2022 - B1 - xyt助教模拟卷

4:37 ~6:02.

1. $-2x+y-x^2+y^2+z+\sin z=0 \quad (0,0,0) \Rightarrow z=f(x,y)$.

(1) 令 $F(x,y,z)=-2x+y-x^2+y^2+z+\sin z$.

$\therefore F(0,0,0)=0$

还需检查在(0,0,0)附近是否有定义

$$\left.\begin{aligned} F_x&=-2-2x\\ F_y&=1+2y\\ F_z&=1+\cos z\end{aligned}\right\}\text{易知均连续.}$$

$$\Rightarrow \begin{cases} F_x(0,0,0)=-2\\ F_y(0,0,0)=1\\ F_z(0,0,0)=2\neq 0\end{cases}$$

由隐函数存在定理

$z=f(x,y)$ 可唯一确定

(2) $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}=1$

$\dfrac{\partial z}{\partial y}=-\dfrac{F_y}{F_z}=-\dfrac{1}{2}$.

$\therefore f(x,y)=f(0,0)+\dfrac{\partial z}{\partial x}\cdot\Delta x+\dfrac{\partial z}{\partial y}\cdot\Delta y+\cancel{o(\rho^2)}\ o(\rho)$ 泰勒展开此项同前幂次

$=x-\dfrac{1}{2}y+\cancel{o(\rho^2)}\ o(\rho)$

其中，$\rho=\sqrt{x^2+y^2}$.

2. (1) $\displaystyle\lim_{x\to0}\frac{\frac{x^2}{2}+1-\sqrt{1+x^2}}{\sin(x^2)(\cos x-e^{x^2})}$

$\displaystyle=\lim_{x\to0}\frac{\frac{x^2}{2}+1-(1+\frac{1}{2}x^2-\frac{1}{8}x^4)}{x^2(1-\frac{1}{2}x^2-(1+x^2))}$

$\displaystyle=\lim_{x\to0}\frac{\frac{1}{8}x^4}{x^2(-\frac{3}{2}x^2)}=\frac{1}{8}\cdot(-\frac{2}{3})=-\frac{1}{12}$.

$(1+x^2)^{\frac{1}{2}}=1+\frac{1}{2}x^2+\frac{1}{2}\cdot\frac{1}{2}\cdot(-\frac{1}{2})(x^2)^2$

$=1+\frac{1}{2}x^2-\frac{1}{8}x^4$

(2) $\displaystyle\lim_{x\to0}\left(\frac{1+\int_0^x e^{t^2}dt}{e^x-1}-\frac{1}{\sin x}\right)$

$\displaystyle=\lim_{x\to0}\left(\frac{1+e^{(\theta x)^2}\cdot x}{e^x-1}-\frac{1}{\sin x}\right)$. $\theta\in(0,1)$

$e^{(\theta x)^2}=1+\theta^2x^2+o(x^4)$.

原式 $\displaystyle=\lim_{x\to0}\left(\frac{1+(\theta^2x^2+1)\cdot x}{e^x-1}-\frac{1}{\sin x}\right)$

$\displaystyle=\lim_{x\to0}\frac{(1+x)\sin x-e^x+1}{(e^x-1)\sin x}=\lim_{x\to0}\frac{(1+x)(x-\frac{1}{6}x^3)-(x+\frac{1}{2}x^2)}{x^2}$

$\displaystyle=\lim_{x\to0}\frac{\frac{1}{2}x^2}{x^2}=\frac{1}{2}$.

$\frac{1}{0}-\frac{1}{0}$

$\displaystyle\lim_{x\to0}\left(\frac{1+\int_0^x e^{t^2}dt}{e^x-1}-\frac{1}{\sin x}\right)$

$\displaystyle=\lim_{x\to0}\frac{\sin x(1+\int_0^x e^{t^2}dt)-e^x+1}{\sin x(e^x-1)}$

$\displaystyle=\lim_{x\to0}\frac{\sin x+\sin x\int_0^x e^{t^2}dt-e^x+1}{x^2}\quad\left(\frac{0+0\cdot0-1+1}{0}\right)$

$\overset{洛}{=}\displaystyle\lim_{x\to0}\frac{\cos x+\cos x\int_0^x e^{t^2}dt+e^{x^2}\sin x-e^x}{2x}$

