Distributional convergence rates for local dependent random variables

Stochastic Webinar, May 22, 2024

Zhuosong Zhang

Department of Statistics and Data Science
Southern University of Science and Technology


南方科技大学

Distributional convergence rates

- Let $X_1, \ldots, X_n$ be a sample, and let $W_n = W(X_1, \ldots, X_n)$ be a statistic of interest.
- In statistics, we want to calculate the p-value: $P(W_n \geq x)$.
- As the true distribution of W_n is unknown, we usually use its limiting distribution to estimate the p-value: Assume that $W_n \xrightarrow{d} Z$, then

$$P(W_n \geq x) \to P(Z \geq x)\,.$$

- How close?

Errors for distributional approximation

- In this talk, we focus on the following two kinds of distances.
- Absolute error: **Kolmogorov distance —**

$$d_{\mathrm{Kol}}(W,Z) := \sup_{z\in\mathbb{R}} |\, P(W \le z) - P(Z \le z) \,| = \sup_{z\in\mathbb{R}} |\, P(W > z) - P(Z > z) \,| .$$

- Relative error: **Cramér-type moderate deviation —**

$$\frac{P(W > z)}{P(Z > z)} = 1 + \text{relative error}, \quad \text{uniformly in } z \in [0, c_n] .$$

Classical results

- Let $X_1, \ldots, X_n$ be independent random variables with $E(X_i) = 0$ and $\mathrm{Var}(X_i) < \infty$ for all $1 \leq i \leq n$, and let

$$W_n = \frac{1}{B_n} \sum_{i=1}^{n} X_i,$$

where

$$B_n^2 = \sum_{i=1}^{n} EX_i^2.$$

- Classical CLT: $W_n \xrightarrow{d} Z \sim N(0,1)$ under the Lindeberg condition.

Classical results: Berry–Esseen bound

- **Berry (1941)** and **Esseen (1942)**: Assume that $E|X_i|^3 < \infty$, then

$$d_{\mathrm{Kol}}(W_n, Z) \le \frac{C}{B_n^3} \sum_{i=1}^{n} E|X_i|^3. \qquad (1)$$

- In particular, if $X_1, \ldots, X_n$ are i.i.d. with $EX_1 = 0$ and $EX_1^2 = \sigma^2$, then $B_n = \sqrt{n}\sigma$, and Eq. (1) reduces to

$$d_{\mathrm{Kol}}(W_n, Z) \le Cn^{-1/2} E|X_1|^3.$$

Classical results: Cramér-type bounds

- **Cramér (1938):** If $Ee^{t_0|X_1|} < \infty$ for some $t_0 > 0$, then

$$\frac{P(W_n \geq x)}{1 - \Phi(x)} = 1 + O(1)\frac{1 + x^3}{\sqrt{n}}$$

uniformly in $0 \leq x = O(n^{1/6})$.

- **Linnik (1961):** If $Ee^{t_0|X_1|^{1/2}} < \infty$ for some $t_0 > 0$, then

$$\frac{P(W_n \geq x)}{1 - \Phi(x)} \to 1$$

uniformly for $x \geq 0$ and $x = o(n^{1/6})$.

Self-normalized sums

- In practice, the true variance σ^2 is unknown, we consider Student's t-statistic

$$T_n = \frac{S_n}{\sqrt{n}\hat{\sigma}},$$

where $S_n = \sum_{i=1}^n X_i$, $\hat{\sigma} = \left(\frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2\right)^{1/2}$, $\bar{X} = S_n/n$.

- The self-normalized sum S_n/V_n, where $V_n^2 = \sum_{i=1}^n X_i^2$.

t-statistics and self-normalized sums

- Student's t-statistic and self-normalized sum are related:

$$T_n = \frac{S_n}{V_n}\left(\frac{n-1}{n-(S_n/V_n)^2}\right)^{1/2}.$$

- Then

$$\{T_n \geq t\} = \left\{\frac{S_n}{V_n} \geq t\left(\frac{n}{n+t^2-1}\right)^{1/2}\right\}.$$

Berry–Esseen bounds

- **Bentkus and Götze (1996):** If $E|X_1|^3 < \infty$,

$$d_{\mathrm{Kol}}(T_n, Z) \le \frac{CE|X_1|^3}{\sqrt{n}\sigma^3}.$$

- **Shao (1999):** If $E|X_1|^3 < \infty$,

$$\frac{P(S_n/V_n \ge x)}{1-\Phi(x)} \to 1, \quad \text{uniformly in } x \in [0, o(n^{1/6})).$$

- **Jing, Shao and Wang (2003):** If $E|X_1|^3 < \infty$,

$$\frac{P(S_n/V_n \ge x)}{1-\Phi(x)} = 1 + O(1)\frac{(1+x^3)}{\sqrt{n}\sigma^3}, \quad \text{for } 0 \le x \le n^{1/6}\sigma/(E|X_1|^3)^{1/3}.$$

