Many-Objective Evolutionary Algorithms Based on Coordinated Selection Strategy

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***Abstract*—Selection strategy, including mating selection and environmental selection, is a key ingredient in the design of evolutionary multiobjective optimization algorithms. Existing approaches, which have shown competitive performance in low-dimensional multiobjective optimization problems with two or three objectives, often encounter considerable challenges in many-objective optimization, where the number of objectives exceeds 3. This paper first provides a comprehensive analysis on the selection strategies in the current evolutionary many-objective optimization algorithms. Afterward, we propose a coordinated selection strategy to improve the performance of evolutionary algorithms in many-objective optimization. This selection strategy considers three crucial factors: 1) the new mating selection criterion considers both the quality of each selected parent and the effectiveness of the combination of selected parents; 2) the new environmental selection criterion directly focuses on the performance of the whole population rather than single individual alone; and 3) both selection steps are complement to each other and the coordination between them in the evolutionary process can achieve a better performance than each of them used individually. Furthermore, in order to handle the curse of dimensionality in many-objective optimization problems, a new convergence measure by distance and a new diversity measure by angle are developed in both selection steps. Experimental results on both DTLZ and WFG benchmark functions demonstrate the superiority of the proposed algorithm in comparison with six state-of-the-art designs in terms of both solution quality and computational efficiency.**

***Index Terms*—Environmental selection, evolutionary computation, many-objective optimization, mating selection.**

I. INTRODUCTION

MULTIOBJECTIVE optimization problems (MOPs) contains more than one objective to be optimized simultaneously [1]. In MOP, because of the conflicting nature among objectives, usually no single optimal solution, but a group of tradeoff solutions known as Pareto optimal solutions, exists. Currently, multiobjective evolutionary algorithms (MOEAs) can successfully solve MOPs with two or three objectives by its powerful meta-heuristics search ability and population-based framework. However, when MOEAs are used to solve many-objective optimization problems (MaOPs) where the number of objectives is generally greater than 3, their performance deteriorates appreciably due to the curse of dimensionality [2], [3]. As MaOPs are widely existed in real-world applications [4]–[6], nowadays a lot of research efforts have been focused on developing new algorithms and technologies to solve MaOPs.

As we learned over years, there are three main steps of evolutionary process in the design of MOEAs: 1) mating selection; 2) recombination; and 3) environmental selection. Mating selection is responsible for selecting high quality parents in order to generate good offspring. Then, recombination combines parents together and generates quality offspring by crossover and mutation. Finally, environmental selection evaluates all parents and offspring by a predefined selection criterion, and preserves the better solutions into the next generation. However, in MaOPs, the increasing number of objectives brings several challenges to the design of many-objective evolutionary algorithms (MaOEAs) in conducting each step.

First, the Pareto optimality, which is effective to facilitate the convergence of the population in a low-dimensional search space [7], [8], loses the selection pressure in both mating selection and environmental selection in a high-dimensional space. This is because the proportion of nondominated individuals in a population rises quickly with the increasing number of objectives [9]. In literature, there are a significant number of publications trying to overcome the ineffectiveness of Pareto optimality. For example, ε-dominance [10], grid dominance [11], and fuzzy Pareto dominance [9] modify the original Pareto dominance relationship to adapt into a higher dimensional space. In hypervolume estimation algorithm for multiobjective optimization (HypE) [12], instead of using Pareto optimality, hypervolume indicator is applied to assign fitness values to each solution in both mating and environmental selection. Furthermore, the decomposition approaches, such as multiple single objective Pareto sampling (MSOPS) [13], MOEA based on decomposition (MOEA/D) [14], and evolutionary many-objective optimization algorithm based on dominance and decomposition (MOEA/DD) [15], decompose an MOP into a number of scalar optimization subproblems and optimize them simultaneously. In doing so, there is no need of using Pareto optimality in the evolutionary process of this type of methods.

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Second, the search space of an MaOP is extremely large, which considerably weakens the effect of the evolutionary operators in the recombination step, such as crossover and mutation [16]. From [17], two distant parent solutions are likely to produce offspring solutions that are also distinct from parents. Therefore, even two high quality (e.g., nearly converged) solutions may generate offspring far from the true Pareto front. In reference-point based many-objective NSGA-II (NSGA-III) [17], the value of crossover's distribution index is set to be large so that the offspring is more similar to parents, which avoids the abovementioned problem. Meanwhile, in the decomposition methods such as MOEA/D [14], the recombination is restricted so that only close solutions (e.g., both solutions are in the same neighborhood) can be combined to generate new offspring.

Third, as suggested in [15], the conflicting nature between convergence and diversity becomes more aggravated as the number of objectives increases. Therefore, in both mating and environmental selection, it is very difficult to design a selection criterion which is able to balance convergence and diversity of the whole population. For example, a diversity enhancement operator may prefer selecting the poorly converged solutions [18]. On the other hand, some convergence improvement indicators may repeatedly select solutions in the same crowded area. In literature, some designs aim at handling this difficulty. The diversity enhancement mechanism of NSGA-III [17] stresses on the diversity performance of solutions while the convergence of the population is solely maintained by environmental selection. In [19], a shift-based density estimation strategy transfers solution's convergence performance into diversity performance and the selection is based on the comparison of solutions' diversity performance. Recently, Li *et al.* [15] exploited both dominance- and decomposition-based approaches to strike a balance between the convergence and diversity in the evolutionary process.

Additionally, there is another variant of designs called preference-based method. By satisfying a group of preferences, the population is pushed toward the true Pareto front while preserving diversity among all solutions. For example, preference-inspired coevolutionary algorithm using goals (PICEA-g) [20] applies the concept of coevolutionary by using a family of preferences to compare solutions.

Nowadays, evolutionary many-objective optimization has been gaining increasing attention among researchers. However, in literature, there is no approach designed to simultaneously address all three challenges on MaOEAs' selection strategy as discussed above. In this paper, we propose a new algorithm, called MaOEA based on coordinated selection strategy (MaOEA-CSS), whose selection strategy directly tackles three challenges in the selection and recombination of the evolutionary process. The main contribution of this paper is summarized as follows.

1) An analysis of mating selection, recombination, and environmental selection strategies in state-of-the-art many-objective optimization algorithms is made.
2) The new mating selection criterion considers both the quality of each selected parent and the effectiveness of the combination of selected parents.
3) The new environmental selection criterion directly focuses on the performance of the whole population rather than single individual alone.
4) The relation between mating selection and environmental selection is investigated the first time in literature. According to this relation, in the design of our new algorithm, we coordinate both selection steps. That is, selection criterions in both selection steps are complement to each other so that they are combined to obtain a good performance for the algorithm.
5) In order to handle the curse of dimensionality in MaOPs, a new convergence measure by distance and a new diversity measure by angle are developed in both selection steps.

The remaining sections complete the presentation of this paper. Section II provides a comprehensive analysis of selection strategies in the state-of-the-art MaOEAs. Our proposed selection strategy and its details are explained in Section III. In Section IV, we elaborate on the experimental results given selected benchmark problems. Finally, the conclusion is drawn in Section V along with pertinent observations.

II. Analysis of Selection Strategies in MaOEAs

In this section, first we analyze the selection strategies in some popular MaOEAs, including MOEAs which are original designed to solve MOPs and then extended to solve MaOPs. According to this analysis, all selection strategy are classified into four different types, each of which corresponds to one class of MaOEAs. After that, we exploit each type of selection strategies and explain the measurement approaches adopted in the respective selection scheme. Finally, the difficulties of each selection strategy in solving MaOPs are summarized.

A. Selection Strategies of Popular MaOEAs

In this section, we select eleven different MaOEAs for analysis. First, ε-domination based MOEA (ε-MOEA) [10], territory defining multiobjective evolutionary algorithms (TDEAs) [21], MSOPS [13], and MOEA/D [14] are originally developed to solve low-dimensional MOPs and have been exploited to solve high-dimensional MaOPs. In ε-MOEA, ε-dominance and grid framework provide mainly convergence measurement and diversity measurement in its selection criterion, respectively. Similarly, TDEA defines a territory framework for diversity evaluation in its selection criterion while keeping the Pareto optimality as the convergence evaluation. On the other hand, MSOPS runs multiple single objective optimizations in parallel, where each aggregated optimization is determined by its own weight vector. Afterward, the algorithm constructs a matrix of target vectors to rank solutions. Additionally, MOEA/D's selection criterion is based on its decomposition structure where an MaOP is decomposed into multiple subproblems and a group of well distributed weight vectors corresponds to these subproblems. Then, the convergence performance of each solution is determined by how it solves its associated subproblem and each solution has equal diversity performance after environmental selection since each one is also associated with a different weight vector, which is well distributed with respect to others.

