2021 · B1   22:32 ~24:00

1. $\forall x\in\mathbb{R},\ \exists\,\theta(x)\in(0,1)$. s.t. $\arctan x=\dfrac{x}{1+(\theta(x))^2x^2}$

证:

$\arctan x$ 在 $(0,+\infty)$ 上可导

$\therefore \arctan x-\arctan 0=\arctan'(\xi)\cdot(x-0)$

$\therefore \arctan x=\dfrac{x}{1+\xi^2}$

令 $\theta(x)=\dfrac{\xi}{x}\in(0,1)$. 即符合条件.

特别地，当 $x=0$ 时，$\theta(x)$ 可任取 $\mathbb{R}$上任一实数.

2. (1) $\displaystyle\lim_{x\to0}\frac{\tan^4x}{\sqrt{1-\frac{x\sin x}{2}}-\sqrt{\cos x}}$

原式 $=\displaystyle\lim_{x\to0}\frac{x^4}{\left(1-\frac{x}{2}\cdot\left(x-\frac{1}{3}x^3\right)\right)^{\frac12}-\left(1-\frac12x^2+\frac1{24}x^4\right)^{\frac12}}$

$$\left(1-\frac{x\sin x}{2}\right)^{\frac12}=1-\frac12\cdot\frac{x}{2}\cdot\left(x-\frac16x^3\right)+\frac12\cdot\frac12\cdot\left(-\frac12\right)\cdot\left[\frac{x}{2}\left(x-\frac16x^3\right)\right]^2+o(x^4)$$
$$=1-\frac14x^2+\frac1{24}x^4-\frac18\left(\frac{x^2}{4}\right)^2+o(x^4)$$
$$=1-\frac14x^2+\frac1{96}x^4+o(x^4)$$

$$(\cos x)^{\frac12}=\left(1-\frac12x^2+\frac1{24}x^4\right)^{\frac12}=1+\frac12\left(-\frac12x^2+\frac1{24}x^4\right)+\frac12\cdot\frac12\cdot\left(-\frac12\right)\left(-\frac12x^2+\frac1{24}x^4\right)^2+o(x^4)$$
$$=1-\frac14x^2+\frac1{48}x^4-\frac18\left(\frac14x^4\right)+o(x^4)$$
$$=1-\frac14x^2-\frac1{96}x^4+o(x^4)$$

$\therefore$ 原式 $=\displaystyle\lim_{x\to0}\frac{x^4}{\frac1{96}x^4+\frac1{96}x^4+o(x^4)}=48$

(2) $n\in\mathbb{N}^+$, $\forall\,1\le k\le n$, $a_k>0$,

$\displaystyle\lim_{x\to0}\left(\frac{\sum_{k=1}^{n}a_k^x}{n}\right)^{\frac1x}$ $a_k^x-1\sim x\ln a_k$

原式 $=\displaystyle\exp\left[\lim_{x\to0}\frac1x\left[\ln\frac{\sum_{k=1}^{n}(a_k^x-1)+n}{n}\right]\right]$

$=\displaystyle\exp\left[\lim_{x\to0}\frac1x\left[\ln\left(1+\frac{x\cdot\sum_{k=1}^{n}\ln a_k}{n}\right)\right]\right]$

$=\displaystyle\exp\left[\lim_{x\to0}\frac1x\cdot\frac{x\cdot\sum_{k=1}^{n}\ln a_k}{n}\right]$

$=\displaystyle\exp\left[\frac{\sum_{k=1}^{n}\ln a_k}{n}\right]$

$=\displaystyle\sqrt[n]{\prod_{k=1}^{n}a_k}$.

3. $n\in\mathbb{N}^+$, $n\ge2$,

$f(x)=\dfrac{1-2x+5x^2}{(1-2x)(1+x^2)}$ 在 $x=0$ 处，$2n+1$ 阶泰勒公式.

