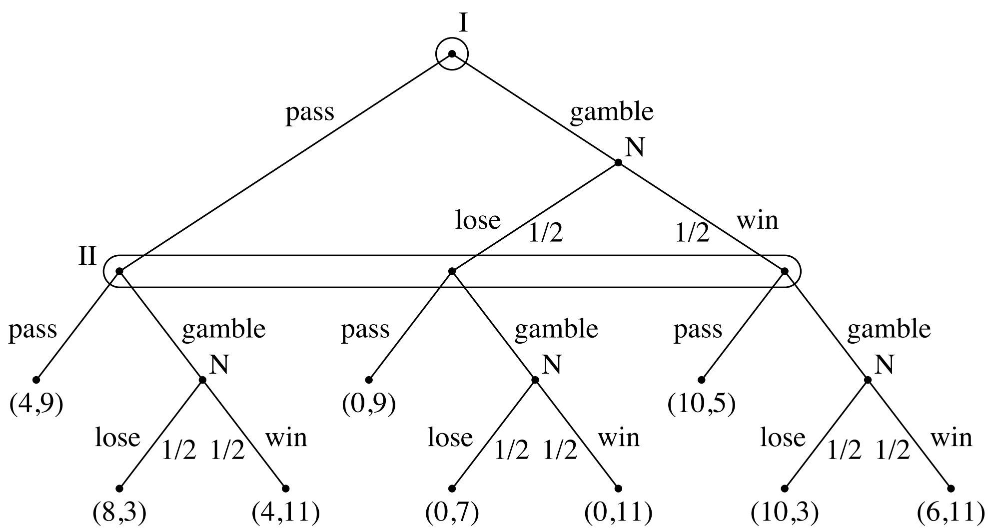
Solutions to Exercises of Section III.1.

1. The bimatrix is $\begin{array}{c c} & \begin{matrix} c & \quad d \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{pmatrix} (13/4, 3) & (22/4, 3/4) \\ (4, 10/4) & (21/4, 2) \end{pmatrix} \end{array}$.

2. (a) Player I's maxmin strategy is $(1, 0)$ (i.e. row 1) guaranteeing him the safety level $v_I = 1$. Player II's maxmin strategy is $(1, 0)$ (i.e. column 1) guaranteeing her the safety level $v_{II} = 1$.

(b) Player I's maxmin strategy is $(1/2, 1/2)$ guaranteeing him the safety level $v_I = 5/2$. Player II's maxmin strategy is $(3/5, 2/5)$ guaranteeing her the safety level $v_{II} = 8$.

3. (a) There are many ways to draw the Kuhn tree. Here is one. The payoffs are in units of 100 dollars.



(b) The bimatrix is:

$$\begin{array}{c c} & \begin{matrix} \text{gamble} & \text{pass} \end{matrix} \\ \begin{matrix} \text{gamble} \\ \text{pass} \end{matrix} & \begin{pmatrix} (4, 8) & (5, 7) \\ (6, 7) & (4, 9) \end{pmatrix} \end{array}$$

(c) Player I's safety level is $v_I = 14/3$. Player II's safety level is $v_{II} = 23/3$. Both maxmin strategies are $(2/3, 1/3)$.

4. Let Q denote the proportion of students in the class (excluding yourself) who choose row 2. If you choose row 2 you win $Q \cdot 6$ on the average. If you choose row 1, you win 4. So you should choose row 2 only if you predict that at least 2/3 of the rest of the class will choose row 2.

In my classes, only between 15% and 35% of the students chose row 2. If your classes are like mine, you should choose row 1.

Solutions to Exercises of Section III.2.

1. Let $\boldsymbol{p}_0$ denote the maxmin strategy of Player I, and let $(\boldsymbol{p}, \boldsymbol{q})$ be any strategic equilibrium. Then, $v_I \le \boldsymbol{p}_0'\boldsymbol{A}\boldsymbol{q}$ since use of $\boldsymbol{p}_0$ guarantees Player I at least v_I no matter what Player II does. But also $\boldsymbol{p}_0'\boldsymbol{A}\boldsymbol{q} \le \boldsymbol{p}'\boldsymbol{A}\boldsymbol{q}$ since $\boldsymbol{p}$ is a best response to $\boldsymbol{q}$. This shows $v_I \le \boldsymbol{p}'\boldsymbol{A}\boldsymbol{q}$. Then $v_{II} \le \boldsymbol{p}'\boldsymbol{A}\boldsymbol{q}$ follows from symmetry.

2(a) The safety levels are $v_I = 2$ and $v_{II} = 16/5$. The corresponding MM strategies are $(0, 1)$ (the second row) for Player I, and $(1/5, 4/5)$ (the equalizing strategy on $\boldsymbol{B}$) for Player II. The unique SE is the PSE in the lower left corner with payoff $(2, 4)$. It may be found by removing strictly dominated rows and columns.

(b) The safety levels are $v_I = 2$ and $v_{II} = 5/2$. The corresponding MM strategies are $(0, 1)$ for Player I, and $(1/2, 1/2)$ for Player II. There are no pure SE's, and the unique SE is the one using the equalizing strategies $(1/4, 3/4)$ for Player I on Player II's payoff matrix, and $(1/2, 1/2)$ for Player II on Player I's payoff matrix. The vector payoff is $(5/2, 5/2)$. Note that Player II's equalizing strategy is not an optimal strategy on Player I's matrix.

