§6

1. $\hat{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\hat{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$
\begin{aligned}
[\hat{\sigma}_y, \hat{\sigma}_z] &= \hat{\sigma}_y\hat{\sigma}_z - \hat{\sigma}_z\hat{\sigma}_y \\
&= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} = 2i\hat{\sigma}_x
\end{aligned}
$$

2. $|\psi_1\rangle = \frac{\sqrt{3}}{2}|u\rangle + \frac{1}{2}|d\rangle$

(1) $\because$ $|u\rangle$, $|d\rangle$ 分别为 $\hat{\sigma}_z$ 本征态，本征值分别为 1, -1

$\therefore$ 有 $(\frac{\sqrt{3}}{2})^2 = \frac{3}{4}$ 几率向上（z正方向），有 $(\frac{1}{2})^2 = \frac{1}{4}$ 几率向下（z负方向）

(2) $\hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ 本征态 $\begin{cases} |f\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{本征值为} 1 \\ |b\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \text{本征值为} -1 \end{cases}$

$\therefore |\psi_1\rangle = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \alpha|f\rangle + \beta|b\rangle \Rightarrow \begin{cases} \frac{1}{\sqrt{2}}(\alpha+\beta) = \frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}}(\alpha-\beta) = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} \alpha = \frac{1+\sqrt{3}}{2\sqrt{2}} \\ \beta = \frac{\sqrt{3}-1}{2\sqrt{2}} \end{cases}$

$\therefore$ 有 $\|\alpha\| = \frac{2+\sqrt{3}}{4}$ 几率向前（x正方向），$\|\beta\| = \frac{2-\sqrt{3}}{4}$ 几率向后（x负方向）

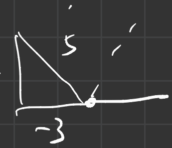
(3). $\vec{n} = \{\frac{4}{5}, 0, -\frac{3}{5}\}$

$\therefore \vec{n} = \frac{4}{5}\hat{\sigma}_x - \frac{3}{5}\hat{\sigma}_z = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

以立体角表示，$\vec{n} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$

解得 $\begin{cases} \theta = \arctan(-\frac{4}{3}) + \pi \\ \varphi = 0 \end{cases}$ $\therefore$

$\therefore |n^+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{5}} \\ \sqrt{\frac{4}{5}} \end{pmatrix}$，本征值为1



$\theta \in [0, \pi)$

$|n^-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -e^{i\varphi}\cos\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{4}{5}} \\ -\sqrt{\frac{1}{5}} \end{pmatrix}$，本征值为-1

$\sin\theta = \frac{4}{5} = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$\cos\theta = \frac{-3}{5} = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

$= 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}$

$\therefore \cos\frac{\theta}{2} = \sqrt{\frac{1}{2}\cdot\frac{2}{5}} = \sqrt{\frac{1}{5}}$

$\sin\frac{\theta}{2} = \sqrt{\frac{1}{2}\cdot\frac{8}{5}} = \sqrt{\frac{4}{5}}$

$\therefore$ 沿$\vec{n}$正方向几率

$= (\langle n^+|\psi\rangle)^2 = \left[\left(\frac{1}{\sqrt5}, \frac{2}{\sqrt5}\right)\begin{pmatrix} \frac{\sqrt3}{2} \\ \frac{1}{2} \end{pmatrix}\right]^2$

$= \left(\frac{\sqrt3}{2\sqrt5} + \frac{2}{2\sqrt5}\right)^2 = \left(\frac{2+\sqrt3}{2\sqrt5}\right)^2$

$= \frac{7+4\sqrt3}{20}$

$\therefore$ 沿$\vec{n}$负方向几率

$= (\langle n^-|\psi\rangle)^2 = \left[\left(\frac{2}{\sqrt5}, -\frac{1}{\sqrt5}\right)\begin{pmatrix} \frac{\sqrt3}{2} \\ \frac{1}{2} \end{pmatrix}\right]^2$

$= \left(\frac{2\sqrt3 - 1}{2\sqrt5}\right)^2$

$= \frac{13 - 4\sqrt3}{20}$

3. $|\psi_2\rangle = \frac{\sqrt3}{2}|u\rangle - \frac{i}{2}|d\rangle$

(1) $\therefore$ 沿z正方向的几率为 $\left(\frac{\sqrt3}{2}\right)^2 = \frac{3}{4}$

沿z负方向的几率为 $\left(-\frac{1}{2}\right)^2 = \frac{1}{4}$

(2) 设 $|\psi_2\rangle = \alpha|f\rangle + \beta|b\rangle$

$\therefore \alpha = \langle f|\psi_2\rangle = \frac{1}{\sqrt2}(1 \quad 1)\begin{pmatrix} \frac{\sqrt3}{2} \\ -\frac{i}{2} \end{pmatrix} = \frac{1}{\sqrt2}\cdot\frac{\sqrt3 - i}{2}$

$\therefore$ 沿x正方向的几率为 $\|\alpha\| = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$

$\therefore \beta = \langle b|\psi_2\rangle = \frac{1}{\sqrt2}(1 \quad -1)\begin{pmatrix} \frac{\sqrt3}{2} \\ -\frac{i}{2} \end{pmatrix} = \frac{\sqrt3 + i}{2\sqrt2}$

$\therefore$ 沿x负方向的几率为 $\|\beta\| = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$

(3) 设 $|\psi_2\rangle = \alpha|n^+\rangle + \beta|n^-\rangle$

$$\therefore \alpha = \langle n^+|\psi_2\rangle = \frac{1}{\sqrt5}\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}\frac{\sqrt3}{2}\\ -\frac{i}{2}\end{pmatrix} = \frac{\sqrt3-2i}{2\sqrt5}$$

$$\beta = \langle n^-|\psi_2\rangle = \frac{1}{\sqrt5}\begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}\frac{\sqrt3}{2}\\ -\frac{i}{2}\end{pmatrix} = \frac{2\sqrt3+i}{2\sqrt5}$$

$\therefore$ 沿 $\vec n$ 正方向的几率为 $\|\alpha\| = \frac{3+4}{20} = \frac{7}{20}$

沿 $\vec n$ 负方向的几率为 $\|\beta\| = \frac{12+1}{20} = \frac{13}{20}$

(4) $$\vec n\cdot\hat\sigma = \begin{pmatrix}\cos\theta & \sin\theta\, e^{-i\varphi}\\ \sin\theta\, e^{-i\varphi} & -\cos\theta\end{pmatrix} = \begin{pmatrix}-\frac35 & \frac45\\ \frac45 & \frac35\end{pmatrix}$$

$\therefore$ 期望值 $= \langle\psi_2|\vec n\cdot\hat\sigma|\psi_2\rangle = \begin{pmatrix}\frac{\sqrt3}{2} & \frac{i}{2}\end{pmatrix}\begin{pmatrix}-\frac35 & \frac45\\ \frac45 & \frac35\end{pmatrix}\begin{pmatrix}\frac{\sqrt3}{2}\\ -\frac{i}{2}\end{pmatrix}$

$$= \begin{pmatrix}\frac{\sqrt3}{2} & \frac{i}{2}\end{pmatrix}\begin{pmatrix}\frac{-3\sqrt3-4i}{10}\\ \frac{4\sqrt3-3i}{10}\end{pmatrix}$$

$$= \frac{-9-4\sqrt3 i}{20} + \frac{4\sqrt3 i+3}{20}$$

$$= \frac{-6}{20} = \frac{-3}{10} \qquad \left(= \frac{-13}{20} + \frac{7}{20}\right)$$