$\therefore\ \underbrace{(1-2x)(1+x^2)}_{p(x)}f(x)=5x^2-2x+1$

$\therefore\ (-2x^3+x^2-2x+1)f(x)=5x^2-2x+1$

$\therefore\ (-6x^2+2x-2)f(x)+(-2x^3+x^2-2x+1)f'(x)=10x-2.$

$(-2x^3+x^2-2x+1)'=-6x^2+2x-2$

$(\cdots)''=-12x+2$

$(\cdots)'''=-12$

$(\cdots)''''=0.$

$p'(x)f(x)+p(x)f'(x)=10x-2$

$p''(x)f(x)+2p'(x)f'(x)+p(x)f''(x)=10$

$p'''(x)f(x)+3p''(x)f'(x)+3p'(x)f''(x)+p(x)f'''(x)=0$

正解: 待定系数法分解

$f(x)=\dfrac{A}{1-2x}+\dfrac{Bx+C}{1+x^2}$

$\begin{cases}A-2B=5\\B-2C=-2\\A+C=1\end{cases}\Rightarrow\begin{cases}A=1\\B=-2\\C=0\end{cases}$

$\Rightarrow f(x)=\dfrac{1}{1-2x}+\dfrac{-2x}{1+x^2}$

$$f'(x)=\frac{(10x-2)(-2x^3+x^2-2x+1)-(-6x^2+2x-2)(5x^2-2x+1)}{(1-2x)^2(1+x^2)^2}$$

$$=-20x^4+14x^3-22x^2+14x^{-2}-(-30x^4+22x^3-16x^2+6x-2)$$

$$=10x^4-8x^3-6x^2+8x$$

$$=2x(5x^3-4x^2-3x+4)$$

$$\frac{1}{1-2x}=(1-2x)^{-1}+(1+x^2)^{-1}\cdot(-2x)$$

$$=1+\sum_{k=1}^{n}(2x)^k+o(x^n)$$

$$=1+\sum_{k=1}^{2n+1}(2x)^k+o(x^{2n+1})$$

$$=1+2x+\sum_{k=1}^{n}2^kx^k+2^{k+1}x^{k+1}+o(x^{2n+1})$$

$$\frac{1}{1+x^2}=(1+x^2)^{-1}$$

$$=1+\sum_{k=1}^{n}(-1)^k\cdot(x^2)^k$$

$$=1+\sum_{k=1}^{n}(-1)^k\cdot x^{2k}+o(x^{2n+1}).$$

$$\therefore\ \frac{-2x}{1+x^2}=-2x-2\sum_{k=1}^{n}(-1)^{-k}\cdot x^{2k+1}+o(x^{2n+1})$$

$$\therefore\ 原式=1+\sum_{k=1}^{n}2^kx^k+2^{k+1}x^{k+1}-2\cdot(-1)^k\cdot x^{2k+1}$$

$$+o(x^{2n+1})$$

$$=1+\sum_{k=1}^{n}2^kx^k+2(2^k-(-1)^k)x^{k+1}$$

$$+o(x^{2n+1})$$

4. $f(x,y,z)=\begin{cases}\dfrac{xyz}{x^2+y^2+z^2}, & (x,y,z)\neq(0,0,0)\\ 0, & =\end{cases}$

(1) $f_x(0,0,0)=f_y(0,0,0)=f_z(0,0,0)=0.$

(2) 若可微 $df(0,0,0)=\underbrace{f_x(0,0,0)}_{=0}dx+f_y(0,0,0)dy+f_z(0,0,0)dz$

$\Rightarrow f(x,y,z)-f(0,0,0)=o(\rho)$

$\Rightarrow \lim\limits_{(x,y,z)\to(0,0,0)}\dfrac{xyz}{(x^2+y^2+z^2)^{\frac{3}{2}}}=0$

当 (x,y,z) 沿 $\begin{cases}x=t\\ y=t\\ z=t\end{cases}$ 趋近 $(0,0,0)$ 时，原式 $=\dfrac{t^3}{(\sqrt{3}t)^3}=\dfrac{1}{3\sqrt{3}}$

沿 $\begin{cases}x=0\\ y=0\\ z=t\end{cases}$ 趋近 $(0,0,0)$ 时，原式 $=0$

∴ 极限不存在.

