

§7

$$1. (a) \textcircled{1} U_s(t) = \begin{pmatrix} \cos \frac{\pi t}{4} & -i \frac{\sqrt{3}}{2} \sin \frac{\pi t}{4} & -\frac{1}{2} \sinh \frac{\pi t}{4} \\ \frac{1}{2} \sinh \frac{\pi t}{4} & \cos \frac{\pi t}{4} & +i \frac{\sqrt{3}}{2} \sinh \frac{\pi t}{4} \end{pmatrix}$$

$$|\phi(0)\rangle = |u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\phi(t)\rangle = U_s(t) |\phi(0)\rangle = \begin{pmatrix} \cos \frac{\pi t}{4} & -i \frac{\sqrt{3}}{2} \sin \frac{\pi t}{4} & -\frac{1}{2} \sinh \frac{\pi t}{4} \\ \frac{1}{2} \sinh \frac{\pi t}{4} & \cos \frac{\pi t}{4} & +i \frac{\sqrt{3}}{2} \sinh \frac{\pi t}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos \frac{\pi t}{4} & -i \frac{\sqrt{3}}{2} \sin \frac{\pi t}{4} \\ \frac{1}{2} \sinh \frac{\pi t}{4} & \cos \frac{\pi t}{4} \end{pmatrix}$$

$$= \left(\cos \frac{\pi t}{4} - i \frac{\sqrt{3}}{2} \sin \frac{\pi t}{4} \right) |u\rangle + \frac{1}{2} \sinh \frac{\pi t}{4} |d\rangle$$

$$\textcircled{2} \text{ 令 } |\phi(t_f)\rangle = e^{i\theta} \left(\frac{\sqrt{3}}{2} |u\rangle + \frac{1}{2} |d\rangle \right)$$

$$\therefore \begin{cases} \cos \frac{\pi t_f}{4} - i \frac{\sqrt{3}}{2} \sinh \frac{\pi t_f}{4} = \frac{\sqrt{3}}{2} e^{i\theta} \\ \frac{1}{2} \sinh \frac{\pi t_f}{4} = \frac{1}{2} e^{i\theta} \end{cases}$$

两例取模

$$\therefore \begin{cases} \cos^2 \frac{\pi t_f}{4} + \frac{3}{4} \sinh^2 \frac{\pi t_f}{4} = \frac{3}{4} \\ \frac{1}{4} \sinh^2 \frac{\pi t_f}{4} = \frac{1}{4} \end{cases}$$

$$\therefore \pi t_f = (4k+2)\pi, k \in \mathbb{N}, t_f > 0$$

$$\therefore t_{f \min} = 2 \quad |\phi(t_f)\rangle = -\frac{\sqrt{3}}{2} i |u\rangle + \frac{1}{2} |d\rangle$$

$$b) |\phi(0)\rangle = |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\therefore |\phi(t)\rangle = U_s(t) |\phi(0)\rangle = \begin{pmatrix} -\frac{1}{2} \sinh \frac{\pi t}{4} \\ \cos \frac{\pi t}{4} + i \frac{\sqrt{3}}{2} \sinh \frac{\pi t}{4} \end{pmatrix}$$

$$= -\frac{1}{2} \sinh \frac{\pi t}{4} |u\rangle + \left(\cos \frac{\pi t}{4} + i \frac{\sqrt{3}}{2} \sinh \frac{\pi t}{4} \right) |d\rangle$$

代入 $t = t_f = 2$

$$\therefore |\phi(t_f)\rangle = -\frac{1}{2} |u\rangle + \frac{\sqrt{3}}{2} i |d\rangle$$

$$c) |\phi(0)\rangle = \frac{3}{5} |u\rangle + \frac{4}{5} i |d\rangle$$

代入前两问之结果

$$\therefore |\phi(t_f)\rangle = -\frac{4+3\sqrt{3}}{10} i |u\rangle + \frac{3-4\sqrt{3}}{10} |d\rangle$$

2.

