§3

1. (1) $z_1 = \sqrt{3} - i \qquad z_2 = 4 + 2i$

$\therefore z_2^* = 4 - 2i$

$\therefore z_3 = z_1 + z_2^* = 4 + \sqrt{3} - 3i$

(2) $\therefore z_1^* = \sqrt{3} + i$

$\therefore z_4 = z_1^* \times z_2 = 4\sqrt{3} - 2 + (4 + 2\sqrt{3})i$

(3) $$z_5 = z_1 / z_2 = \frac{\sqrt{3} - i}{4 + 2i} = \frac{\sqrt{3} - i}{4 + 2i} \cdot \frac{4 - 2i}{4 - 2i}$$

$$= \frac{(4\sqrt{3} - 2) - (4 + 2\sqrt{3})i}{20} = \frac{(2\sqrt{3} - 1) - (2 + \sqrt{3})i}{10}$$

$$= \frac{2\sqrt{3} - 1}{10} - \frac{2 + \sqrt{3}}{10} i$$

(4) $r = |z_1| = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$

$$\tan\theta = \frac{Im(z_1)}{Re(z_1)} = \frac{-1}{\sqrt{3}}$$

$$\therefore \theta = \arctan \frac{-1}{\sqrt{3}} = -\frac{\pi}{6}$$

2. (1) $|\psi_1\rangle = \frac{1}{5}\begin{pmatrix} 3 \\ 4i \end{pmatrix} \qquad |\psi_2\rangle = \begin{pmatrix} 2+2i \\ 1-3i \end{pmatrix}$

$$\langle\psi_1|\psi_1\rangle = (\tfrac{1}{5})^2 \cdot [3^2 - (4i)^2] = \frac{1}{25}(9 + 16) = 1$$

$$\langle\psi_2|\psi_2\rangle = (2-2i \quad 1+3i)\begin{pmatrix} 2+2i \\ 1-3i \end{pmatrix} = (4+4) + (1+9) = 18$$

(2) $\langle\psi_1| = \frac{1}{5}(3 \quad -4i)$

$$\langle\psi_1|\psi_2\rangle = \frac{1}{5}(3 \quad -4i) \cdot \begin{pmatrix} 2+2i \\ 1-3i \end{pmatrix}$$

$$= \frac{1}{5}[6 + 6i - 4i - 12]$$

$$= \frac{1}{5}(-6 + 2i) = -\frac{6}{5} + \frac{2}{5}i$$

$$\therefore \langle\psi_2|\psi_1\rangle = \langle\psi_1|\psi_2\rangle^* = -\frac{6}{5} - \frac{2}{5}i$$

3. (1) $|\bar{e}_1\rangle = \frac{1}{13}\begin{pmatrix} 5 \\ 12i \end{pmatrix}$ $|\bar{e}_2\rangle = \frac{1}{13}\begin{pmatrix} 12 \\ -5i \end{pmatrix}$

i. 正交性

$\langle \bar{e}_1 | = \frac{1}{13}(5, -12i)$

$\langle \bar{e}_1 | \bar{e}_2 \rangle = \frac{1}{169}(60 + 60i^2) = 0$

ii. 归一性

$\langle \bar{e}_1 | \bar{e}_1 \rangle = \frac{1}{169}(25 - 144i^2) = 1$

$\langle \bar{e}_2 | \bar{e}_2 \rangle = \frac{1}{169}(144 - 25i^2) = 1$

(2) $|\psi_2\rangle = \begin{pmatrix} 2+2i \\ 1-3i \end{pmatrix}$

$|\psi_2\rangle = a|\bar{e}_1\rangle + b|\bar{e}_2\rangle$

$\therefore \langle \bar{e}_2 | \psi_2 \rangle = a\langle \bar{e}_2 | \bar{e}_1 \rangle + b = b$

$\therefore \frac{1}{13}(12 \quad 5i)\begin{pmatrix} 2+2i \\ 1-3i \end{pmatrix} = b$

$\therefore b = \frac{1}{13}(24 + 24i + 5i + 15)$

$= \frac{1}{13}(39 + 29i) = 3 + \frac{29}{13}i$

同理 $\langle \bar{e}_1 | \psi_2 \rangle = a$

$\therefore \frac{1}{13}(5 \quad -12i)\begin{pmatrix} 2+2i \\ 1-3i \end{pmatrix} = a$

$\therefore a = \frac{1}{13}(10 + 10i - 12i - 36)$

$= \frac{1}{13}(-26 - 2i)$

$= -2 - \frac{2}{13}i$