# Notes on Graph Laplacian, Fourier Transform, and Spectral Filtering

## Contents



# 1 Graph Laplacian

Let a graph be $G = (V, E)$ with $|V| = N$. A graph signal is a function

$$f : V \to \mathbb{R}, \qquad f = (f(1), f(2), \ldots, f(N))^\top.$$

## 1.1 Adjacency, Degree and Laplacian

Let $W$ be the adjacency matrix:

$$W_{ij} = \begin{cases} w_{ij} > 0, & (i,j) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Degree matrix:

$$D = \mathrm{diag}(d_1, \ldots, d_N), \qquad d_i = \sum_j W_{ij}.$$

The **combinatorial graph Laplacian**:

$$L = D - W.$$

## 1.2 Nodewise interpretation

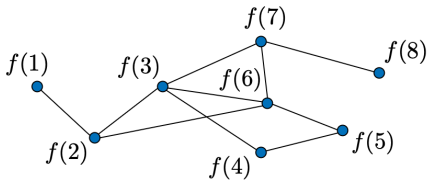
The $i$-th entry of $Lf$ is

$$(Lf)(i) = \sum_{j=1}^{N} W_{ij}\,(f(i) - f(j)).$$

Hence:

- If $f(i)$ is very different from its neighbors $\{f(j)\}$, then $(Lf)(i)$ becomes large. - If $f$ is locally smooth around node $i$, then $(Lf)(i) \approx 0$.

### Graph Laplacian




graph signal $f : \mathcal{V} \to \mathbb{R}$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \\ f(6) \\ f(7) \\ f(8) \end{pmatrix}$$

$$\begin{pmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \\ f(6) \\ f(7) \\ f(8) \end{pmatrix}^T \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \\ f(6) \\ f(7) \\ f(8) \end{pmatrix}$$

$$Lf(i) = \sum_{j=1}^{N} W_{ij}(f(i) - f(j)) \qquad f^T L f = \frac{1}{2} \sum_{i,j=1}^{N} W_{ij}\,(f(i) - f(j))^2$$

a measure of "smoothness"

## 1.3 Global smoothness measure

A common global smoothness functional is

$$f^\top L f = \frac{1}{2} \sum_{i,j=1}^{N} W_{ij} (f(i) - f(j))^2.$$

Thus:

- $f^\top L f$ is small $\iff$ $f$ varies slowly over edges. - $f^\top L f$ large $\iff$ signal has many "jumps" across edges.

### 1.4 Spectral decomposition of $L$

Since $L$ is symmetric and positive semidefinite, it has an orthonormal set of eigenvectors

$$L = X \Lambda X^\top,$$

where:

- $X = [\chi_0 \ \chi_1 \ \dots \ \chi_{N-1}]$ with $X^\top X = I$, - $\Lambda = \mathrm{diag}(\lambda_0, \dots, \lambda_{N-1})$, with

$$0 = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{N-1}.$$

Eigenvectors play the role of "graph Fourier modes."

## Graph Laplacian


- $L$ has a complete set of orthonormal eigenvectors: $L = \chi \Lambda \chi^T$

$$L = \begin{bmatrix} | & & | \\ \chi_0 & \cdots & \chi_{N-1} \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_{N-1} \end{bmatrix} \begin{bmatrix} — & \chi_0^T & — \\ & \cdots & \\ — & \chi_{N-1}^T & — \end{bmatrix}$$

$$\chi \qquad\qquad \Lambda \qquad\qquad \chi^T$$

- Eigenvalues are usually sorted increasingly: $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_{N-1}$

# 2 Graph Fourier Transform

### 2.1 Relation to classical Fourier transform

Classical Fourier transform uses eigenfunctions of the 1D Laplace operator $-\nabla^2$:

$$(-\nabla^2)\, e^{j\omega x} = \omega^2 e^{j\omega x}.$$

Thus sinusoids are eigenfunctions of the operator.

On graphs, replace the Laplace operator with graph Laplacian:

$$L\chi_\ell = \lambda_\ell \chi_\ell.$$

### 2.2 Graph Fourier transform (GFT)

Given graph signal $f \in \mathbb{R}^N$, its GFT coefficients are:

$$\hat{f}(\ell) = \langle \chi_\ell, f \rangle = \chi_\ell^\top f.$$

This is a projection of $f$ onto eigenvector $\chi_\ell$.

