $p(a_1 \text{ winning}) = p(b_2 < b_1) = p(v' < b_1)$

$$= \min(b_1, 1) \quad (\text{since } v' \text{ uniformly distributed})$$

Note that $v' \sim U[0,1]$, so there is no point to bid anything > 1, since $p(\text{winning})$ remains the same as bid $= 1$ (i.e. $p(\text{winning}) = 1$)

$$\therefore \min(b_1, 1) = b_1$$

- $p(a_1 \text{ losing}) = 1 - p(\text{winning})$

$$= 1 - \min(b_1, 1)$$

$$= 1 - b_1$$

- $E[a_1 \text{ utility} \mid \text{winning}] = 0 \times p + (1-p)E[v - b_2]$

$$= (1-p)E[v - v']$$

↳ where v' lies in the range $[0, b_1)$

$$= (1-p)(v - E[v'])$$

$$= (1-p)\left(v - \frac{b_1}{2}\right)$$

since $v' \sim U[0,1]$

$$= (1-p)\left(v - \frac{b_1}{2}\right)$$

- $E[a_1 \text{ utility} \mid \text{losing}] = (v - b_1)p + 0 \times (1-p)$

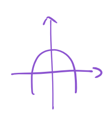
$$= p(v - b_1)$$

So $E[a_1 \text{ utility}] = b_1 \cdot (1-p)\left(v - \frac{b_1}{2}\right) + (1-b_1)\cdot(p)\cdot(v - b_1)$

$$= \frac{1}{2}\left(b_1^2(3p-1) - 2b_1(2vp + p - v) + 2pv\right)$$

Want to maximize this function w.r.t. b_1

Notice that crit. point is max only if $3p - 1 < 0$

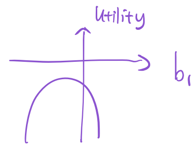


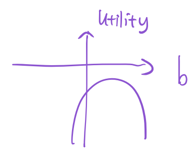
$$p < 1/3$$

Case $p < \frac{1}{3}$:

$$\frac{\partial E[a_1 \text{ utility}]}{\partial b_1} = b_1(3p-1) - (2vp + p - v) = 0$$

$$\Rightarrow b_1 = \frac{2vp+p-v}{3p-1} \ ?$$

Notice that our polynomial could have shape : 

or 

In this case, max lies in $b_1 < 0$ or $b_1 > 1$

$\therefore$ Pick $b_1 = 0$ or $b_1 = 1$

- Condition s.t. $\dfrac{2vp+p-v}{3p-1} < 0$:

$$\Rightarrow 2vp+p-v > 0$$

$$\Rightarrow p(2v+1) > v$$

$$\Rightarrow p > \frac{v}{2v+1}$$

Apply constraints $0 \leq p < \frac{1}{3}$, $0 \leq v \leq 1$:

Then $\left[0 < p < \frac{1}{3},\ 0 \leq v < \frac{-p}{2p-1}\right]$ is solution to this system

(i.e. a solution exists)

- Condition s.t. $\dfrac{2vp+p-v}{3p-1} > 1$:

$$\Rightarrow 2vp+p-v < 3p-1$$

$$\Rightarrow 1-v < 2p-2vp$$

$$\Rightarrow \frac{1-v}{2(1-v)} < p$$

$$\Rightarrow \frac{1}{2} < p$$

Notice that $p < \frac{1}{3}$, so this case is impossible

Sub-case $\dfrac{2vp+p-v}{3p-1} < 0$:

pick $b_1 = 0$

Sub-case $\frac{2vp+p-v}{3p-1} \geq 0$:

pick $b_1 = \frac{2vp+p-v}{3p-1}$

Case $p = \frac{1}{3}$:

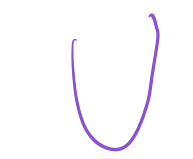
$$E[a_1 \text{ utility}] = -b_1(2vp+p-v) + pv$$
$$= -b_1\left(\frac{2}{3}v + \frac{1}{3} - v\right) + \frac{v}{3}$$
$$= -b_1\left(\frac{1}{3} - \frac{v}{3}\right) + \frac{v}{3}$$

$v \sim U[0,1]$, so $0 \leq v \leq 1$, so $\frac{1}{3} - \frac{v}{3} \geq 0$

$\Rightarrow$ linear function w/ negative slope

$\therefore$ max is achieved @ minimum b_1

$\Rightarrow b_1 = 0$



Case $p > \frac{1}{3}$:

Then we have a polynomial that concaves up

$\therefore$ for $b_1 \in [0,1]$, max is located on either 0 or 1.

- $b_1 = 0$:

$$E[a_1 \text{ utility}] = \cancel{b_1 \cdot (1-p)\left(v - \frac{b_1}{2}\right)} + (1-\cancel{b_1}) \cdot (p) \cdot (v - \cancel{b_1}) \Big|_{b_1=0}$$
$$= pv$$

- $b_1 = 1$:

  $$E[a_1 \text{ utility}] = (1-p)(v-\tfrac{1}{2}) = v - \tfrac{1}{2} - pv + \tfrac{p}{2}$$

  Sub-case $pv > v - \frac{1}{2} - pv + \frac{p}{2}$:

  Then pick $b_1 = 0$

  Sub-case $pv < v - \frac{1}{2} - pv + \frac{p}{2}$:

  Then pick $b_1 = 1$