(c) The safety levels are $v_I = 0$ and $v_{II} = 0$. The corresponding MM strategies are $(1, 0)$ for Player I, and $(1, 0)$ for Player II. There is no PSE. The unique SE is the one using equalizing strategies, $(3/4, 1/4)$ for Player I and $(1/2, 1/2)$ for Player II, with payoff vector $(0, 0)$.

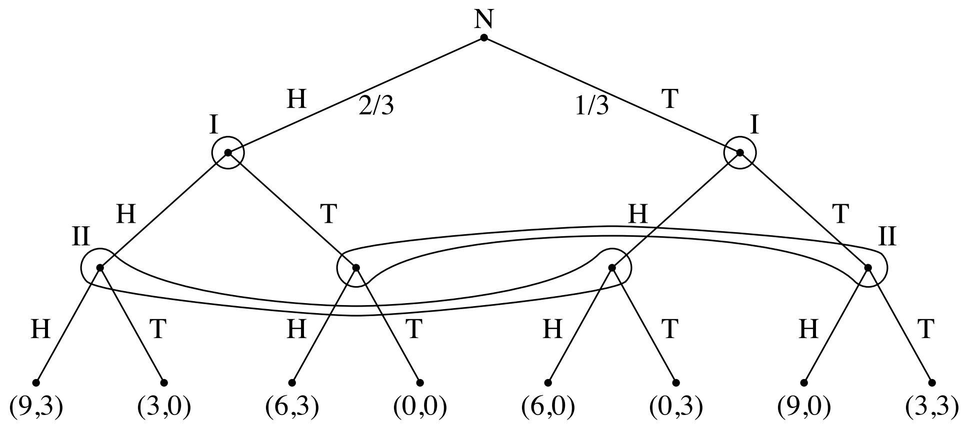
3. (a) The bimatrix is

$$\begin{array}{l} \\ \text{chicken} \\ \text{iron nerves} \end{array} \begin{array}{c} \begin{array}{cc} \text{chicken} & \text{iron nerves} \end{array} \\ \begin{pmatrix} (1,1) & (-1,2) \\ (2,-1) & (-2,-2) \end{pmatrix} \end{array}$$

(b) I's matrix has a saddle point with value $v_I = -1$, achievable if I uses the top row. Similarly, II's MM strategy is the first column, with value $v_{II} = -1$. Thus the safety levels are -1 for both players. Yet if both players play their MM strategies, the happy result is that they both receive $+1$.

(c) There are two PSE's: the lower left corner and the upper right corner. The third SE involves mixed strategies and may be found using equalization. The mixed strategy (1/2,1/2) for II is an equalizing strategy for I's matrix (even though it is not optimal there). Against this strategy, the average payoff to I is zero. Similarly, the strategy (1/2,1/2) for I is equalizing for II's matrix, giving an average payoff of zero. Thus, $((1/2, 1/2), (1/2, 1/2))$ is a mixed SE with payoff vector, (0,0).

4.(a)



(b)

$$\begin{array}{c} \\ \text{HH} \\ \text{HT} \\ \text{TH} \\ \text{TT} \end{array} \begin{array}{c} \begin{array}{cccc} \text{HH} & \text{HT} & \text{TH} & \text{TT} \end{array} \\ \begin{pmatrix} (8,2^*) & (8^*,2^*) & (2,1) & (2,1) \\ (9^*,2) & (7,3^*) & (5,0) & (3^*,1) \\ (6,2) & (2,0) & (4,3^*) & (0,1) \\ (7,2^*) & (1,1) & (7^*,2^*) & (1,1) \end{pmatrix} \end{array}$$

(c) There are two PSE's, those with double asterisks.

5.(a) We star the entries of the A matrix that are maxima of their column and entries of the B matrix that are maxima of their row.

$-3\ ,-4$	$2^*,-1$	$0\ ,\ \ 6^*$	$1^*,\ \ 1$
$2^*,\ \ 0$	$2^*,\ \ 2^*$	$-3\ ,\ \ 0$	$1^*,-2$
$2^*,-3$	$-5\ ,\ \ 1^*$	$-1\ ,-1$	$1^*,-3$
$-4\ ,\ \ 3^*$	$2^*,-5$	$1^*,\ \ 2$	$-3\ ,\ \ 1$

The only doubly starred entry occurs in the second row, second column, and hence the unique PSE is $\langle 2,2\rangle$.

(b) Starring the entries in a similar manner leads to the matrix

$0\ ,\ \ 0$	$1^*,-1$	$1^*,\ \ 1^*$	$-1\ ,\ \ 0$
$-1\ ,\ \ 1^*$	$0\ ,\ \ 1^*$	$1^*,\ \ 0$	$0^*,\ \ 0$
$1^*,\ \ 0$	$-1\ ,-1$	$0\ ,\ \ 1^*$	$-1\ ,\ \ 1^*$
$1^*,-1$	$-1\ ,\ \ 0^*$	$1^*,-1$	$0^*,\ \ 0^*$
$1^*,\ \ 1^*$	$0\ ,\ \ 0$	$-1\ ,-1$	$0^*,\ \ 0$

We find there are three doubly starred squares, and hence three PSE's, namely, $\langle 5,1\rangle$ and $\langle 1,3\rangle$ and $\langle 4,4\rangle$.