⇒ 不可微.

5. ✓

6. $F(x,y,z)=x^3+(y^2-1)z^3-xyz.$

(1) 在点 $(1,1)$ 近有唯一隐函数 $z=(x,y)$, s.t. $F(x,y,z)=0$. $z(1,1)=1.$

显然，F 过 $(1,1,1)$

又 $\left.\begin{cases}F_x=3x^2-yz\\ F_y=z^3\cdot 2y-xz\\ F_z=(y^2-1)\cdot 3z^2-xy\end{cases}\right\}$ 连续

∵ $F_z=(y^2-1)\cdot 3z^2-xy$

$F_z(1,1)=-1\neq 0.$

由隐函数存在定理，知存在唯一隐函数 $z=z(x,y)$ 符合题意.

(2) $F_x(1,1,1)=2$

$F_y(1,1,1)=1$

$F_z(1,1,1)=-1$

$$z'_x=-\frac{F_x}{F_z}=2$$

$$z'_y=-\frac{F_y}{F_z}=1$$

$\therefore \operatorname{grad} z=(2,1)$

$-\operatorname{grad} z=(-2,-1)$

$\therefore E=\left(-\frac{2}{\sqrt5},-\frac{1}{\sqrt5}\right)$

(3) $\therefore N=(-1,-2,2)$.

$(E,0)=\left(-\frac{2}{\sqrt5},-\frac{1}{\sqrt5},0\right)$

$$\therefore \cos\langle\vec N,(E,0)\rangle=\frac{\vec N\cdot(E,0)}{|\vec N|\cdot|(E,0)|}=\frac{\frac{2}{\sqrt5}+\frac{2}{\sqrt5}}{3\cdot1}=\frac{4}{3\sqrt5}=\frac{4\sqrt5}{15}$$

7.

证 $\because$ n个点与原点连线构成n个三角形. 设圆心角依次为$\theta_1,\cdots\theta_n$.

则
$$\begin{cases} S=\frac12\sum\limits_{k=1}^{n}\sin\theta_k \\ \sum\limits_{k=1}^{n}\theta_k=2\pi \quad (0<\theta_i<2\pi,\ i=1,\cdots n.) \end{cases}$$

下证. 若 $\theta_1+\theta_2=k$ $(k\in(0,4\pi),\ \theta_{1,2}\in(0,2\pi))$ 则 $\sin\theta_1+\sin\theta_2\le 2\cdot\sin\frac{k}{2}$

$$\begin{aligned}\sin\theta_1+\sin\theta_2&=\sin\left(\frac{\theta_1+\theta_2}{2}-\frac{\theta_1-\theta_2}{2}\right)+\sin\left(\frac{\theta_1+\theta_2}{2}-\frac{\theta_1-\theta_2}{2}\right)\\&=2\sin\frac{\theta_1+\theta_2}{2}\cdot\cos\frac{\theta_1-\theta_2}{2}\\&=2\sin\frac{k}{2}\cdot\cos\frac{\theta_1-\theta_2}{2}\le 2\sin\frac{k}{2}.\end{aligned}$$

又 $\theta_1=\theta_2$ 时可以取等.

$\therefore$ 证毕.

对n个θ_i依次用此定理, 知当且仅当$\theta_1=\cdots=\theta_n$时, $\sum\limits_{k=1}^{n}\sin\theta_k$最大

此时 $\sum\limits_{k=1}^{n}\sin\theta_k=n\cdot\sin\frac{2\pi}{n}$

$$\therefore S_{max}=\frac12\left(\sum_{k=1}^{n}\sin\theta_k\right)_{max}=\frac{n}{2}\cdot\sin\frac{2\pi}{n}.$$