Second, NSGA-III [17], grid-based evolutionary algorithm (GrEA) [11], HypE [12], and PICEA-g [20] are the most popular algorithms specifically crafted to solve MaOPs. NSGA-III's selection strategy incorporates Pareto optimality for convergence measurement and a decomposition based diversity preservation operator for diversity measurement. In GrEA, both convergence and diversity criterion in selection are based on grid. Three grid based criteria, specifically grid ranking, grid crowding distance, and grid coordinate point distance, can quantitatively compare solutions' convergence and diversity performance. Furthermore, hypervolume indicator in HypE assigns each solution a fitness value as the selection criterion, which reflects both convergence and diversity performance of this solution. In PICEA-g, the fitness of solutions is determined by the number of preferences it meets while the fitness of preferences is related to how many times it is satisfied by solutions.

Most recently, there are some novel algorithms published: MOEA/DD [15], knee point driven evolutionary algorithm (KnEA) [22], and improved two-archive algorithm (Two_Arch2) [23]. MOEA/DD's selection strategy is based on the combination of both Pareto optimality and decomposition structure so as to balance the convergence and diversity in the evolutionary process. In KnEA, a knee point is identified first and the distance between each solution and the knee point measures the convergence performance of that solution. Beside the knee point, a weighted distance measure is designed as the diversity measurement in selection. Finally, Two_Arch2 uses two archives, each of which focuses on convergence and diversity, respectively. A quality indicator I_ε is applied for fitness assignment of each solution in the convergence archive while the predefined boundary solution and the similarity degree analysis measure the diversity performance of each solution in the diversity archive, Table I lists all these eleven MaOEA's selection strategies.

Based on different selection strategies, each MaOEA also chooses different recombination strategy. MOEA/D and MOEA/DD share the same strategy that only parents close to each other (e.g., in the same neighborhood) can be combined. On the other hand, NSGA-III applies modified crossover operator with a larger value of distribution index. In ε-MOEA, TDEA, and Two_Arch2 with two archives, each archive provides one parent, then both of them from different archives combine. Finally, there is no recombination restriction in KnEA, HypE, and GrEA.

TABLE I
LIST OF MAOEAS' SELECTION STRATEGIES

Algorithm	Mating Selection	Environmental Selection
ε-MOEA	**Population** Pareto dominance **Archive** Random	**Population** Pareto dominance **Archive** ε-dominance Euclidean distance
TDEA	**Population** Pareto dominance **Archive** Random	**Population** Pareto dominance **Archive** Pareto dominance Territory checking
MOEA/D	Random	Scalar optimization
NSGA-III	Random	Pareto dominance Niche count value Perpendicular distance
GrEA	Pareto dominance Grid dominance Density estimation	Pareto dominance Grid ranking Grid crowding distance Grid coordinate distance
HypE	Hypervolume indicator	Pareto dominance Hypervolume indicator
MOEA/DD	Random	Pareto dominance Niche count value Boundary intersection
KnEA	Pareto dominance Knee point criterion Weighted distance	Pareto dominance Knee point criterion Weighted distance
Two_Arch2	Random	**Convergence archive** quality indicatorI_ε **Diversity archive** Pareto dominance Boundary solution Similarity degree
PICEA-g	Random	Preference satisfaction Pareto dominance
MSOPS	Random	Aggregated optimization

B. Four Different Classes of Selection Strategies

In this section, based on the explanation of each algorithm's selection strategy, we classify all different selection strategies into four distinct classes, each of which determines one type of MaOEAs. The summary of these discussions is listed in Table II.

The first strategy, applied in MOEA/D, MOEA/DD, NSGA-III, and MSOPS is based on the decomposition structure. Each solution's convergence performance is determined by how it solves its corresponding single objective optimization subproblem. For diversity measurement, MOEA/D treats all solutions as equal since only one solution belongs to each weight vector in MOEA/D. On the other hand, MOEA/DD and NSGA-III allows multiple solutions to be associated with the same weight vector. Therefore, solutions' diversity performance can be evaluated by two aspects. First, the niche count of each weight vector measures how crowded of solutions in this vector. For example, if three solutions are associated to this vector, its niche count is 3. The larger the number of niche count, the more crowded of these solutions, therefore the worse diversity performance of them. Second, if two solutions share the same niche count value, some scalar optimization approaches are used to further differentiate them, e.g., perpendicular distance in NSGA-III and boundary intersection value in MOEA/DD. In this type of selection strategy, both convergence and diversity criteria discussed above are only used in environmental selection while there is no selection criterion in mating selection, where parents are just randomly chosen to combine. The quality of new solutions is ensured by the

TABLE II
SUMMARY OF FOUR TYPES OF SELECTION STRATEGIES

Type	Step	Description
Decomposition MOEA/D MOEA/DD NSGA-III MSOPS	Mating Selection	Random
	Recombination	In the same neighborhood
	Environmental Selection	Pareto dominance Density estimation Scalar optimization approach
Grid-based GrEA ε-MOEA	Mating Selection	Modified Pareto dominance
	Recombination	No restrictions
	Environmental Selection	Modified Pareto dominance Density/distance estimation
Double Archives TDEA ε-MOEA Two_Arch2	Mating Selection	**Population:** Pareto dominance **Archive:** Random
	Recombination	No restrictions
	Environmental Selection	**Population:** Pareto dominance **Archive:** (Modified) Pareto dominance Density/distance estimation
New Fitness Assignment KnEA HypE PICEA-g	Mating Selection	Pareto dominance Quality indicator or target point Random
	Recombination	No restrictions
	Environmental Selection	Pareto dominance Quality indicator or target point Density/distance estimation Preference satisfaction

restriction that only parents from the same neighbor can be combined together.

The second strategy is grid based approaches. The grid modifies Pareto optimality and provides a framework for diversity measurement. GrEA and ε-MOEA are of this type. For convergence criterion, the grid relaxes the Pareto optimality and differentiates solutions into different degrees of dominance, such as grid dominance in GrEA and ε-dominance in ε-MOEA. For diversity criterion, the distance measurement inside the grid and the density measure among grids are used. Meanwhile, there is no recombination restriction for this type of strategy in that the mating selection according to modified Pareto dominance has selected good parents for comparison.

The third strategy exists in MaOEAs with double archives, such as TDEA, ε-MOEA, and Two_Arch2. In different archives, different selection criteria are used. For example, in Two_Arch2, selection criterion in convergence archive mainly focuses on selecting the solution with better convergence performance while in diversity archive selecting better diversified solutions is emphasized. Here, the grid-based algorithm, ε-MOEA, *also* belongs to this class, since one archive of ε-MOEA stores nondominated solutions based on ε-dominance criterion while the other stores all EA population. At mating selection, each archive provides a solution for combination, where the mating selection strategy in each archive is different from the other. Similarly, environmental selection strategies of both archives are different. The idea behind this strategy is that different solutions selected from different archives contain different aspects of good characteristics, then the combination of them can generate good solutions inheriting all these characteristics. In summary, this strategy mainly focuses on the structure of the selection rather than the exactly selection criterion.

The last selection strategy exploits new fitness assignment methods, such as quality indicator (e.g., hypervolume indicator in HypE), predefined target points (e.g., knee point in KnEA), or preference (e.g., preference indicators in PICEA-g) to assign each solution a fitness value, which directly reflects both convergence and diversity performance of each solution, the selection criterion is based on the distance between each solution and the predefined target points.