6.(a) $v_I = 0$ and $v_{II} = 2/3$.

(b) There is a unique PSE at row 2, column 1, with payoff vector (0,1).

(c) The mixed strategy $(1/3, 2/3)$ is the unique equalizing strategy for Player I. Column 1 is an equalizing strategy for II, but so is the mixture, $(0, 2/3, 1/3)$. More generally, any mixture of the form $(1-p, 2p/3, p/3)$ for $0 \le p \le 1$ is an equalizing strategy for II. Therefore, any of the strategy pairs, $(1/3, 2/3)$ for I and $(1-p, 2p/3, p/3)$ for $0 \le p \le 1$ for II, gives a strategic equilibrium. There are also some non-equalizing strategy pairs forming a strategic equilibrium, namely $(p, 1-p)$ for $0 \le p \le 1/3$ for I, and column 1 for II.

7. We are given $a_{1j} < \sum_{i=2}^m x_i a_{ij}$ for all j, where $x_i \ge 0$ and $\sum_{i=2}^m x_i = 1$. Suppose $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ is a strategic equilibrium. Then

$$\sum_j \sum_i p_i^* a_{ij} q_j^* \ge \sum_j \sum_i p_i a_{ij} q_j^* \quad \text{for all } \boldsymbol{p} = (p_1, \ldots, p_m). \tag{$*$}$$

We are to show $p_1^* = 0$. Suppose to the contrary that $p_1^* > 0$. Then

$$\begin{aligned}
\sum_j \sum_i p_i^* a_{ij} q_j^* &= \sum_j [p_1^* a_{1j} q_j^* + \sum_{i=2}^m p_i^* a_{ij} q_j^*] \\
&< \sum_j [p_1^* (\sum_{i=2}^m x_i a_{ij}) q_j^* + \sum_{i=2}^m p_i^* a_{ij} q_j^*] \quad \text{(strict inequality)} \\
&= \sum_j \sum_{i=2}^m (p_1^* x_i + p_i^*) a_{ij} q_j^* = \sum_j \sum_i p_i a_{ij} q_j^*
\end{aligned}$$

where $p_1 = 0$ and $p_i = p_1^* x_i + p_i^*$ for $i = 2, \ldots, m$. But The p's are nonnegative and add to one, so this contradicts (*).

8. (a) We have

$$A = \begin{pmatrix} 3 & 2 & 3 \\ 6 & 0 & 3 \\ 4 & 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 6 & 4 & 5 \end{pmatrix}$$

In A, the third row second col is a saddle point. So $v_I = 3$ and the third row is a maxmin strategy for Player I. In B, the row 3 is dominated by row 1, and col 2 is an equal probability mixture of col 1 and col 3. With these removed, the resulting 2 by 2 matrix has value $v_{II} = 2.5$. The maxmin strategy for Player II is $(1/4, 0, 3/4)$. (Another maxmin strategy for II is $(0, 1/2, 1/2.)$

(b) There are no PSE's. In A, row 1 is strictly dominated by row 3 and may be removed from consideration. Then in B, col 2 is strictly dominated by col 3 and may be removed. In the resulting 2 by 2 bimatrix game, there is a unique SE. It is given by equalizing strategies, $(1/3, 2/3)$ for I and $(1/3, 2/3)$ for II. In the original game, the unique SE is $(0, 1/3, 2/3)$ for Player I and $(1/3, 0, 2/3)$ for Player II. The equilibrium payoff is $(4, 4\frac{1}{3})$.

9. (a) At II's information set, a dominates b, so that vertex is worth $(1,0)$. Then at I's information set, B dominates A, so the PSE found by backward induction is (B,a), having payoff $(1,0)$. This is a subgame perfect PSE.

(b)

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} \quad a \quad & \quad b \quad \end{array} \\ \begin{pmatrix} (0,1) & (0,1) \\ (1,0) & (-10,-1) \end{pmatrix} \end{array}$$

(c) There are two PSE's, the lower left and the upper right. The lower left, (B,a), is the subgame perfect PSE. The upper right, (A,b), corresponds to the the PSE where Player I plays A because he believes Player II will play b. This is not subgame perfect because at Player II's vertex, it is not an equilibrium for Player II to play b.

10. There were 18 answers for Player I and 17 for Player II. The data is as follows.

I:	Stop at	No.	Score	II:	Stop at	No.	Score
	(1,1)	10	17		(0,3)	8	34
	(98,98)	2	882		(97,100)	2	806
	(99,99)	4	887		(98,101)	7	804
	never	2	880				

Scores for Player I ranged from 17 for those who selected $(1,1)$, to 887 for those who selected (99,99). Scores for Player II ranged from 34 for those who chose (0,3) to 806 for those who chose (97,100). Total scores ranged from 51 to 1693. Those who scored above 1000 received 5 points. Those who scored between 500 and 1000 received 3 points. Those who scored less than 100 recieved 1 point.

Solutions to Exercises of Section III.3.

