

## Solutions to Exercises of Section II.1.

1. The new payoff matrix is

$$\begin{array}{|c|c|} \hline -1 & +2 \\ \hline +2 & -4 \\ \hline \end{array}$$

If player I uses the mixed strategy  $(p, 1 - p)$ , the expected payoff is  $-1p + 2(1 - p)$  if II uses column 1, and  $-2p - 4(1 - p)$  if II uses column 2. Equating these, we get  $-p + 2(1 - p) = 2p - 4(1 - p)$  and solving for  $p$  gives  $p = 2/3$ . Use of this strategy guarantees that player I wins 0 on the average no matter what II does. Similarly, if II uses the same mixed strategy  $(2/3, 1/3)$ , II is guaranteed to win 0 on the average no matter what I does. Thus, 0 is the value of the game. Since the value of a game is zero, the game is fair by definition.

2. The payoff matrix is

$$\begin{array}{cc} & \begin{array}{cc} \text{red 2} & \text{black 7} \end{array} \\ \begin{array}{c} \text{black Ace} \\ \text{red 8} \end{array} & \begin{pmatrix} -2 & 1 \\ 8 & -7 \end{pmatrix} \end{array}$$

Solving  $-2p + 8(1 - p) = p - 7(1 - p)$  gives  $p = 5/6$  as the probability that Player I should use the black Ace. Similarly,  $q = 5/9$  is the probability that Player II should use the red 2. The value is  $-1/3$ .

3. If Professor Moriarty stops at Canterbury with probability  $p$  and continues to Dover with probability  $1 - p$ , then his average payoff is  $100p$  if Holmes stops at Canterbury, and is  $-50p + 100(1 - p)$  if Holmes continues to Dover. Equating these payoffs gives  $250p = 100$ , or  $p = 2/5$ . Use of this mixed strategy guarantees Moriarty an average payoff of  $100p = 40$ .

On the other hand, if Holmes stops at Canterbury with probability  $q$  and continues to Dover with probability  $1 - q$ , then his average payoff is  $100q - 50(1 - q)$  if Moriarty stops at Canterbury, and is  $100(1 - q)$  if Holmes continues to Dover. Equating these gives  $q = 3/5$ . Use of this strategy holds Moriarty to an average payoff of 40.

The value of the game is 40, and so the game favors Moriarty. But, as related by Dr. Watson in *The Final Problem*, Holmes outwitted Moriarty once again and held the diabolical professor to a draw.

4. Without the side payment, the game matrix in cents is

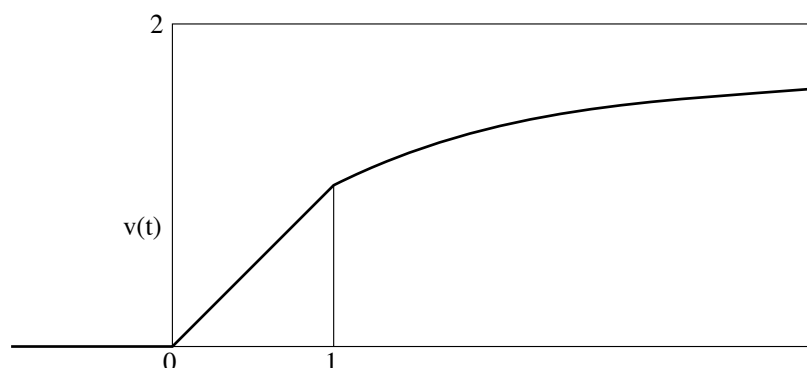
$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 55 & 10 \\ 10 & 110 \end{pmatrix} \end{array}$$

Let  $p$  be the probability that Player I (Alex) uses row 1. Equating his payoffs if Player II uses cols 1 or 2 gives  $55p + 10(1 - p) = 10p + 110(1 - p)$ , or  $145p = 100$  or  $p = 20/29$ . If Player I uses  $(20/29, 9/29)$ , his average payoff is  $55(20/29) + 10(1 - 9/29) = 1190/29$ . Since this is  $41\frac{1}{29}$ , a side payment of 42 cents overcompensates slightly. With the side payment, the game is in Olaf's favor by  $28/29$  of one cent.

## Solutions to Exercises of Section II.2.

1. The value is  $-4/3$ . The mixed strategy,  $(2/3, 1/3)$ , is optimal for I, and the mixed strategy  $(5/6, 1/6)$  is optimal for II.

2. If  $t \leq 0$ , the strategy pair  $\langle 1, 1 \rangle$  is a saddle-point, and the value is  $v(t) = 0$ . If  $0 \leq t \leq 1$ , the strategy pair  $\langle 2, 1 \rangle$  is a saddle-point, and the value is  $v(t) = t$ . If  $t > 1$ , there is no saddle-point; I's optimal strategy is  $((t-1)/(t+1), 2/(t+1))$ , II's optimal strategy is  $(1/(t+1), t/(t+1))$ , and the value is  $v(t) = 2t/(t+1)$ .