There are various measurement approaches called upon in each selection strategy. Each selection strategy could exploit more than one measurement approach and some selection strategies could adopt the same approach. The first approach is the modified Pareto dominance, such as grid Pareto dominance in GrEA and ε-dominance in ε-MOEA. This approach aims at relaxing the Pareto dominance to make one individual dominate another easier in a high-dimensional space. The second approach is the scalar optimization approach widely used in decomposition based methods, such as MOEA/D, MOEA/DD, and NSGA-III. This type of approaches, including weighted sum [24], Tchebycheff approach [24], normal boundary intersection [25], and achievement scalarizing function (ASF) [17], transfers all objective values in an MaOP to a single aggregated value, which then becomes the fitness value of each solution reflecting its convergence performance. The third approach is density/distance estimation, such as niche count [26], perpendicular distance [23], and similarity degree [23]. This type of approaches measures the density of one subpopulation, or the distance from a solution to the target point or weight vector. The last approach is quality indicator, which also includes some target points criterion and preference. Knee point criterion, hypervolume indicator, and ε-indicator I_ε [23] belong to this type. This type of approaches converts the performance of the solution into a scalar value as its fitness value. It becomes easy to classify each individual into different levels.

C. *Difficulties of Each Selection Strategy in Solving MaOPs*

Although each selection strategy has established itself as an effective approach in solving MaOPs under some limited conditions, it still suffers many deficiencies when applied to other cases.

In the decomposition-based strategy, there remain lots of difficulties for parameter settings in the evolutionary process. First, the setting of the number of weight vectors must satisfy two conflicting goals-the effective combination of solutions (i.e., too small number of weight vectors makes solutions in the same neighborhood much distant from each other, thus the combination of them cannot generate good offspring) and the computational workload (i.e., heavy computational load is caused by the large number of vectors). Meanwhile, the neighborhood size has an impact on the balance of convergence and diversity, where a small neighborhood increasing the spread of a population while a large one enhancing the convergence speed of a population toward the Pareto front. Finally, the choice of aggregation method still needs to be considered carefully. For example, according to [25], weighted sum method performs well in convex problems, while Tchebycheff method is better suited in solving nonconvex problems. Furthermore, this selection strategy is only applied in the environmental selection while random selection is done in mating selection. If each individual has equal convergence and diversity performance before mating selection, then random selection is suitable. However, in MOEA/DD and NSGA-III, solutions have different Pareto optimality levels and different density (niche count) values. That is, solutions are in the different degrees of convergence and diversity. Therefore, random selection cannot ensure high quality solutions to be selected for combination.

Second, similar to the first strategy, the grid based strategy needs the parameter setting of gird, such as grid size in GrEA and hyperbox size in ε-MOEA.

Third, in double archive strategy, there is still random selection in mating selection but all solutions are not equal to each other. Furthermore, it is possible that solutions from different archives are very far from each other. However, this strategy does not have any combination restriction to avoid two distant solutions to be combined together. Meanwhile, solutions that are better in one selection criterion cannot ensure it is also better in the other criterion. For example, in Two_Arch2, one solution with good convergence but bad diversity my still be selected from convergence archive.

Finally, for the last selection strategy, in the new fitness assignment methods using the quality indicator, as noted from [27], no single indicator alone can faithfully assign the fitness value to each solution, which can only provide some specific, but incomplete, quantifications of performance. Furthermore, in mating selection, if the selection criterion is based on their fitness values, there is a large probability that one individual with the best fitness value is selected many times for combination. For fitness methods with predefined target points, appropriate scalable target vector generation mechanism is needed [28]. For fitness methods based on preference, the process of choosing an appropriate preference model may be problem-dependent [29] and how to define a suitable family of preferences to full representation of the Pareto front is a challenge problem [20].

In addition to the above discussions, all strategies highlighted above share three common problems. First, all mating selection criteria only consider selecting high quality parents. However, the effectiveness of the combination of selected parents is not taken into account. In most cases, even high quality parents cannot ensure that their combination can also generate good offspring. Second, no strategy analyzes the relationship between mating selection and environmental selection. Moreover, there is no coordination and complementary design between mating selection and environmental selection. In some strategy, the selection criteria in both selections are repeated, such as the new fitness assignment strategy. In others such as decomposition strategy, selection criterion is only applied in environmental selection. Third, all selection criteria only operate on individual but not on the whole population. No criterion is designed from the perspective of the whole population and tries to directly evaluate the performance of the whole population. In the next section, a new MaOEA is proposed to handle the challenges arising from MaOPs using its improved selection strategy, which overcomes all disadvantages of existing selection strategies and does not require any parameter setting.

Additionally, there is another classification on MaOEAs, providing sufficient details in the spirit how each algorithm is been designed: decomposition (MSOPS, MOEA/D, and MOEA/DD), grid-based (GrEA and ε-MOEA), indicator (HypE), co-evolution (PICEA-g), predefined targets (KnEA and NSGA-III), double archives (TDEA and Two_Arch2), and modified density (SDE and DMO). As our focus has been placed solely on the classification of "*selection strategies*" according to the structure of each selection strategy, we choose not to use the above classification. For example, in grid-based approaches, selection strategies are directly applied in the objective space which has been divided into multiple hyperboxes. The decomposition strategy can only be used after the many-objective problem is divided into a group of single-objective ones. Furthermore, all fitness assignment methods try to incorporate other elements (e.g., quality indicator, predefined targets, or preference) into selection strategies. Finally, in the type of double archives strategy, different archives preserve different types of solutions. These rationales justify the choice we made in classification of MaOEAs.

III. Proposed Method

The proposed algorithm, MaOEA-CSS, contains several unique features. First, the coordination of mating selection and environmental selection is incorporated into the structure of the algorithm. A coordination mechanism that seamlessly integrates both selection criteria provides a better performance than each of them used alone during the evolutionary process. Second, in both selection criteria in order to address the curse of dimensionality, the convergence measurement is based on distance while the diversity measurement is by angle. Third, the new mating selection criterion considers both the quality of each selected parent and the effectiveness of the combination of selected parents. Finally, the new environmental selection criterion directly focuses on the performance of the whole population rather than on the single individual.

A. Coordination Mechanism

As we know, the main goal of mating selection is to select well converged and diversified parents so that the combination of them can generate good offspring while environmental selection tries to preserve the best N solution among the combination of parents and offspring, where N is the population size. Although both selections aim at selecting solutions with good convergence and diversity performance, there are some differences between them. First, the mating selection always picks up one winner from a pair of solutions through binary tournament while the environmental selection retains a group of solutions by applying some criteria. Second, there is randomness in the mating selection, e.g., in mating selection, each time two solutions are randomly selected and put into binary tournament. However, in the environmental selection, the decision process is determined and every solution must be taken into account. Third, the mating selection is more focusing on effectively generating better genes through synthesis of excellent older genes from the old population. On the other hand, environmental selection tries to pass all excellent genes to the next generation by keeping all good solutions. Finally, the mating selection is not the necessary step in some algorithms' design but environmental selection must exist in all algorithms.

Therefore, according to differences between both selections, we need to choose different selection criterion for each selection step. That is, both selections should complement to each other so as to achieve the best performance in the evolutionary process. In our proposed method, the coordination mechanism contains the following characteristics. First, mating selection criterion judges the performance of each individual (solution): it compares two individuals directly and selects one winner from them. Environmental selection criterion evaluates the performance of the whole population: it decides which N solutions altogether can achieve the best convergence and diversity performance among the combination of all old solutions and new solutions. Second, the mating selection must consider the effectiveness of the combination. That is, the combination of selected high quality parents should generate good offspring. Third, after environmental selection, each solution has different convergence and diversity degrees (or different rank values) among N best solutions. Then, in the next generation's mating selection, the selection criterion should be based on these degrees. That is, selection criterion in each step should not conflict to each other. Fourth, in the mating selection, the design of selection criterion should consider avoiding one condition that one solution is chosen many times as parent, which could harms the diversity of the evolutionary process. Therefore, the mating selection criterion should also subject to some randomness. Because the environmental selection criterion constantly keeps best solutions, there is no need to worry about losing good solutions in the next generation. Finally, although both selections have different roles in the evolutionary process, they still have the same goal: select solutions with good convergence and diversity performance. Therefore, the convergence and diversity measurements in both steps should be comparable.

