PC3233 Assignment 3

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Feb 2024

1 Stern-Gerlach Experiment

From question, we have the following information:

$$\begin{cases} \frac{\mathrm{d}B}{\mathrm{d}z} = 10^3 \text{ T/m} \\ l_1 = 4 \times 10^{-2} \text{ m} \\ l_2 = 10 \times 10^{-2} \text{ m} \\ d = 2 \times 10^{-3} \text{ m} \\ v = 500 \text{ m/s} \\ M = 1.79 \times 10^{-25} \text{ kg} \end{cases} \tag{1}$$

From lecture, we know that potential energy of a magnetic dipole in a magnetic field is given by:

$$E_{\text{pot}} = -\vec{\mu} \cdot \vec{B} \tag{2}$$

And the Lorentz force is given by:

$$\begin{aligned} \vec{F} &= -\vec{\nabla} E_{\text{pot}} \\ &= \vec{\nabla}(\vec{\mu} \cdot \vec{B}) \\ &= \mu_z \frac{\mathrm{d}B}{\mathrm{d}z} \hat{z} \\ &= \mu_z 10^3 \hat{z} \end{aligned}$$

Therefore, using Newton's second law, we can find the acceleration of the particle:

$$\begin{aligned} \vec{F} &= M\vec{a} \\ \mu_z 10^3 &= Ma \\ a &= \frac{\mu_z 10^3}{M} \end{aligned}$$

Since the acceleration is constant and perpendicular to the initial velocity, v, we can use the following kinematic equation to find the time taken for the particle to travel from distance of l_1 and l_2 in the x-direction, t_1 and t_2 respectively:

$$
\begin{aligned}
t_1 &= \frac{l_1}{v} = \frac{4 \times 10^{-2}}{500} = 8 \times 10^{-5} \text{ s} \\
t_2 &= \frac{l_2}{v} = \frac{10 \times 10^{-2}}{500} = 2 \times 10^{-4} \text{ s}
\end{aligned}
$$

Then, with the initial velocity in z-direction of v_{z_i} which is 0 m/s, we express the distance travelled in the z-direction as a function of μ_z:

$$
\begin{aligned}
\frac{d}{2} &= s_{y1} + s_{y2} && s = v_i t + \frac{1}{2}at^2 \\
\frac{d}{2} &= v_{z_i} t_1 + \frac{1}{2}at_1^2 + v_{z_f} t_2 && \\
\frac{d}{2} &= \frac{1}{2}at_1^2 + v_{z_f} t_2 && v_{z_f} = v_{z_i} + at_1 \\
\frac{d}{2} &= \frac{1}{2}at_1^2 + (at_1)t_2 && \text{where } a = \frac{\mu_z 10^3}{M} \\
d &= \mu_z \frac{10^3}{M}\left(t_1^2 + 2t_1 t_2\right) &&
\end{aligned}
$$

Rearrange to get μ_z, then substitute the values to get the answer:

$$
\begin{aligned}
\mu_z &= \frac{dM}{10^3\left(t_1^2 + 2t_1 t_2\right)} \\
&= \frac{2 \times 10^{-3} \times 1.79 \times 10^{-25}}{10^3\left((8 \times 10^{-5})^2 + 2(8 \times 10^{-5})(2 \times 10^{-4})\right)} \\
&= 9.323 \times -24 \text{ A m}^2
\end{aligned}
$$

For second part, why doesn't the nuclear spin affect the experiemnt?

The deflection depends on the strength of magnetic dipole moment, μ.

For electron, the magnetic dipole moment is given by:

$$
\mu_e \approx 2\mu_B = \frac{e\hbar}{2m_e} \qquad \text{where } \mu_B = \frac{e\hbar}{2m_e} \tag{3}
$$

For nuclear, the magnetic dipole moment is given by:

$$
\mu_{\text{nuclear}} \approx \mu_N = \frac{e\hbar}{2m_p} \qquad \text{where } \mu_N = \frac{e\hbar}{2m_p} \tag{4}
$$

Then,

$$\frac{\mu_N}{\mu_e} = \frac{m_e}{m_p} \approx \frac{1}{1836} \ll 1 \tag{5}$$

So the deflection of the proton is much smaller than the electron, and hence the nuclear spin does not affect the experiment.

2 Coupling of Angular Momentum

2.1 Part a

For uncoupled bases, J_1^2, J_2^2, J_{z1} and J_{z2} are the commuting observable which can be simultaneously diagonalized with basis specified by the quantum numbers j_1, j_2, m_1 and m_2 respectively. Hence, j_1, j_2, m_1 and m_2 are good quantum numbers.