1.(a) I's strategy space is $X = [0, \infty)$ and II's strategy space is $Y = [0, \infty)$. If I chooses $q_1 \in X$ and II chooses $q_2 \in Y$, the payoffs to I and II are

$$u_1(q_1, q_2) = q_1(a - q_1 - q_2)^+ - c_1 q_1, \qquad u_2(q_1, q_2) = q_2(a - q_1 - q_2)^+ - c_2 q_2$$

respectively. To find a PSE, we set derivatives to zero:

$$\frac{\partial}{\partial q_1} u_1(q_1, q_2) = a - 2q_1 - q_2 - c_1 = 0, \qquad \frac{\partial}{\partial q_2} u_2(q_1, q_2) = a - q_1 - 2q_2 - c_2 = 0.$$

The unique solution is (q_1^*, q_2^*), where

$$q_1^* = (a + c_2 - 2c_1)/3, \qquad q_2^* = (a + c_1 - 2c_2)/3.$$

Since we have assumed $c_1 < a/2$ and $c_2 < a/2$, both these production points are positive. Thus (q_1^*, q_2^*) is a PSE. Its payoff vector is $((a + c_2 - 2c_1)^2/9, (a + c_1 - 2c_2)^2/9)$.

(b) I's profit is $v_1(x, y) = x(17 - x - y) - x - 2 = x(16 - x - y) - 2$. II's profit is $v_2(x, y) = y(17 - x - y) - 3y - 1 = y(14 - x - y) - 1$. For fixed y, I should choose x so that $\partial v_1/\partial x = 16 - 2x - y = 0$. For fixed x, II should choose y so that $\partial v_2/\partial y = 14 - x - 2y = 0$. The equilibrium point is achieved if these two equations are satisfied simultaneously. This gives $x = 6$ and $y = 4$. The equilibrium payoff is $(36 - 2, 16 - 1) = (34, 15)$.

2. We assume $c < a$ — otherwise no company will produce anything. The payoff functions are

$$u_i(q_1, q_2, q_3) = q_i P(q_1 + q_2 + q_3) - c q_i = q_i[(a - q_1 - q_2 - q_3)^+ - c]$$

for $i = 1, 2, 3$. Assuming $q_1 + q_2 + q_3 < a$, there will be equilibrium production if the following three equations are satisfied:

$$\frac{\partial}{\partial q_i} u_i(q_1, q_2, q_3) = a - q_i - q_1 - q_2 - q_3 - c = 0$$

for $i = 1, 2, 3$. This solution is easily found to be $q_i = (a - c)/4$ for $i = 1, 2, 3$. This is the equilibrium production. The total production is $(3/4)(a - c)$, compared to $(2/3)(a - c)$ for the duopoly production, and $(1/2)(a - c)$ for the monopoly production.

3. The profit functions are

$$u_1(p_1, p_2) = (a - p_1 + bp_2)^+(p_1 - c) \quad \text{and} \quad u_2(p_1, p_2) = (a - p_2 + bp_1)^+(p_2 - c).$$

Knowing Player I's choice of p_1, Player II would choose p_2 to maximize $u_2(p_1, p_2)$. As in the Bertrand model with differentiated products, we find

$$\frac{\partial}{\partial p_2} u_2(p_1, p_2) = a - 2p_2 + bp_1 + c = 0 \quad \text{and hence} \quad p_2(p_1) = (a + bp_1 + c)/2.$$

Knowing Player II will use $p_2(p_1)$, Player I would choose p_1 to maximize $u_1(p_1, p_2(p_1))$. We have

$$\frac{\partial}{\partial p_1} u_1(p_1, p_2(p_1)) = a - p_1 + (b/2)(2 + bp_1 + c) - (p_1 - c)(2 - b^2)/2 = 0.$$

Hence, solving for p_1 and substituting into p_2 gives

$$p_1^* = \frac{a(2+b) + c(2+b-b^2)}{2(2-b^2)} \qquad \text{and} \qquad p_2^* = \frac{a+c}{2} + \frac{b}{2} \cdot \frac{a(2+b) + c(2+b-b^2)}{2(2-b^2)}$$

as the PSE.

Both p_1^* and p_2^* are greater than $(a + c)/(2 - b)$, so both players charge more than in the Bertrand model. Surprisingly, both players receive more from the sequential PSE than they do from the PSE of the Bertrand model. However, Player I receives more than Player II. (This model is suspect. Do not assume these results hold in general.)

4. (a) $u(Q) = QP(Q) - Q$, so $u'(Q) = P(Q) + QP'(Q) - 1 = (3/4)Q^2 - 10Q + 25$. This quadratic function has roots $Q = 10/3$ and $Q = 10$. The maximum of $u(Q)$ on the interval $[0, 10]$ is at $Q = 10/3$, so this is the monopoly production. The monopoly price is $P(10/3) = 109/9$ and the return of this production is $u(10/3) = 1000/27 = 37+$.

