PC3233 Assignment 4

March 12, 2024

1 Dipole-Dipole Interaction

1.1 Part A

The interaction energy, E, between two magnetic dipole, $\vec{\mu}_1$ and $\vec{\mu}_2$, is given by:

$$E = +\frac{\mu_0}{4\pi}\frac{\vec{\mu}_1 \cdot \vec{\mu}_2 - 3(\vec{\mu}_1 \cdot \hat{r})(\vec{\mu}_2 \cdot \hat{r})}{r^3} \tag{1}$$

Since the energy is magnetic potential energy is given by:

$$E = -\vec{mu} \cdot \vec{B} \tag{2}$$

Therefore, we can extract the magnetic field of magnetic dipole, $\vec{B}_i$, from the interaction energy, E:

$$\vec{B_i(r)} = -\frac{\mu_0}{4\pi}\frac{3(\vec{\mu}_i \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{\mu}_i}{r^3} \tag{3}$$

Hence, in order for $E = 0$ for any given $|\vec{r}|$, B_i must be orthornormal to $\vec{\mu}_j$.

1.2 Part B

For parallel moments, we can expand the dot product of E as:

$$E = \frac{\mu_0}{4\pi}\frac{\mu_1\mu_2}{r^3}(1 - 3\cos^2\theta) \tag{4}$$

Then, we can obtain the following:

$$\begin{cases} E_{min} = -\frac{\mu_0}{2\pi}\frac{\mu_1\mu_2}{r^3} & \text{if } \theta = 0, \pi \\ \\ E_{max} = +\frac{\mu_0}{4\pi}\frac{\mu_1\mu_2}{r^3} & \text{if } \theta = \frac{\pi}{2} \end{cases} \tag{5}$$

1.3 Part C

For case (b) with $|\vec{r}| = 2$Å. We can calculate the interaction energy, E, as follows using the formula we obtain from the previous section:

For electron-electron case, we have:

$$\begin{cases} E_{min} = -\dfrac{\mu_0}{2\pi}\dfrac{\mu_B^2}{r^3} = -6.4505 \times 10^{-24}\text{J} \quad \text{if } \theta = 0, \pi \\ \\ E_{max} = +\dfrac{\mu_0}{4\pi}\dfrac{\mu_B^2}{r^3} = 3.22527 \times 10^{-24}\text{J} \quad \text{if } \theta = \frac{\pi}{2} \end{cases} \tag{6}$$

For proton-proton case ($\mu_{proton} = 1.4 \times 10^{-26}$ Åm^2), we have:

$$\begin{cases} E_{min} = -\dfrac{\mu_0}{2\pi}\dfrac{\mu_{proton}^2}{r^3} = -4.90 \times 10^{-30}\text{J} \quad \text{if } \theta = 0, \pi \\ \\ E_{max} = +\dfrac{\mu_0}{4\pi}\dfrac{\mu_{proton}^2}{r^3} = 2.45 \times 10^{-30}\text{J} \quad \text{if } \theta = \frac{\pi}{2} \end{cases} \tag{7}$$

To compute the magnetic field, $\vec{B}$, we can rearranged the formula we obtained in part (a):

$$\begin{aligned} E = -\vec{\mu} \cdot \vec{B} &\implies E = -|\mu||B| \cos\theta \\ &\implies |B| = \frac{E}{|\mu|} \qquad \text{since } \mu \text{ always align to the } \vec{B} \end{aligned}$$

Hence, we can compute the magnetic field, B, for the electron-electron as follows:

$$\begin{cases} |B| = \dfrac{E}{|\mu|} = 0.4016\text{T} \quad \text{if } \theta = 0, \pi \\ \\ |B| = \dfrac{E}{|\mu|} = 0.2007\text{T} \quad \text{if } \theta = \frac{\pi}{2} \end{cases} \tag{8}$$

For proton-proton case, we have:

$$\begin{cases} |B| = \dfrac{E}{|\mu|} = 3.5 \times 10^{-4}\text{T} \quad \text{if } \theta = 0, \pi \\ \\ |B| = \dfrac{E}{|\mu|} = 1.75 \times 10^{-4}\text{T} \quad \text{if } \theta = \frac{\pi}{2} \end{cases} \tag{9}$$

1.4 Part D

Express the interaction energy, E, in the previous section in terms of J, cm^{-1}, and μK.