B. Convergence and Diversity Measures

In mating selection, the convergence performance of each solution is measured using the modification of ASF [30]. The ASF value of solution i is defined as

$$\mathrm{ASF}\left(i, w^i\right) = \max_{k=1:M}\left\{\left(f_k^i - z_k^*\right)\Big/ w_k^i\right\} \tag{1}$$

where z_k^* is the best value found so far among all solutions for objective k and the obtained ideal point $z^* = (z_1^*, z_2^*, \ldots, z_M^*)$ is constructed. f_k^i is the objective value of solution i in the kth objective.w^i is the favorable weight vector [31] corresponding to solution i and its kth element w_k^i is defined as

$$w_k^i = \frac{f_k^i}{\sum_{l=1:M} f_l^i}. \tag{2}$$

Here, if w_k^i equals to zero, then it is set be 10^{-6}. Similarly as favorable weight defined in [31], w^i makes solution i superior over others within its weight values. Then, the ASF value can reflect the best convergence degree of each solution. A smaller ASF value of solution i implies its better convergence performance. In literature, this ASF value has been widely applied for convergence measure [16], [17], [32].

In environmental selection, the Euclidean distance between each parent individual and the obtained ideal point z^*, is applied for convergence measure. The larger the distance, the worse the convergence performance is. From [28], it is mainly convergence measure in penalty-based boundary intersection (PBI) approach [14] and weighted sum approach [24] which have been widely used in decomposition methods [14], [15], [28]. The performance of these decomposition methods have shown that Euclidean distance is an effective way for convergence measure.

The diversity measurement is based on each solution's minimum angle degree, which is calculated as follows. First, the angle A_{ij} between solution i and j in an M-dimensional objective space is defined as the angle between the line connecting i and z^* and the line connecting j and z^*

$$A_{ij} = \arccos \frac{\sum_{k=1}^{M}\left[\left(f_k^i - z_k^*\right) * \left(f_k^j - z_k^*\right)\right]}{\sqrt{\sum_{k=1}^{M}\left(f_k^i - z_k^*\right)^2} * \sqrt{\sum_{k=1}^{M}\left(f_k^j - z_k^*\right)^2}} \tag{3}$$

where M is the number of objectives, f_k^i is the objective value of individual i in the kth objective, and z_k^* is the objective value of z^* in the kth objective.

Then, the minimum angle degree of i is

$$A_i^{\min} = \min_{j \in P, j \neq i} A_{ij} \tag{4}$$

where P represents the whole population. The larger the value of $A_i^{\min}$, the better the diversity performance of i is.

For each solution, its minimum angle degree reflects its diversity performance. From Tables I and II, traditional diversity measurements, such as crowding distance [7] and kth nearest distance [8], are still based on distance measures between solutions. However, there are two drawbacks. First, the diversity performance of each solution can be disturbed by its convergence degree. Under the distance measure, some solutions with very poor convergence may obtain better evaluation for its diversity performance. For instance, as stated in [15] and [18], the dominance resistant solutions are preferred by this diversity measurement. Second, this distance measure only considers the distribution among solutions but it cannot provide any information about the spread of solutions. That is, we are not sure whether every part of the true Pareto front has been covered by solutions. In literature, PBI approach tries to solve both drawbacks by separating convergence distance and diversity distance under the decomposition structure. Here, each solution is first associated with one weight vector,

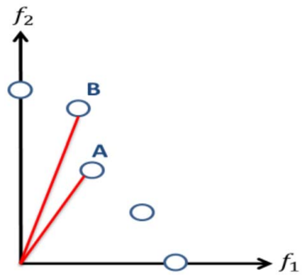

Fig. 1. One example in 2-D objective space.

then the distance between it and this vector is its diversity distance, which is used to evaluate the diversity performance of the solution. However, this diversity measure can only reflect how far between it and its associated weight vector. The distance between two solutions associated with different vectors is not taken into account in this measurement.

In our proposed method, diversity based on angle can adequately solve these difficulties. It is not influenced by any convergence degree of the solution. The smaller the minimum angle degree, the worse the diversity performance of the solution is. Let us look at an example of comparison between measurements based on angle and based on distance. In a 2-D objective space shown in Fig. 1, under the distance measure, the dominance resistant solution B has the best diversity performance among all of them. However, by the angle measure, solutions A and B have the minimum angle, which means A and B are closest to each other than any other pairs of solutions. Thus both of them have bad diversity.

C. Environmental Selection

In our design, the environmental selection focuses on the convergence and diversity performance of the whole population. That is, we keep N solutions so that those N solutions together achieve the best convergence and diversity performance than any other combination of N solutions. The process of environmental selection is summarized as follows. Among the combination of parents and offspring, each time we find a pair of solutions with the minimum angle. Then, for this pair of solutions, we need to identify one representative solution (vector) between them, which not only has more contribution to the convergence performance of the whole population, but also improves the diversity of the population. If the difference of two closest solutions' Euclidean distance is larger than a predefined threshold value, eliminate the one with the larger Euclidean distance to the obtained ideal point. Otherwise, eliminate the solution with smaller angle between it and other solutions. Continue eliminating solutions until the number of solutions is equal to N. For example, in Fig. 1, we first find that a pair of solutions A and B contains the minimum angle and the difference of their Euclidean distance is large, then we compare the Euclidean distances of them. Because solution B has larger distance to the obtained ideal point than that of A, B is eliminated and A is kept.

Most importantly, our environmental selection method can balance the convergence and diversity for the whole population. If two solutions are very close to each other in angle, then their converge contribution to the whole population are repeated. Therefore, we only keep the solution with better convergence between them. Elimination of one solution does not make the convergence performance of the whole population worse but improve the diversity of the whole population. Thus, our design decreases the conflicting degree between convergence and diversity, both of which can be improved simultaneously.

D. Mating Selection

In order to get high quality offspring, two issues need to be attended properly. First, in mating selection, both parents should be in high quality. After environmental selection, all solutions kept into the next generation have been distributed as apart as possible from each other. But each of them still has different diversity degrees since each one contains different minimum angle value. Meanwhile, all solutions may also contain different convergence degree. Here, ASF is used to measure the convergence degree of each solution along its special direction which is determined by its favorable weight vector. Therefore, the high quality parents should have smaller ASF value and larger minimum angle value among all solutions.

Both Euclidean distance and ASF measure the solution's convergence degree in complement ways. Euclidean distance measures the convergence degree of each solution based on its distance to the obtained ideal point, while ASF reflects the convergence degree of each solution along the specific direction corresponding to a weight vector.

Second, the combination of them should be effective. In an MaOP, the major difficulty for effective combination is the extremely large objective space [16]. In order to cope with it, decomposition based methods restrict the combination only in the same neighborhood. However, there is no guaranty that two solutions in the same neighbor are not too far away from each other since the neighbor is defined according to the position of weight vectors but not the solution itself. Furthermore, it is still possible that two solutions are very close to each other and their combination cannot generate offspring with good diversity. In our design, we solve this problem in two aspects. First, it is clear that two solutions are not too far away from each other when both of them are well converged. Therefore, in mating selection, we try to find solutions with smaller ASF values as parents. Second, if both solutions have good diversity performance, they are not too close to each other and their combination can ensure the diversity of offspring. Thus, mating selection also tries to pick up solutions with larger minimum angles. Moreover, the balance between convergence and diversity can also be achieved if each parent contains both characteristics simultaneously.

Based on the above discussions, the process of mating selection is summarized as below. Binary tournament selection plays the major role to select good parents for recombination. First, each solution's ASF value and minimum angle are calculated first. Then, randomly pick up two solutions (e.g., solutions A and B). If A has both smaller ASF value and larger minimum angle value than B, A is selected as the winner. On the other hand, if B has both smaller ASF value and larger minimum angle value than A, B is selected as the winner. If A is better in one measure but worse in the other, we treat both of them equally and randomly select one solution. After mating selection, one winner is chosen. Then, this winner is selected as a parent with a probability, which is called accepting probability and calculated based on this winner's ASF value. The accepting probability p for solution i is defined as

$$p = 1 - \frac{\text{ASF\_Rank}(i)}{N} + \varepsilon \tag{5}$$

where ASF_Rank(i) is the rank value of solution i's ASF value, the larger the ASF_Rank(i), the larger the ASF value and the worse the convergence performance of i is. $\varepsilon = 0.0002$ is applied to ensure $p > 0$. Therefore, the better the convergence degree of the solution, the larger the probability it is selected as the parent. If the winner is not selected, a solution in the population is randomly chosen. It is possible that some solutions with very poor convergence performance but acceptable diversity performance survive in the environmental selection and are put into the next generation. Because of this, we incorporate this probability into mating selection in order to restrict these much ill-converged solutions to be chosen as the parent. Finally, two solutions with well convergence and diversity performance are within the suitable distance.