(b) $u_1(q_1, q_2) = q_1 P(q_1 + q_2) - q_1$, so

$$\begin{aligned}\frac{\partial}{\partial q_1} u_1(q_1, \frac{5}{2}) &= P(q_1 + \frac{5}{2}) + q_1 p'(q_1 + \frac{5}{2}) - 1 \\ &= \frac{3}{4}(q_1^2 - 10q_1 + \frac{75}{4})\end{aligned}$$

This has roots $q_1 = 5/2$ and $q_1 = 15/2$. The maximum occurs at $q_1 = 5/2$, and for $q_1 > 15/2$, $u_1(q_1, 5/2) = 0$. This shows that the optimal reply to $q_2 = 5/2$ is $q_1 = 5/2$. But the situation is symmetric, so the optimal reply of firm 2 to $q_1 = 5/2$ of firm 1, is $q_2 = 5/2$ also. This shows that $q_1 = q_2 = 5/2$ is a PSE.

5. (a) If Firm 2 knows Firm 1 is producing q_1, then Firm 2 will produce $q_2 \in [0, a]$ to maximize $q_2(a - q_1 - q_2)^+ - c_2 q_2$. This gives

$$q_2(q_1) = \begin{cases} (a - q_1 - c_2)/2 & \text{if } q_1 < a - c_2 \\ 0 & \text{if } q_1 \geq a - c_2 \end{cases}$$

as in Equation (13). Therefore Firm 1 wull choose to produce $q_1 \in [0, a]$ to maximize the payoff

$$u_1(q_1) = \begin{cases} q_1(a - 2c_1 + c_2)/2 - q_1^2/2 & \text{if } q_1 < a - c_2 \\ q_1(a - c_1) - q_1^2 & \text{if } q_1 \geq a - c_2. \end{cases}$$

The two functions, $f_1(q) = q(a-2c_1+c_2)/2-q^2/2$ and $f_2(q) = q(a-c_1)-q^2$, are quadratic and agree at $q = 0$ and at $q = a - c_2$. Since the difference, $f_1(q) - f_2(q)$ is also quadratic, and the slope of $f_1(q)$ at 0 is less than the slope of $f_2(q)$ at 0, we have $f_1(q) < f_2(q)$ for all $q \in (0, a - c_2)$ and $f_1(q) > f_2(q)$ for $q \in (a - c_2, a)$. Therefore, if $f_1'(a - c_2) \leq 0$, that is if $c_2 \leq (a+2c_1)/3$, then the maximum of $u_1(q_1)$ occurs at $f_1'(q_1) = 0$ or at $q_1 = 0$, namely at $q_1 = (a - 2c_1 + c_2)^+$. If $f'(a - c_2) > 0$ and $f_2'(a - c_2) \leq 0$, that is, if $c_2 > (1a + 2c_1)/3$ and $c_2 < (a + c_1)/2$, the maximum occurs at $q_1 = a - c_2$. If $f'(a - c_2) > 0$ and $f_2'(a - c_2) > 0$, that is, if $c_2 > (a + c_1)/2$, the maximum occurs at $f_2'(q_1) = 0$, namely at $q_1 = (a - c_1)/2$.

In summary we have four cases:
(1) If $c_1 > (a + c_2)/2$, then $q_1 = 0$ and $q_2 = (a - c_2)/2$.
(2) If $c_1 < (a+c_2)/2$ and $c_2 < (a+2c_1)/3$, then $q_1 = (a-2c_1+c_2)/2$ and $q_2 = (a-q_1-c_2)/2$.
(3) If $c_2 > (a + 2c_1)/3$ and $c_2 < (a + c_1)/2$, then $q_1 = a - c_2$ and $q_2 = 0$.
(4) If $c_2 > (a + c_1)/2$, then $q_1 = (a - c_1)/2$ and $q_2 = 0$.

(b) The payoff functions are

$$\begin{aligned} u_1(q_1, q_2) &= q_1(17 - q_1 - q_2) - q_1 - 2 \\ u_2(q_1, q_2) &= q_2(17 - q_1 - q_2) - 3q_2 - 1. \end{aligned}$$

The optimal production for Firm 2 satisfies $(\partial/\partial q_2)u_2(q_1, q_2) = 17 - q_1 - 3 - 2q_2 = 0$. So $q_2 = (14 - q_1)/2$. Then,

$$u_1(q_1, q_2(q_1)) = q_1(10 - (q_1/2)) - q_1 - 2$$

from which we find the equilibrium productions to be

$$q_1 = 9 \qquad \text{and} \qquad q_2 = 2.5.$$

the equilibrium price is $P = 17 - 9 - (5/2) = 11/2$, and the equilibrium payoffs are

$$u_1 = 9 \cdot 11/2 - 9 - 2 = 37.5 \qquad \text{and} \qquad u_2 = 2.5 \cdot 11/2 - 3 \cdot 2.5 = 4.25.$$

6. The three payoffs are

$$\begin{aligned} u_1(q_1, q_2, q_3) &= q_1(a - c - q_1 - q_2 - q_3) \\ u_2(q_1, q_2, q_3) &= q_2(a - c - q_1 - q_2 - q_3) \\ u_3(q_1, q_2, q_3) &= q_3(a - c - q_1 - q_2 - q_3) \end{aligned}$$

