

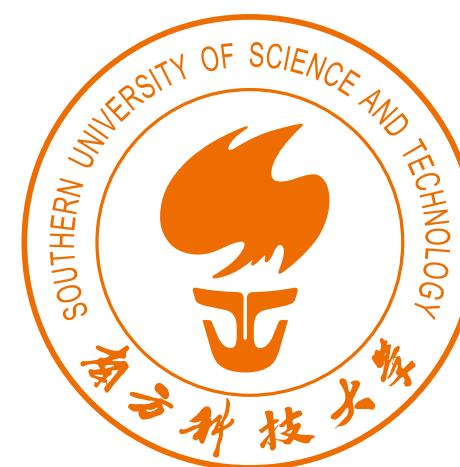
# Distributional convergence rates for local dependent random variables

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第三教学楼

# Distributional convergence rates

- Let  $X_1, \dots, X_n$  be a sample, and let  $W_n = W(X_1, \dots, X_n)$  be a statistic of interest.
- In statistics, we want to calculate the  $p$ -value:  $P(W_n \geq x)$ .
- As the true distribution of  $W_n$  is unknown, we usually use its limiting distribution to estimate the  $p$ -value: Assume that  $W_n \xrightarrow{d} Z$ , then

$$P(W_n \geq x) \rightarrow P(Z \geq x).$$

- How close?



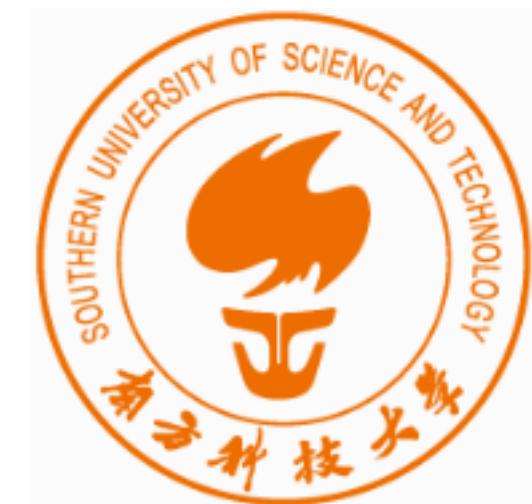
# Errors for distributional approximation

- In this talk, we focus on the following two kinds of distances.
- Absolute error: **Kolmogorov distance** —

$$\begin{aligned} d_{\text{Kol}}(W, Z) &:= \sup_{z \in \mathbb{R}} |P(W \leq z) - P(Z \leq z)| \\ &= \sup_{z \in \mathbb{R}} |P(W > z) - P(Z > z)|. \end{aligned}$$

- Relative error: **Cramér-type moderate deviation** —

$$\frac{P(W > z)}{P(Z > z)} = 1 + \text{relative error}, \quad \text{uniformly in } z \in [0, c_n].$$



# Classical results

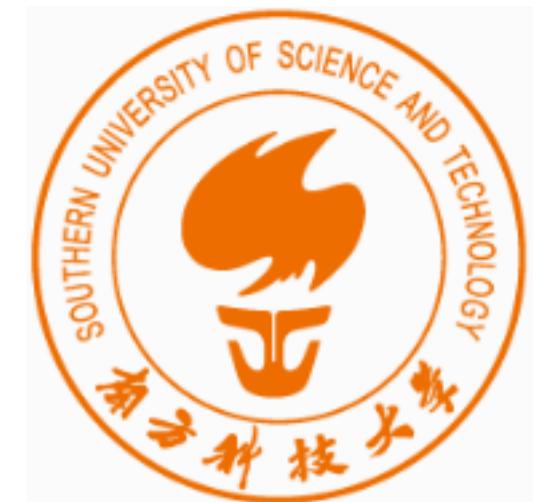
- Let  $X_1, \dots, X_n$  be independent random variables with  $E(X_i) = 0$  and  $\text{Var}(X_i) < \infty$  for all  $1 \leq i \leq n$ , and let

$$W_n = \frac{1}{B_n} \sum_{i=1}^n X_i,$$

where

$$B_n^2 = \sum_{i=1}^n E X_i^2.$$

- Classical CLT:  $W_n \xrightarrow{d} Z \sim N(0,1)$  under the Lindeberg condition.



# Classical results: Berry–Esseen bound

- **Berry (1941)** and **Esseen (1942)**: Assume that  $E|X_i|^3 < \infty$ , then

$$d_{\text{Kol}}(W_n, Z) \leq \frac{C}{B_n^3} \sum_{i=1}^n E|X_i|^3. \quad (1)$$

- In particular, if  $X_1, \dots, X_n$  are i.i.d. with  $EX_1 = 0$  and  $EX_1^2 = \sigma^2$ , then  $B_n = \sqrt{n}\sigma$ , and Eq. (1) reduces to

$$d_{\text{Kol}}(W_n, Z) \leq Cn^{-1/2}E|X_1|^3.$$



# Classical results: Cramér-type bounds

- **Cramér (1938):** If  $Ee^{t_0|X_1|} < \infty$  for some  $t_0 > 0$ , then

$$\frac{P(W_n \geq x)}{1 - \Phi(x)} = 1 + O(1) \frac{1 + x^3}{\sqrt{n}}$$

uniformly in  $0 \leq x = O(n^{1/6})$ .

- **Linnik (1961):** If  $Ee^{t_0|X_1|^{1/2}} < \infty$  for some  $t_0 > 0$ , then

$$\frac{P(W_n \geq x)}{1 - \Phi(x)} \rightarrow 1$$

uniformly for  $x \geq 0$  and  $x = o(n^{1/6})$ .



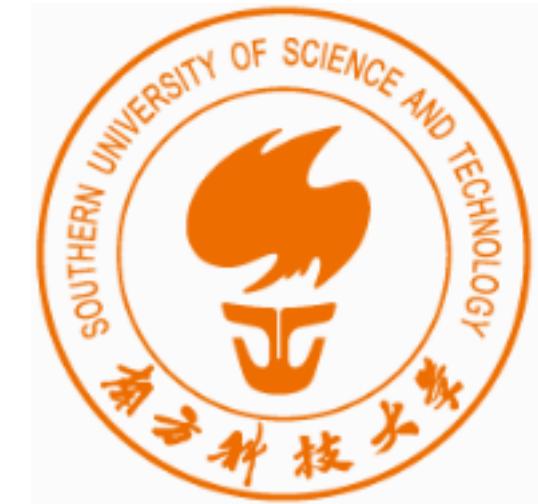
# Self-normalized sums

- In practice, the true variance  $\sigma^2$  is unknown, we consider Student's  $t$ -statistic

$$T_n = \frac{S_n}{\sqrt{n}\hat{\sigma}},$$

where  $S_n = \sum_{i=1}^n X_i$ ,  $\hat{\sigma} = \left( \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 \right)^{1/2}$ ,  $\bar{X} = S_n/n$ .

- The self-normalized sum  $S_n/V_n$ , where  $V_n^2 = \sum_{i=1}^n X_i^2$ .



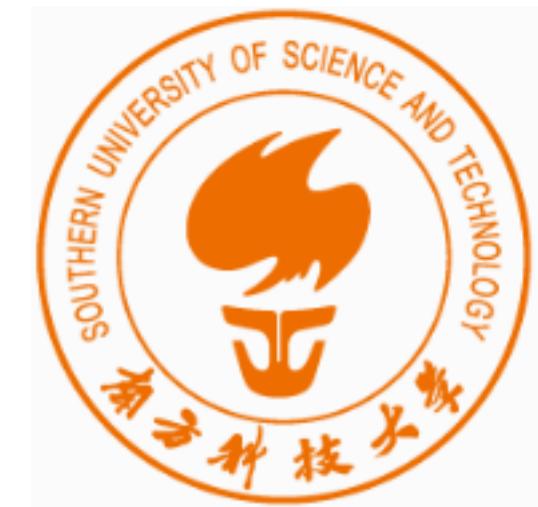
# $t$ -statistics and self-normalized sums

- Student's  $t$ -statistic and self-normalized sum are related:

$$T_n = \frac{S_n}{V_n} \left( \frac{n-1}{n - (S_n/V_n)^2} \right)^{1/2}.$$

- Then

$$\{T_n \geq t\} = \left\{ \frac{S_n}{V_n} \geq t \left( \frac{n}{n + t^2 - 1} \right)^{1/2} \right\}.$$