E. Framework of the Proposed Algorithm

Algorithm 1 shows the general framework of MaOEA-CSS. In the initialization step, N initial solutions are generated. Meanwhile, their ASF values to the obtained ideal point and minimum angles are calculated. Then, mating selection chooses some parents with compromised convergence and diversity for offspring generation, where the distribution index of the crossover is set to 30 in order to make the offspring similar as parents. Finally, the offspring is used to update the parent population according to environmental selection criterion.

Algorithm 1 Framework of MaOEA-CSS

Input: population size (N), the number of generations (num), threshold value(t)

Step 1: Initialization
1) Generate an initial population with N individuals
2) Construct the obtained ideal point $z^* = \left(z_1^*, \ldots, z_M^*\right)^T$
3) The transformed objective value: $f_i^{'} = f_i - z_i^*$
4) Angle calculation: calculate minimum angle degree between each solution and all others.
5) ASF value calculation: calculate ASF value of each solution

Step 2: Mating Selection
1) Randomly pick up two solutions, if one solution has both smaller ASF value and larger minimum angle value than the other, it is selected as the winner; if it has both larger ASF value and smaller minimum angle value, the other is selected as the winner.
2) Otherwise, if one solution is better in one aspect and worse in the other aspect, both solutions are treated equal and randomly pick up one solution as the winner
3) Probability: $p = 1 - \frac{\text{ASF\_Rank}(i)}{N} + \varepsilon$
   if rand(1) $<p$
      The winner is selected as the parent
   else
      One solution from the population is randomly selected
   end

Step 3: Recombination
1) The distribution index of SBX crossover operator is set as 30
2) The distribution index of polynomial mutation is set to be 20

Step 4: Environmental Selection
1) Among the combination of parents and offspring, each time find a pair of individuals with the minimum angle
2) if the difference of their Euclidean distance $>t$,
      Eliminate one with the larger Euclidean distance
   else
      Eliminate the solution with smaller angle to others
   end
3) Go back to 1) until the number of individuals is equal to N
4) If the number of generations is larger than or equal to num, stop; otherwise, go back to Step 2.

Output: a group of solutions

IV. Experimental Results

A. Benchmark Problems

Two sets of widely used scalable many-objective benchmark functions are chosen for empirical studies. DTLZ test instances includes DTLZ1–DTLZ7 [32] and WFG toolkit has WFG1–WFG9 [33]. In the experiment, chosen MaOEAs are tested in 5-D and 10-D objective spaces of these benchmark problems. According to the recommendations in [33], the number of decision variables in DTLZ test instances is set as $n = M + k - 1$, where $k = 5$ for DTLZ1, $k = 10$ for DTLZ2-6, and $k = 20$ for DTLZ7. M denotes the number of objectives. As suggested in [34], in WFG test instances, the number of decision variables is set as $n = k + l$, where the number of position-related variables $k = M - 1$ and the number of distance-related variables $l = 20$.

Both DTLZ and WFG test problems contain a variety of problem characteristics presenting various degrees of complications to test underlying MaOEAs. The characteristics of all test instances are summarized in Table III. Since each dimension of WFG problems' objective space contains different scale, in the proposed method, at each generation of the evolutionary process, before calculation of Euclidian and

TABLE III
SUMMARY OF CHARACTERISTICS OF ALL TEST INSTANCES [33]

Characteristics	Test instances
Multi-modal	DTLZ1, DTLZ3, DTLZ6
Biased	DTLZ4, WFG1, WFG7, WFG8, WFG9
Non-separable	WFG2, WFG3, WFG6, WFG8, WFG9
Deceptive	WFG5, WFG9
Degenerate	DTLZ5, DTLZ6,DTLZ7, WFG3
Disconnect	DTLZ6, WFG2
Linear	DTLZ1, WFG3
Convex	WFG2
Mixed	DTLZ7, WFG1
Concave	DTLZ2, DTLZ3, DTLZ4, WFG4, WFG5, WFG6, WFG7, WFG8, WFG9

TABLE IV
IGD PERFORMANCE OF CHOSEN MAOEAS ON DTLZS

	M	MaOEA-CSS	MOEA/D	HypE	GrEA	NSGA-III	ε-MOEA	PICEA-g
DTLZ1	5	0.0982(0.0024)	0.1537(0.0021)+	0.1623(0.0514)+	**0.0891(0.0009)-**	0.1476(0.0043)+	0.1368(0.0105)+	0.1282(0.0035)+
	10	**0.1841(0.0123)**	0.2109(0.1317)=	0.2216(0.0483)+	0.2173(0.0621)+	0.2033(0.0684)=	0.2517(0.0693)+	0.2069(0.0215)=
DTLZ2	5	**0.1910(0.0412)**	0.2314(0.0099)+	0.4306(0.0121)+	0.2034(0.0024)=	0.2451(0.0053)+	0.2292(0.0178)+	0.2025(0.0029)=
	10	**0.4937(0.0624)**	0.5255(0.0418)+	0.4964(0.0289)=	0.4995(0.0202)=	0.5076(0.0143)=	0.5081(0.0367)=	0.5101(0.0064)=
DTLZ3	5	0.6122(0.0337)	0.5377(0.0009)-	0.8307(0.3498)+	**0.2824(0.0056)-**	0.4734(0.0316)-	0.6207(0.0093)=	0.5391(0.4811)-
	10	0.8691(0.0142)	0.8724(0.0024)=	0.9922(0.0297)+	0.8703(0.0114)=	**0.7483(0.3099)-**	1.4073(0.0503)+	0.8799(0.0672)=
DTLZ4	5	**0.2207(0.0019)**	0.5752(0.0915)+	0.4818(0.0435)+	0.2554(0.0663)+	0.2374(0.1103)=	0.3226(0.1067)+	0.2351(0.0802)=
	10	**0.4361(0.0081)**	0.8361(0.0705)+	0.6557(0.0673)+	0.4598(0.0072)+	0.4927(0.1125)+	0.5847(0.0708)+	0.4557(0.0579)+
DTLZ5	5	**0.0165(0.0031)**	0.0472(0.0519)+	0.1425(0.0376)+	0.0554(0.0112)+	0.1316(0.065)+	0.0786(0.0304)+	0.0277(0.0016)+
	10	**0.0211(0.0025)**	0.1797(0.0983)+	0.1034(0.0621)+	0.0934(0.0674)+	0.1449(0.0917)+	0.1799(0.0692)+	0.1412(0.0031)+
DTLZ6	5	**0.1285(0.0139)**	0.1537(0.2675)+	0.7213(0.1824)+	0.1454(0.0932)+	0.7538(0.2213)+	1.1045(0.1906)+	0.1471(0.0612)+
	10	0.8177(0.0122)	**0.5292(0.0921)-**	2.1275(0.7135)+	0.9925(0.5872)+	1.4094(0.2527)+	2.2191(0.2492)+	1.0032(0.0583)+
DTLZ7	5	0.5088(0.0378)	0.7471(0.3037)+	0.8043(0.5096)+	**0.3781(0.0348)-**	0.5519(0.3483)=	0.6009(0.3794)+	0.9812(0.1013)+
	10	**1.0127(0.2432)**	1.3446(0.7385)+	1.4803(0.9123)+	1.0972(0.0426)=	1.2311(0.7254)+	1.3866(0.9471)+	1.4101(0.1294)+
+/=/-			10/2/2	15/1/0	7/4/3	8/4/2	12/2/0	8/5/1

angle measures, we have normalized the objective values of each solution, where its ith objective value is divided by $2i$, $i = 1, \dots, M$.