Setting $\partial u_3(q_1, q_2, q_3)/\partial q_3$ to zero and solving gives

$$q_3(q_1, q_2) = (a - c - q_1 - q_2)/2$$

as the optimal production for Firm 3. Then, evaluating

$$u_2(q_1, q_2, q_3(q_1, q_2)) = q_2(a - c - q_1 - q_2 - \frac{a - c - q_1 - q_2}{2}) = q_2(\frac{a - c - q_1 - q_2}{2})$$

and setting $\partial q_2(q_1, q_2, q_3(q_1, q)_2))/\partial q_3$ to zero and solving gives

$$q_2(q_1) = (a - c - q_1)/4$$

as the optimal production for Firm 2. We find $q_3(q_1, q_2(q_1)) = (a - c - q_1)/4$. Then, evaluating

$$u_1(q_1, q_2(q_1), q_3(q_1, q_2(q_1))) = q_1(a - c - q_1 - \frac{a - c - q_1}{2} - \frac{a - c - q_1}{4}) = q_1(\frac{a - c - q_1}{4})$$

as the optimal production for Firm 1. Setting the derivative of this to zero, solving and substituting into the productions of Firm 2 and Firm 3, gives

$$q_1 = (a - c)/2, \qquad q_2 = (a - c)/4, \qquad q_3 = (a - c)/8,$$

as the strategic equilibrium.

7. (a) Setting the partial derivatives to zero,

$$\begin{aligned} \frac{\partial M_1}{\partial x} &= V\frac{y}{(x + y)^2} - C_1 = 0 \\ \frac{\partial M_2}{\partial y} &= V\frac{x}{(x + y)^2} - C_2 = 0, \end{aligned}$$

we see that $C_1 x = C_2 y$, from which it is easy to solve for x and y:

$$\begin{aligned} x &= VC_2/(C_1 + C_2)^2 \\ y &= VC_1/(C_1 + C_2)^2 \end{aligned}$$

The profits are

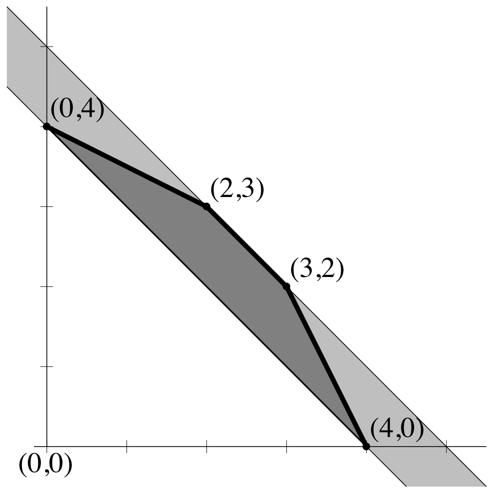
$$\begin{aligned} M_1 &= C_2^2/(C_1 + C_2)^2 \\ M_2 &= C_1^2/(C_1 + C_2)^2 \end{aligned}$$

(b) If $V = 1$, $C_1 = 1$ and $C_2 = 2$, we find

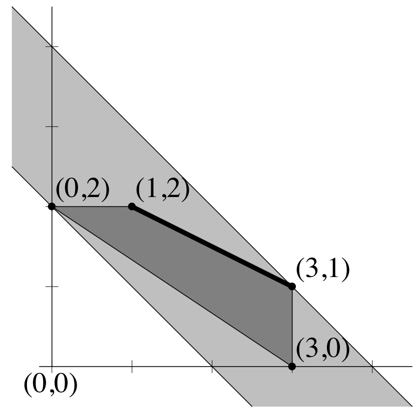
$$\begin{aligned} x &= 2/9 \qquad M_1 = 4/9 \\ y &= 1/9 \qquad M_2 = 1/9 \end{aligned}$$

Solutions to Exercises of Section III.4.

1. (a)

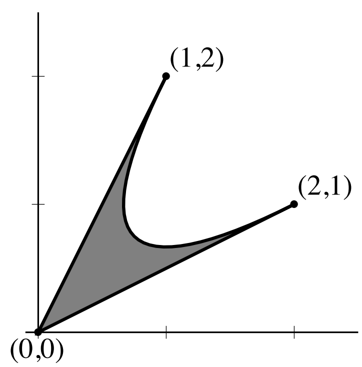


(b)



The light shaded region is the TU-feasible set. The dark shaded region is the NTU-feasible region. The NTU-Pareto optimal outcomes are the vectors along the heavy line. The TU-Pareto outcomes are the upper right lines of slope -1.

(c)



The curve joining (1,2)and (2,1) has parametric form $(x, y) = (1-2a+3a^2, 2-4a+3a^2)$ for $0 \le a \le 1$.

2. (a) The cooperative strategy is ((1,0),(1,0)) with sum $\sigma = 7$. The difference matrix $\boldsymbol{A} - \boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$ has a saddle point at the upper right with value $\delta = 0$. So $\boldsymbol{p}^* = (1, 0)$ and $\boldsymbol{q}^* = (0, 1)$ are the threat strategies and the disagreement point is $(0, 0)$. The TU-solution is $\boldsymbol{\varphi} = ((\sigma + \delta)/2, (\sigma - \delta)/2) = (7/2, 7/2)$. Since the cooperative strategy gives payoff $(4, 3)$, this requires a side payment of $1/2$ from I to II.