# Berry–Esseen bounds

- **Bentkus and Götze (1996):** If  $E|X_1|^3 < \infty$ ,

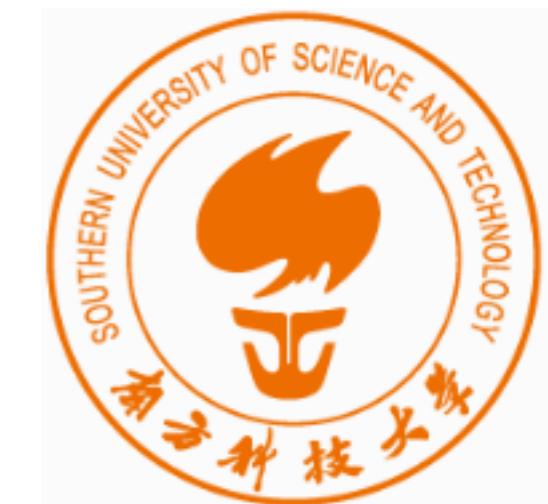
$$d_{\text{Kol}}(T_n, Z) \leq \frac{CE|X_1|^3}{\sqrt{n}\sigma^3}.$$

- **Shao (1999):** If  $E|X_1|^3 < \infty$ ,

$$\frac{P(S_n/V_n \geq x)}{1 - \Phi(x)} \rightarrow 1, \quad \text{uniformly in } x \in [0, o(n^{1/6})].$$

- **Jing, Shao and Wang (2003):** If  $E|X_1|^3 < \infty$ ,

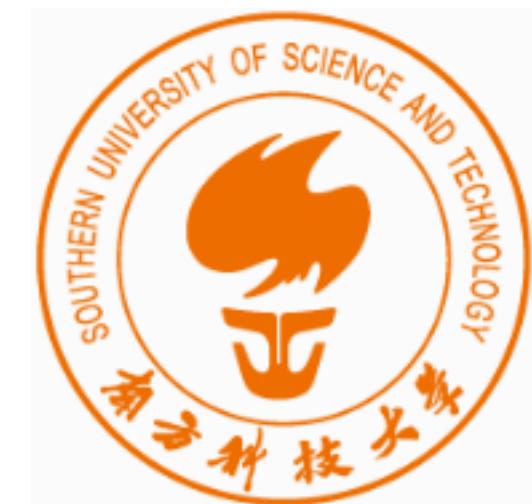
$$\frac{P(S_n/V_n \geq x)}{1 - \Phi(x)} = 1 + O(1) \frac{(1 + x^3)}{\sqrt{n}\sigma^3}, \quad \text{for } 0 \leq x \leq n^{1/6}\sigma/(E|X_1|^3)^{1/3}.$$



# Local dependence

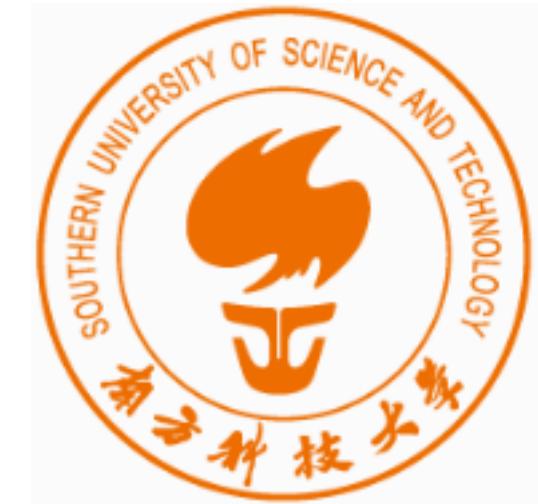
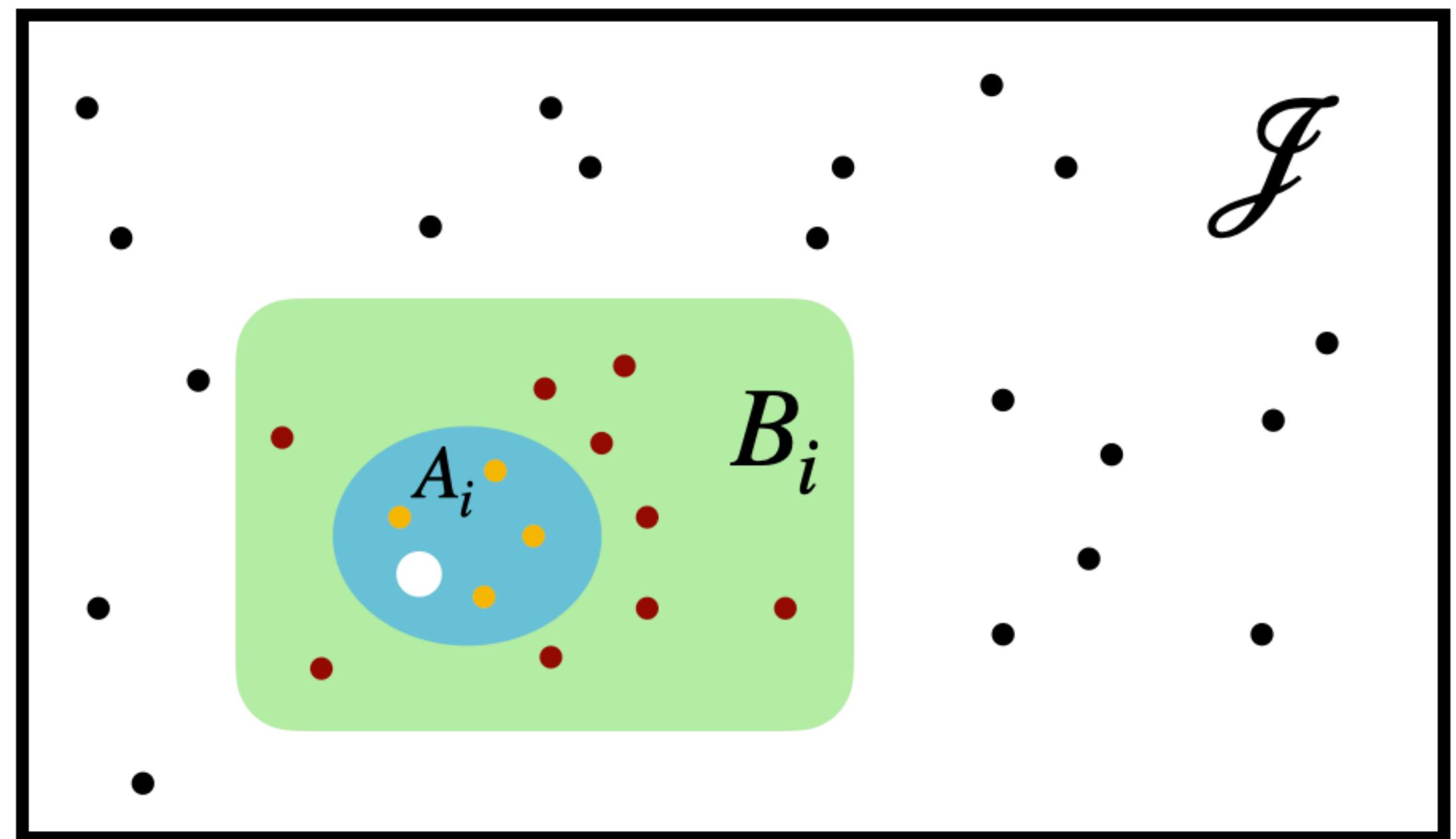
- In practice, we often observe dependent random variables. Examples include:
- **$m$ -dependent random variables:** A discrete-time stochastic process  $(X_i)_{i \in \mathbb{Z}}$  is  $m$ -dependent, if for all  $k$ ,  $(X_i)_{i \leq k}$  are independent of  $(X_i)_{i \geq k+m+1}$ .
- **Graph dependency:** Let  $(X_n)_{n \in \mathcal{V}}$  be a random field index on a graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , so that for any disjoint subsets  $\mathcal{V}_1$  and  $\mathcal{V}_2$  with no edges between  $\mathcal{V}_1$  and  $\mathcal{V}_2$ , then  $(X_i)_{i \in \mathcal{V}_1}$  and  $(X_j)_{j \in \mathcal{V}_2}$  are independent.
- **$U$ -statistics:** Let  $h$  be a kernel function, and let

$$U = \frac{1}{\binom{n}{k}} \sum_{i_1 < \dots < i_k} h(X_{i_1}, \dots, X_{i_k})$$



# Local dependence conditions

- **Chen and Shao (2004):** Consider a random field  $\{X_i, i \in J\}$ :
  - ① For each  $i \in J$ , there exists  $A_i \subset J$  such that  $X_i$  is independent of  $\{X_j, j \in A_i^c\}$ .
  - ② For each  $i \in J$ , there exists  $B_i \subset J$  such that  $A_i \subset B_i$  and  $\{X_j, j \in A_i\}$  is independent of  $\{X_j, j \in B_i^c\}$ .



