§5

1. $|\psi_1\rangle = \frac{5}{13}|u\rangle + \frac{12}{13} i |d\rangle$

$|\psi_2\rangle = \frac{12}{13}|u\rangle - \frac{5}{13}|d\rangle$

$\hat{\sigma}_x |u\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |d\rangle$

$\hat{\sigma}_x |d\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |u\rangle$

$\hat{\sigma}_y |u\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|d\rangle$

$\hat{\sigma}_y |d\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i|u\rangle$

(1) $\therefore\ |\psi_3\rangle = \hat{\sigma}_x |\psi_1\rangle = \frac{12}{13} i |u\rangle + \frac{5}{13}|d\rangle$

$\therefore\ |\psi_4\rangle = \hat{\sigma}_y |\psi_2\rangle = \frac{12}{13} i |d\rangle + \frac{5}{13} i |u\rangle$

(2) $\langle\psi_1|\hat{\sigma}_x|\psi_1\rangle = \left(\frac{5}{13}\langle u| - \frac{12}{13} i \langle d|\right)\left(\frac{12}{13} i |u\rangle + \frac{5}{13}|d\rangle\right)$

$= \frac{60}{169} i - \frac{60}{169} i = 0$

(3) $\langle\psi_2|\hat{\sigma}_y|\psi_4\rangle = \left(\frac{12}{13}\langle u| - \frac{5}{13}\langle d|\right)\left(\frac{5}{13} i |u\rangle + \frac{12}{13} i |d\rangle\right)$

$= \frac{60}{169} i - \frac{60}{169} i = 0$

2. $\vec{n} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$

$|n_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$

$\therefore\ \vec{n}\cdot\hat{\sigma}\,|n_+\rangle = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$

$= \begin{pmatrix} \cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2} \\ \sin\theta\cos\frac{\theta}{2}\, e^{i\varphi} - \cos\theta\sin\frac{\theta}{2}\, e^{i\varphi} \end{pmatrix}$

$= \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\varphi} \end{pmatrix} = |n_+\rangle$

3. 磁场沿z方向

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{\sigma}_z |u\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |u\rangle$$

$$\hat{\sigma}_z |d\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -|d\rangle$$

$$\therefore \hat{\sigma}_z |\varphi\rangle = \hat{\sigma}_z \left(\frac{3}{5} |u\rangle + \frac{4i}{5} |d\rangle \right) = \frac{3}{5} |u\rangle - \frac{4i}{5} |d\rangle$$

∴ 上旋点概率：$\left(\frac{3}{5}\right)^2 = \frac{9}{25}$，约 144 个

下旋点概率：$\left(\frac{4i}{5}\right)^2 = \frac{16}{25}$，约 256 个

4. $|\psi\rangle = \frac{\sqrt{3}}{2} |u\rangle + \frac{1}{2} |d\rangle$

即须使 $|\psi\rangle$ 是 $\vec{n} \cdot \hat{\sigma}$ 的本征态

设 $\vec{n} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$

则 $\vec{n} \cdot \hat{\sigma}$ 的本征态为 $|n_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$ $\quad |n_-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -e^{i\varphi}\cos\frac{\theta}{2} \end{pmatrix}$

$\therefore |\psi\rangle \propto |n_+\rangle$ 或 $|\psi\rangle \propto |n_-\rangle$

$$\therefore \frac{\sqrt{3}}{1} = \frac{\cos\frac{\theta}{2}}{e^{i\varphi}\sin\frac{\theta}{2}} \quad \text{或} \quad \frac{\sqrt{3}}{1} = \frac{\sin\frac{\theta}{2}}{-e^{i\varphi}\cos\frac{\theta}{2}}$$

即 $\tan\frac{\theta}{2} = \frac{\sqrt{3}}{3} e^{-i\varphi}$ 或 $\sqrt{3} \cdot (-e^{i\varphi}) = \tan\frac{\theta}{2}$

① $\begin{cases} \varphi = 0 \\ \theta = \frac{\pi}{3} \end{cases}$ 或 $\begin{cases} \varphi = \pi \\ \theta = \frac{4\pi}{3} + 2k\pi \text{（舍）} \end{cases}$ ② $\begin{cases} \varphi = 0 \\ \theta = -\frac{2\pi}{3} + 2k\pi \text{（舍）} \end{cases}$ 或 $\begin{cases} \varphi = \pi \\ \theta = \frac{2\pi}{3} \end{cases}$

$$\therefore \hat{n} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \text{ 或 } \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}$$