$\left(\dfrac{1+1\cdot0+1\cdot0-1}{0}\right)$

$\overset{洛}{=}\displaystyle\lim_{x\to0}\frac{-\sin x-\sin x\int_0^x e^{t^2}dt+\cos x\cdot e^{x^2}+e^{x^2}\cdot2x\cdot\sin x+\cos x e^{x^2}-e^x}{2}$

$\left(\dfrac{0-0\cdot0+1\cdot1+1\cdot0\cdot0+1\cdot1-1}{2}\right)$

$=\dfrac{1}{2}$

3. (1) $L:\begin{cases}x+y+z-3=0\\ x-2y-z+2=0.\end{cases}$

$y+z=3$
$2y+z=2$. $y=-1$

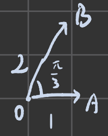
$(1,1,1)\times(1,-2,-1)=\begin{vmatrix} i & j & k\\ 1 & 1 & 1\\ 1 & -2 & -1\end{vmatrix}=(1,2,-3)$

又显然 $P_0(0,-1,4)$ 在 L 上

$\therefore L: \dfrac{x-0}{1} = \dfrac{y+1}{2} = \dfrac{z-4}{-3}$

$\therefore$ 平面: $(x-1)+2(y-2)-3(z-3)=0$

即 $x+2y-3z+4=0$

(2) 

题目订正: $\vec{OP}=(1-\lambda)\vec{OA}, \vec{OQ}=\lambda\vec{OB}$

令 $A(1,0)$ $B(1,\sqrt{3})$.

$\therefore \vec{OP}=(1-\lambda,0)$ $\lambda\in[0,1]$

$\vec{OQ}=(\lambda,\sqrt{3}\lambda)$

$\therefore \vec{PQ}=\vec{OQ}-\vec{OP}$

$=(2\lambda-1,\sqrt{3}\lambda)$

$\therefore |\vec{PQ}|=\sqrt{4\lambda^2-4\lambda+1+3\lambda^2}$

$=\sqrt{7\lambda^2-4\lambda+1}$

令 $f(\lambda)=7\lambda^2-4\lambda+1$

$f'(\lambda)=14\lambda-4$

令 $f'(\lambda)=0 \Rightarrow \lambda=\frac{4}{14}=\frac{2}{7}$.

二次函数知, $\lambda=\frac{2}{7}$时, $f(\lambda)$取最小.

此时, $|\vec{PQ}|=\sqrt{f(\lambda)}=\sqrt{7\cdot\frac{4}{49}-\frac{8}{7}+1}$

$=\sqrt{\dfrac{28-56+49}{49}}=\sqrt{\dfrac{21}{49}}=\dfrac{\sqrt{21}}{7}$

4. $f(x,y)=\begin{cases}\dfrac{y^2}{x^2+y^2}, & y\neq 0\\ 1, & y=0\end{cases}$

(1) 在 $(0,0)$ 两偏导.

$f_x=\lim\limits_{\Delta x\to 0}\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x}=0$

$f_y=\lim\limits_{\Delta y\to 0}\dfrac{f(0,\Delta y)-f(0,0)}{\Delta y}=\dfrac{1-1}{\Delta y}=0$

$f_x(0,0)$、$f_y(0,0)$ 均存在且等于0.

(2) 若全微分存在.

则 $\Delta f-df=o(\rho^2)$

$\Rightarrow f(\Delta x,\Delta y)-f(0,0)-f_x(0,0)\cdot\Delta x-f_y(0,0)\cdot\Delta y=o(\rho^2)$.

$\Rightarrow f(\Delta x,\Delta y)-f(0,0)=o(\rho^2)$

$\dfrac{y^2}{x^2+y^2}$

考虑. $\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)-1}{x^2+y^2}$

当 (x,y) 沿 $y=0$ 趋近时, 原式 $=\lim\limits_{x\to 0}\dfrac{1-1}{x}=0$.

当 (x,y) 沿 $x=ky$ 趋近时, 原式 $=\lim\limits_{x\to 0}\dfrac{\frac{y^2}{(k^2+1)y^2}-1}{(k^2+1)y^2}=\lim\limits_{y\to 0}\dfrac{\frac{1}{k^2+1}-1}{(k^2+1)y^2}$ 发散.