Similarly, for coupled bases, J^2, J_z, J_1^2, J_2^2 are the commuting observable which can be simultaneously diagonalized with basis specified by the quantum numbers j, m, j_1 and j_2 respectively. Hence, j, m, j_1 and j_2 are good quantum numbers.

2.2 Part b

Prove the commutation relation:

$$\left[J^2, J_i^2\right] = 0 \tag{6}$$

We know that J^2 is given by $J^2 = J_1^2 + J_2^2 + 2J_1 \cdot J_2$. Then, we can write the commutator as:

$$\begin{aligned}\left[J^2, J_i^2\right] &= \left[J_1^2 + J_2^2 + 2J_1 \cdot J_2, J_i^2\right] \\ &= \left[J_1^2, J_i^2\right] + \left[J_2^2, J_i^2\right] + \left[2J_1 \cdot J_2, J_i^2\right]\end{aligned}$$

Both $\left[J_1^2, J_i^2\right]$ and $\left[J_2^2, J_i^2\right]$ are zero either because J_i^2 is acting on a non-i space J^2 operator or J_i^2 is acting on the itself. Hence,

$$\begin{aligned}\left[J_1^2, J_i^2\right] + \left[J_2^2, J_i^2\right] + \left[2J_1 \cdot J_2, J_i^2\right] &= 0 + 0 + 2[J_1 \cdot J_2, J_i^2] \\ &= 2([J_1, J_i^2]J_2 + J_1[J_2, J_i^2])\end{aligned}$$

Similar argument to above can be made for the commutator $[J_1, J_i^2]$ and $[J_2, J_i^2]$ either because J_i^2 is acting on a non-i space J operator or $\left[J_i^2, J_i\right] = [J_i, J_i]J_i + J_i[J_i, J_i] = 0$. Hence,

$$\begin{aligned}2([J_1, J_i^2]J_2 + J_1[J_2, J_i^2]) &= 2(0 + 0) \\ &= 0\end{aligned}$$

Therefore, $\left[J^2, J_i^2\right] = 0$.

2.3 Part c

Given two angular momenta with given quantum numbers j_1 and j_2, the possible quantum numbers for J^2 is:

$$j \in \{|j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2\} \tag{7}$$

And the possible quantum numbers for J_z is:

$$m \in \{-j, -j + 1, \ldots, j - 1, j\} \tag{8}$$

Below is the illustration for J^2 when $j_1 = 1$ and $j_2 = 1$:

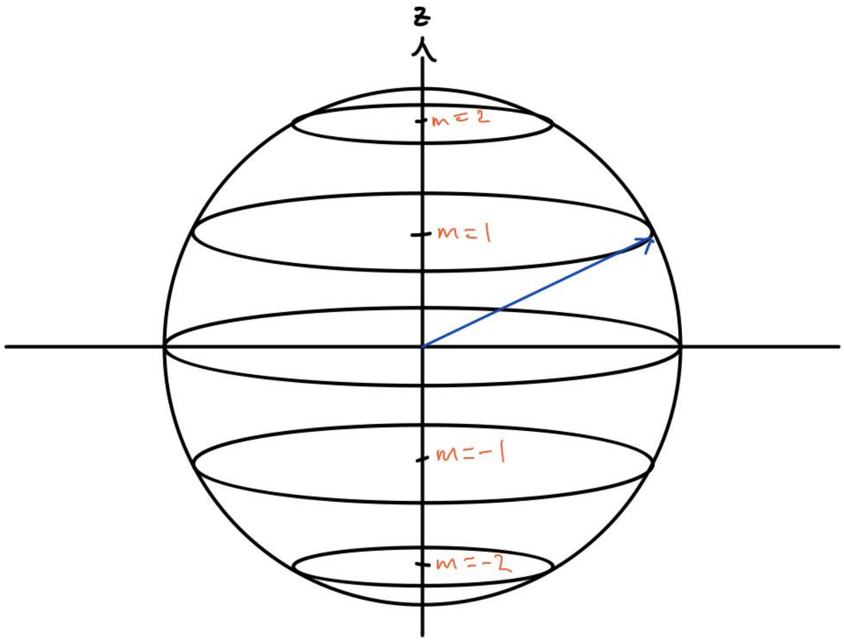

Figure 1: illustration for J^2 when $j_1 = 1$ and $j_2 = 1$

2.4 Part d

Given the general basis transformation from uncoupled to coupled basis:

$$|j_1, j_2, j, m\rangle = \sum_{j_1', j_2'} \sum_{m_1, m_2} |j_1', j_2', m_1, m_2\rangle \langle j_1', j_2', m_1, m_2 | j_1, j_2, j, m\rangle \tag{9}$$