B. Parameter Setting of Chosen MaOEAs

We choose six state-of-the-art MaOEAs, including MOEA/D, HypE, GrEA, NSGA-III, ε-MOEA, and PICEA-g for comparison. Each MaOEA represents one type of algorithms under the classification in Table II. The fitness assignment of HypE is according to quality indicator. GrEA is the grid based method. The design of NSGA-III emphases on diversity, while MOEA/D is decomposition based approach. ε-MOEA modifies the original Pareto dominance and PICEA-g applies preference.

In MOEA/D and NSGA-III, due to the combinatorial nature of uniformly distributed weight vectors, the population size cannot be arbitrarily specified. It is set as 126 in 5-D problems and 220 in 10-D problems. In order to draw a fair comparison, the population sizes in all MaOEAs considered are set at the same numbers as in MOEA/D. The stopping criterion is set at 1000 generations in 5-D problems and 1500 in 10-D problems. Initial populations are generated randomly from the search space in all MaOEAs chosen. The simulated binary crossover (SBX) and polynomial mutation are used. In all MaOEAs, the distribution indexes in SBX is set as 30 as suggested by [17] and in the polynomial mutation is set to be 20. The crossover rate is 1.00, while the mutation rate is $1/n$, where n is the number of decision variables. For GrEA, the parameter setting of grid size follows [11] in DTLZ problems and [15] in WFG problems. In ε-MOEA, the setting of ε in DTLZ problems follows [11] while the setting in WFG test problems follows [20]. In MOEA/D, the number of the weight vectors in the neighborhood of each weight vector is set to be 20 [14]. For HypE, according to [12], 10 000 sampling points are used. In PICEA-g [18], the number of goals is set to be $M \times 100$, where M is the number of objectives. For MaOEA-CSS, the threshold value t is set as 0.005 in DTLZ1, 0 in DTLZ2–DTLZ6, 0.3 in DTLZ7, 0.005 in WFG1–WFG3, and 0 in WFG4–WFG9. In DTLZ2–DTLZ6 and WFG4–WFG9, the threshold value is set as $t = 0$ since these problems contain concave Pareto fronts (e.g., $f_1^2 + f_2^2 + \cdots + f_M^2$), where all solutions on the true front have the same Euclidean distance.

C. Performance Metrics

In this experiment, two performance metrics are chosen. Inverted generational distance (IGD) concerns how well is the Pareto-optimal front represented by the obtained approximation front [35]. Hypervolume Indicator (also called S-metric) considers both closeness and diversity of obtained approximation front by calculating hypervolume [36], [37]. For IGD, the number of sampled reference points on the true Pareto front is set as 12 650 in 5-D DTLZ problems and 24 310 in 10-D DTLZ problems, which are calculated as suggested by [14]. Here, the way to generate these points in DTLZ1–DTLZ4 follows the approach in [15]. In DTLZ5–DTLZ7 with irregular Pareto front, we randomly sample these large numbers of points on the true Pareto front. In S-metric, for WFG1–WFG9, as suggested by [20], we choose the ith objective value of the reference point to be $2i + 1$.

D. Experiment Findings on Performance Metrics

In this section, Tables IV and V present performance metrics results on IGD in DTLZ problems and S-metric in WFG problems, given mean values and standard deviations (in brackets) for comparing MaOEA-CSS with six chosen MaOEAs in 5-D and 10-D objective spaces. These results (mean and standard deviation) are the average value of 30 independent trials under the same experimental setup.

In order to statistically analyze results of both IGD and S-metric, the Mann–Whitney–Wilcoxon rank-sum test [32] is applied to quantify whether one of the both fronts by independent observations tends to have better performance than the other in a statistically meaningful sense, especially when the performance metric values of two approximation fronts are very close to each other or even indifferentiable. This test compares MaOEA-CSS with respect to each chosen MaOEAs over 30 independent runs in all benchmark problems. The statistical results are also shown in Tables IV and V, following each competing MaOEAs' metric values except MaOEA-CSS. Here,

TABLE V
S-METRIC PERFORMANCE OF CHOSEN MAOEAS ON WFGS

	M	MaOEA-CSS	MOEA/D	HypE	GrEA	NSGA-III	ε-MOEA	PICEA-g
WFG 1	5	**4.538E3 (8.391E2)**	4.185E3 (1.985E2)=	3.670E3 (3.216E2)+	3.852E3 (1.783E2)+	3.904E3 (1.219E2)+	3.879E3 (1.435E2)+	4.063E3 (1.367E2)+
	10	4.314E9 (7.183E8)	4.317E9 (1.981E8)=	3.762E9 (1.076E8)+	4.297E9 (2.027E8)=	3.897E9 (2.435E8)+	3.824E9 (1.268E8)+	**4.421E3 (1.992E8)-**
WFG 2	5	**7.142E3 (8.782E2)**	6.721E3 (5.199E2)+	6.319E3 (1.743E2)+	6.674E3 (2.786E2)+	6.927E3 (1.023E2)=	6.845E3 (3.789E2)+	6.947E3 (3.004E2)=
	10	7.365E9 (7.763E8)	**7.547E9 (6.451E8)-**	7.298E9 (3.179E8)=	7.403E9 (3.974E8)=	7.259E9 (4.337E8)=	6.901E9 (2.432E8)+	7.402E9 (3.173E8)=
WFG 3	5	**5.835E3 (6.772E2)**	5.518E3 (4.875E2)=	5.238E3 (2.456E2)+	5.592E3 (2.789E2)-	5.601E3 (2.719E2)=	5.112E3 (2.763E2)+	5.574E3 (2.608E2)-
	10	**7.378E9 (5.642E8)**	6.402E9 (1.992E8)+	6.452E9 (1.098E8)+	6.637E9 (2.217E8)+	6.857E9 (3.753E8)+	6.274E9 (1.906E8)+	6.511E9 (1.536E8)+
WFG 4	5	**7.673E3 (5.964E2)**	6.836E3 (5.613E2)+	6.909E3 (1.328E2)+	7.170E3 (3.986E2)+	7.017E3 (3.441E2)+	7.206E3 (2.539E2)+	7.194E3 (2.254E2)+
	10	**9.374E9 (7.285E8)**	8.577E9 (2.443E8)+	7.823E9 (1.875E8)+	8.883E9 (2.754E8)+	8.613E9 (3.137E8)+	7.562E9 (1.975E8)+	8.739E9 (1.624E8)+
WFG 5	5	**8.085E3 (5.812E2)**	7.205E3 (2.511E2)+	7.315E3 (4.596E2)+	7.874E3 (2.242E2)=	7.339E3 (2.956E2)+	7.429E3 (2.473E2)+	7.543E3 (2.393E2)+
	10	**9.811E9 (9.482E8)**	8.571E9 (4.235E8)+	8.613E9 (3.581E8)+	9.195E9 (2.738E8)+	8.688E9 (2.132E8)+	8.607E9 (2.126E8)+	8.725E9 (1.718E8)+
WFG 6	5	**7.342E3 (3.450E2)**	6.084E3 (1.551E2)+	5.869E3 (1.135E2)+	6.276E3 (1.528E2)+	6.756E3 (5.148E2)+	6.633E3 (1.874E2)+	6.933E3 (1.472E2)+
	10	**9.993E9 (4.279E8)**	8.537E9 (3.015E8)+	7.856E9 (1.461E8)+	8.742E9 (1.336E8)+	9.018E9 (1.129E8)+	7.675E9 (1.236E8)+	9.999E9 (1.662E8)=
WFG 7	5	**5.940E3 (2.382E2)**	5.272E3 (1.556E2)+	5.037E3 (3.453E2)+	5.785E3 (3.743E2)=	5.363E3 (2.657E2)+	5.642E3 (3.996E2)=	5.841E3 (2.653E2)=
	10	6.521E9 (6.773E8)	6.369E9 (5.272E8)=	5.982E9 (2.471E8)+	6.455E9 (4.628E8)=	**6.776E9 (3.475E8)-**	5.764E9 (3.681E8)+	6.678E9 (2.917E8)=
WFG 8	5	7.277E3 (8.919E2)	6.737E3 (5.312E2)+	6.179E3 (3.024E2)+	6.803E3 (4.795E2)+	6.821E3 (3.205E2)+	7.023E3 (3.427E2)=	**7.712E3 (3.247E2)-**
	10	**7.914E9 (9.554E8)**	7.448E9 (5.489E8)+	7.216E9 (2.398E8)+	7.783E9 (3.892E8)=	7.202E9 (4.746E8)+	7.217E9 (2.375E8)+	7.315E9 (2.361E8)+
WFG 9	5	7.129E3 (6.116E2)	6.751E3 (2.723E2)+	6.533E3 (1.563E2)+	6.706E3 (1.695E2)+	6.946E3 (2.317E2)=	6.991E3 (2.437E2)=	**7.854E3 (4.109E2)-**
	10	**8.875E9 (5.796E8)**	7.835E9 (3.784E8)+	8.146E9 (1.872E8)+	8.311E9 (2.475E8)+	7.970E9 (2.173E8)+	8.062E9 (4.246E8)+	8.015 (2.439E8)+
+/=/-			13/4/1	17/1/0	11/6/1	13/4/1	15/3/0	9/5/4

