PC3233 Assignment 2

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1 The Radial Equation

Consider the following potential and differentiation:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + 2(\frac{1}{r} - \kappa)\frac{\mathrm{d}u}{\mathrm{d}r} + \left(\frac{2a - 2\kappa}{r} - \frac{l(l+1)}{r^2}\right) u = 0 \tag{1}$$

Ansatz: Assume the general solution of $u(r) = \sum_j b_j r^j$ Then, we have

$$\begin{cases} u(r) = \sum_{j=0} b_j r^j \\ \frac{\mathrm{d}u}{\mathrm{d}r} = \sum_{j=0} b_j j r^{j-1} \\ \frac{\mathrm{d}^2 u}{\mathrm{d}r^2} = \sum_{j=0} b_j j(j-1) r^{j-2} \end{cases} \tag{2}$$

Since any r^j term with $j < 0$ is zero, we can rewrite the differentiation as

$$\begin{cases} u(r) = \sum_{j=0} b_j r^j \\ \frac{\mathrm{d}u}{\mathrm{d}r} = \sum_{j=1} b_j j r^{j-1} = \sum_{j=0} b_{j+1}(j+1) r^j \\ \frac{\mathrm{d}^2 u}{\mathrm{d}r^2} = \sum_{j=2} b_j j(j-1) r^{j-2} = \sum_{j=0} b_{j+2}(j+2)(j+1) r^j \end{cases} \tag{3}$$

Substituting the result (3) into the original equation (1), we have

$$\sum_{j=0} b_{j+2}(j+2)(j+1) r^j + 2(\frac{1}{r} - \kappa) \sum_{j=0} b_{j+1}(j+1) r^j + \left(\frac{2a - 2\kappa}{r} - \frac{l(l+1)}{r^2}\right) \sum_{j=0} b_j r^j = 0$$

Then expanding the summation, we have

$$\sum_{j=0} b_{j+2}(j+2)(j+1)r^j + 2\sum_{j=0} b_{j+1}(j+1)r^{j-1} - 2\kappa\sum_{j=0} b_{j+1}(j+1)r^j + (2a-2\kappa)\sum_{j=0} b_j r^{j-1} - l(l+1)\sum_{j=0} b_j r^{j-2} = 0$$

Similarly, we can shift the index of the summation to get for any r^j with $j < 0$, we have

$$\sum_{j=0} b_{j+2}(j+2)(j+1)r^j + 2\sum_{j=1} b_{j+1}(j+1)r^{j-1} - 2\kappa\sum_{j=0} b_{j+1}(j+1)r^j + (2a-2\kappa)\sum_{j=1} b_j r^{j-1} - l(l+1)\sum_{j=2} b_j r^{j-2} = 0$$

Then, we can rewrite the summation as

$$\sum_{j=0} b_{j+2}(j+2)(j+1)r^j + 2\sum_{j=0} b_{j+2}(j+2)r^j - 2\kappa\sum_{j=0} b_{j+1}(j+1)r^j + (2a-2\kappa)\sum_{j=0} b_{j+1} r^j - l(l+1)\sum_{j=0} b_{j+2} r^j = 0$$

Then, we can combine the summation and factor out the r^j term to get

$$\sum_{j=0} \left[b_{j+2}(j+2)(j+1) + 2b_{j+2}(j+2) - 2\kappa b_{j+1}(j+1) + (2a-2\kappa)b_{j+1} - l(l+1)b_{j+2}\right] r^j = 0 \quad (4)$$

In order for the equation to hold for all r^j, we must have

$$b_{j+2}(j+2)(j+1) + 2b_{j+2}(j+2) - 2\kappa b_{j+1}(j+1) + (2a-2\kappa)b_{j+1} - l(l+1)b_{j+2} = 0 \quad (5)$$

For all j. Then, we can rewrite the equation as

$$b_{j+2}\left[(j+2)(j+1) + 2(j+2) - l(l+1)\right] = -2\kappa b_{j+1}(j+1) + (2a-2\kappa)b_{j+1} \quad (6)$$

Then, we can factor out the b_{j+2} and b_{j+1} term to get

$$b_{j+2}\left[(j+2)(j+1) + 2(j+2) - l(l+1)\right] = b_{j+1}[(2a-2\kappa) - 2\kappa(j+1)] \quad (7)$$

Then, we can simplify the equation to get

$$\begin{aligned} b_{j+2} &= 2b_{j+1}\frac{(a-\kappa) - \kappa(j+1)}{(j+2)(j+1) + 2(j+2) - l(l+1)} && (8)\\ &= 2b_{j+1}\frac{k(j+2) - a}{(j+2)(j+3) - l(l+1)} && (9)\\ & && (10) \end{aligned}$$