# Berry–Esseen bounds for local dependence

- Assume that  $EX_i = 0$ ,  $Var(X_i) < \infty$  for each  $i \in J$ .
- Let  $S = \sum_{i \in J} X_i$  and assume that  $Var(S) = \sigma^2$ .
- Let  $N_i = \{j \in J : B_i \cap B_j \neq \emptyset\}$  and  $\kappa := \max_{i \in J} |N_i|$ .
- **Chen and Shao (2004):** Let  $Y_i = \sum_{j \in A_i} X_j$ , and assume that  $E|X_i|^p < \infty$  where  $2 < p \leq 4$ ,

$$\begin{aligned} \sup_{z \in \mathbb{R}} |P(S/\sigma \leq z) - \Phi(z)| &\leq (13 + 11\kappa) \sum_{i \in J} (E|X_i/\sigma|^{3 \wedge p} + E|Y_i/\sigma|^{3 \wedge p}) \\ &\quad + 2.5 \left( \kappa \sum_{i \in J} (E|X_i/\sigma|^p + E|Y_i/\sigma|^p) \right)^{1/2} \end{aligned}$$



# Cramér-type moderate deviations

**Theorem (Liu and Zhang, 2023, AAP)**

Under the local dependence conditions, assume that  $\sigma = 1$ , and

$$\mathbb{E}\{\exp(a_n \kappa |X_j|)\} \leq b.$$

holds. Then,

$$\left| \frac{\mathbb{P}[S \geq z]}{1 - \Phi(z)} - 1 \right| \leq C\delta_n (1 + z^3)$$

for  $0 \leq z \leq ca_n^{1/3} \min\{1, \kappa^{-1/3}(1 + \theta_n)^{-2/3}\}$ , where  $C$  and  $c$  are absolute constants and  $\delta_n = \kappa^2 a_n^{-1} (1 + \theta_n^6)$  and  $\theta_n = b^{1/2} n^{1/2} a_n^{-1}$ .



# Self-normalized sums for local dependence

- Consider local dependent random variables.
- Define

$$S = \sum_{i \in J} X_i, \quad V^2 = \left( \sum_{i \in J} (X_i Y_i - \bar{X} \bar{Y}) \right)_+, \quad W = S/V,$$

where  $\bar{X}$  is the sample mean of  $X_i$ 's, and  $\bar{Y}$  is the sample mean of  $Y_i$ 's.

- Berry–Esseen bound for  $W$ .



# Some notation

- Let  $\sigma^2 = \text{Var}(S)$ , and let  $\kappa$  be any number such that

$$\kappa \geq \max_{i \in J} \{ |j : B_i \cap A_j \neq \emptyset|, |j : i \in B_j| \}.$$

- Let

$$\beta_0 = \sum_{i \in J} P(|X_i| > \sigma/\kappa),$$

$$\beta_1 = \frac{1}{\sigma^2} \sum_{i \in J} E\{|X_i|^2 I(|X_i| > \sigma/\kappa)\},$$

$$\beta_2 = \frac{1}{\sigma^3} \sum_{i \in J} E\{|X_i|^3 I(|X_i| \leq \sigma/\kappa)\},$$

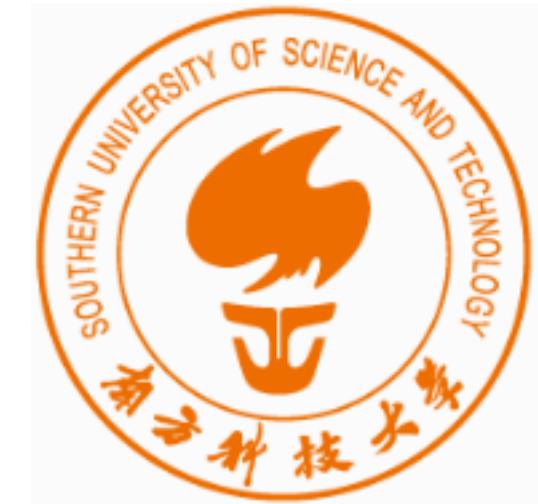
$$\theta = \sum_{i \in J} \sum_{j \in A_i} E|X_i X_j I(|X_i| \leq \sigma/\kappa, |X_j| \leq \sigma/\kappa)|.$$



# Berry–Esseen bound

- **Zhang (2024, SCM):** We have

$$d_{\text{Kol}}(W, Z) \leq C \left\{ (1 + \theta) \kappa^2 \beta_3 + \kappa \beta_2 + \beta_0 \right\} + C \kappa^{1/2} (\theta + 1) |J|^{-1/2}.$$



# Remarks

- We now give some remarks on  $\theta$  and  $\beta_j$ 's.
- By Chebyshev's inequality, we have

$$\beta_0 \leq \kappa^2 \beta_2.$$

- If  $E|X_i|^3 < \infty$ , then

$$\kappa\beta_2 + \kappa^2\beta_3 \leq \frac{\kappa^2}{\sigma^3} \sum_{i \in J} E|X_i|^3.$$

- For the Berry–Esseen bound: If  $E|X_i|^3 < \infty$ ,

$$d_{\text{Kol}}(W, Z) \leq C\kappa^3 \sum_{i \in J} E|X_i|^3 + C\kappa^{1/2} |J|^{-1/2}.$$



# Applications: $m$ -dependence

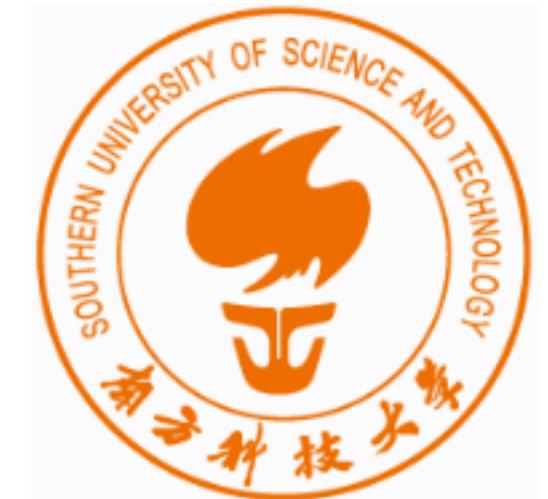
- Classical Berry–Esseen bounds for normalized  $m$ -dependent random variables:
  - **Shergin (1979):** Let  $\{X_i\}_{1 \leq i \leq n}$  be a sequence of  $m$  dependent random variables, let

$$S_n = \sum_{i=1}^n X_i, \text{ and } B_n^2 = \text{Var}(S_n).$$

- Then,

$$d_{Kol}(S_n/B_n, Z) \leq C(m+1)^2 B_n^{-3} \sum_{k=1}^n E|X_i|^3.$$

- However, there are few results on their self-normalized sums.



# Applications: $m$ -dependence

- Let  $\{X_i\}_{1 \leq i \leq n}$  be an  $m$ -dependent field with  $EX_i = 0$  and assume that  $E|X_i|^3 < \infty$ . Let  $Y_i = \sum_{j \in A_i} X_j$ . Assume that  $\sigma^2 := \sum_{i \in \mathcal{J}} E\{X_i Y_i\} > 0$ . Let  $W$  be the self-normalized sum defined by  $W = S/V$ , where

$$V = \sqrt{\left( \sum_{i \in \mathcal{V}} (X_i Y_i - \bar{X} \bar{Y}) \right)_+}.$$

- Zhang (2024, SCM):** We have

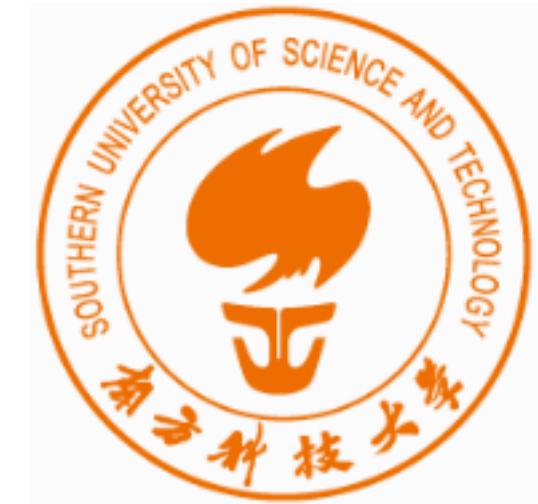
$$d_{\text{Kol}}(W, Z) \leq C(m+1)^3 \left( 1 + (m+1) |n^{1/3} \gamma^{2/3}| \right) (\gamma + n^{-1/2}),$$

where  $\gamma = \sum_{i \in \mathcal{J}} \mathbb{E}|X_i|^3 / \sigma^3$ .



# Application: Graph dependency

- **Graph dependency:** Let  $(X_n)_{n \in \mathcal{V}}$  be a random field index on a graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , so that for any disjoint subsets  $\mathcal{V}_1$  and  $\mathcal{V}_2$  with no edges between  $\mathcal{V}_1$  and  $\mathcal{V}_2$ , then  $(X_i)_{i \in \mathcal{V}_1}$  and  $(X_j)_{j \in \mathcal{V}_2}$  are independent.
- **Normalized sum**  $S/\sigma$ , where  $\sigma^2 = \text{Var}(S)$ :
  - **Baldi and Rinott (1989) (also, Rinott, 1994)** proved an optimal Berry–Esseen bound for bounded random variables.
  - **Chen and Shao (2004)** proved a Berry–Esseen bound under some moment conditions.
  - Our result can be applied to the self-normalized sum.