由此，知 $f(x,y)$ 在 $(0,0)$ 附近不可微.

5. $f(x,y)=2x^3-3x^2-6xy(x-y-1)$ 在 R^2 所有极值点.

易知 $f(x,y)$ 在 R^2 上连续

又 $f_x=6x^2-6x-6y(x-y-1)-6xy\cdot 1$

$=6x^2-6x-6xy+6y^2+6y-6xy$

$=6(x^2-x-2xy+y+y^2)$

$f_y=-6x(x-y-1)-6xy(-1)$

$=-6x^2+6xy+6x+6xy$

$=6(-x^2+2xy+x)$

令 $f_x=f_y=0 \Rightarrow \begin{cases} -x^2+2xy+x=0 \\ x^2-x-2xy+y+y^2=0 \end{cases} \Rightarrow \begin{cases} x=0 \text{ 或 } -x+2y+1=0 \\ y=0 \text{ 或 } -1 \end{cases}$

考虑点 $(0,0)$ $(0,-1)$ $(1,0)$ $(-1,-1)$

$f_x=6(x^2-x-2xy+y+y^2)$

$f_y=6(-x^2+2xy+x)$

$\therefore f_{xx}=6(2x-1-2y) \quad =A$

$f_{yx}=f_{xy}=6(-2x+2y+1) \quad =B$

$f_{yy}=6(2x) \quad =C$

判别式 $B^2-AC=36[(-2x+2y+1)^2-2x(2x-1-2y)]$

$=36[4x^2+4y^2+1-8xy+4y-4x-4x^2+2x+4xy]$

$=36[4y^2+4y-2x+1-4xy]$

$(0,0)$: $B^2-AC>0 \Rightarrow$ 不是极值点

$4-4+2+1-4$

$(0,-1)$: $B^2-AC>0 \Rightarrow$ 不是极值点

$(1,0)$: $B^2-AC<0$，$A>0 \Rightarrow$ 是极小值点

$(-1,-1)$: $B^2-AC<0$，$A<0 \Rightarrow$ 是极大值点.

综上，$f(x,y)$ 的极小点为 $(1,0)$，极大值为 $(-1,-1)$.

6. $a>e \quad 0<x<y<\frac{\pi}{2} \qquad a^y-a^x > a^x \ln a(\cos x-\cos y)$.

柯西中值: $\frac{a^y-a^x}{\cos x-\cos y} = \frac{a^\xi \ln a}{\sin\xi} > a^\xi \ln a > a^x \ln a$.

证: 原式 $\Leftrightarrow \quad a^{y-x}-1 > \ln a(\cos x-\cos y)$

$a^{y-x}-1 > (y-x)\ln a$ （考虑函数 $f(x)=a^x-1-x\ln a$

$f'(0) = \ln a - \ln a = 0$

$f''(0) = a^x \ln^2 a - \ln a$

$= \ln a(\ln a-1) > 0$

$\Rightarrow f(x)$ 在 $(0,+\infty)$ 上单增

$\Rightarrow f(x) > f(0)=0. \quad (x>0)$.

$y-x > \cos x-\cos y$

$\Leftrightarrow y-x > \cos\left(\frac{x+y}{2}+\frac{x-y}{2}\right)-\cos\left(\frac{x+y}{2}-\frac{x-y}{2}\right)$

$\Leftrightarrow y-x > -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

$\Leftrightarrow y-x > 2\sin\frac{y-x}{2}\sin\frac{x+y}{2}$

$\because \quad y-x > 2\sin\frac{y-x}{2}$ （考虑函数 $g(t)=t-2\sin\frac{t}{2}$

$g'(t) = 1-2\cos t\cdot\frac{1}{2} = 1-\cos t \geq 0$

$\Rightarrow g(t)$ 在 $(0,+\infty)$ 上单增

$\Rightarrow g(t) > g(0)=0$.

$\therefore \quad y-x > 2\sin\frac{y-x}{2}\sin\frac{x+y}{2} \quad (x+y<\pi)$

$\Leftrightarrow \quad a^{y-x}-1 > (y-x)\ln a > (\cos x-\cos y)\ln a$

$\Leftrightarrow$ 原命题成立.