symbols "+," "=," and "−" denote that MaOEA-CSS's performance is statistically better than, equivalent to, and worse than a competing algorithm in the corresponding column, respectively. These results are generated by testing the hypothesis that there is no significant difference of performance metric values between MaOEA-CSS and other algorithms. An inferential statistical testing calculates the *p-value* which is the probability that the null hypothesis is true. If it is lower than the chosen significance level, then the null hypothesis can be rejected in favor of an alternative test H_1 [38]. This user-defined significance level is chosen as 5%. Summaries of statistical test results in each benchmark function are shown at the bottom of Tables IV and V.

According to Table IV, MaOEA-CSS achieves nine best IGD scores out of all 14 DTLZ test cases. In 5-D DTLZ1 and DTLZ7, GrEA is the best algorithm and MaOEA-CSS comes as the closest second. Also, in 10-D problem, MaOEA-CSS is only worse than NSGA-III in DTLZ3 and MOEA/D in DTLZ6, while it is better than other algorithms. Furthermore, statistical comparison results at the bottom of the Table IV confirm our observation from performance metrics result that MaOEA-CSS wins most competitions with respect to all competing MaOEAs.

For 5-D WFG problems, from Table V, MaOEA-CSS shows better performance on *S*-metric scores in WFG1–WFG7. In WFG8 and WFG9, it is only worse than PICEA-g. For 10-D WFG problems, MaOEA-CSS performs better than others in WFG3–WFG9 while it is worse than PICEA-g, MOEA/D, and NSGA-III in WFG1, WFG2, and WFG7, respectively. In 10-D WFG2, although the *S*-metric value of MaOEA-CSS is lower than HypE and GrEA, all of them achieve the same performance in the statistical sense. The summary of statistical comparison results in the last row of Table V supports our observation from *S*-metrics result that MaOEA-CSS wins most competitions among all competing MaOEAs in WFG problems.

Specially, in the test problems with multimodel characteristics, such as DTLZ1, DTLZ3, and DTLZ6, among all six test cases, MaOEA-CSS and GrEA win twice while NSGA-III and MOEA/D win once. Although the winning times of MaOEA-CSS and GrEA are equal, MaOEA-CSS performs as the second best algorithm twice (5-D DTLZ1 and 10-D DTLZ6) while GrEA only does once (5-D DTLZ6). Therefore, MaOEA-CSS has the best performance than all others in solving multimodel problems. Furthermore, in bias problems including DTLZ4, WFG1, WFG7, WFG8, and

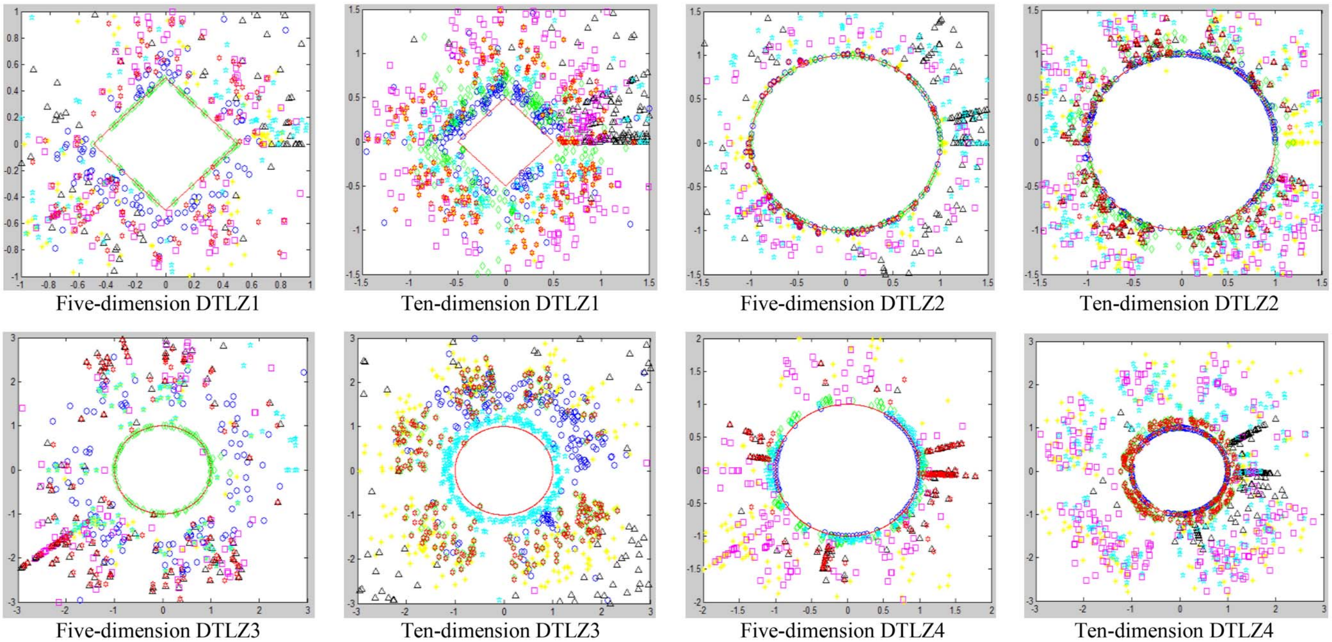

Fig. 2. Visualization of approximated Pareto fronts by all MaOEAs in DTLZ1–DTLZ4.

WFG9, MaOEA-CSS performs much better than other algorithms. Among all ten test instances, it wins six competitions. Similarly, in nonseparable problems WFG2, WFG3, WFG6, WFG8, and WFG9, MaOEA-CSS wins seven times. Then, in deceptive problems WFG5 and WFG9, MaOEA-CSS wins three test cases. The ability to handle with deceptive characteristic is also helpful to solve problems with multi-model or lots of local optima points. Finally, in degenerate problems (e.g., DTLZ5, DTLZ7, and WFG3) and disconnected problems (e.g., DTLZ6 and WFG2), MaOEA-CSS wins seven out of ten instances. In summary, MaOEA-CSS ensures a better or at least competitive performance in comparison with six state-of-the-art MaOEAs at most benchmark problems containing different characteristics from a statistical perspective.

E. Visualization of Experimental Results in DTLZ1–DTLZ4

In this section, the visualization method proposed in [39], which is an approach for the visualization of high-dimensional objective space, is applied to visualize the approximated fronts generated by all MaOEAs considered in both 5-D and 10-D DTLZ1–DTLZ4. This method maps individuals from a high-dimensional space into a 2-D polar coordinate system with pole (0, 0), where each individual is assigned a radial coordinate value and an angular coordinate value. Radial coordinate reflects convergence performance of each individual and is determined by the original objective value of each individual and the shape of the approximate front in the original high-dimensional space. The smaller the radial coordinate value, the closer the distance is to the true Pareto front is. On the other hand, angular coordinate reveals distribution of individuals on the approximate front. It also shows crowdedness in each sub region of the original high-dimensional space. The more number of different angular coordinate values among all individuals, the better distribution and spread of the approximate front is.