# Application: Graph dependency

- **Zhang (2024, SCM):** Let  $\{X_i, i \in \mathcal{V}\}$  be a field of random variables indexed by the vertices of a dependency graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ . Assume that  $EX_i = 0$ . Let  $S = \sum_{i \in \mathcal{V}} X_i$ ,  $Y_i = \sum_{j \in A_i} X_j$  and let  $V = (\sum_{i \in \mathcal{V}} (X_i Y_i - \bar{X} \bar{Y}))_+^{1/2}$ . Put  $W = S/V$  and  $n = |\mathcal{V}|$ . Let  $d$  be the maximal degree of  $\mathcal{G}$ .
- Then, we have

$$d_{Kol}(W, Z) \leq Cd^9 \left(1 + d^6 n^{1/3} \gamma^{2/3}\right) \left(n^{-1/2} + \gamma\right),$$

where  $\gamma = \sum_{i \in \mathcal{V}} E|X_i/\sigma|^3$ , and  $\sigma^2 = \sum_{i \in \mathcal{V}} E\{X_i Y_i\}$ .



# Some remarks on the results

- We proved a Berry–Esseen bound for self-normalized sum for local dependent random variables.
- The convergence rate on the sample size  $n$  is optimal.
- However, the rate on the neighborhood size  $\kappa$  (or  $m$  in the  $m$ -dependence) is not optimal.
- For example,

- Classical results by **Shergin (1979)**:  $d_{Kol}(S_n/B_n, Z) \leq C(m+1)^2 B_n^{-3} \sum_{k=1}^n E|X_i|^3$
- Our result:  $d_{Kol}(W, Z) \leq C(m+1)^3 (1 + (m+1)|n^{1/3}\gamma^{2/3}|) (\gamma + n^{-1/2})$



# $\kappa$ cannot be too large

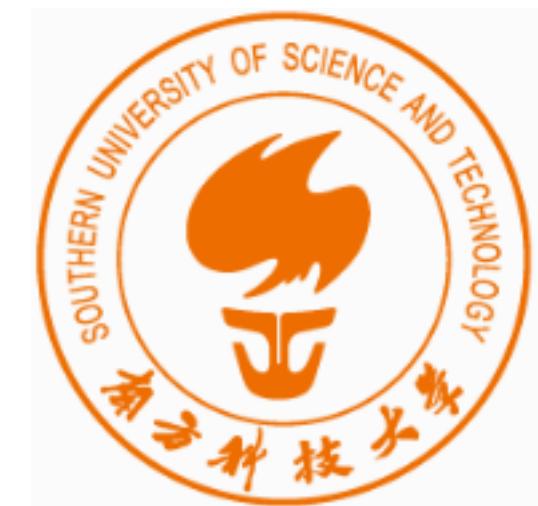
Recall that  $\kappa$  measures the size of the neighborhoods

$$\kappa \geq \max_{i \in J} \{ |j : B_i \cap A_j \neq \emptyset|, |j : i \in B_j| \}$$

and here is our result:

$$d_{\text{Kol}}(W, Z) \leq C \left\{ (1 + \theta) \kappa^2 \beta_3 + \kappa \beta_2 + \beta_0 \right\} + C \kappa^{1/2} (\theta + 1) |J|^{-1/2}.$$

Therefore,  $\kappa$  cannot be too large.

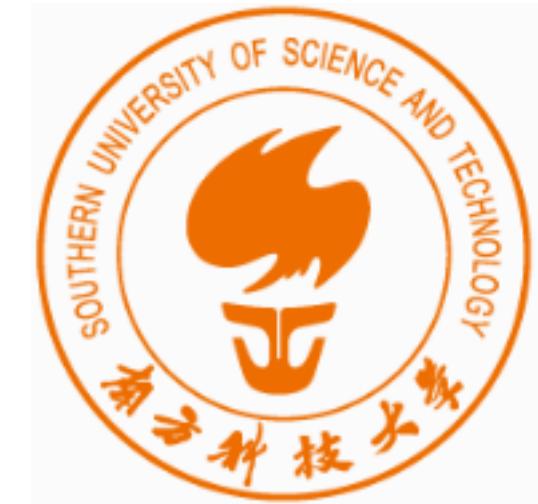


# U-statistics

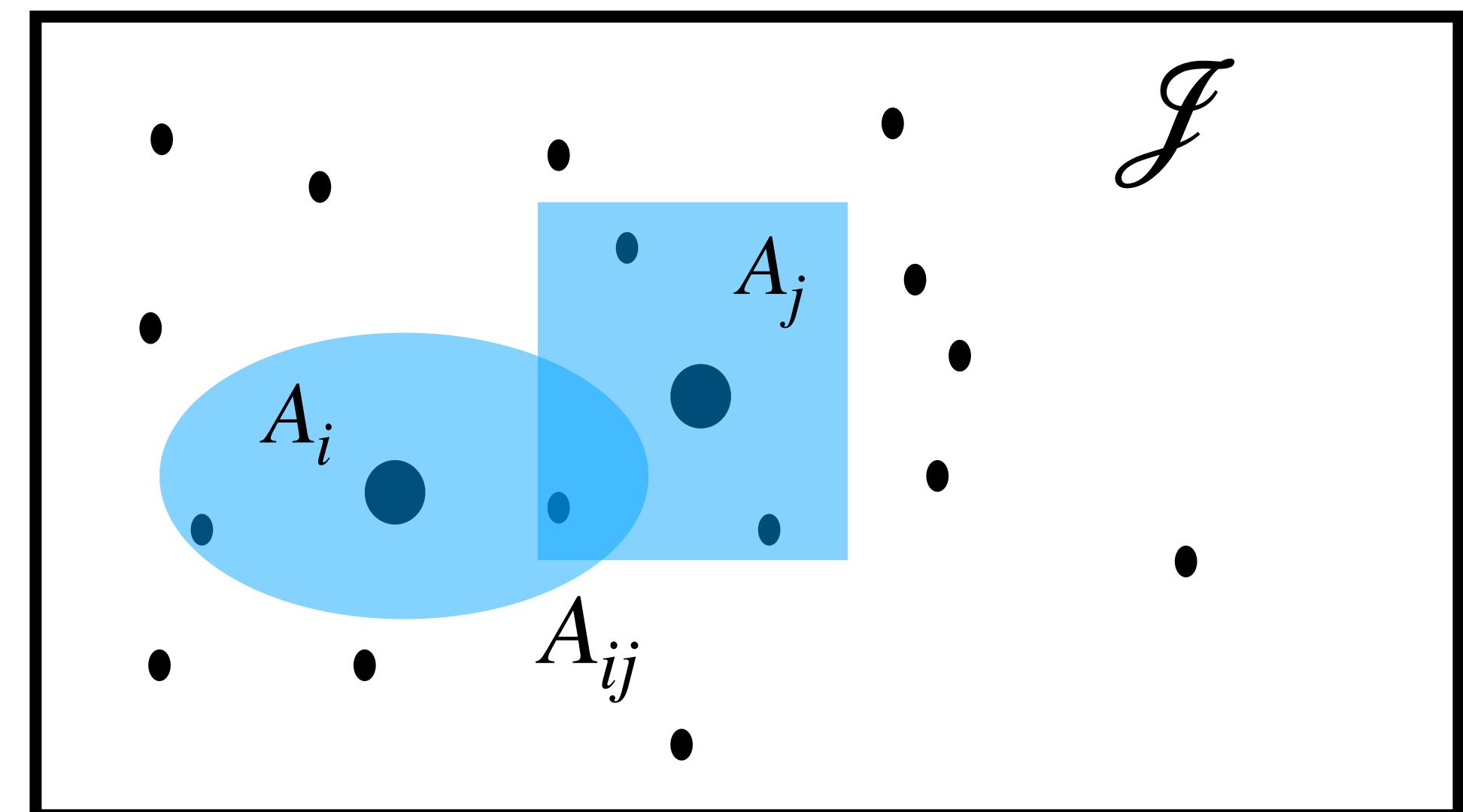
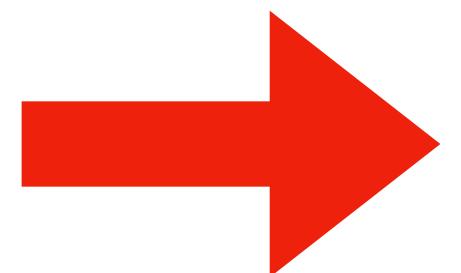
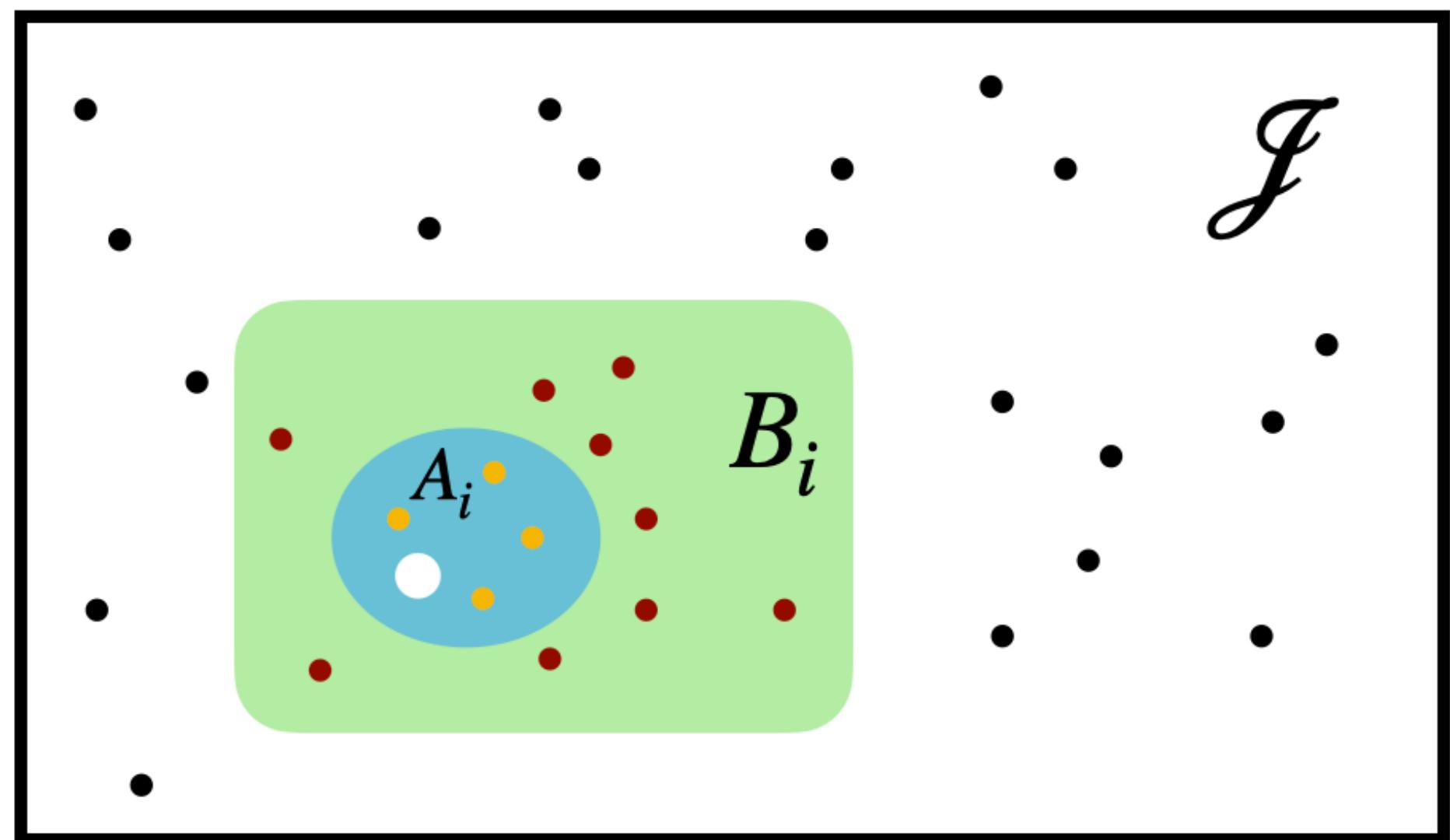
- U-statistics are commonly used in statistics.
- Definition