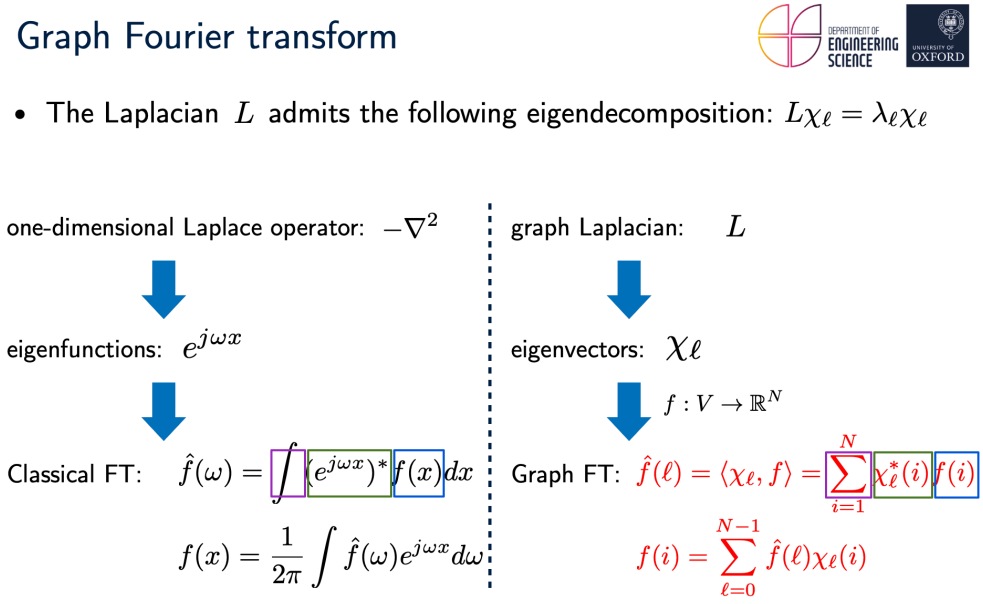
### 2.3 Inverse GFT

Because eigenvectors form an orthonormal basis:

$$f(i) = \sum_{\ell=0}^{N-1} \hat{f}(\ell)\, \chi_\ell(i), \qquad f = X\hat{f}.$$

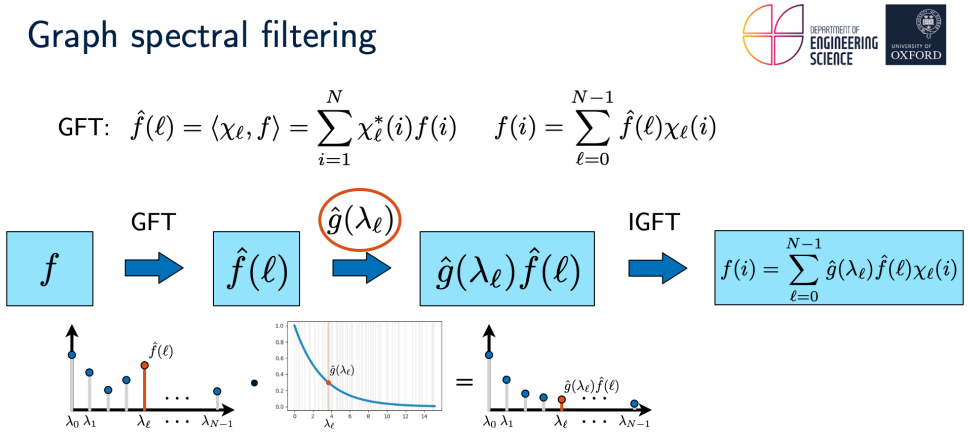
### 2.4 Interpretation

- $\lambda_0 = 0$ corresponds to the constant eigenvector $\rightarrow$ *lowest frequency.* - Larger $\lambda_\ell$ correspond to eigenvectors with more oscillations $\rightarrow$ *higher frequencies.* Thus GFT decomposes $f$ into low/high-frequency variations on the graph.




## 3 Graph Spectral Filtering

### 3.1 Filtering in the spectral domain




A graph filter $g$ is defined by its action on GFT coefficients:

$$\hat{f}(\ell) \mapsto \hat{g}(\lambda_\ell)\hat{f}(\ell).$$

### 3.2 Spectral filtering process

$$f \xrightarrow{\text{GFT}} \hat{f} = X^\top f,$$

$$\hat{f}(\ell) \xrightarrow{\text{spectral multiplication}} \hat{g}(\lambda_\ell)\hat{f}(\ell),$$