Then, rewrite the index $j+2$ as j to get

$$b_j = 2b_{j-1}\frac{\kappa j - a}{(j)(j+1) - l(l+1)} \tag{11}$$

In order for the series to not diverge, Then

$$j \geq l + 1 \implies n \geq l + 1 \tag{12}$$

2 The Hydrogen Atom and Spin

2.1 Part A

"eV" stands for electron-volt, which is a unit of energy. It is defined as the energy gain for an single electron when it is accelerated through an electric potential difference of 1 volt.

$$\text{Work done (In Joule), } W = qV \text{ (In eV)} \tag{13}$$

Then,

$$1\text{eV} = 1.602 \times 10^{-19}\text{J, Joule} \tag{14}$$

2.2 Part B

$$\mu = \frac{m_e m_p}{m_e + m_p} = \frac{9.10938 \times 10^{-31} \cdot 1.67262 \times 10^{-27}}{9.10938 \times 10^{-31} + 1.67262 \times 10^{-27}} = 9.103 \times 10^{-31} \tag{15}$$

Since the energy levels are quantized, the energy levels are given by

$$E_n = -\frac{m_e e^4}{8(h\epsilon_0)^2}\frac{1}{n^2} = -R_y^*\frac{Z^2}{n^2} \tag{16}$$

Where $R_y^* = \frac{m_e e^4}{8(h\epsilon_0)^2}$ is the Rydberg constant for hydrogen-like atoms which is 13.6 eV.

For Hydrogen atom, the Z is 1, since it has only one proton. Then, the energy levels are given by

$$\begin{cases} E_1 = -13.6\frac{1}{1^2} = -13.6\text{eV} \\ E_2 = -13.6\frac{1}{2^2} = -3.4\text{eV} \\ E_3 = -13.6\frac{1}{3^2} = -1.51\text{eV} \end{cases} \tag{17}$$

To calculate the electric current and the magnetic dipole moment, we can use the following equations

$$\begin{cases} \mu_l = I \cdot \vec{A} = I \cdot \pi r^2 \cdot \vec{n} \\ \mu_l = -\frac{u_B}{\hbar} \cdot \vec{L} \end{cases} \tag{18}$$

Then we can rearrange the equation to get

$$I = -\frac{u_B}{\pi r^2}\left|\vec{L}\right| \tag{19}$$

The magnitude of the angular momentum is given by

$$\left|\vec{L}\right| = \sqrt{l(l+1)}\hbar \tag{20}$$

The maximum value of the magnetic dipole moment is given by the maximum value of l which can determine by the quantum number n. Then, the maximum value of l is $n-1$.

Since we know that r_n is given by

$$r_n = \frac{4\pi\epsilon_o\hbar^2}{m_e e^2}n^2 \tag{21}$$

$$\begin{cases} r_1 = 5.29\times 10^{-11}\text{m} \\ r_2 = 4\times 5.29\times 10^{-11}\text{m} = 2.12\times 10^{-10}\text{m} \\ r_3 = 9\times 5.29\times 10^{-11}\text{m} = 4.76\times 10^{-10}\text{m} \end{cases} \tag{22}$$

Then we can find

$$\begin{cases} \mu_{n=1,l=0} = 0 \implies I_{n=1,l=0} = 0 \\ \\ \mu_{n=2,l=1} = \sqrt{2}\mu_B \implies I_{n=2,l=1} = \dfrac{\sqrt{2}\mu_B}{2\pi r_2^2} \\ \\ \mu_{n=3,l=2} = \sqrt{6}\mu_B \implies I_{n=3,l=2} = \dfrac{\sqrt{6}\mu_B}{2\pi r_3^2} \end{cases} \tag{23}$$

2.3 Part C

The magnetic moment of electron is given by

$$\mu_e = -9.284\times 10^{-24}\text{J/T} \tag{24}$$

Since $\mu_e = \frac{e}{2m_e}L = -\frac{erv}{2}$, we can solve for v to get by assuming $r = 10^{-18}$m

$$v = -\frac{2\mu_e}{er} \tag{25}$$

$$= -\frac{2\cdot -9.284\times 10^{-24}}{1.6\times 10^{-19}\cdot 10^{-18}} \tag{26}$$

$$= 1.159\times 10^{14}\text{m/s} \tag{27}$$

$$\tag{28}$$

This is a very high speed, which is exceeded to the speed of light. This is not possible since the speed of light is the maximum speed in the universe.