$$S = \frac{1}{\sigma_n} \sum_{i_1 < \dots < i_k} h(X_{i_1}, \dots, X_{i_k}).$$

- Then, for non-degenerate case,
  - $J = \{(i_1, \dots, i_k) : i_1 < \dots < i_k\} \implies |J| = O(n^k)$
  - $\sigma_n^2 = O(n^{2k-1})$
  - $A_i = O(n^{k-1})$ ,  $B_i = J$ , and hence  $\kappa = |J|$ .
  - Then,  $d_K(S, Z) = O(1)$ ?



# Another local dependence structure



# Decomposable random variables

- Consider a random field  $\{X_i, i \in J\}$ ,
  - ① For each  $i \in J$ , there exists  $A_i \subset J$  such that  $X_i$  is independent of  $\{X_j, j \in A_i^c\}$ .
  - ② For each  $i \in J$  and  $j \in A_i$ , there exists  $A_{ij} \subset J$  such that  $A_i \subset A_{ij}$  and  $\{X_i, X_j\}$  is independent of  $\{X_k, k \in A_{ij}^c\}$ .
  - ③ ...
- This structure was introduced by **Barbour et al. (1989)**.



# Existing results

- **Barbour, Karoiski and Rucisjski (1989)** proved a bound for the Wasserstein distance

$$d_1(W, Z) = \sup_{\|h'\| \leq 1} |Eh(W) - Eh(Z)| \leq \kappa^2 \sum_{i=1}^n E |X_i|^3.$$

- **Fang (2019)** proved a Wasserstein-2 distance under a **three-level** dependence condition, where the Wasserstein- $p$  distance is defined as

$$\begin{aligned} d_p(W, Z) &:= \left( \inf_{\pi \in \Gamma(W, Z)} \int |x - y|^p d\pi(x, y) \right)^{1/p} \\ &\leq C\kappa^{3/2} \left( \sum_{i=1}^n E |X_i|^4 \right)^{1/2} \end{aligned}$$



# Berry–Esseen bounds for decomposable r.v.'s

**Theorem (Cai and Zhang, 2024+)**

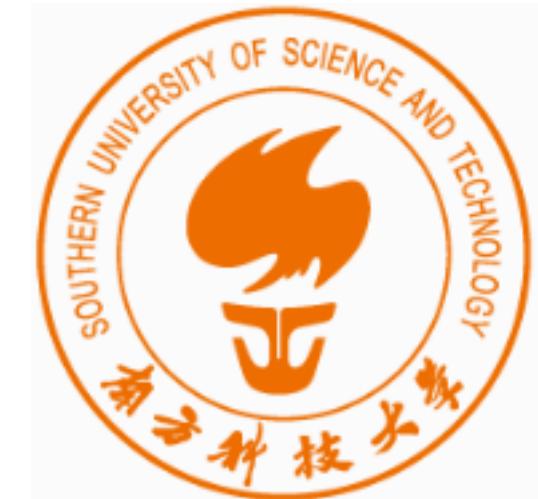
Assume that  $\mathbb{E}\{X_i\} = 0$  for  $i \in [n]$ . Let  $S = \sum_{i \in [n]} X_i$ , and assume that its variance is 1. For any  $i$ , let  $N_i = \{k \in [n]; i \in A_k\}$ . Let  $\kappa$  and  $\tau$  be such that

$$\kappa = \max\{\sup_{i,j} |A_{ij}|, \sup_i |N_i|\}, \quad \text{and} \quad \tau = \max_{k \in [n]} |\{(i, j) : j \in A_i, k \in A_{ij}\}|.$$

We have

$$d_{Kol}(S, Z) \leq C\theta\kappa^2 \sum_{i \in [n]} \|X_i\|_4^3 + C\theta\kappa^{1/2} (\kappa + \tau^{1/2}) \left( \sum_{i \in [n]} \|X_i\|_4^4 \right)^{1/2},$$

where  $\|X\|_p = (E|X|^p)^{1/p}$  and where  $\theta = \frac{\kappa}{\sigma^2} \sum_{i=1}^n \|X_i\|_4^2$ .