IGFT $$f_{\text{out}} = X\,\hat{g}(\Lambda)X^\top f.$$

### 3.3 Filtering as a function of Laplacian

Since

$$L = X\Lambda X^\top,$$

we define the operator

$$g(L) := X\hat{g}(\Lambda)X^\top.$$

Thus spectral filtering becomes the linear operator:

$$f_{\text{out}} = g(L)\, f.$$

### 3.4 Polynomial filters

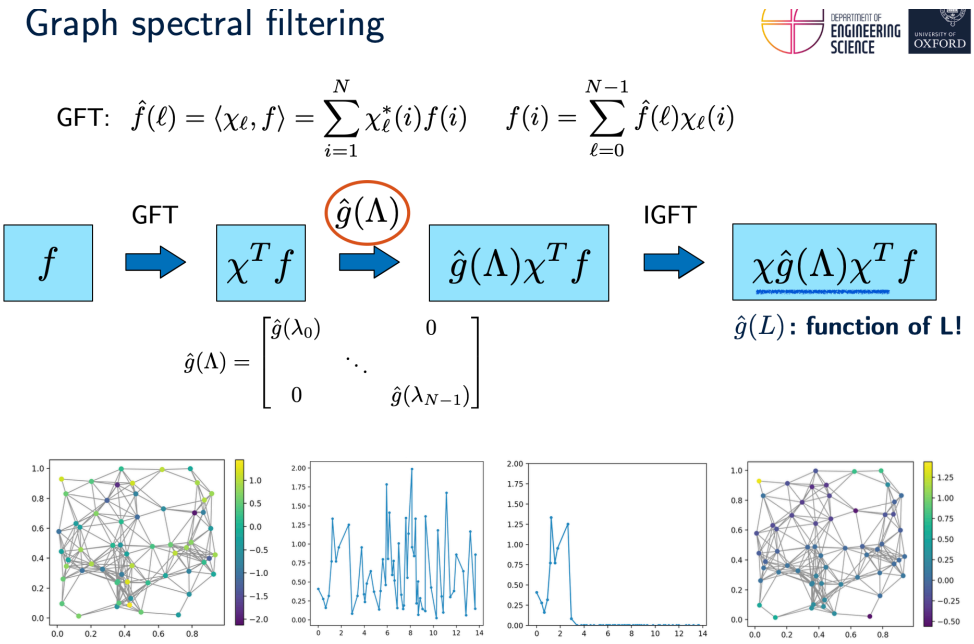
Often $\hat{g}(\lambda)$ is chosen as a polynomial:

$$\hat{g}_\theta(\lambda) = \sum_{j=0}^{K} \theta_j \lambda^j.$$

Then

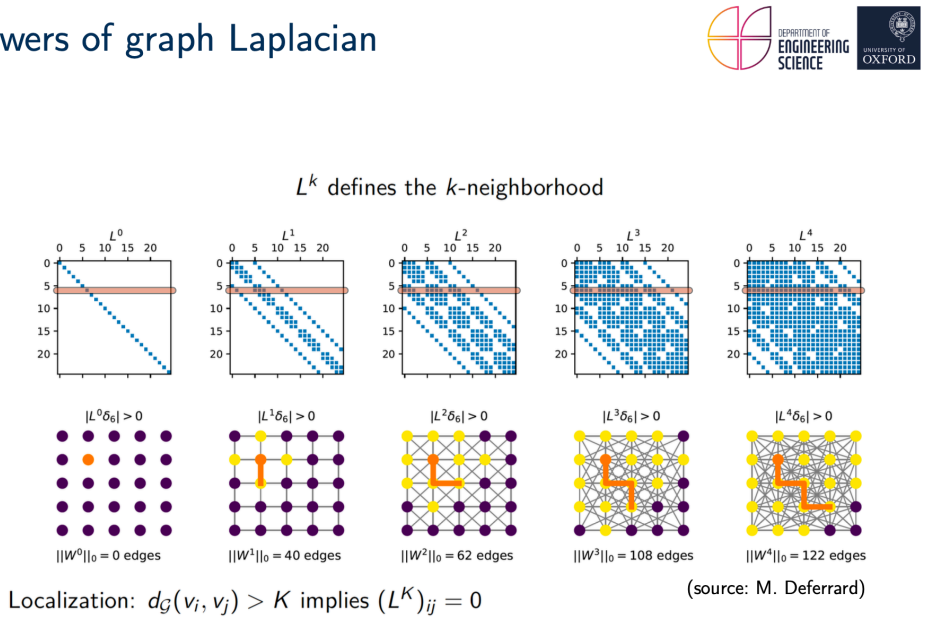
$$g_\theta(L) = \sum_{j=0}^{K} \theta_j L^j.$$

Advantage: no need to compute eigenvectors. Computation depends only on multiplication by $L$.



# 4 Convolution on Graphs

## 4.1 $L^k$ and $k$-hop neighborhoods




Observe:

- $L = D - W$ contains information of 1-hop connections. - Multiplying $Lf$ mixes information from 1-hop neighbors. - $L^2 f$ mixes information from 2-hop neighbors, etc.

Formally, for adjacency matrix $A = W$:

$$(A^k)_{ij} > 0 \quad \Longleftrightarrow \quad \text{there exists a length-}k\text{ walk between } i \text{ and } j.$$

Thus:

- $L^0 f = f$ (0-hop) - $Lf$ aggregates 1-hop neighbor differences - $L^2 f$ aggregates 2-hop information - $L^k f$ spreads information over $k$-hop neighborhoods

This yields a notion of graph convolution: polynomial in $L$.