Local dependence

- In practice, we often observe dependent random variables. Examples include:
- m**-dependent random variables:** A discrete-time stochastic process $(X_i)_{i\in\mathbb{Z}}$ is m-dependent, if for all k, $(X_i)_{i\le k}$ are independent of $(X_i)_{i\ge k+m+1}$.
- **Graph dependency**: Let $(X_n)_{n\in\mathscr{V}}$ be a random field index on a graph $\mathscr{G}=(\mathscr{V},\mathscr{E})$, so that for any disjoint subsets $\mathscr{V}_1$ and $\mathscr{V}_2$ with no edges between $\mathscr{V}_1$ and $\mathscr{V}_2$, then $(X_i)_{i\in\mathscr{V}_1}$ and $(X_j)_{j\in\mathscr{V}_2}$ are independent.
- U**-statistics:** Let h be a kernel function, and let

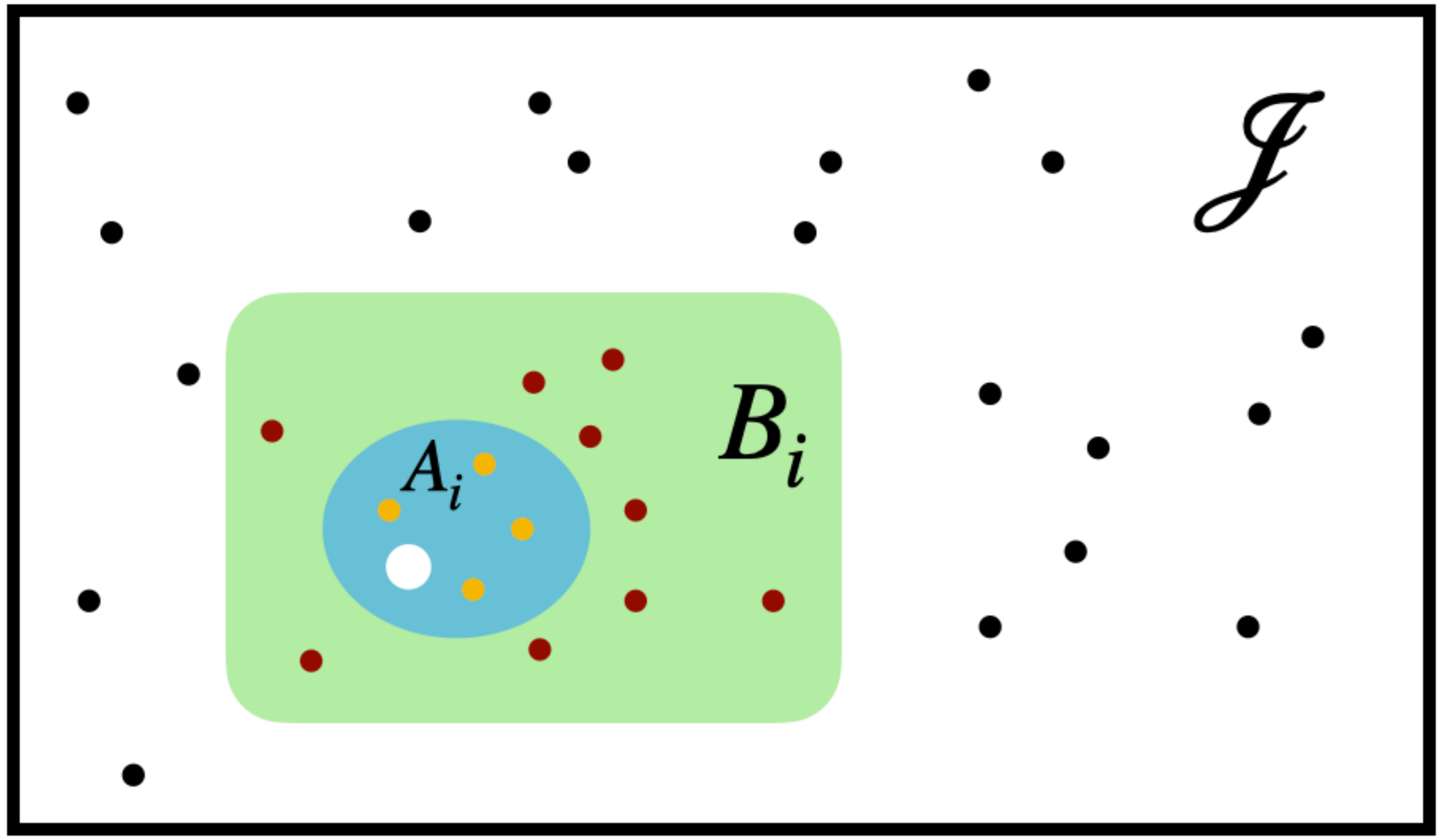
$$U=\frac{1}{\binom{n}{k}}\sum_{i_1<\cdots<i_k}h(X_{i_1},\ldots,X_{i_k})$$

Local dependence conditions

- **Chen and Shao (2004)**: Consider a random field $\{X_i, i \in J\}$:

  ① For each $i \in J$, there exists $A_i \subset J$ such that X_i is independent of $\{X_j, j \in A_i^c\}$.

  ② For each $i \in J$, there exists $B_i \subset J$ such that $A_i \subset B_i$ and $\{X_j, j \in A_i\}$ is independent of $\{X_j, j \in B_i^c\}$.