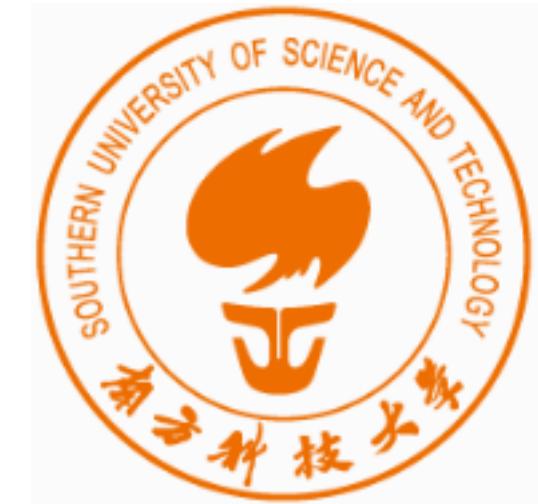
# Remarks

- The bound is sharper w.r.t. the neighborhood size  $\kappa$ .
- The Berry–Esseen bound is of the same order as the Wasserstein distance, with a cost of fourth moment condition.
- Consider U statistics,

$$S = \frac{1}{\sigma_n} \sum_{i_1 < \dots < i_k} h(X_{i_1}, \dots, X_{i_k})$$

- If  $E |h(\mathbf{X})|^4 < \infty$ , then our main results implies

$$d_{\text{Kol}}(S, Z) \leq Cn^{-1/2}.$$



# Self-normalized sums of decomposable r.v.'s

**Theorem (Cai and Zhang, 2024+)**

Let  $S, \kappa, \tau$  be defined as above. Let  $W = S/V$  be the self-normalized sum, where

$$V = \sqrt{\left( \sum_{i \in [n]} (X_i Y_i - \bar{X} \bar{Y}) \right)_+}.$$

We have

$$d_{\text{Kol}}(W, Z) \leq C\theta \left\{ \frac{\kappa^2}{\sigma^3} \sum_{i \in [n]} \|X_i\|_4^3 + \frac{\kappa^{1/2} (\kappa + \tau^{1/2})}{\sigma^2} \left( \sum_{i \in [n]} \|X_i\|_4^4 \right)^{1/2} + \kappa^{1/2} n^{-1/2} \right\},$$

$$\text{where } \theta = \frac{\kappa}{\sigma^2} \sum_{i=1}^n (E |X_i|^4)^{1/2}.$$



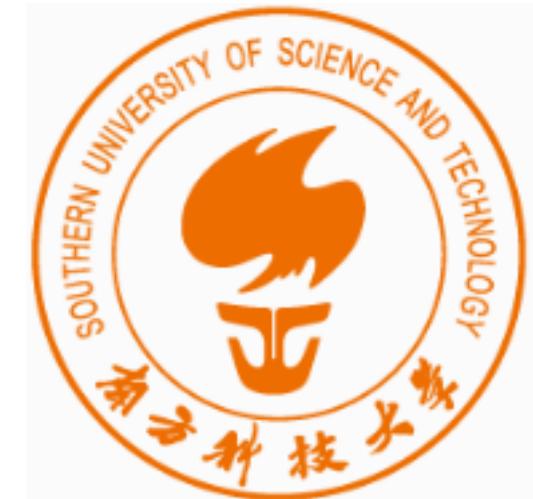
# Applications: Pattern matching

- Text:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
A	A	B	A	A	C	A	A	D	A	A	B	A	A	B	A	A	B

- Pattern: **A A B A**

- Pattern found at **1, 10, and 13**



# Pattern matching

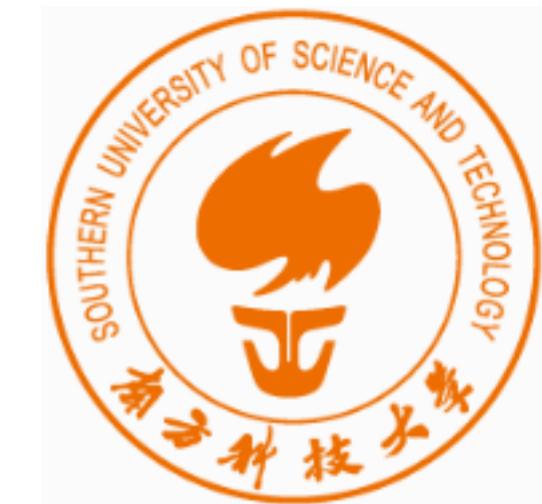
- Text:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
A	A	B	A	A	C	A	A	D	A	A	B	A	A	B	A	A	B

- Pattern: **A A \*\*\* B A**

**there is a gap between the second and the third letter**

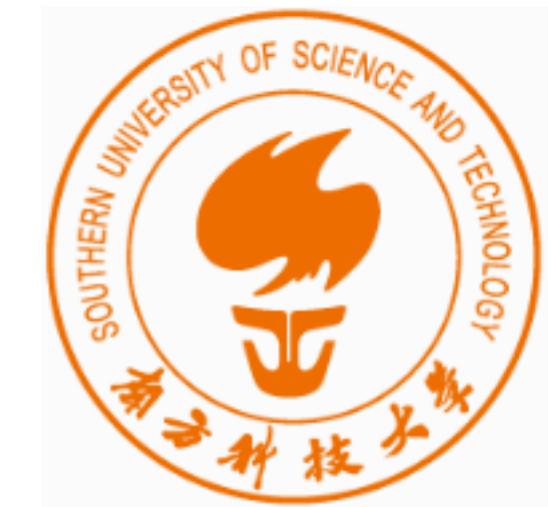
- Pattern found at **10**



# Pattern matching

- Let  $\Xi = \xi_1 \xi_2 \dots \xi_n$  be a random string, consisting of  $n$  i.i.d. or m dependent random letters from a finite alphabet  $\mathcal{A}$
- Let  $w = w_1 \dots w_l$  be a given word
- Let  $d = (d_1, \dots, d_{l-1})$  be the vector of gap lengths in the pattern
- An occurrence of  $w$  is an increasing sequence of indices  $i_1 < \dots < i_l$  in  $[n] = \{1, \dots, n\}$  such that

$$(\xi_{i_1}, \dots, \xi_{i_l}) = (w_1, \dots, w_l)$$



# Constrained U-statistics

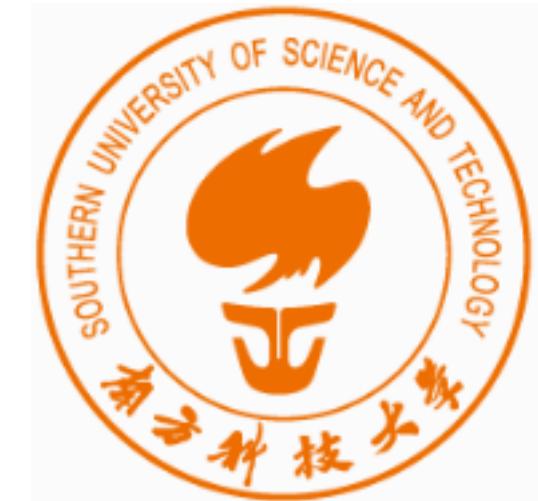
- Let  $X_1, X_2, \dots$  be a sequence of random variables
- Let  $f$  be a measurable kernel function of  $l$  variables.
- Given a constraint  $(d_1, \dots, d_{l-1})$ , we define the constrained U-statistic

$$U = \sum_{\substack{1 \leq i_1 < \dots < i_l \leq n \\ i_{j+1} - i_j \leq d_j}} f(X_{i_1}, \dots, X_{i_l})$$



# Central limit theorems

- **Hoeffding (1948):** CLT for classical U statistics
- **Janson (1991):** CLT for generalized U statistics
- **Shang (2012):** CLT for U statistics of m-dependent random variables
- **Bona (2007):** CLT for constraint U statistics with  $d_j = \infty$
- **Hofer (2017):** CLT for general constraint U statistics
- We want to study the convergence rates for these normal approximations



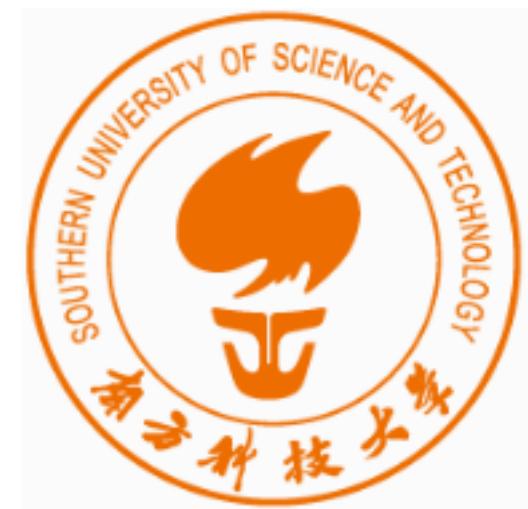
# Applications: Constrained U-stat

- Let  $X_1, X_2, \dots$  be a sequence of random variables
- Let  $f$  be a measurable kernel function of  $l$  variables.
- Given a constraint  $(d_1, \dots, d_{l-1})$ , we define the constrained U-statistic

$$U = \sum_{\substack{1 \leq i_1 < \dots < i_l \leq n \\ i_{j+1} - i_j \leq d_j}} f(X_{i_1}, \dots, X_{i_l})$$

- The normalized statistic is defined as

$$S = \frac{U - E(U)}{\sqrt{Var(U)}}$$



# Existing results

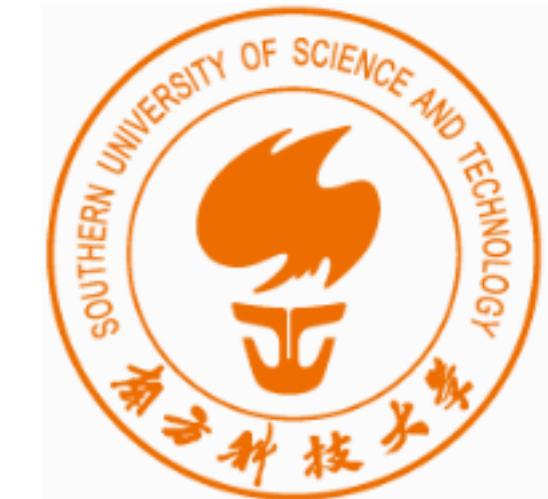
- **Janson (2023):** Let  $X_1, X_2, \dots$  be a sequence of stationary  $m$ -dependent random variables. Let

$$b = 1 + |\{k \in [l-1] : d_k = \infty\}|,$$

if  $\frac{\text{Var}(U_n)}{n^{2b-1}} \rightarrow \sigma^2 > 0$ , and assume that  $\|f\|_\infty \leq C$ , then

$$\sup_{z \in \mathbb{R}} |P(S \leq z) - \Phi(z)| \leq Cn^{-1/2},$$

where  $C$  is a constant depending on  $m, l$  and  $f$ .