# 5 Simplified Parametric Filter and GCN

## 5.1 General first-order spectral filter

Take a first-order polynomial:

$$\hat{g}_\theta(\lambda) = \theta_0 + \theta_1 \lambda.$$

Then

$$g_\theta(L) = \theta_0 I + \theta_1 L.$$

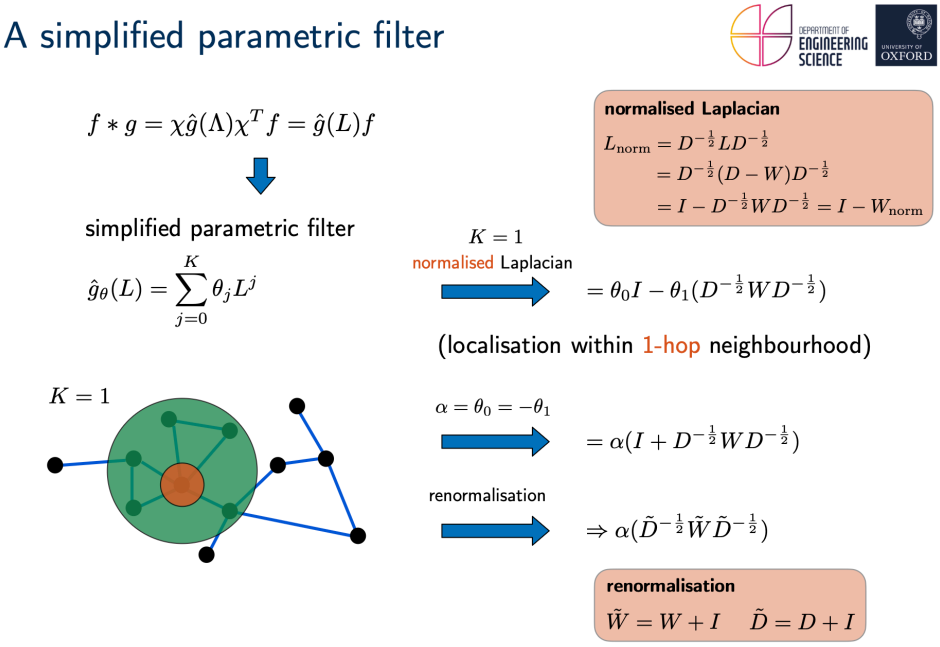
## 5.2 Using normalized Laplacian

Normalized Laplacian:

$$L_{\text{norm}} = I - D^{-1/2} W D^{-1/2}.$$

Substitute into filter:

$$g_\theta(L_{\text{norm}}) = \theta_0 I + \theta_1 (I - D^{-1/2} W D^{-1/2}) = (\theta_0 + \theta_1) I - \theta_1 D^{-1/2} W D^{-1/2}.$$




## 5.3 Kipf–Welling simplification

Impose constraint:

$$\theta_0 = -\theta_1 = \alpha.$$

Then

$$g_\alpha(L_{\text{norm}}) = \alpha(I + D^{-1/2} W D^{-1/2}).$$

This is the key simplification enabling:
- locality (1-hop only), - numerical stability, - avoiding eigen-decomposition.

## 5.4 Resulting propagation rule

For node $i$:

$$y_i = \alpha f_i + \alpha \sum_{j \sim i} \frac{1}{\sqrt{d_i d_j}} f_j.$$

For unweighted graphs with constant degree $d_i = d = 4$ (grid example):

$$y_i = \alpha f_i + \frac{\alpha}{4} \sum_{j \sim i} f_j.$$

## 5.5 GCN form

Add self-loops and re-normalize:

$$\tilde{A} = A + I, \qquad \tilde{D}_{ii} = \sum_j \tilde{A}_{ij}.$$

GCN layer becomes:

$$H^{(l+1)} = \sigma\left(\tilde{D}^{-1/2} \tilde{A} \tilde{D}^{-1/2} H^{(l)} W^{(l)}\right).$$

This is exactly the spatial message-passing form but derived from spectral filtering.

# A simplified parametric filter

$$f * g = \chi \hat{g}(\Lambda) \chi^T f = \hat{g}(L) f$$

simplified parametric filter

$$\hat{g}_\alpha(L) = \alpha(I + D^{-\frac{1}{2}} W D^{-\frac{1}{2}})$$

$$y_i = \alpha f_i + \alpha \frac{1}{\sqrt{d_i}} \sum_{j:(i,j)\in\mathcal{E}} w_{ij} \frac{1}{\sqrt{d_j}} f_j$$

unitary edge weights

$$y_i = \alpha f_i + \frac{1}{4}\alpha \sum_{j:(i,j)\in\mathcal{E}} f_j$$