Berry–Esseen bounds for local dependence

- Assume that $EX_i = 0$, $Var(X_i) < \infty$ for each $i \in J$.
- Let $S = \sum_{i \in J} X_i$ and assume that $Var(S) = \sigma^2$.
- Let $N_i = \{j \in J : B_i \cap B_j \neq \emptyset\}$ and $\kappa := \max_{i \in J} |N_i|$.
- **Chen and Shao (2004)**: Let $Y_i = \sum_{j \in A_i} X_j$, and assume that $E|X_i|^p < \infty$ where $2 < p \leq 4$,

$$\sup_{z \in \mathbb{R}} |P(S/\sigma \leq z) - \Phi(z)| \leq (13 + 11\kappa) \sum_{i \in J} \left(E|X_i/\sigma|^{3 \wedge p} + E|Y_i/\sigma|^{3 \wedge p}\right)$$

$$+2.5 \left(\kappa \sum_{i \in J} \left(E|X_i/\sigma|^p + E|Y_i/\sigma|^p\right) \right)^{1/2}.$$

Cramér-type moderate deviations

Theorem (Liu and Zhang, 2023, AAP)

Under the local dependence conditions, assume that $\sigma = 1$, and

$$\mathbb{E}\{\exp(a_n \kappa |X_j|)\} \leqslant b.$$

holds. Then,

$$\left| \frac{\mathbb{P}[S \geqslant z]}{1 - \Phi(z)} - 1 \right| \leqslant C\delta_n \left(1 + z^3\right)$$

for $0 \leqslant z \leqslant c a_n^{1/3} \min\{1, \kappa^{-1/3}(1 + \theta_n)^{-2/3}\}$, where C and c are absolute constants and $\delta_n = \kappa^2 a_n^{-1} \left(1 + \theta_n^6\right)$ and $\theta_n = b^{1/2} n^{1/2} a_n^{-1}$.

Self-normalized sums for local dependence

- Consider local dependent random variables.
- Define

$$S = \sum_{i \in J} X_i, \quad V^2 = \left(\sum_{i \in J} (X_i Y_i - \bar{X}\bar{Y}) \right)_+, \quad W = S/V,$$

where $\bar{X}$ is the sample mean of X_i's, and $\bar{Y}$ is the sample mean of Y_i's.

- Berry–Esseen bound for W.

Some notation

- Let $\sigma^2 = Var(S)$, and let κ be any number such that

$$\kappa \geq \max_{i \in J} \{ |j : B_i \cap A_j \neq \emptyset|, |j : i \in B_j| \}.$$

- Let

$$\beta_0 = \sum_{i \in J} P(|X_i| > \sigma/\kappa),$$

$$\beta_1 = \frac{1}{\sigma^2} \sum_{i \in J} E\{ |X_i|^2 I(|X_i| > \sigma/\kappa)\},$$

$$\beta_2 = \frac{1}{\sigma^3} \sum_{i \in J} E\{ |X_i|^3 I(|X_i| \leq \sigma/\kappa)\},$$

$$\theta = \sum_{i \in J} \sum_{j \in A_i} E|X_i X_j I(|X_i| \leq \sigma/\kappa, |X_j| \leq \sigma/\kappa)|.$$

Berry–Esseen bound

- **Zhang** (2024, **SCM**): We have

$$d_{\mathrm{Kol}}(W, Z) \leqslant C\left\{(1+\theta)\kappa^2\beta_3 + \kappa\beta_2 + \beta_0\right\} + C\kappa^{1/2}(\theta + 1)\,|\,J\,|^{-1/2}.$$

Remarks

- We now give some remarks on θ and β_j's.
- By Chebyshev's inequality, we have

$$\beta_0 \le \kappa^2 \beta_2.$$

- If $E|X_i|^3 < \infty$, then

$$\kappa\beta_2 + \kappa^2\beta_3 \le \frac{\kappa^2}{\sigma^3}\sum_{i\in J} E|X_i|^3.$$

- For the Berry–Esseen buond: If $E|X_i|^3 < \infty$,

$$d_{\mathrm{Kol}}(W, Z) \le C\kappa^3 \sum_{i\in J} E|X_i|^3 + C\kappa^{1/2}|J|^{-1/2}.$$

Applications: m-dependence

- Classical Berry–Esseen bounds for normalized m-dependent random variables:
  - **Shergin (1979)**: Let $\{X_i\}_{1\le i\le n}$ be a sequence of m dependent random variables, let

$$S_n = \sum_{i=1}^{n} X_i, \text{ and } B_n^2 = Var(S_n).$$

  - Then,

$$d_{Kol}(S_n/B_n, Z) \le C(m+1)^2 B_n^{-3} \sum_{k=1}^{n} E|X_i|^3 .$$

- However, there are few results on their self-normalized sums.