Fig. 2 directly shows the performance of each algorithm in DTLZ1–DTLZ4 by providing the visualization of its approximated fronts. In each subplot of Fig. 2, "red line" represents the true Pareto front, "blue circle" denotes the approximate front obtained by MaOEA-CSS, "yellow star" represents MOEA/D, "black triangle" is HypE, "green diamond" refers to GrEA, "cyan pentagram" corresponds to NSGA-III while "purple square" is ε-MOEA, and "red hexagram" is PICEA-g. In each subplot, under the same DTLZ problem, the mapped 5-D true Pareto front is the same as the mapped 10-D true Pareto front (e.g., the true Parent front of 5-D DTLZ1 is the same as that of 10-D DTLZ1). They appear differently because the scales are different to allow the best visualization of the approximate fronts generated by all seven MaOEAs. In 5-D DTLZ1, GrEA (green diamond) nearly converges to the true Pareto front (red line) and also distributes well along the front. MaOEA-CSS (blue circle) has a little bit worse convergence and diversity performance as GrEA, but it is significantly better than other algorithms. HypE (black triangle), MOEA/D (yellow star), and NSGA-III (cyan pentagram) present very similar performance, which is also reflected by their very close IGD scores in Table IV. In 10-D DTLZ1, MaOEA-CSS (blue circle) is the closest to the true front and nearly enclose the entire front, which means it achieves very best in both convergence and diversity performance as also reflected by its IGD scores. NSGA-III (cyan pentagram), MOEA/D (yellow star), and GrEA (green diamond) with nearly equal IGD scores, still has similar approximated fronts.

In 5-D DTLZ2, MaOEA-CSS (blue circle) and GrEA (green diamond) achieves nearly equal performance and much better than others. In 10-D DTLZ2, MaOEA-CSS (blue circle) still shows both best convergence and diversity performance although its IGD score is similar to others in Table IV.

In 5-D DTLZ3, GrEA (green diamond) attains much better performance than others. Although MaOEA-CSS (blue circle) does not have very well convergence performance, it contains very good diversity performance. In 10-D DTLZ3, NSGA-III (cyan pentagram) achieves the best approximated front while MaOEA-CSS (blue circle), MOEA/D (yellow star), PICEA-g (red hexagram), and GrEA (green diamond) share similar performance, which is also suggested by their IGD scores.

In 5-D DTLZ4, MaOEA-CSS (blue circle) is the overall winner and followed by NSGA-III (cyan hexagram). In 10-D DTLZ4, MaOEA-CSS (blue circle) wins over all other algorithms again and PICEA-g (red hexagram) and GrEA (green diamond) come as the second best designs.

In summary, visualization results in Fig. 2 support our conclusion from the IGD scores in Table IV that MaOEA-CSS wins most competitions with respect to all chosen competing MaOEAs.

F. Comparison Among Three Different Variants

In this section, we investigate the validation of the proposed mating selection and environmental selection approaches by comparing MaOEA-CSS with two variants. In variant I, at the mating selection, the ASF measurement is replaced by the Euclidean distance measure. That is, the convergence measurement in the mating selection is based on the Euclidean distance from each individual to the obtained ideal point rather than ASF value. This variant aims to validate whether ASF is an effective convergence measurement for mating selection. For variant II, the diversity measurement in both mating and environmental selection is based on the distance among different individuals rather than the angle among them. The comparison between this variant and MaOEA-CSS can determine whether the diversity performance can be enhanced by angle. These two variants are tested on DTLZ1–DTLZ7 under the same parameter setting as in Table IV, the Mann–Whitney–Wilcoxon rank-sum test [32] is also applied to check whether there is a significant performance difference between MaOEA-CSS and two variants. The experiment results are shown in Table VI.

Experiment results in Table VI show that both variants I and II perform worse than MaOEA-CSS. One interesting finding is that variant I (Euclidean distance measure replacing ASF in mating selecting) can achieve similar IGD values as those of MaOEA-CSS. However, the computational complexity of Euclidean distance measurement is $O(MN^2)$ while that of ASF value calculation is only $O(MN)$. Therefore, convergence measurement based on ASF can achieve a better convergence performance with a smaller computational effort. On the other hand, variant II cannot achieve a good IGD score which demonstrates that diversity measurement based on angle does improve the algorithm's overall performance.

TABLE VI
COMPARISON OF MAOEA-CSS AND ITS TWO VARIANTS BY IGD

	M	MaOEA-CSS	Variant I	Variant II
DTLZ 1	5	**0.0982(0.0024)**	0.1056(0.0065) =	0.1879(0.0057) +
	10	**0.1921(0.0056)**	0.1898(0.0096) =	0.2475(0.0138) +
DTLZ 2	5	**0.1910(0.0412)**	0.2078(0.0239) =	0.4008(0.0417) +
	10	**0.4937(0.0624)**	0.5142(0.0748) +	0.8905(0.0374) +
DTLZ 3	5	**0.6122(0.0337)**	0.6433(0.0521) +	1.1103(0.0387) +
	10	**0.8691(0.0142)**	0.9921(0.0137) +	1.6995(0.0197) +
DTLZ 4	5	**0.2207(0.0019)**	0.2255(0.0025) =	0.6305(0.0022) +
	10	**0.4361(0.0081)**	0.4441(0.0098) =	0.9825(0.0059) +
DTLZ 5	5	**0.0165(0.0031)**	0.0223(0.0047) =	0.1299(0.0074) +
	10	**0.0211(0.0025)**	0.0275(0.0101) =	0.1447(0.0066) +
DTLZ 6	5	**0.1285(0.0139)**	0.1383(0.0143) +	0.3087(0.0226) +
	10	**0.8177(0.0122)**	0.8309(0.0131) +	1.7720(0.0364) +
DTLZ 7	5	**0.5421(0.0378)**	0.5514(0.0231) =	1.1622(0.0277) +
	10	**1.0739(0.2304)**	1.1124(0.1873) =	2.0146(0.1619) +
+/=/-			5/9/0	14/0/0

G. Computational Complexity Analysis

In this section, given the number of objectives to be M and the population size as N, we show an upper bound of the computational complexity for one generation of MaOEA-CSS. At the initialization step (step 1 in Algorithm 1), objective evaluation of one solution requires $O(M^2)$ computations. Then objective evaluations of all solutions require $O(M^2N)$ computations. Within one generation, there are three main steps: 1) mating selection; 2) recombination; and 3) environmental selection. In mating selection (step 2 in Algorithm 1), angle calculation needs a runtime of $O(N^2)$ while Euclidean distance calculation requires $O(MN)$ calculations. In order to form a mating pool of size N, N binary tournament selections with $O(N)$ computations are needed. Since $M \ll N$ in general, the computation complexity in this step is $O(N^2)$. In recombination step (step 3 in Algorithm 1), the running time required by SBX and polynomial mutation is the largest value among $O(N^2)$, $O(M^2N)$, and $O(Nn)$, where n is the number of decision variables. Because $N > M^2 > n$ in most cases, the total running time in this generation is $O(N^2)$. Finally, at environmental selection step (step 4 in Algorithm 1), in the worst case scenario, $O(N^2)$ computations are needed to select N solutions. Therefore, the total computation complexity of MaOEA-CSS is $O(N^2)$.

V. CONCLUSION

Selection strategy, including mating selection and environmental selection, is a critical issue in the design of evolutionary multiobjective optimization. In order to handle with all difficulties arisen in MaOPs, an advanced strategy is essential for MaOEAs. In this paper, we first provide a comprehensive analysis on the selection strategy in the current MaOEAs. Afterward, we propose a new selection strategy to improve the performance significantly. This selection strategy considers three crucial factors: 1) the new mating selection criterion considers both the quality of each selected parent and the effectiveness of the combination of selected parents; 2) the new environmental selection criterion directly focuses on the performance of the whole population rather than on single individual alone; and 3) both selection steps are complement to each other and the coordination of them in the evolutionary process can achieve a better performance than each of them used alone. Furthermore, in order to handle the curse of dimensionality in MaOPs, a new convergence measurement by distance and a new diversity measurement by angle are developed in both selection steps.

From the experimental results and analysis, the proposed MaOEA-CSS ensures a better performance in both convergence and diversity given a large number of benchmark problems with various problem characteristics. The performance improvement is gained directly by this advanced selection strategy with the new convergence and diversity measurements. In the future research, this advanced selection strategy will be further extended in solving constrained [40], [41] and dynamic [42] MaOPs. Furthermore, in order to solve real-world problems, our new selection strategy with decision maker's preference incorporated can be used to search for a subset of Pareto-optimal solutions mandated by the decision maker.

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