信息学中的概率统计：期末考试答案

题目一

1. 当 $x \le \theta$，$F(x) = 0$。当 $x \ge \theta$，

$$F(x) = 1 - P(X \ge x) = 1 - \int_x^\infty \frac{2\theta^2}{t^3} dt = 1 - \frac{\theta^2}{x^2}。$$

因此，

$$F(x) = \begin{cases} 0 & x \le \theta \\ 1 - \dfrac{\theta^2}{x^2} & x \ge \theta \end{cases}。$$

2.

$$E(X) = \int_\theta^\infty \frac{2\theta^2}{x^2} dx = 2\theta,$$

因此

$$\hat{\theta}_1 = \frac{\overline{X}}{2}。$$

由于 $E(X) = E(\overline{X}) = 2\theta$，因此 $\hat{\theta}_1$ 是无偏和渐进无偏估计量。

3.

$$L(\theta) = \frac{2^n \theta^{2n}}{\prod_{i=1}^n x_i^3} \cdot 1_{\min\{x_1, x_2, \ldots, x_n\} \ge \theta}$$

注意到当 $\min\{x_1, x_2, \ldots, x_n\} \ge \theta$，$L(\theta)$ 为关于 θ 的增函数，因此

$$\hat{\theta}_2 = \min\{X_1, \ldots, X_n\}。$$

4. 当 $x \le \theta$, $F_{\hat{\theta}_2}(x) = 0$。当 $x \ge \theta$，

$$F_{\hat{\theta}_2}(x) = 1 - P(\hat{\theta}_2 \ge x) = 1 - (P(X_i \ge x))^n = 1 - (\theta/x)^{2n}。$$

$$E(\hat{\theta}_2) = \int_0^\infty (1 - F_{\hat{\theta}_2}(x)) dx = \theta + \int_\theta^\infty (\theta/x)^{2n} dx = \frac{2n}{2n-1}\theta。$$

因此 $\hat{\theta}_2$ 不是无偏估计量，但是渐进无偏估计量。当 $x \le \theta^2$，$F_{\hat{\theta}_2}(x) = P(\hat{\theta}_2^2 \le x) = 0$。当 $x \ge \theta^2$，

$$1 - F_{\hat{\theta}_2^2}(x) = 1 - P(\hat{\theta}_2^2 \ge x) = 1 - P(\hat{\theta}_2 \ge \sqrt{x}) = \frac{\theta^{2n}}{x^n}。$$

当 $n \ge 2$，

$$E(\hat{\theta}_2^2) = \int_0^\infty (1 - F_{\hat{\theta}_2^2}(x)) dx = \theta^2 + \int_{\theta^2}^\infty \frac{\theta^{2n}}{x^n} dx = \frac{n}{n-1}\theta^2。$$

$$\mathrm{MSE}(\hat{\theta}_2) = E(\hat{\theta}_2^2) + \theta^2 - 2\theta E(\hat{\theta}_2) = \frac{\theta^2}{(n-1)(2n-1)},$$

因此是一致估计量。

5. 令枢轴量 $G=\hat{\theta}_2/\theta$，

$$F_G(x)=P(G\le x)=1-P(G\ge x)=1-P(\hat{\theta}/\theta\ge x)=1-P(\hat{\theta}_2\ge\theta x)=\begin{cases}0 & x\le 1\\ 1-x^{-2n} & x\ge 1\end{cases},$$

令 $1-x^{-2n}=1-\alpha$，则有 $x=\alpha^{-\frac{1}{2n}}$。因此 $P(\hat{\theta}_2/\theta\le\alpha^{-\frac{1}{2n}})=1-\alpha$，也即 $\hat{\theta}_L=\alpha^{\frac{1}{2n}}\hat{\theta}_2$。

6. 当 $\theta=\theta_0$，

$$P(\hat{\theta}_2\ge c)=\begin{cases}1 & c\le\theta_0\\ (\theta_0/c)^{2n} & c\ge\theta_0\end{cases}。$$

因此，

$$c\ge\theta_0\alpha^{-\frac{1}{2n}}。$$

题目二

1.

$$\begin{cases}\frac{\partial}{\partial\alpha}Q_\gamma(\alpha,\beta)=-2\sum_{i=1}^n(y_i-\beta x_i-\alpha)=0\\ \frac{\partial}{\partial\beta}Q_\gamma(\alpha,\beta)=-2\sum_{i=1}^n x_i(y_i-\beta x_i-\alpha)+2\gamma\beta\end{cases}$$

沿用课上关于 $\overline{x},\overline{y},s_{xx}$ 和 s_{xy} 的定义，有

$$\sum y_i-\hat{\beta}_\gamma\sum x_i-n\hat{\alpha}_\gamma=0,$$

也即

$$\hat{\alpha}_\gamma=\overline{y}-\hat{\beta}_\gamma\overline{x}。$$

$$\sum x_iy_i-\hat{\beta}_\gamma\sum x_i^2-n\overline{x}\cdot\overline{y}+\hat{\beta}_\gamma n(\overline{x})^2-\gamma\hat{\beta}_\gamma=0,$$

