6 第六次作业

题目 1

(5 分) 证明 $[\hat{\sigma}_x, \hat{\sigma}_z] = -2i\hat{\sigma}_y$.

$$[\hat{\sigma}_x, \hat{\sigma}_z] = \hat{\sigma}_x\hat{\sigma}_z - \hat{\sigma}_z\hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = -2i\hat{\sigma}_y$$

题目 2

(15 分) 给定一个自旋态

$$|\psi_1\rangle = \frac{\sqrt{3}}{2}|u\rangle + \frac{1}{2}|d\rangle$$

和一个方向

$$\vec{n} = \{4/5, 0, -3/5\}$$

1. (5 分) 测得自旋沿 z 方向向上和向下的几率分别是多少?
2. (5 分) 测得自旋沿 x 正方向和负方向的几率分别是多少?
3. (5 分) 测得自旋沿 $\vec{n}$ 正方向和负方向的几率分别是多少?

1. $P_u = |\langle u \mid \psi_1\rangle|^2 = \dfrac{3}{4}, P_d = |\langle d \mid \psi_1\rangle|^2 = \dfrac{1}{4}$
2. x 轴: $|f\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle), |b\rangle = \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle)$

$$P_+ = |\langle f \mid \psi_1\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)\right|^2 = \frac{1}{4}(2+\sqrt{3})$$

$$P_- = |\langle b \mid \psi_1\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)\right|^2 = \frac{1}{4}(2-\sqrt{3})$$

3. $\vec{n}\cdot\vec{\sigma} = \frac{4}{5}\sigma_x - \frac{3}{5}\sigma_z = \frac{4}{5}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{3}{5}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

假设其本征值为 λ，则有 $\begin{vmatrix} -\frac{3}{5}-\lambda & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5}-\lambda \end{vmatrix} = \lambda^2 - \dfrac{9}{25} - \dfrac{16}{25} = \lambda^2 - 1 = 0 \to \lambda = \pm 1$

假设其本征态为 $V = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$，则有 $\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -\frac{3}{5}\alpha + \frac{4}{5}\beta \\ \frac{4}{5}\alpha + \frac{3}{5}\beta \end{bmatrix} = \pm\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$$\lambda_1 = +1 \to \alpha = \frac{1}{2}\beta \to V_+ = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{5}}(|u\rangle + 2|d\rangle) = |n_+\rangle$$

$$\lambda_2 = -1 \to \alpha = -2\beta \to V_- = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{5}}(2|u\rangle - |d\rangle) = |n_-\rangle$$

$$P_{n_+} = |\langle n_+ \mid \psi_1\rangle|^2 = \left|\frac{1}{2\sqrt{5}}(\sqrt{3}+2)\right|^2 = \frac{1}{20}(7+4\sqrt{3})$$

$$P_{n_-} = |\langle n_- \mid \psi_1\rangle|^2 = \left|\frac{1}{2\sqrt{5}}(2\sqrt{3}-1)\right|^2 = \frac{1}{20}(13-4\sqrt{3})$$

题目 3

（20 分）给定一个自旋态

$$|\psi_2\rangle = \frac{\sqrt{3}}{2}|u\rangle - \frac{1}{2}i|d\rangle$$

和一个方向

$$\vec{n} = \{4/5, 0, -3/5\}$$

1. （5 分）测得自旋沿 z 方向向上和向下的几率分别是多少?
2. (5 分) 测得自旋沿 x 正方向和负方向的几率分别是多少?
3. （5 分）测得自旋沿 $\vec{n}$ 正方向和负方向的几率分别是多少?
4. （5 分）计算期待值 $\langle\psi_2|\vec{n}\cdot\hat{\sigma}|\psi_2\rangle$.

1. $P_u = |\langle u \mid \psi_2\rangle|^2 = \frac{3}{4}, P_d = |\langle d \mid \psi_2\rangle|^2 = \frac{1}{4}$
2. $|f\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle), |b\rangle = \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle)$

   $$P_+ = |\langle f \mid \psi_2\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)\right|^2 = \frac{1}{2}$$

   $$P_- = |\langle b \mid \psi_2\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\right|^2 = \frac{1}{2}$$
3. $|n_+\rangle = \frac{1}{\sqrt{5}}(|u\rangle + 2|d\rangle), |n_-\rangle = \frac{1}{\sqrt{5}}(2|u\rangle - |d\rangle)$

   $$P_{n_+} = |\langle n_+ \mid \psi_2\rangle|^2 = \left|\frac{\sqrt{3}}{2\sqrt{5}} - \frac{1}{\sqrt{5}}i\right|^2 = \frac{7}{20}$$

   $$P_{n_-} = |\langle n_- \mid \psi_2\rangle|^2 = \left|\frac{\sqrt{3}}{\sqrt{5}} + \frac{1}{2\sqrt{5}})\right|^2 = \frac{13}{20}$$
4. $\langle\psi_2|\hat{n}\cdot\hat{\sigma}|\psi_2\rangle = \frac{1}{20}\begin{bmatrix}\sqrt{3} & i\end{bmatrix}\begin{bmatrix}-3 & 4\\ 4 & 3\end{bmatrix}\begin{bmatrix}\sqrt{3}\\ -i\end{bmatrix} = \frac{1}{20}\begin{bmatrix}\sqrt{3} & i\end{bmatrix}\begin{bmatrix}-3\sqrt{3} & -4i\\ 4\sqrt{3} & -3i\end{bmatrix} = -\frac{3}{10}$