3　第三次作业

题目 1

考虑两个复数 $z_1=\sqrt{3}-i$ 和 $z_2=4+2i$

1. (2 分) 求 $z_3=z_1+z_2^*$;
2. (2 分) 求 $z_4=z_1^*\times z_2$;
3. (2 分) 求 $z_5=z_1/z_2$;
4. (4 分) 将 z_1 用模和幅角表达 $z_1=re^{i\theta}$, 求 r 和 θ.

1. $z_3=z_1+z_2^*=\sqrt{3}-i+4-2i=\sqrt{3}+4-3i$
2. $z_4=z_1^*\times z_2=(\sqrt{3}+i)\times(4+2i)=(4\sqrt{3}-2)+(2\sqrt{3}+4)i$
3. $z_5=\dfrac{z_1}{z_2}=\dfrac{\sqrt{3}-i}{4+2i}=\dfrac{(\sqrt{3}-i)(4-2i)}{(4+2i)(4-2i)}=\dfrac{4\sqrt{3}-2-2i\left(\sqrt{3}+2\right)}{20}=\dfrac{2\sqrt{3}-1}{10}-\dfrac{\sqrt{3}+2}{10}i$
4. $r=\sqrt{\sqrt{3}^2+(-1)^2}=2\to z_1=2\left(\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}i\right)\sim r(\cos\theta+i\sin\theta)\to z_1=2e^{-i\frac{\pi}{6}}\begin{cases}r=2\\ \theta=-\dfrac{\pi}{6}(+2k\pi,k\in\mathbb{Z})\end{cases}$

题目 2

在二维希尔伯特空间中有两个向量:

$$|\psi_1\rangle=\frac{1}{5}\begin{bmatrix}3\\4i\end{bmatrix},\quad |\psi_2\rangle=\begin{bmatrix}2+2i\\1-3i\end{bmatrix}\tag{1}$$

1. (6 分) 计算 $\langle\psi_1|\psi_1\rangle$ 和 $\langle\psi_2|\psi_2\rangle$;
2. (6 分) 计算 $\langle\psi_1|\psi_2\rangle$ 和 $\langle\psi_2|\psi_1\rangle$.

$\langle\psi_1|=\frac{1}{5}\begin{bmatrix}3&-4i\end{bmatrix},\langle\psi_2|=\begin{bmatrix}2-2i&1+3i\end{bmatrix}$

1. $\langle\psi_1|\psi_1\rangle=\dfrac{1}{25}\begin{bmatrix}3&-4i\end{bmatrix}\begin{bmatrix}3\\4i\end{bmatrix}=\dfrac{1}{25}(9+16)=1,$

   $\langle\psi_2|\psi_2\rangle=\begin{bmatrix}2-2i&1+3i\end{bmatrix}\begin{bmatrix}2+2i\\1-3i\end{bmatrix}=((2-2i)(2+2i)+(1+3i)(1-3i))=18$

2. $\langle\psi_1|\psi_2\rangle=\dfrac{1}{5}\begin{bmatrix}3&-4i\end{bmatrix}\begin{bmatrix}2+2i\\1-3i\end{bmatrix}=\dfrac{1}{5}\left(3(2+2i)+(-4i)(1-3i)\right)=\dfrac{1}{5}(-6+2i),$

   $\langle\psi_2|\psi_1\rangle=\frac{1}{5}\begin{bmatrix}2-2i&1+3i\end{bmatrix}\begin{bmatrix}3\\4i\end{bmatrix}=\frac{1}{5}\left((2-2i)\cdot3+(1+3i)\cdot4i\right)=-\frac{1}{5}(6+2i)$

题目 3

在二维希尔伯特空间里定义两个向量:

$$|\bar{e}_1\rangle = \frac{5}{13}|e_1\rangle + \frac{12}{13}i|e_2\rangle = \frac{1}{13}\begin{bmatrix}5\\12i\end{bmatrix}, \quad |\bar{e}_2\rangle = \frac{12}{13}|e_1\rangle - \frac{5}{13}i|e_2\rangle = \frac{1}{13}\begin{bmatrix}12\\-5i\end{bmatrix} \tag{2}$$

1. (8 分) 证明这两个向量正交归一;
2. (10 分) 由于它们正交归一, 所以它们是二维希尔伯特空间的正交基矢。在这组正交基下，题 2 中的 $|\psi_2\rangle$ 可以展开成如下形式:

$$|\psi_2\rangle = a|\bar{e}_1\rangle + b|\bar{e}_2\rangle \tag{3}$$

求 a 和 b.

1. $\langle\bar{e}_1 \mid \bar{e}_1\rangle = \frac{1}{13}\left[\begin{array}{cc}5 & -12i\end{array}\right]\frac{1}{13}\begin{bmatrix}5\\12i\end{bmatrix} = \frac{25+144}{169} = 1$

   $\langle\bar{e}_2 \mid \bar{e}_2\rangle = \frac{1}{13}\left[\begin{array}{cc}12 & 5i\end{array}\right]\frac{1}{13}\begin{bmatrix}12\\-5i\end{bmatrix} = \frac{144+25}{169} = 1$

   $\langle\bar{e}_1 \mid \bar{e}_2\rangle = \frac{1}{13}\left[\begin{array}{cc}5 & -12i\end{array}\right]\frac{1}{13}\begin{bmatrix}12\\-5i\end{bmatrix} = \frac{60-60}{169} = 0$

2. $\begin{cases}(\alpha): |\bar{e}_1\rangle = \frac{5}{13}|e_1\rangle + \frac{12}{13}i|e_2\rangle \\ (\beta): |\bar{e}_2\rangle = \frac{12}{13}|e_1\rangle - \frac{5}{13}i|e_2\rangle\end{cases}$

$$(\alpha)\times 5 + (\beta)\times 12 \to 5|\bar{e}_1\rangle + 12|\bar{e}_2\rangle = 13|e_1\rangle \tag{4}$$

$$|e_1\rangle = \frac{5}{13}|\bar{e}_1\rangle + \frac{12}{13}|\bar{e}_2\rangle \tag{5}$$

$$(\alpha)\times 12 - (\beta)\times 5 \to 12|\bar{e}_1\rangle - 5|\bar{e}_2\rangle = 13i|e_2\rangle \tag{6}$$

$$|e_2\rangle = -\frac{12}{13}i|\bar{e}_1\rangle + \frac{5}{13}i|\bar{e}_2\rangle \tag{7}$$

$$|\psi_2\rangle = \begin{bmatrix}2+2i\\1-3i\end{bmatrix} = (2+2i)|e_1\rangle + (1-3i)|e_2\rangle \tag{8}$$

$$= \underbrace{\left[(2+2i)\frac{5}{13} - (1-3i)\frac{12}{13}i\right]}_{a}|\bar{e}_1\rangle + \underbrace{\left[(2+2i)\frac{12}{13} + (1-3i)\frac{5}{13}i\right]}_{b}|\bar{e}_2\rangle \tag{9}$$

$$a = \frac{10-36}{13} + \frac{10-12}{13}i = -2 - \frac{2}{13}i \tag{10}$$

$$b = \frac{24+15}{13} + \frac{24+5}{13}i = 3 + \frac{29}{13}i \tag{11}$$