Applications: m-dependence

- Let $\{X_i\}_{1\leq i\leq n}$ be an m-dependent field with $EX_i = 0$ and assume that $E|X_i|^3 < \infty$. Let $Y_i = \sum_{j\in A_i} X_i$. Assume that $\sigma^2 := \sum_{i\in\mathcal{J}} E\left\{X_iY_i\right\} > 0$. Let W be the self-normalized sum defined by $W = S/V$, where

$$V = \sqrt{\left(\sum_{i\in\mathcal{V}} \left(X_iY_i - \bar{X}\bar{Y}\right)\right)_+}.$$

- **Zhang (2024, SCM):** We have

$$d_{\mathrm{Kol}}(W, Z) \leqslant C(m+1)^3\left(1 + (m+1)|n^{1/3}\gamma^{2/3}\right)\left(\gamma + n^{-1/2}\right),$$

where $\gamma = \sum_{i\in\mathcal{J}} \mathbb{E}|X_i|^3/\sigma^3$.

Application: Graph dependency

- **Graph dependency**: Let $(X_n)_{n\in\mathscr{V}}$ be a random field index on a graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, so that for any disjoint subsets $\mathscr{V}_1$ and $\mathscr{V}_2$ with no edges between $\mathscr{V}_1$ and $\mathscr{V}_2$, then $(X_i)_{i\in\mathscr{V}_1}$ and $(X_j)_{j\in\mathscr{V}_2}$ are independent.
- **Normalized sum** S/σ, where $\sigma^2 = Var(S)$:
  - **Baldi and Rinott (1989) (also, Rinott, 1994)** proved an optimal Berry–Esseen bound for bounded random variables.
  - **Chen and Shao (2004)** proved a Berry–Esseen bound under some moment conditions.
- Our result can be applied to the self-normalized sum.

Application: Graph dependency

- **Zhang (2024, SCM)**: Let $\left\{X_i, i \in \mathscr{V}\right\}$ be a field of random variables indexed by the vertices of a dependency graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$. Assume that $EX_i = 0$. Let $S = \sum_{i \in \mathscr{V}} X_i$, $Y_i = \sum_{j \in A_i} X_j$ and let $V = \left(\sum_{i \in \mathscr{V}} (X_i Y_i - \bar{X}\bar{Y})\right)_+^{1/2}$. Put $W = S/V$ and $n = |\mathscr{V}|$. Let d be the maximal degree of $\mathscr{G}$.
- Then, we have

$$d_{Kol}(W, Z) \leqslant Cd^9 \left(1 + d^6 n^{1/3} \gamma^{2/3}\right) \left(n^{-1/2} + \gamma\right),$$

where $\gamma = \sum_{i \in \mathscr{V}} E|X_i/\sigma|^3$, and $\sigma^2 = \sum_{i \in \mathscr{V}} E\{X_i Y_i\}$.

Some remarks on the results

- We proved a Berry–Esseen bound for self-normalized sum for local dependent random variables.
- The convergence rate on the sample size n is optimal.
- However, the rate on the neighborhood size κ (or m in the m-dependence) is not optimal.
- For example,
  - Classical results by **Shergin (1979)**: $d_{Kol}(S_n/B_n, Z) \leq C\boxed{(m+1)^2} B_n^{-3} \sum_{k=1}^{n} E|X_i|^3$
  - Our result: $d_{Kol}(W, Z) \leqslant C\boxed{(m+1)^3}\left(1 + (m+1)\,|\,n^{1/3}\gamma^{2/3}\right)\left(\gamma + n^{-1/2}\right)$

κ cannot be too large

Recall that κ measures the size of the neighborhoods

$$\kappa \geq \max_{i \in J} \{ \, |j : B_i \cap A_j \neq \emptyset \, | \, , |j : i \in B_j | \, \}$$

and here is our result:

$$d_{\mathrm{Kol}}(W, Z) \leqslant C \left\{ (1+\theta)\kappa^2 \beta_3 + \kappa \beta_2 + \beta_0 \right\} + C\kappa^{1/2}(\theta + 1) \, | \, J \, |^{-1/2} .$$

Therefore, κ cannot be too large.