(b) The cooperative strategy is ((0,1),(0,1)) with sum $\sigma = 9$. The difference matrix $\boldsymbol{A} - \boldsymbol{B} = \begin{pmatrix} 2 & 4 \\ 0 & -3 \end{pmatrix}$ has a saddle point at the upper left with value $\delta = 2$. So $\boldsymbol{p}^* = (1, 0)$

and $\boldsymbol{q}^* = (1,0)$ are the threat strategies and the disagreement point is $(5,3)$. The TU-solution is $\boldsymbol{\varphi} = ((\sigma+\delta)/2, (\sigma-\delta)/2) = (11/2, 7/2)$. Since the cooperative strategy gives payoff $(3,6)$, this requires a side payment of $5/2$ from II to I.

3. (a) The cooperative strategy is ((1,0,0,0),(0,0,1,0)) with sum $\sigma = 6$. The difference matrix is

$$\boldsymbol{A} - \boldsymbol{B} = \begin{pmatrix} 1 & 3 & -6 & 0 \\ 2 & 0 & -3 & 3 \\ 5 & -6 & 0 & 4 \\ -7 & 7 & -1 & -4 \end{pmatrix}$$

By the matrix game solver, the value is $\delta = -3/7$ and the threat strategies are $\boldsymbol{p}^* = (0,0,4/7,3/7)$ and $\boldsymbol{q}^* = (0,1/14,13/14,0)$. So the TU-solution is $\boldsymbol{\varphi} = ((\sigma+\delta)/2, (\sigma-\delta)/2) = (3 - \frac{3}{14}, 3 + \frac{3}{14})$. The disagreement point is $(-27/98, 15/98)$. The cooperative strategy gives payoff $(0,6)$ so this requires a side payment of $2 + \frac{11}{14}$ from II to I.

(b) We have $\sigma = 2$. One cooperative strategy is $((0,0,0,0,1),(1,0,0,0))$. The difference matrix is

$$\begin{pmatrix} 0 & 2 & 0 & -1 \\ -2 & -1 & 1 & 0 \\ 1 & 0 & -1 & -2 \\ 2 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

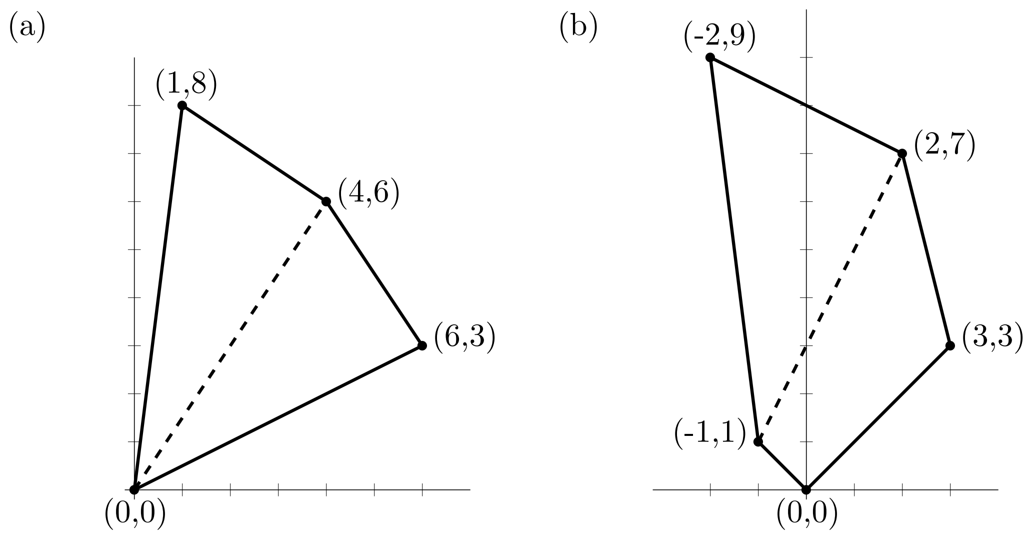
There is a saddle point at the lower right corner. The value is $\delta = 0$. The TU-solution is $(1,1)$. The threat strategies are $(0,0,0,0,1)$ and $(0,0,0,1)$. The disagreement point is $(0,0)$. There is no side payment.

4. (a) The set of Pareto optimal points is the parabolic arc, $y = 4 - x^2$, from $x = 0$ to $x = 2$. We seek the point on this arc that maximizes the product $xy = x(4-x^2) = 4x - x^3$. Setting the derivative with respect to x to zero gives $4 - 3x^2 = 0$, or $x = 2/\sqrt{3}$. The corresponding value of y is $y = 4 - (4/3) = 8/3$. Hence the NTU solution is $(\bar{u}, \bar{v}) = (2/\sqrt{3}, 8/3)$.

(b) This time we seek the Pareto optimal point that maximizes the product $x(y-1) = x(3-x^2) = 3x - x^3$. Setting the derivative with respect to x to zero gives $3 - 3x^2 = 0$, or $x = 1$. The corresponding value of y is $y = 4 - 1 = 3$. Hence the NTU solution is $(\bar{u}, \bar{v}) = (1, 3)$.