For electron-electron case, we have:

$$\begin{cases} E_{min} = -\frac{\mu_0}{2\pi}\frac{\mu_B^2}{r^3} = -6.4505 \times 10^{-24}\text{J} = -32.47\text{cm}^{-1} = -467211\mu K & \text{if } \theta = 0, \pi \\ \\ E_{max} = +\frac{\mu_0}{4\pi}\frac{\mu_B^2}{r^3} = 3.22527 \times 10^{-24}\text{J} = 16.24\text{cm}^{-1} = 233605\mu K & \text{if } \theta = \frac{\pi}{2} \end{cases} \tag{10}$$

For proton-proton case, we have:

$$\begin{cases} E_{min} = -\frac{\mu_0}{2\pi}\frac{\mu_{proton}^2}{r^3} = -4.90 \times 10^{-30}\text{J} = -2.4667\text{cm}^{-1} = -0.355\mu K & \text{if } \theta = 0, \pi \\ \\ E_{max} = +\frac{\mu_0}{4\pi}\frac{\mu_{proton}^2}{r^3} = 2.45 \times 10^{-30}\text{J} = 1.2334\text{cm}^{-1} = 0.177\mu K & \text{if } \theta = \frac{\pi}{2} \end{cases} \tag{11}$$

2 Mathematical Relationships

Consider the following matrix:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{12}$$

2.1 Analytic Expression of Eigenvalues

To diagonalize the matrix, we solve for the eigenvalues, λ, by solving the following equation:

$$\det(M - \lambda I) = 0 \tag{13}$$

Then, we have:

$$\det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

Then compute the determinant:

$$\begin{aligned} (a-\lambda)(d-\lambda) - bc &= 0 \\ \lambda^2 - (a+d)\lambda + ad - bc &= 0 \end{aligned}$$

Since the eigenvalues are the roots of the characteristic equation, which is quadratic, we get the analytic expression for λ using the quadratic formula:

$$\lambda = \frac{-(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} \tag{14}$$

2.2 Generalized $n \times n$ Eignevalues

The degree of polynomial of the characteristic equation is n, where n is the dimension of the matrix.

According to Abel-Ruffini theorem, there is no general solution for the roots of a polynomial of degree $n \geq 5$.

For degree $n = 2, 3, 4$, we can solve for the roots of the characteristic equation using the quadratic, cubic, and quartic formulae respectively.

3 Atoms in Magnetic Field

3.1 Part A

Given a Li^{++} with $I = 1$. The energy levels due to fine structure splitting is given by:

$$E_{n,j} = E_n[1 + \frac{Z^2\alpha^2}{n}(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n})] \tag{15}$$

Hence we can determeine the following:

$$\begin{cases} E_{1,\frac{1}{2}} & = E_1[1 + \frac{Z^2\alpha^2}{4}] \\ \\ E_{2,\frac{1}{2}} & = E_2[1 + \frac{5Z^2\alpha^2}{16}] \\ \\ E_{2,\frac{3}{2}} & = E_3[1 + \frac{Z^2\alpha^2}{16}] \end{cases} \tag{16}$$

The energy levels shift due to Hyperfine strucutre is given by:

$$\Delta H_{HF} = \frac{g_i \mu_k B}{2\sqrt{j(j+1)}}[F(F+1) - I(I+1) - j(j+1)] \tag{17}$$