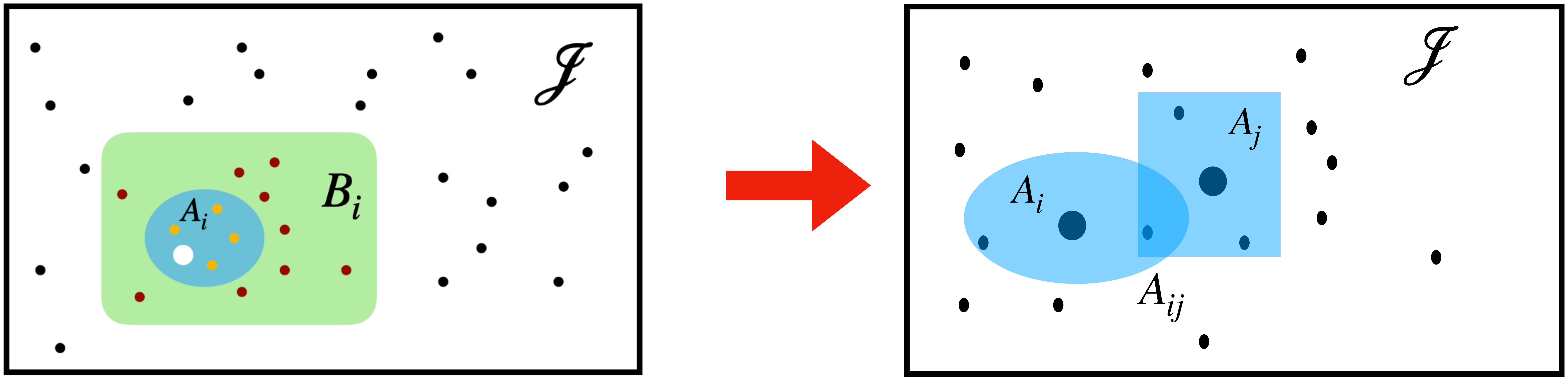
U-statistics

- U-statistics are commonly used in statistics.
- Definition

$$S = \frac{1}{\sigma_n} \sum_{i_1 < \ldots < i_k} h(X_{i_1}, \ldots, X_{i_k}).$$

- Then, for non-degenerate case,
  - $J = \{(i_1, \ldots, i_k) : i_1 < \ldots < i_k\} \implies |J| = O(n^k)$
  - $\sigma_n^2 = O(n^{2k-1})$
  - $A_i = O(n^{k-1}), B_i = J$, and hence $\kappa = |J|$.
  - Then, $d_K(S, Z) = O(1)$?

Another local dependence structure



Decomposable random variables

- Consider a random field $\{X_i, i \in J\}$,

  ① For each $i \in J$, there exists $A_i \subset J$ such that X_i is independent of $\{X_j, j \in A_i^c\}$.

  ② For each $i \in J$ and $j \in A_i$, there exists $A_{ij} \subset J$ such that $A_i \subset A_{ij}$ and $\{X_i, X_j\}$ is independent of $\{X_k, k \in A_{ij}^c\}$.

  ③ ...

- This structure was introduced by **Barbour et al. (1989).**

Existing results

- **Barbour, Karoiski and Rucisjski (1989)** proved a bound for the Wasserstein distance

$$d_1(W,Z) = \sup_{\|h'\|\le 1} |Eh(W) - Eh(Z)| \le \kappa^2 \sum_{i=1}^{n} E|X_i|^3 .$$

- **Fang (2019)** proved a Wasserstein-2 distance under a **three-level** dependence condition, where the Wasserstein-p distance is defined as

$$d_p(W,Z) := \left(\inf_{\pi\in\Gamma(W,Z)} \int |x-y|^p \, d\pi(x,y) \right)^{1/p}$$

$$\le C\kappa^{3/2} \left(\sum_{i=1}^{n} E|X_i|^4 \right)^{1/2}$$

Berry–Esseen bounds for decomposable r.v.'s

Theorem (Cai and Zhang, 2024+)

Assume that $\mathbb{E}\{X_i\} = 0$ for $i \in [n]$. Let $S = \sum_{i\in[n]} X_i$, and assume that its variance is 1. For any i, let $N_i = \left\{k \in [n]; i \in A_k\right\}$. Let κ and τ be such that

$$\kappa = \max\{\sup_{i,j} |A_{ij}|, \sup_i |N_i|\}, \quad \text{and} \quad \tau = \max_{k\in[n]} |\{(i,j) : j \in A_i, k \in A_{ij}\}|.$$

We have

$$d_{Kol}(S,Z) \le C\theta\kappa^2 \sum_{i\in[n]} \|X_i\|_4^3 + C\theta\kappa^{1/2}\left(\kappa + \tau^{1/2}\right)\left(\sum_{i\in[n]} \|X_i\|_4^4\right)^{1/2},$$

where $\|X\|_p = \left(E|X|^p\right)^{1/p}$ and where $\theta = \dfrac{\kappa}{\sigma^2}\sum_{i=1}^{n} \|X_i\|_4^2$.

Remarks

- The bound is sharper w.r.t. the neighborhood size κ.
- The Berry–Esseen bound is of the same order as the Wasserstein distance, with a cost of fourth moment condition.
- Consider U statistics,

$$S = \frac{1}{\sigma_n} \sum_{i_1 < \ldots < i_k} h(X_{i_1}, \ldots, X_{i_k})$$

- If $E|h(\mathbf{X})|^4 < \infty$, then our main results implies

$$d_{\mathrm{Kol}}(S, Z) \le Cn^{-1/2}.$$

Self-normalized sums of decomposable r.v.'s

Theorem (Cai and Zhang, 2024+)

Let S, κ, τ be defined as above. Let $W = S/V$ be the self-normalized sum, where

$$V = \sqrt{\left(\sum_{i\in[n]} \left(X_i Y_i - \bar{X}\bar{Y}\right) \right)_+}.$$

We have

$$d_{\mathrm{Kol}}(W, Z) \leq C\theta \left\{ \frac{\kappa^2}{\sigma^3} \sum_{i\in[n]} \|X_i\|_4^3 + \frac{\kappa^{1/2}\left(\kappa + \tau^{1/2}\right)}{\sigma^2} \left(\sum_{i\in[n]} \|X_i\|_4^4 \right)^{1/2} + \kappa^{1/2} n^{-1/2} \right\},$$

where $\theta = \dfrac{\kappa}{\sigma^2} \displaystyle\sum_{i=1}^{n} (E\,|\,X_i\,|^4)^{1/2}$.

