2022-B1-期末

01:57 ~ 3:07

1. (1) $\lim\limits_{x\to 0}\dfrac{2\cos x-2+x^2}{x^4}$

$= \lim\limits_{x\to 0}\dfrac{-2(1-\cos x)+x^2}{x^4}$

$= \lim\limits_{x\to 0}\dfrac{-2\cdot(\frac{1}{2}x^2-\frac{1}{24}x^4)+x^2}{x^4}$

$= \lim\limits_{x\to 0}\dfrac{\frac{1}{12}x^4}{x^4} = \frac{1}{12}$.

(2) $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^5y^3}{x^8+y^8}$

令 (x,y) 沿 $y=kx$ 趋近 $(0,0)$.

原式 $= \lim\limits_{x\to 0}\dfrac{k^3x^8}{x^8+k^8x^8} = \dfrac{k^3}{1+k^8}$ 与 k 有关.

k 取不同值时，极限不一致，由此知全面极限不存在.

(3) $\lim\limits_{(x,y)\to(0,0)}(x+\sin y)\cos\dfrac{1}{|x|+|y|}$

$\because 0 \le \left|(x+\sin y)\cos\dfrac{1}{|x|+|y|}\right| \le |x+\sin y|$

令 $(x,y)\to(0,0)$，则 $\lim\limits_{(x,y)\to(0,0)}|x+\sin y|=0$

由夹逼定理知原式极限存在且等于0

2. $[-1,1]$ $\quad f(x)=x^{\frac{2}{3}}-(x^2-1)^{\frac{1}{3}}$. $\quad$ 最小值.

易知 $f(x)$ 在 $[-1,1]$ 上连续，在 $(-1,1)\setminus\{0\}$ 上可导.

又 $f(-1) = 1-0=1$ $\quad\quad f(0)=1$

$f(1) = 1-0=1$ $\quad\quad \sqrt[3]{\frac{1}{2}}-\left(-\frac{1}{2}\right)^{\frac{1}{3}}$

$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} - \frac{1}{3}(x^2-1)^{-\frac{2}{3}}\cdot 2x$

令 $f'(x)=0$

$\Rightarrow \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3}\cdot x\cdot(x^2-1)^{-\frac{2}{3}}$

$\Rightarrow x^{-\frac{1}{3}} = x\cdot(x^2-1)^{-\frac{2}{3}}$

$\Rightarrow x^{-1} = x^3\cdot(x^2-1)^{-2}$

$\because x\neq 0,\ x\neq\pm 1$

则 $x^4=(x^2-1)^2$

$x^2=\pm(x^2-1)$

$\Rightarrow 2x^2=1 \Rightarrow x=\pm\frac{\sqrt{2}}{2}$

又 $f(\frac{\sqrt{2}}{2}) = f(-\frac{\sqrt{2}}{2}) = 2\sqrt[3]{\frac{\sqrt{2}}{2}} > 1$

知 $f(x)$ 最小值为1，在 $-1, 0, 1$ 取得

$$\left(\begin{array}{l}\lim\limits_{x\to 1-0} f(x) = -\infty \\ \lim\limits_{x\to -1+0} f(x) = +\infty\end{array}\right)$$

3. (1) $a,b\in\mathbb{R}$, $b\neq 0$, $f(x,y)=\arctan\frac{x}{y}$ 在(a,b)处泰勒多项式

$$f_x=\frac{1}{(\frac{x}{y})^2+1}\cdot\frac{1}{y}=\frac{y}{x^2+y^2}$$

$$f_y=\frac{1}{(\frac{x}{y})^2+1}\cdot\left(-\frac{x}{y^2}\right)=\frac{-x}{(x^2+y^2)}$$

$$f_{xx}=\frac{-y\cdot 2x}{(x^2+y^2)^2}=\frac{-2xy}{(x^2+y^2)^2}$$

$$f_{xy}=\frac{(x^2+y^2)-y\cdot 2y}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}=f_{yx}$$

$$f_{yy}=\frac{x\cdot 2y}{(x^2+y^2)^2}=\frac{2xy}{(x^2+y^2)^2}$$

此处参考答案没有系数

$\therefore f(x,y)$ 在(a,b)处泰勒多项式

$$=\arctan\frac{a}{b}+\frac{1}{a^2+b^2}\big(b\cdot(x-a)-a\cdot(y-b)\big)+\frac{1}{(a^2+b^2)^2}\left(-ab(x-a)^2+(a^2-b^2)(x-a)(y-b)+ab(y-b)^2\right)$$