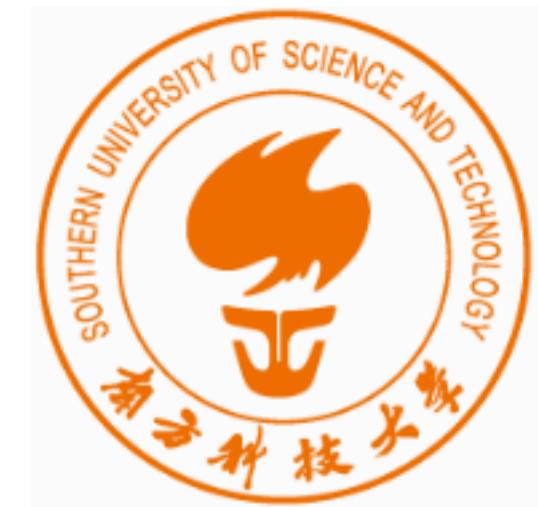
# Our results

**Cai and Zhang (2024+):** Let  $S$  be defined as above, and assume that

$E |f(X_{i_1}, \dots, X_{i_l})|^4 < \infty$  for all  $i_1, \dots, i_l$ . Then

$$d_{Kol}(S, Z) \leq Cn^{-1/2},$$

where  $C$  is a constant depending on  $m, l$  and  $f$ . Moreover, a same bound also holds for the self-normalized sum.



# Stein's method

- Stein (1972):

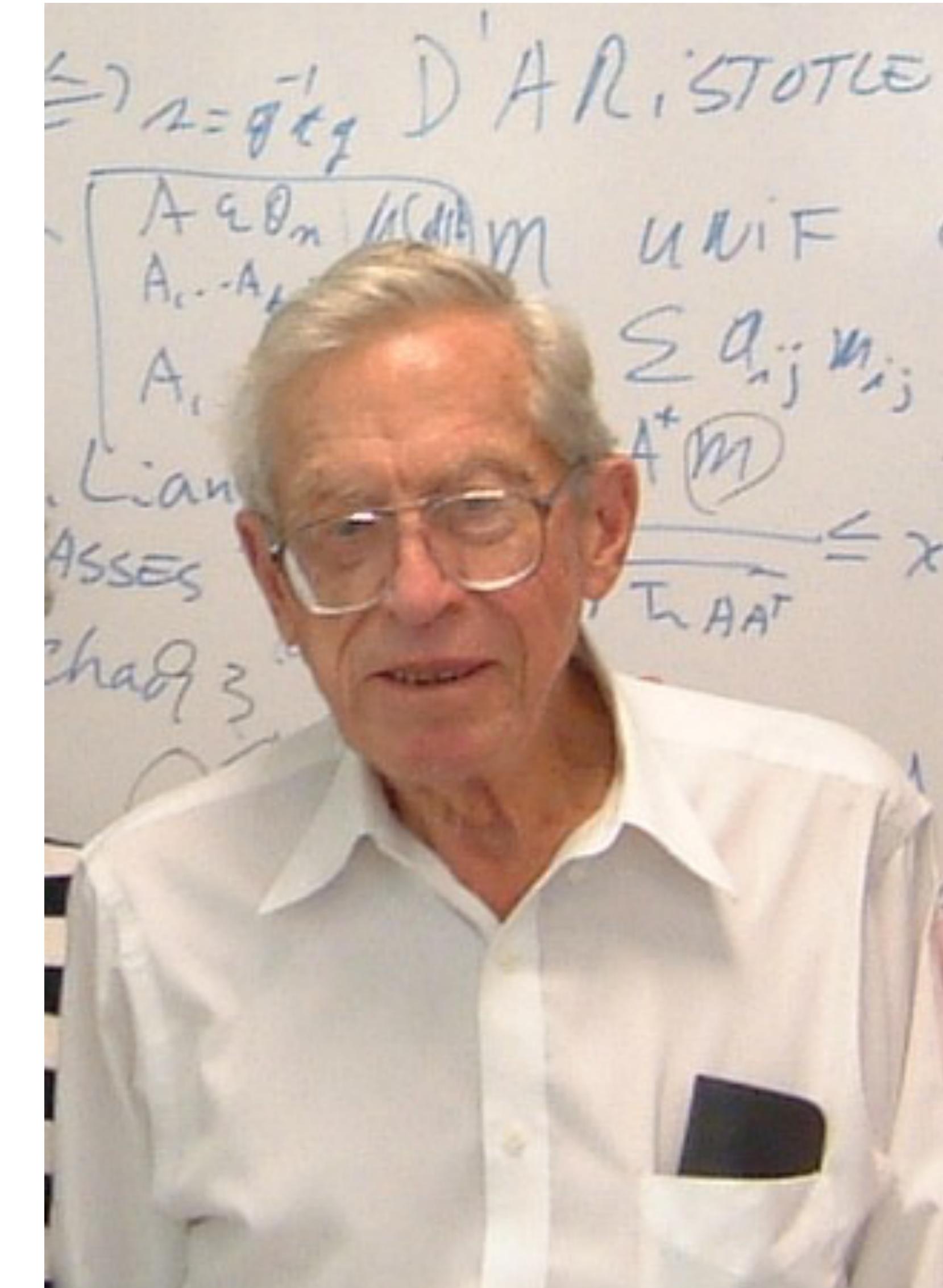
$$W \sim N(0,1) \text{ iff } \mathbb{E}f'(W) = \mathbb{E}Wf(W)$$

- Stein equation:

$$f'(w) - wf(w) = h(w) - Eh(Z)$$

- Let  $f_h$  be its solution, then

$$Eh(W) - Eh(Z) = \mathbb{E}[f'_h(W)] - \mathbb{E}[Wf_h(W)]$$



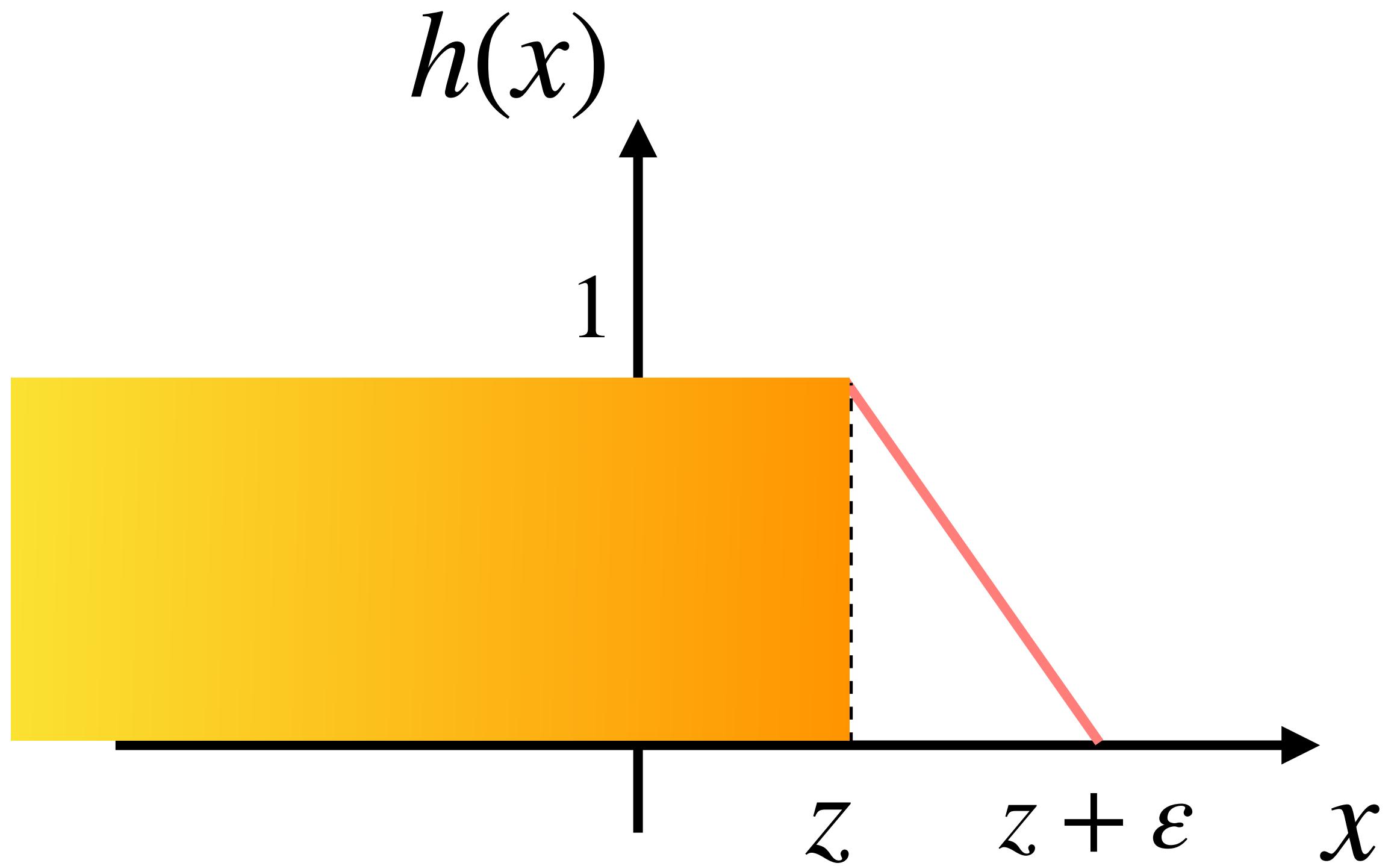
# A smoothing function

- Consider

$$h_{z,\varepsilon}(w) = \begin{cases} 1 & \text{if } w \leq z \\ 1 + \varepsilon^{-1}(z - w) & \text{if } z < w \leq z + \varepsilon \\ 0 & \text{if } w > z + \varepsilon \end{cases}$$

Then,

$$1_{\{w \leq z\}} \leq h_{z,\varepsilon}(w) \leq 1_{\{w \leq z+\varepsilon\}}.$$



# A direct bound

We have

$$\begin{aligned} |\mathbb{P}(S \leq z) - \Phi(z)| &\leq 0.4\epsilon + \left| \mathbb{E}h_{z,\epsilon}(S) - \mathbb{E}h_{z,\epsilon}(Z) \right| \\ &\leq 0.4\epsilon + \mathbb{E} \left| \sum_i X_i Y_i - E(X_i Y_i) \right| + \frac{1}{2} \sum_{i \in [n]} \mathbb{E} \left| X_i Y_i^2 (1 + S) \right| \\ &\quad + \frac{1}{\epsilon} \sum_{i \in [n]} \left\{ \int_{-\infty}^{+\infty} \int_{t \wedge 0}^{t \vee 0} E\{\mathbf{1}(z \leq S_i + u \leq z + \epsilon) \cdot |\hat{K}_i(t)|\} du dt \right\}. \end{aligned}$$

where  $\hat{K}_i(t) = X_i(1_{0 \leq t \leq Y_i} - 1_{Y_i < t \leq 0})$ , and  $S_i = S - Y_i$  is independent of  $X_i$ .



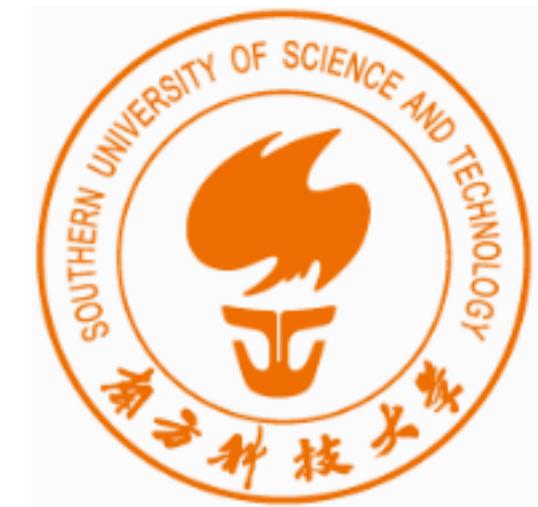
# Some take-home messages

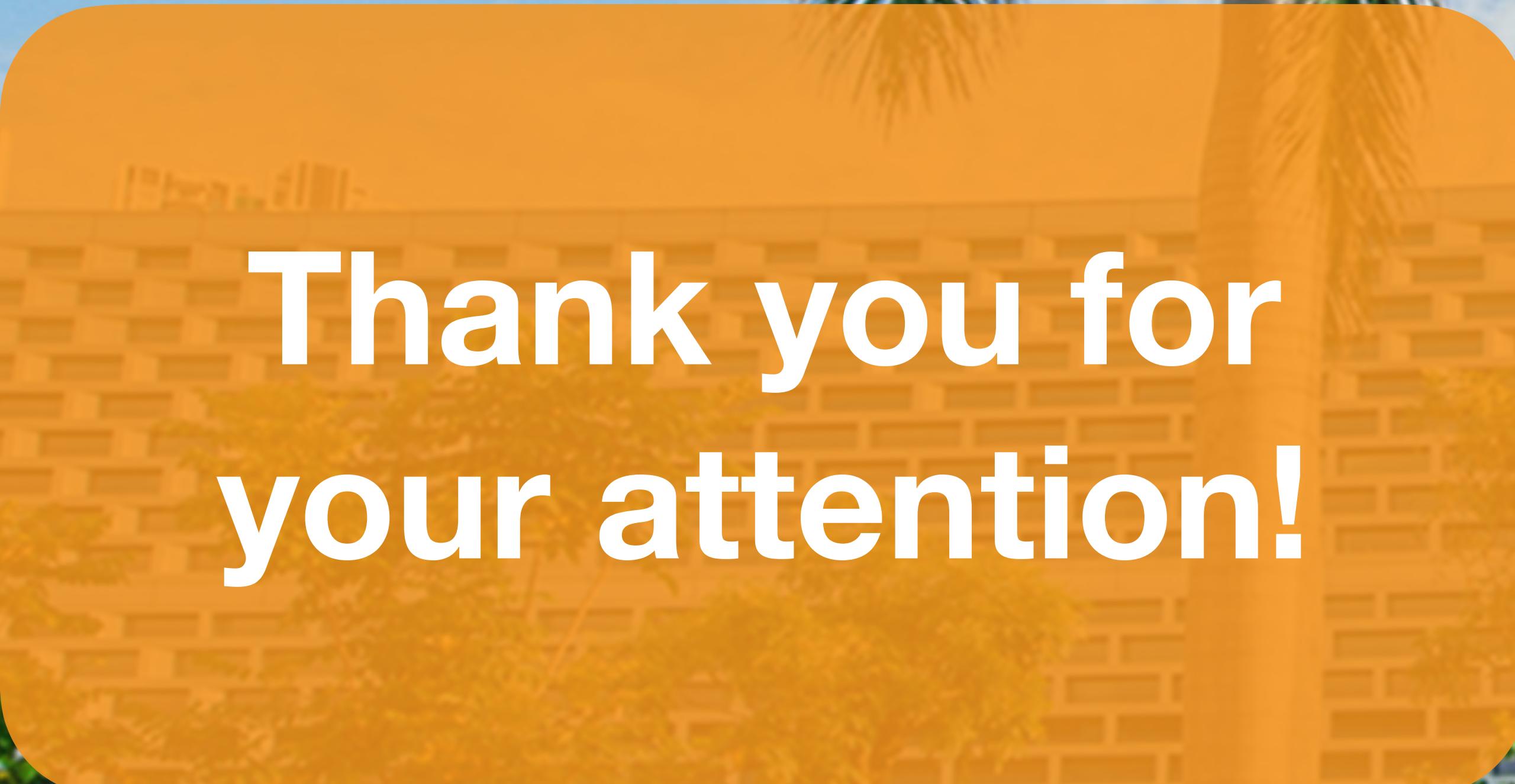
- Berry–Esseen bounds and Cramér-type moderate deviations are important in statistics.
- Optimal convergence rates for local dependent random variables are established.
- Our results are based on Stein's method, concentration inequality approach and a recursive argument.



# Selected References

- ① **Cai and Zhang (2024+).** Berry–Esseen bounds for decomposable random variables. *Working paper.*
- ② **Chen and Shao (2004).** Normal approximation under local dependence. *Annals of Probability.*
- ③ **Liu and Zhang (2023).** Cramér-type moderate deviations under local dependence. *Annals of Applied Probability.*
- ④ **Zhang (2024).** Berry–Esseen bounds for self-normalized sums of local dependent random variables. *To appear in Science China Mathematics*





Thank you for  
your attention!



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