Applications: Pattern matching

- Text:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
A	A	B	A	A	C	A	A	D	A	A	B	A	A	B	A	A	B

- Pattern: A A B A

- **Pattern found at 1, 10, and 13**

Pattern matching

- Text:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
A	A	B	A	A	C	A	A	D	A	A	B	A	A	B	A	A	B

- Pattern: A A *** B A

  there is a gap between the second and the third letter

- Pattern found at 10

Pattern matching

- Let $\Xi = \xi_1\xi_2\cdots\xi_n$ be a random string, consisting of n i.i.d. or m dependent random letters from a finite alphabet $\mathscr{A}$
- Let $\mathbf{w} = w_1\cdots w_l$ be a given word
- Let $d = (d_1, \ldots, d_{l-1})$ be the vector of gap lengths in the pattern
- An occurrence of w is an increasing sequence of indices $i_1 < \ldots < i_l$ in $[n] = \{1,\ldots,n\}$such that

$$(\xi_{i_1}, \ldots, \xi_{i_l}) = (w_1, \ldots, w_l)$$

Constrained U-statistics

- Let $X_1, X_2, \ldots$ be a sequence of random variables
- Let f be a measurable kernel function of l variables.
- Given a constraint $(d_1, \ldots, d_{l-1})$, we define the constrained U-statistic

$$U = \sum_{\substack{1 \le i_1 < \ldots < i_l \le n \\ i_{j+1} - i_j \le d_j}} f(X_{i_1}, \ldots, X_{i_l})$$

Central limit theorems

- **Hoeffding (1948):** CLT for classical U statistics
- **Janson (1991):** CLT for generalized U statistics
- **Shang (2012):** CLT for U statistics of m-dependent random variables
- **Bona (2007):** CLT for constraint U statistics with $d_j = \infty$
- **Hofer (2017):** CLT for general constraint U statistics
- We want to study the convergence rates for these normal approximations

Applications: Constrained U-stat

- Let $X_1, X_2, \ldots$ be a sequence of random variables
- Let f be a measurable kernel function of l variables.
- Given a constraint $(d_1, \ldots, d_{l-1})$, we define the constrained U-statistic

$$U = \sum_{\substack{1 \le i_1 < \ldots < i_l \le n \\ i_{j+1} - i_j \le d_j}} f(X_{i_1}, \ldots, X_{i_l})$$

- The normalized statistic is defined as

$$S = \frac{U - E(U)}{\sqrt{Var(U)}}$$

Existing results

- **Janson (2023):** Let $X_1, X_2, \ldots$ be a sequence of stationary m-dependent random variables. Let

$$b = 1 + |\{k \in [l-1] : d_k = \infty\}|,$$

if $\dfrac{Var(U_n)}{n^{2b-1}} \to \sigma^2 > 0$, and assume that $\|f\|_\infty \leq C$, then

$$\sup_{z \in \mathbb{R}} |P(S \leq z) - \Phi(z)| \leq Cn^{-1/2},$$

where C is a constant depending on m, l and f.

Our results

Cai and Zhang (2024+): Let S be defined as above, and assume that $E|f(X_{i_1},\ldots,X_{i_l})|^4 < \infty$ for all $i_1,\ldots,i_l$. Then

$$d_{Kol}(S,Z) \le Cn^{-1/2},$$

where C is a constant depending on m, l and f. Moreover, a same bound also holds for the self-normalized sum.

Stein's method

- **Stein (1972)**:

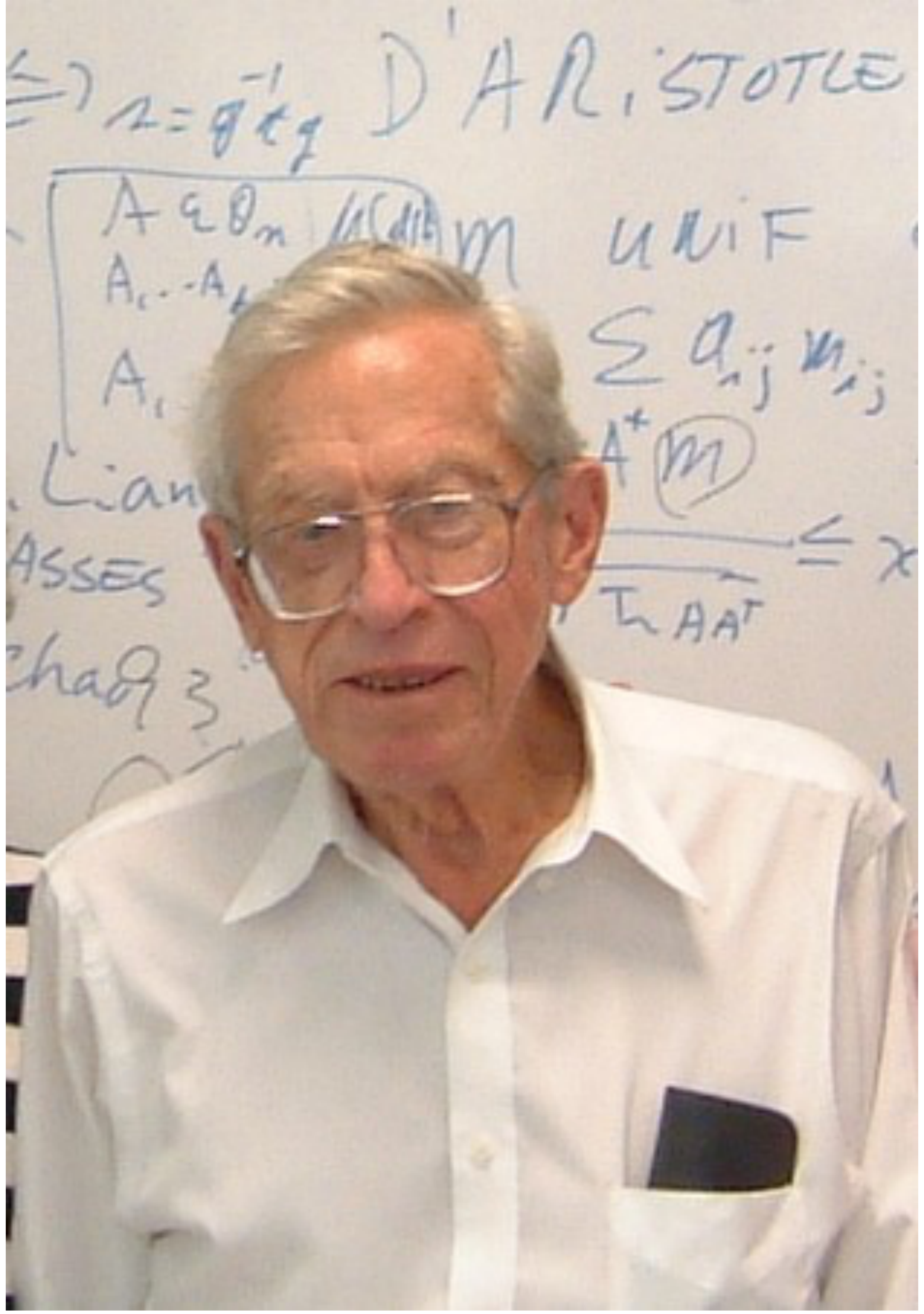
$$W \sim N(0,1) \text{ iff } \mathbb{E} f'(W) = \mathbb{E}\, Wf(W)$$

- Stein equation:

$$f'(w) - wf(w) = h(w) - Eh(Z)$$

- Let f_h be its solution, then

$$Eh(W) - Eh(Z) = \mathbb{E}[f_h'(W)] - \mathbb{E}[Wf_h(W)]$$



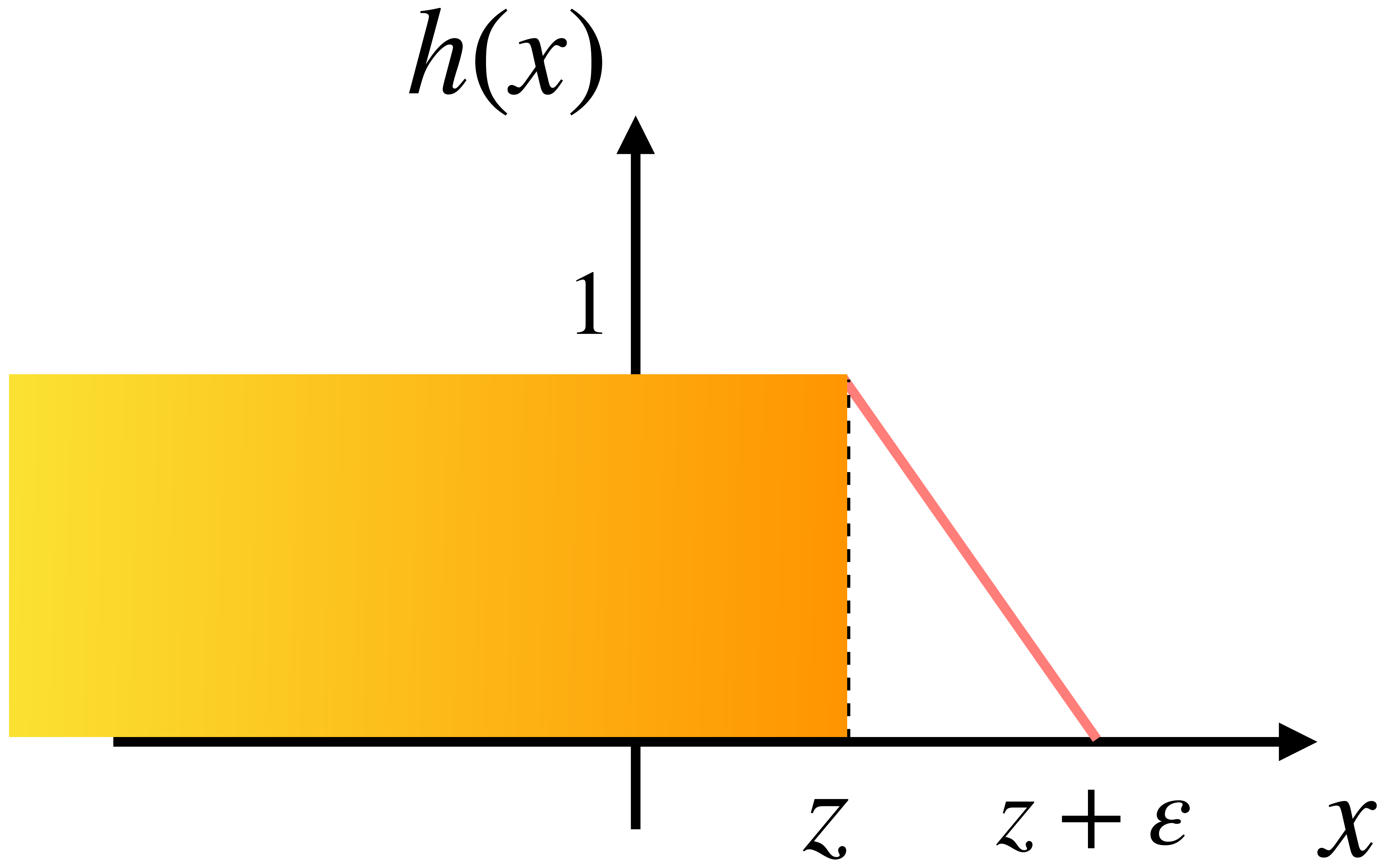
A smoothing function

- Consider

$$h_{z,\varepsilon}(w) = \begin{cases} 1 & \text{if } w \le z \\ 1 + \varepsilon^{-1}(z - w) & \text{if } z < w \le z + \varepsilon \\ 0 & \text{if } w > z + \varepsilon \end{cases}$$

Then,

$$\mathbf{1}_{\{w \le z\}} \le h_{z,\varepsilon}(w) \le \mathbf{1}_{\{w \le z+\varepsilon\}}.$$



A direct bound

We have

$$
\begin{aligned}
|\mathbb{P}(S \le z) - \Phi(z)| &\le 0.4\varepsilon + \left|\mathbb{E}h_{z,\varepsilon}(S) - \mathbb{E}h_{z,\varepsilon}(Z)\right| \\
&\le 0.4\varepsilon + \mathbb{E}\left|\sum_i X_iY_i - E(X_iY_i)\right| + \frac{1}{2}\sum_{i\in[n]}\mathbb{E}\left|X_iY_i^2(1+S)\right| \\
&\quad + \frac{1}{\varepsilon}\sum_{i\in[n]}\left\{\int_{-\infty}^{+\infty}\int_{t\wedge 0}^{t\vee 0} E\{\mathbf{1}(z \le S_i + u \le z+\varepsilon)\cdot|\hat{K}_i(t)|\}\mathrm{d}u\ \mathrm{d}t\right\}.
\end{aligned}
$$

where $\hat{K}_i(t) = X_i(1_{0\le t\le Y_i} - 1_{Y_i<t\le 0})$, and $S_i = S - Y_i$ is independent of X_i.

Concentration inequalities

Let $A \subset [n]$ and $B \subset [n]$ be two arbitrary index sets. Let $S_A = S - \sum_{j \in N_A^c} X_j$, and let

$\xi_A = h\left(X_A\right) \geqslant 0, \eta_B = a - c \sum_{m \in B} \left|X_m\right|$ and $\zeta_B = b + c \sum_{m \in B} \left|X_m\right|$, where $a, b, c \in \mathbb{R}$ satisfying

that $a \leq b$ and $c \geqslant 1$. Then,

$$\mathbb{E}\left\{\xi_A \mathbf{1}\left(\eta_B \leq S_A \leq \zeta_B\right)\right\} \leq 846 \, \| \xi_A \|_{4/3} \cdot \{0.005(b-a) + \delta_1 + \delta_2 + \delta_3 + \delta_4\}$$

where

$$\delta_1 = c \sum_{m \in B} \| X_m \|_4, \quad \delta_2 = c \sum_{m \in B} \sum_{k \in N_m} \| X_k \|_4 \| X_m \|_4,$$

$$\delta_3 = c\kappa^2 |A|^2 \sum_{i=1}^{n} \| X_i \|_4^3, \quad \delta_4 = c\kappa^{1/2}\left(\kappa + \tau^{1/2}\right) |A|^{1/2} \left(\sum_{i=1}^{n} \| X_i \|_4^4\right)^{1/2}.$$

Some take-home messages

- Berry–Esseen bounds and Cramér-type moderate deviations are important in statistics.
- Optimal convergence rates for local dependent random variables are established.
- Our results are based on Stein's method, concentration inequality approach and a recursive argument.

Selected References

① **Cai and Zhang (2024+).** Berry–Esseen bounds for decomposable random variables. *Working paper.*

② **Chen and Shao (2004).** Normal approximation under local dependence. *Annals of Probability.*

③ **Liu and Zhang (2023).** Cramér-type moderate deviations under local dependence. *Annals of Applied Probability.*

④ **Zhang (2024).** Berry–Esseen bounds for self-normalized sums of local dependent random variables. *To appear in Science China Mathematics*

Thank you for your attention!