7. $f(x)=x\sin(x^2-2x) \qquad x=1$ 局部泰勒公式.

$f^{(n)}(x) = n\cdot\sin^{(n-1)}(x^2-2x)+x\cdot\sin^{(n)}(x^2-2x)$

$\sin^{(n)}(x^2-2x) = \sin^{(n-1)}(x^2-2x+\frac{\pi}{2})\cdot(2x-2)$.

$= \cos^{(n-1)}(x^2-2x)\cdot(2x-2)$

$f^{(n)}(1)$.

$\because x\to1 \quad \therefore x^2-2x+1\to0$

$\because \sin(t) = \sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}x^{2k+1}+o(x^{2n+2})$

$\therefore \sin(x^2-2x) = \sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}(x^2-2x)^{2k+1}+o\left((x^2-2x)^{2k+2}\right)$

$= \sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}x^{2k+1}\cdot(x-2)^{2k+1}+o\left((x^2-2x)^{2k+2}\right)$

$\therefore x\sin(x^2-2x) = \sum$

令 $y=x-1$

$f(x)=g(y)=(y+1)\sin(y^2-1)$

$=(y+1)\left[\sin y^2\cos 1-\cos y^2\sin 1\right]$

$=(y+1)\left[\cos 1\cdot\left(\sum_{k=1}^{n}\frac{(-1)^{k-1}}{(2k-1)!}(y^2)^{2k-1}+o(y^{4n})\right)-\sin 1\cdot\left(1+\sum_{k=1}^{n}\frac{(-1)^k}{(2k)!}(y^2)^{2k}+o(y^{4n+2})\right)\right]$

$=(y+1)\left\{-\sin 1+\sum_{k=1}^{n}\left[\frac{(-1)^{k-1}}{(2k-1)!}y^{4k-2}\left(\cos 1+\frac{1}{2k}y^2\cdot\sin 1\right)\right]+o(y^{4n})\right\}$

$=-\sin 1\cdot x+\sum_{k=1}^{n}\left[\frac{(-1)^{k-1}}{(2k-1)!}\left[\cos 1\cdot\left[(x-1)^{4k-2}+(x-1)^{4k-1}\right]+\frac{\sin 1}{2k}\left[(x-1)^{4k}+(x-1)^{4k+1}\right]\right]\right]+o\left((x-1)^{4n+1}\right)$

8. 记 $\eta=\frac{P+Q}{2}$，将 $f(x)$ 在 η 处展开

$$f(x)=f(\eta)+f'(\eta)\cdot(x-\eta)+\frac{1}{2}f''(\xi)(x-\eta)^2$$

ξ 介于 η、x 之间

记 $\varphi(t)=f(\eta+t)-f(\eta-t)$

$$\therefore \varphi(t)=2f(\eta)+\frac{1}{2}f''(\lambda_1)t^2+\frac{1}{2}f''(\lambda_2)t^2 \quad \left(\eta-t<\lambda_1<\eta<\lambda_2<\eta+t,\ t<\frac{P-Q}{2}\right).$$

$\eta-t$ η $\eta+t$

$$S=\frac{1}{2}t\cdot\left|-f'(\eta)t+\frac{1}{2}f''(\lambda_1)t^2\right|+\frac{1}{2}t\left|f'(\eta)t+\frac{1}{2}f''(\lambda_2)t^2\right|$$

$$\geq \frac{1}{2}t\cdot\left(\frac{1}{2}f''(\lambda_1)t^2-|f'(\eta)t|\right)+\frac{1}{2}t\left(\frac{1}{2}f''(\lambda_2)t^2-|f'(\eta)t|\right)$$

$$=\frac{1}{4}t^3\left[f''(\lambda_1)+f''(\lambda_2)\right]-t^2f'(\eta)$$

$\frac{1}{2}(x-x_0)^2$

$f'(\eta)=$

$f''(x)\geq 1$

$f'(x)\geq f'(x_0)+\ \ (x-x_0)$

$f(x)\geq f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2}(x-x_0)^2$

固定 a，c，将 S 看作 b 的函数

$S(a)=S(c)=0 \Rightarrow$ 最大值内点.

$\Rightarrow S'(b)=0$

$\Rightarrow S''(b)\leq$

$\Rightarrow$ 泰勒.