因此

$$\hat{\beta}_\gamma=\frac{s_{xy}}{s_{xx}+\gamma}。$$

2. 根据课上推导，

$$\hat{\beta}_\gamma=\frac{s_{xy}}{s_{xx}+\gamma}=\frac{\sum(x_i-\overline{x})(\beta x_i+\epsilon_i)}{s_{xx}+\gamma}=\sum\epsilon_i\frac{(x_i-\overline{x})}{s_{xx}+\gamma}+\frac{\sum_i(x_i-\overline{x})\cdot\beta(x_i-\overline{x})}{s_{xx}+\gamma}=\sum\epsilon_i\frac{(x_i-\overline{x})}{s_{xx}+\gamma}+\beta\cdot\frac{s_{xx}}{s_{xx}+\gamma}。$$

因此，

$$E(\hat{\beta}_\gamma)=\beta\cdot\frac{s_{xx}}{s_{xx}+\gamma},$$

$$\mathrm{Var}(\hat{\beta}_\gamma)=\sum\mathrm{Var}(\epsilon_i)\frac{(x_i-\overline{x})^2}{(s_{xx}+\gamma)^2}=\sigma^2\sum_i\frac{(x_i-\overline{x})^2}{(s_{xx}+\gamma)^2}=\sigma^2\cdot\frac{s_{xx}}{(s_{xx}+\gamma)^2},$$

因此 $\hat{\beta}_\gamma$ 为有偏估计量。

$$\mathrm{MSE}(\hat{\beta}_\gamma)=\mathrm{Var}(\hat{\beta}_\gamma)+(\mathrm{Bias}(\hat{\beta}_\gamma))^2=\frac{\sigma^2s_{xx}+\beta^2\gamma^2}{(s_{xx}+\gamma)^2}。$$

$$\hat{\alpha}_\gamma=\overline{y}-\hat{\beta}_\gamma\overline{x}=\alpha+\frac{1}{n}\sum\epsilon_i+(\beta-\hat{\beta}_\gamma)\overline{x}=\alpha+\sum\epsilon_i\left(\frac{1}{n}-\frac{(x_i-\overline{x})\cdot\overline{x}}{s_{xx}+\gamma}\right)+\beta\cdot\frac{\gamma\overline{x}}{s_{xx}+\gamma}。$$

$$E(\hat{\alpha}_\gamma)=\alpha+\beta\cdot\frac{\gamma\overline{x}}{s_{xx}+\gamma},$$

$$\mathrm{Var}(\hat{\alpha}_\gamma) = \sigma^2 \sum_{i=1}^n \left(\frac{1}{n} - \frac{(x_i - \overline{x}) \cdot \overline{x}}{s_{xx} + \gamma}\right)^2 = \sigma^2 \sum_{i=1}^n \left(\frac{1}{n^2} + \left(\frac{(x_i - \overline{x}) \cdot \overline{x}}{s_{xx} + \gamma}\right)^2\right) = \sigma^2 \left(\frac{1}{n} + \frac{(\overline{x})^2 s_{xx}}{(s_{xx} + \gamma)^2}\right)。$$

$$\mathrm{MSE}(\hat{\alpha}_\gamma) = \mathrm{Var}(\hat{\alpha}_\gamma) + (\mathrm{Bias}(\hat{\alpha}_\gamma))^2 = \sigma^2 \left(\frac{1}{n} + \frac{(\overline{x})^2 s_{xx}}{(s_{xx} + \gamma)^2}\right) + \frac{(\beta\gamma\overline{x})^2}{(s_{xx} + \gamma)^2}。$$

题目三

1. 回顾：若随机变量 $A \sim N(\mu, \sigma^2)$，则

$$M_A(t) = E\left(e^{At}\right) = e^{\mu t + \frac{\sigma^2 t^2}{2}}。$$

另外，有 $X \sim N(\mu_1, \sigma_1^2), Y \sim N(\mu_2, \sigma_2^2)$。

因此

$$E(U) = E\left(e^X\right) = M_X(1) = e^{\mu_1 + \frac{\sigma_1^2}{2}}, E(V) = E\left(e^Y\right) = M_Y(1) = e^{\mu_2 + \frac{\sigma_2^2}{2}},$$

$$E(U^2) = E\left(e^{2X}\right) = M_X(2) = e^{2\mu_1 + 2\sigma_1^2}, E(V^2) = E\left(e^{2Y}\right) = M_Y(2) = e^{2\mu_2 + 2\sigma_2^2},$$

$$\mathrm{Var}(U) = e^{2\mu_1 + 2\sigma_1^2} - e^{2\mu_1 + \sigma_1^2} = e^{2\mu_1 + \sigma_1^2}\left(e^{\sigma_1^2} - 1\right), \mathrm{Var}(V) = e^{2\mu_2 + 2\sigma_2^2} - e^{2\mu_2 + \sigma_2^2} = e^{2\mu_2 + \sigma_2^2}\left(e^{\sigma_2^2} - 1\right)。$$

