PC3233 AM1 Assignment 1

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1 Angular Momenta

1.1 Prove $\left[J^2, J_i\right] = 0$

Proof. Since $J^2 = J_x^2 + J_y^2 + J_z^2 = \sum_{j=x,y,z} J_j^2$.

$$\begin{aligned}
\left[J^2, J_i\right] &= \left[\sum_j J_j^2, J_i\right] \\
&= \sum_j \left[J_j^2, J_i\right] \\
&= \sum_j \left[J_j, J_i\right] J_j + J_j \left[J_j, J_i\right]
\end{aligned}$$

Since $\left[J_j, J_i\right] = i\hbar\epsilon_{ijk} J_k$, where $\hbar$ is set to be 1. Then,

$$\left[J_j, J_i\right] = \begin{cases} 0 & \text{if } i = j \\ i\epsilon_{jik} J_k & \text{if } i \neq j \end{cases}$$

Therefore,

$$\begin{aligned}
\sum_j \left[J_j, J_i\right] J_j + J_j \left[J_j, J_i\right] &= 0 + i\epsilon_{yxz} J_z J_y + i\epsilon_{yxz} J_y J_z + i\epsilon_{zxy} J_y J_z + i\epsilon_{zxy} J_z J_y \\
&= i(-1) J_z J_y + i(-1) J_y J_z + i(1) J_y J_z + i(1) J_z J_y \\
&= 0
\end{aligned}$$

Hence, $\left[J^2, J_i\right] = 0$. □

1.2 Prove $\left[J^2, J_\pm\right] = 0$

Proof. Since $J_\pm = J_x \pm iJ_y$,

$$\begin{aligned}
\left[J^2, J_\pm\right] &= \left[J^2, J_x \pm iJ_y\right] \\
&= \left[J^2, J_x\right] \pm i\left[J^2, J_y\right] \\
&= 0 \pm 0 \quad \because \text{From the Solution 1.1} \\
&= 0
\end{aligned}$$

Hence, $\left[J^2, J_\pm\right] = 0$. □

1.3 Prove $[J_z, J_\pm] = \pm J_\pm$

Proof.

$$
\begin{aligned}
[J_z, J_\pm] &= \left[J_z, J_x \pm iJ_y\right] \\
&= [J_z, J_x] \pm i\left[J_z, J_y\right] \\
&= iJ_y \pm i(-iJ_x) \quad \because \left[J_x, J_y\right] = i\epsilon_{xyz}J_z \\
&= iJ_y \pm J_x \\
&= \pm(\pm iJ_y + J_x) \\
&= \pm(J_x \pm iJ_y) \\
&= \pm J_\pm
\end{aligned}
$$

Hence, $[J_z, J_\pm] = \pm J_\pm$. □

1.4 Prove $J^2 = J_\pm J_\mp + J_z^2 \mp J_z$

Goal: Simplify the RHS of the equation to get J^2.

Proof.

$$
\begin{aligned}
J_\pm J_\mp + J_z^2 \mp J_z &= (J_x \pm iJ_y)(J_x \mp iJ_y) + J_z^2 \mp J_z \\
&= J_x^2 \mp iJ_xJ_y \pm iJ_yJ_x + J_y^2 + J_z^2 \mp J_z \\
&= J_x^2 + J_y^2 + J_z^2 \mp iJ_xJ_y \pm iJ_yJ_x \mp J_z \\
&= J^2 \mp iJ_xJ_y \pm iJ_yJ_x \mp J_z \\
&= \begin{cases} J^2 - iJ_xJ_y + iJ_yJ_x - J_z \\ J^2 + iJ_xJ_y - iJ_yJ_x + J_z \end{cases} \\
&= \begin{cases} J^2 + i(J_yJ_x - J_xJ_y) - J_z \\ J^2 + i(J_xJ_y - J_yJ_x) + J_z \end{cases} \\
&= \begin{cases} J^2 + i\left[J_y, J_x\right] - J_z \\ J^2 + i\left[J_x, J_y\right] + J_z \end{cases} \\
&= \begin{cases} J^2 + i(-iJ_z) - J_z \\ J^2 + i(iJ_z) + J_z \end{cases} \\
&= \begin{cases} J^2 + J_z - J_z \\ J^2 - J_z + J_z \end{cases} \\
&= \begin{cases} J^2 \\ J^2 \end{cases} \\
&= J^2
\end{aligned}
$$

Hence, $J^2 = J_\pm J_\mp + J_z^2 \mp J_z$. □

[1]:
```python
# import library
import sympy as sym
from IPython.display import display, Latex
```

2 Legendre polynomials

2.1

Write a program in your favorite math software to obtain the **first ten** Legendre Polynomials $P_l(x)$ using Rodrigues' formula

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}[(x^2-1)^l]$$

[2]:
```python
x, l = sym.symbols("x l")
factorial = sym.factorial
diff = sym.diff

def legendre_polynomails(l, x):
    coefficient = 1 / (2 ** l * factorial(l))
    polynomial_expression = diff((x ** 2 - 1) ** l, x, l)
    res = sym.factor(coefficient * polynomial_expression)
    display(Latex(f"$P_{l}(x) = {sym.latex(res)}$"))
    return res

for i in range(10):
    legendre_polynomails(i, x)
```

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{3x^2-1}{2}$
$P_3(x) = \frac{x\left(5x^2-3\right)}{2}$
$P_4(x) = \frac{35x^4-30x^2+3}{8}$
$P_5(x) = \frac{x\left(63x^4-70x^2+15\right)}{8}$
$P_6(x) = \frac{231x^6-315x^4+105x^2-5}{16}$
$P_7(x) = \frac{x\left(429x^6-693x^4+315x^2-35\right)}{16}$
$P_8(x) = \frac{6435x^8-12012x^6+6930x^4-1260x^2+35}{128}$
$P_9(x) = \frac{x\left(12155x^8-25740x^6+18018x^4-4620x^2+315\right)}{128}$

2.2

The Legendre polynomials satisfy the recursion relation:

$$lP_l(x) = (2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x)$$

Write a program in your favorite math software to determine $P_2(x)$ through $P_{10}(x)$ (starting with $P_0(x) = 1$ and $P_1(x) = x$) using the above recursion relation.

[3]:

```
term = 10

P0 = 1
display(Latex(f"$P_{0}(x) = {P0}$"))
P1 = x
display(Latex(f"$P_{1}(x) = {P1}$"))
for l in range(2, term):
    tem = (2 * l - 1) * x * P1 - (l - 1) * P0
    result = sym.factor(tem / l)
    display(Latex(f"$P_{l}(x) = {sym.latex(result)}$"))
    P0 = P1
    P1 = result
```

$P_0(x) = 1$
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