5. (a) The fixed threat point is $(u^*, v^*) = (0,0)$. The set of Pareto optimal points consists of the two line segments, from (1,8) to (4,6) and from (4,6) to (6,3). The first has slope, $-2/3$, and the second has slope, $-3/2$. The slope of the line from (0,0) to (4,6) is $3/2$, exactly the negative of the slope of the second line. Thus, $(\bar{u}, \bar{v}) = (4,6)$ is the NTU-solution. The equilibrium exchange rate is $\lambda^* = 3/2$.

(b) Both matrices, A and B, have saddle-points at the first row, second column. Therefore, the fixed threat point is $(u^*, v^*) = (-1, 1)$. The set of Pareto optimal points consists of the two line segments, from $(-2,9)$ to (2,7) and from (2,7) to (3,3). The first has slope, $-1/2$, and the second has slope, -4. The slope of the line from $(-1,1)$ to $(2,7)$ is 2. Since this is between the negatives of the two neighboring slopes, the NTU-solution is $(\bar{u},\bar{v}) = (2,7)$. The equilibrium exchange rate is $\lambda^* = 2$.



6. (a) Clearly, the NTU-solution must be on the line joining $(1,4)$ and $(5,2)$. For the TU-solution to be equal to the NTU-solution, we suspect that the slope of the λ-transformed line, from $(\lambda,4)$ to $(5\lambda,2)$, would be equal to -1. Since the slope of this line is $-2/(4\lambda)$, we have $\lambda^* = 1/2$, in which case the game matrix becomes $\begin{pmatrix} (5/2,2) & (0,0) \\ (0,0) & (1/2,4) \end{pmatrix}$. This gives $\sigma = 9/2$ and $\delta = 0$, so that the TU-solution is $(9/4,9/4)$. The NTU-solution of the original matrix is obtained from this by dividing the first coordinate by λ^*, so that $\boldsymbol{\varphi} = (9/2,9/4)$.

(b) The lambda-transfer matrix is $\begin{pmatrix} (3\lambda,2) & (0,5) \\ (2\lambda,1) & (\lambda,0) \end{pmatrix}$. We see

$$\sigma(\lambda) = \begin{cases} 5 & \text{if } \lambda \le 1 \\ 3\lambda + 2 & \text{if } \lambda \ge 1 \end{cases}.$$

The difference matrix is

$$\lambda\boldsymbol{A} - \boldsymbol{B} = \begin{pmatrix} 3\lambda - 2 & -5 \\ 2\lambda - 1 & \lambda \end{pmatrix}.$$

This matrix has a saddle point no matter what be the value of $\lambda > 0$. If $0 < \lambda \le 1$, there is a saddle at $\langle 2,1\rangle$. If $\lambda \ge 1$, there is a saddle at $\langle 2,2\rangle$. Thus,

$$\delta(\lambda) = \begin{cases} 2\lambda - 1 & \text{if } 0 < \lambda \le 1 \\ \lambda & \text{if } \lambda \ge 1. \end{cases}$$

From (10),

$$\boldsymbol{\varphi}(\lambda) = \Big(\frac{\sigma(\lambda)+\delta(\lambda)}{2\lambda}, \frac{\sigma(\lambda)-\delta(\lambda)}{2}\Big).$$

For $\lambda = 1$, we find $\boldsymbol{\varphi}(\lambda) = (3, 2)$. This is obviously feasible since it is the upper left entry of the originalbimatrix. So the NTU-solution is (3,2), and $\lambda^* = 1$ is the equilibrium exchange rate.

7. (a) If Player II uses column 2, Player I is indifferent as to what he plays. If I uses $(1-p, p)$, Player II prefers column 2 to column 1 if $89(1-p)+98p \le 90(1-p)$, that is if $99p \le 1$. I's safety level is $\mathrm{Val}\begin{pmatrix} 11 & 10 \\ 2 & 10 \end{pmatrix} = 10$, and II's safety level is $\mathrm{Val}\begin{pmatrix} 89 & 98 \\ 90 & 0 \end{pmatrix} = 89 + \frac{1}{11}$. At the equilibrium with $p = 1/99$, the payoff vector is $(10, 90(98/99)) = (10, 89 + \frac{1}{11})$. Thus both players only get their safety levels.

(b) Working together, Player I and II can achieve $\sigma = 100$. The difference matrix, $\boldsymbol{D} = \boldsymbol{A} - \boldsymbol{B}$, has value

$$\delta = \mathrm{Val}\begin{pmatrix} -78 & -80 \\ -96 & 10 \end{pmatrix} = \frac{-78 \cdot 10 - 80 \cdot 98}{10 - 78 + 80 + 96} = -\frac{235}{3} = -78\frac{1}{3}.$$

Therefore, the TU solution is $\boldsymbol{\varphi} = ((\sigma + \delta)/2, (\sigma - \delta)/2) = (10\frac{5}{6}, 89\frac{1}{6})$. This is on the line segment joining the top two payoff vectors of the game matrix, and so is in the NTU feasible set. Player I's threat strategy is $(106/108, 2/108) = (53/54, 1/54)$. Player II's threat strategy is $(90/108, 18/108) = (5/6, 1/6)$.

Part of the reason Player I is so strong in this game is that even if Player I carries out his threat strategy, $(53/54, 1/54)$, the best Player II can do against it is to play column 1, when the payoff to the players is $((10\frac{5}{6}, 89\frac{1}{6}, 0)$, the same as given by the NTU solution.