(2) $a<b$. $n\in\mathbb{N}^+$, $f:(a,b)\to\mathbb{R}$ $n+1$阶可导. $T:(a,b)\times(a,b)\to\mathbb{R}$

$$T(x,y)=f(x)-f(y)-\sum_{k=1}^{n}\frac{f^{(k)}(y)}{k!}(x-y)^k.$$

$$\frac{\partial T(x,y)}{\partial y}=-f'(y)-\left(\sum_{k=1}^{n}\frac{f^{(k)}(y)}{k!}(x-y)^k\right)'_y$$

$$=-f'(y)-\sum_{k=1}^{n}\left[\frac{f^{(k+1)}(y)}{k!}\cdot(x-y)^k+\frac{f^{(k)}(y)}{k!}\cdot k\cdot(x-y)^{k-1}\cdot(-1)\right]$$

$$=-f'(y)-\sum_{k=1}^{n}\frac{f^{(k+1)}(y)}{k!}(x-y)^k+\sum_{k=0}^{n-1}\frac{f^{(k+1)}(y)}{k!}(x-y)^k$$

$$=-f'(y)-\frac{f^{(n+1)}(y)}{n!}(x-y)^n+f'(y)$$

$$=-\frac{f^{(n+1)}(y)}{n!}(x-y)^n.$$

4. $\forall p\in\mathbb{R}$, $\exists$ 开邻域 U,W, $y=f(x)$. $x\in U$, $f(x)\in W$. $x^p+y^p-2xy=0$.

记 $F(x,y)=x^p+y^p-2xy$, 显然 $F(1,1)=0$

$F_x=px^{p-1}-2y$

$F_y=py^{p-1}-2x$

若 $F_y(1,1)=0 \Rightarrow p-2=0\Rightarrow p=0.$

① $p=2$

此时 $F(x,y)=x^2+y^2-2xy=(x-y)^2$

知 $U,W=\mathbb{R}$, $y=f(x)=x$ 满足题意.

② $p\neq 2$

此时,由隐函数存在定理, F_x,F_y在$(1,1)$处连续,且 $F_y(1,1)\neq 0$

知存在隐函数 $y=f(x)$, s.t. $F(x,f(x))\equiv 0$, 且 $f'(x)=\frac{-F_x}{F_y}$, 满足题意.

5. $\mathbb{R}^3$ $V(x,y,z)=\left(\frac{2y}{z}\right)^x$. $(1,\frac{1}{2},1)$

$\therefore \ln V = x\ln\left(\frac{2y}{z}\right)$

$\frac{1}{V}V_x = \ln\left(\frac{2y}{z}\right)$

$\Rightarrow V_x = V\cdot\ln\left(\frac{2y}{z}\right)$

$\therefore V_x(1,\frac{1}{2},1) = 1\cdot\ln 1 = 0$

$V_y(1,\frac{1}{2},1) = (2y)' = 2$

$V_z(1,\frac{1}{2},1) = \left(\frac{1}{z}\right)' = -\frac{1}{z^2} = -1$

$\therefore \operatorname{grad}V = (0,2,-1)$

$-\operatorname{grad}V = (0,-2,1)$

$\therefore$ 单位向量为 $(0,-\frac{2}{\sqrt5},\frac{1}{\sqrt5})$.

6. $K: x+2y+3z=6$. $d_{HK}=1$. $H\to K=M.\in\Delta ABC$.

(1) $A(6,0,0)$

$B(0,3,0)$

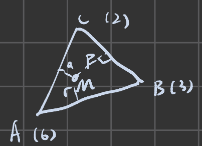
$C(0,0,2)$

$\vec{AB}=(-6,3,0)$

$\vec{AC}=(-6,0,2)$

$$S_{\Delta ABC}=\frac{1}{2}\left|\vec{AB}\times\vec{AC}\right|=\frac{1}{2}\left\|\begin{vmatrix} i & j & k\\ -6 & 3 & 0\\ -6 & 0 & 2\end{vmatrix}\right\|=\frac{1}{2}\cdot\left|(6,12,18)\right|=\frac{1}{2}\cdot 6\cdot\left|(1,2,3)\right|=3\sqrt{14}$$

(2)



(好像 p, q, r 顺序有些问题，但懒得改了).