We want to prove that the Clebsch-Gordan coefficients $\langle j_1^{'}, j_2^{'}, m_1, m_2 | j_1, j_2, j, m\rangle$ is not zero if and only if:

(i) $m = m_1 + m_2$

(ii) $j_1^{'} = j_1$ and $j_2^{'} = j_2$

To prove (i), we act the null operator $J_z - J_z^{(1)} - J_z^{(2)} = 0$ in between $\langle j_1^{'}, j_2^{'}, m_1, m_2 | j_1, j_2, j, m\rangle$:

$$\langle j_1^{'}, j_2^{'}, m_1, m_2 | \left(J_z - J_z^{(1)} - J_z^{(2)}\right) | j_1, j_2, j, m\rangle = 0 \tag{10}$$

Then, we can evaluate the above equation to get:

$$(m - m_1 - m_2) \langle j_1^{'}, j_2^{'}, m_1, m_2 | j_1, j_2, j, m\rangle = 0 \tag{11}$$

Proof. i.

Case 1: $m \neq m_1 + m_2$
If $m \neq m_1 + m_2 \implies m - m_1 - m_2 \neq 0$, then the above equation is satisfied if and only if the Clebsch-Gordan coefficient, $\langle j_1^{'}, j_2^{'}, m_1, m_2 | j_1, j_2, j, m\rangle = 0$

Case 2: $m = m_1 + m_2$
If $m = m_1 + m_2 \implies m - m_1 - m_2 = 0$, then the Clebsch-Gordan coefficient, $\langle j_1^{'}, j_2^{'}, m_1, m_2 | j_1, j_2, j, m\rangle$ can be any arbitary value.

Therefore, we have proved (i).

With a similar idea, we can prove (ii) by acting the null operator $J^{2(1)} - J^{2(1)} = 0$ in between $\langle j_1^{'}, j_2^{'}, m_1, m_2 | j_1, j_2, j, m\rangle$:

$$\langle j_1^{'}, j_2^{'}, m_1, m_2 | \left(J^{2(1)} - J^{2(1)}\right) | j_1, j_2, j, m\rangle = 0 \tag{12}$$

Then, we can evaluate the above equation to get:

$$\left(\hbar j_1^{'}(j_1^{'} + 1) - \hbar j_1(j_1 + 1)\right) \langle j_1^{'}, j_2^{'}, m_1, m_2 | j_1, j_2, j, m\rangle = 0 \tag{13}$$

Proof. ii.a

Case 1: $j_1^{'} \neq j_1$
Then $\hbar j_1^{'}(j_1^{'} + 1) - \hbar j_1(j_1 + 1) \neq 0$, then the above equation is satisfied if and only if the Clebsch-Gordan coefficient, $\langle j_1^{'}, j_2^{'}, m_1, m_2 | j_1, j_2, j, m\rangle = 0$

Case 2: $j_1^{'} = j_1$
Then $\hbar j_1^{'}(j_1^{'} + 1) - \hbar j_1(j_1 + 1) = 0$, then the Clebsch-Gordan coefficient, $\langle j_1^{'}, j_2^{'}, m_1, m_2 | j_1, j_2, j, m\rangle$ can be any arbitary value.

Without loss of generality, we can prove for $j_2^{'} = j_2$ as well by null operator $J^{2(1)} - J^{2(1)} = 0$ in between $\langle j_1^{'}, j_2^{'}, m_1, m_2 | j_1, j_2, j, m\rangle$. (*Proof. ii.b*)

To combine the two conclusions from $(ii.a)$ and $(ii.b)$, we need to consider four cases:

1. $j_1^{'} \neq j_1$ and $j_2^{'} \neq j_2$, then the Clebsch-Gordan coefficient, $\langle j_1^{'}, j_2^{'}, m_1, m_2 | j_1, j_2, j, m\rangle = 0$

2. $j_1^{'} = j_1$ and $j_2^{'} \neq j_2$, then the Clebsch-Gordan coefficient, $\langle j_1^{'}, j_2^{'}, m_1, m_2 | j_1, j_2, j, m\rangle = 0$.
3. $j_1^{'} \neq j_1$ and $j_2^{'} = j_2$, then the Clebsch-Gordan coefficient, $\langle j_1^{'}, j_2^{'}, m_1, m_2 | j_1, j_2, j, m\rangle = 0$.
4. $j_1^{'} = j_1$ and $j_2^{'} = j_2$, then the Clebsch-Gordan coefficient, $\langle j_1^{'}, j_2^{'}, m_1, m_2 | j_1, j_2, j, m\rangle$ can be any arbitary value.

Hence, we have proved (ii).