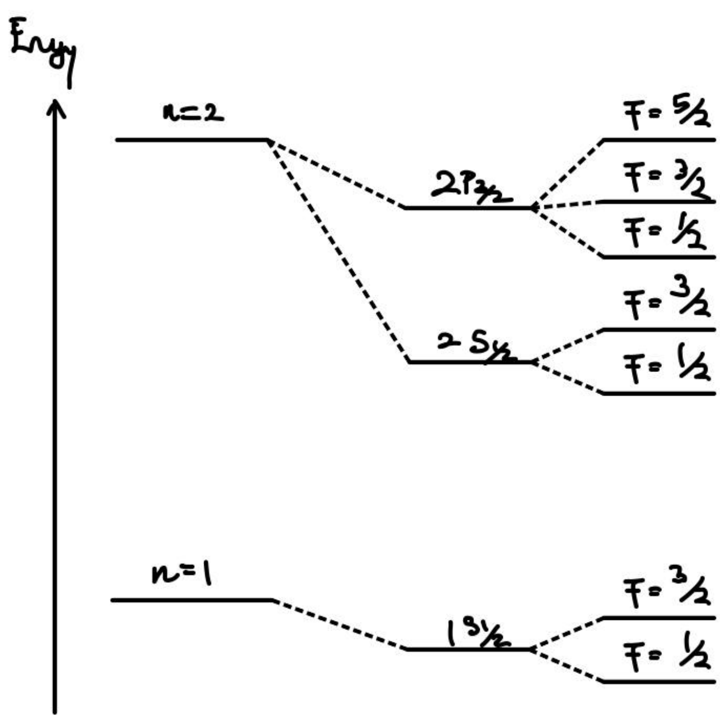
The possible of value of total angular mmomentum, is given by:

$$F = |j - I|, |j - I + 1|, \ldots, |j + I| \tag{18}$$

Hence, by setting $g_i \mu_k B = 1$, we can compute the energy levels shift due to hyperfine structure.

$$
\begin{cases}
\Delta H_{HF} = \dfrac{-2\sqrt{3}}{3} & \text{if } F = \frac{1}{2}, j = \frac{1}{2}, 1s \\[2ex]
\Delta H_{HF} = \dfrac{2\sqrt{6}}{3} & \text{if } F = \frac{3}{2}, j = \frac{1}{2}, 1s \\[2ex]
\Delta H_{HF} = -\dfrac{2\sqrt{3}}{3} & \text{if } F = \frac{1}{2}, j = \frac{1}{2}, 2s \\[2ex]
\Delta H_{HF} = \dfrac{\sqrt{3}}{3} & \text{if } F = \frac{3}{2}, j = \frac{1}{2}, 2s \\[2ex]
\Delta H_{HF} = -\dfrac{\sqrt{15}}{3} & \text{if } F = \frac{1}{2}, j = \frac{3}{2}, 2p \\[2ex]
\Delta H_{HF} = \dfrac{2\sqrt{15}}{15} & \text{if } F = \frac{3}{2}, j = \frac{3}{2}, 2p \\[2ex]
\Delta H_{HF} = -\dfrac{\sqrt{15}}{5} & \text{if } F = \frac{5}{2}, j = \frac{3}{2}, 2p
\end{cases}
\tag{19}
$$

Hence we can draw the energy level diagram for the fine and hyperfine structure states as below.



3.2 Part B: Compute Lande Factor

For Lande factor, g_j, is given by:

$$g_j = 1 + \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)} \tag{20}$$

Hence, for g_j for all fine structure states up to $n = 2$ is given in the below:

$$\begin{cases} g_j = \dfrac{1}{3} & \text{if } j = \frac{1}{2}, l = 1, s = \frac{1}{2} \\ \\ g_j = \dfrac{4}{3} & \text{if } j = \frac{3}{2}, l = 1, s = \frac{1}{2} \end{cases} \tag{21}$$

3.3 Part C: Zeeman Effect

The Zeeman effect energy splitting, ΔE, is given by:

$$\Delta E = \mu_j g_j \mu_B B \tag{22}$$

Hence for lowest Zeeman states, we have:

$$\Delta E = m_j g_j \mu_B B = \Delta m_j \frac{1}{3} \mu_B B = 1.0304 \times 10^{-24} \text{J} \tag{23}$$