$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)$，因此

$$E(UV) = e^{X+Y} = M_{X+Y}(1) = e^{\mu_1 + \mu_2 + \frac{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}{2}},$$

也即

$$\mathrm{Cov}(U, V) = E(UV) - E(X)E(Y) = (e^{\rho\sigma_1\sigma_2} - 1)e^{\mu_1 + \mu_2 + \frac{\sigma_1^2 + \sigma_2^2}{2}},$$

相关系数为

$$\frac{e^{\rho\sigma_1\sigma_2} - 1}{\sqrt{e^{\sigma_2^2} - 1}\sqrt{e^{\sigma_2^2} - 1}}。$$

2. 注意到雅可比矩阵的行列式为 $\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} = \frac{1}{uv} > 0$。因此，当 $u, v > 0$，

$$f_{U,V}(u, v) = f_{X,Y}(\ln u, \ln v) \cdot \frac{1}{uv},$$

否则 $f_{U,V}(u, v) = 0$。注意到 $X \sim N(\mu_1, \sigma_1^2)$，当 $u > 0$，

$$f_U(u) = f_X(\ln u) \cdot \frac{1}{u} = \frac{1}{\sqrt{2\pi}\sigma_1 u} e^{-\frac{(\ln u - \mu_1)^2}{2\sigma_1^2}},$$

否则 $f_U(u) = 0$。类似，

$$f_V(v) = f_Y(\ln v) \cdot \frac{1}{v} = \frac{1}{\sqrt{2\pi}\sigma_2 v} e^{-\frac{(\ln v - \mu_2)^2}{2\sigma_2^2}}。$$

3. $V = v$ 等价于 $Y = \ln v$，因此给定 $V = v$，X 的条件分布服从

$$N\left(\mu_1 + \rho \cdot \frac{\sigma_1}{\sigma_2} \cdot (\ln v - \mu_2), \sigma_1^2(1 - \rho^2)\right),$$

其条件密度函数为

$$f(x|v) = \frac{1}{\sqrt{2\pi}\sigma_1\sqrt{1 - \rho^2}} \exp\left[-\frac{(x - (\mu_1 + \rho \cdot \sigma_1/\sigma_2 \cdot (\ln v - \mu_2)))^2}{2\sigma_1^2(1 - \rho^2)}\right],$$

因此

$$f(u|v)=\frac{1}{\sqrt{2\pi}\cdot\sigma_1\sqrt{1-\rho^2}\cdot u}\exp\left[-\frac{(\ln u-(\mu_1+\rho\cdot\sigma_1/\sigma_2\cdot(\ln v-\mu_2)))^2}{2\sigma_1^2(1-\rho^2)}\right]。$$

应用第一问中的结论，有

$$E(U\mid V=v)=e^{\mu_1+\rho\cdot\frac{\sigma_1}{\sigma_2}\cdot(\ln v-\mu_2)+\frac{\sigma_1^2(1-\rho^2)}{2}}。$$

题目四

1.

$$f(x)=\begin{cases}\frac{2}{\pi(1+x^2)} & x\geq 0\\ 0 & x\leq 0\end{cases}。$$

2. 令 E_i 表示 $Y_i\geq 100n$。

$$P(E_i)=\int_{100n}^{\infty}\frac{2}{\pi(1+x^2)}dx\leq\int_{100n}^{\infty}\frac{1}{x^2}dx=\frac{1}{100n}。$$

Union bound，有

$$P\left(\bigcup E_i\right)\leq n\cdot\frac{1}{100n}=0.01,$$

也即

$$P(Y_1\leq 100n\cap Y_2\leq 100n\cap\cdots Y_n\leq 100n)\geq 0.99。$$

注意若有对于任意 $1\leq i\leq n$，$Y_i\leq 100n$，则有 $\sum_{i=1}^n Y_i\leq 100n^2$。

3.

$$P(Y_i\leq 0.01)=\int_0^{0.01}\frac{2}{\pi(1+x^2)}dx\leq 0.01,$$

$$E\left(\sum_{i=1}^n 1_{Y_i\leq 0.01}\right)\leq 0.01n。$$

Markov,

$$P\left(\left(\sum_{i=1}^n 1_{Y_i\leq 0.01}\right)\geq 0.1n\right)\leq 0.1,$$

也即

$$P\left(\left(\sum_{i=1}^n 1_{Y_i\leq 0.01}\right)\leq 0.1n\right)\geq 0.9。$$

若

$$\sum_{i=1}^n 1_{Y_i\leq 0.01}\leq 0.1n,$$

则有

$$\sum_{i=1}^n Y_i\geq 0.9n\cdot 0.01\geq 0.009n。$$

另一种做法：

$$P(Y_i\geq 0.01n)=\int_{0.01n}^{\infty}\frac{2}{\pi(1+x^2)}dx\geq\int_{0.01n}^{\infty}\frac{1}{10x^2}dx\geq\frac{10}{n}。$$

因此

$$P\left(\bigcup_{i=1}^{n} Y_i \geq 0.1n\right) \geq 1-(1-10/n)^n \geq 1-e^{-10} \geq 2/3。$$

4. $n \ln n$。