$|AB| = |(-6,3,0)| = 3\sqrt5$

$|BC| = |(0,-3,2)| = \sqrt{13}$

$|CA| = |(6,0,-2)| = 2\sqrt{10}$

$\therefore S(p,q,r) = 3\sqrt{14}+\frac{3\sqrt5}{2}\cdot\sqrt{p^2+1}+\frac{\sqrt{13}}{2}\sqrt{q^2+1}+\sqrt{10}\cdot\sqrt{r^2+1}$.

(3). $\begin{cases} p>0\\ q>0\\ r>0\\ 3\sqrt5 p+\sqrt{13}q+2\sqrt{10}r=6\sqrt{14}\end{cases}$ (考虑 $S_{\Delta ABC}$).

(4) 记 $\varphi(p,q,r)=3\sqrt5p+\sqrt{13}q+2\sqrt{10}r-6\sqrt{14}=0$

$\therefore F(p,q,r)=S(p,q,r)-\lambda\varphi(p,q,r)$.

$\because S(p,q,r)$ 为条件极值

$\therefore S(p,q,r) = 3\sqrt{14}+\frac{3\sqrt5}{2}\cdot\sqrt{p^2+1}+\frac{\sqrt{13}}{2}\sqrt{q^2+1}+\sqrt{10}\cdot\sqrt{r^2+1}$.

$$\therefore \begin{cases} F_p = \dfrac{3\sqrt{5}}{2} \cdot \dfrac{p}{\sqrt{p^2+1}} - 3\sqrt{5}\lambda = 0 \\ F_q = \dfrac{\sqrt{13}}{2} \cdot \dfrac{q}{\sqrt{q^2+1}} - \sqrt{13}\lambda = 0 \\ F_r = \sqrt{10} \cdot \dfrac{r}{\sqrt{r^2+1}} - 2\sqrt{10}\lambda = 0 \\ F_\lambda = 3\sqrt{5}p + \sqrt{13}q + 2\sqrt{10}r - 6\sqrt{14} = 0 \end{cases}$$

$$\Rightarrow \frac{p}{\sqrt{p^2+1}} = \frac{q}{\sqrt{q^2+1}} = \frac{r}{\sqrt{r^2+1}} = 2\lambda, \quad \lambda > 0$$

考虑函数 $f(x) = \dfrac{x}{\sqrt{x^2+1}} - 2\lambda$

$$f(x) = 0$$

$$\Leftrightarrow x^2 = 4\lambda^2(x^2+1)$$

$$\Leftrightarrow (1-4\lambda^2)x^2 = 4\lambda^2$$

$$\Leftrightarrow x = \frac{2\lambda}{\sqrt{1-4\lambda^2}}$$ 唯一解

$$\Rightarrow p = q = r$$

$$\Rightarrow p = q = r = \frac{6\sqrt{14}}{3\sqrt{5}+\sqrt{13}+2\sqrt{10}} = \frac{6\sqrt{14}}{k}$$

$$\Rightarrow S(p,q,r) = 3\sqrt{14} + \frac{1}{2}k \cdot \sqrt{p^2+1}$$

$$= 3\sqrt{14} + \frac{1}{2}k \cdot \sqrt{\frac{504}{k^2}+1}$$

$$= 3\sqrt{14} + \frac{1}{2}\sqrt{504+k^2}$$

其中 $k = 3\sqrt{5}+\sqrt{13}+2\sqrt{10}$.

$$k^2 = 45+13+40+6\sqrt{65}+4\sqrt{130}+60\sqrt{2}.$$

$$= 88+6\sqrt{65}+4\sqrt{130}+60\sqrt{2}$$

$$\Rightarrow S(p,q,r) = 3\sqrt{14} + \frac{1}{2}\sqrt{592+6\sqrt{65}+4\sqrt{130}+60\sqrt{2}}.$$

稳定点：$\left(\dfrac{6\sqrt{14}}{3\sqrt{5}+\sqrt{13}+2\sqrt{10}}, \dfrac{6\sqrt{14}}{3\sqrt{5}+\sqrt{13}+2\sqrt{10}}, \dfrac{6\sqrt{14}}{3\sqrt{5}+\sqrt{13}+2\sqrt{10}}\right).$