3. Suppose that  $\langle x, y \rangle$  and  $\langle u, v \rangle$  are saddle-points. Look at the four numbers  $a_{x,y}$ ,  $a_{x,v}$ ,  $a_{u,v}$ , and  $a_{u,y}$ . We must have  $a_{x,y} \leq a_{x,v}$  since  $a_{x,y}$  is the minimum in its row. Also,  $a_{x,v} \leq a_{u,v}$  since  $a_{u,v}$  is the maximum of its column. Keep going:  $a_{u,v} \leq a_{u,y}$  since  $a_{u,v}$  is the minimum of its row and  $a_{u,y} \leq a_{x,y}$  since  $a_{x,y}$  is the maximum of its column. We have

$$a_{x,y} \leq a_{x,v} \leq a_{u,v} \leq a_{u,y} \leq a_{x,y}.$$

Since this begins and ends with the same number, we must have equality throughout:  $a_{x,y} = a_{x,v} = a_{u,v} = a_{u,y} = a_{x,y}$ . (This argument also works if  $x = u$  or  $y = v$ .)

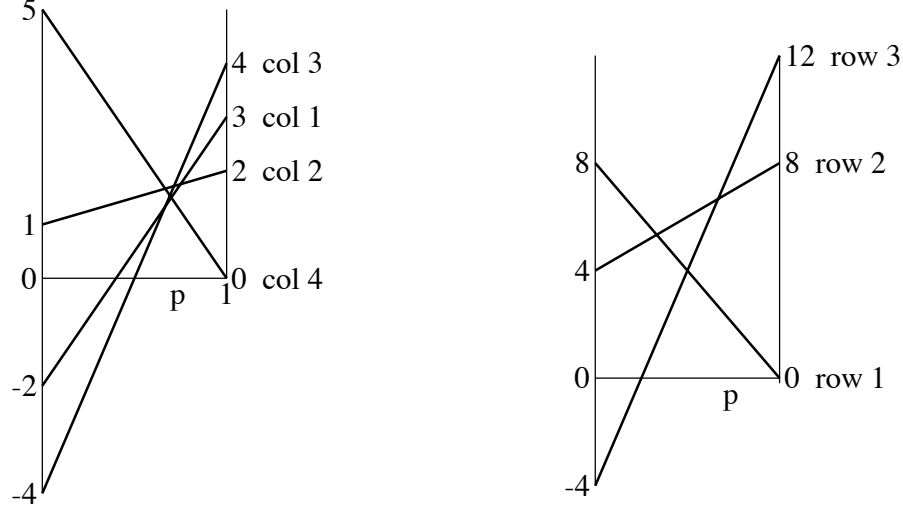
4. (a) Column 2 dominates column 1; then row 3 dominates row 4; then column 4 dominates column 3; then row 1 dominates row 2. The resulting submatrix consists of rows 1 and 3 vs. columns 2 and 4. Solving this 2 by 2 game and moving back to the original game we find that the value is  $3/2$ , I's optimal strategy is  $p = (1/2, 0, 1/2, 0)$  and II's optimal strategy is  $q = (0, 3/8, 0, 5/8)$ .

(b) Column 2 dominates column 4; then  $(1/2)\text{row } 1 + (1/2)\text{row } 2$  dominates row 3; then  $(1/2)\text{col } 1 + (1/2)\text{col } 2$  dominates col 3. The resulting 2 by 2 game is easily solved. Moving back to the original game we find that the value is  $30/7$ , I's optimal strategy is  $(2/7, 5/7, 0)$  and II's optimal strategy is  $(3/7, 4/7, 0, 0)$ .

5. (a) From the graph on the left, we guess that Player II uses columns 1 and 4. Solving this 2 by 2 subgame gives

$$\begin{matrix} & \begin{matrix} 1/2 & 1/2 \end{matrix} \\ \begin{matrix} 7/10 \\ 3/10 \end{matrix} & \begin{pmatrix} 3 & 0 \\ -2 & 5 \end{pmatrix} \end{matrix} \quad \text{Value} = 1.5$$

We conjecture I's optimal strategy is (.7,.3), II's optimal strategy is (.5,0,0,.5), and the value is 1.5. Let us check how well I's strategy works on columns 2 and 3. For column 2,  $2(.7) + 1(.3) = 1.7$  and for column 3,  $4(.7) - 4(.3) = 1.6$ , both greater than 1.5. This strategy guarantees I at least 1.5 so our conjecture is verified.



(b)  $(3/8)\text{col } 1 + (5/8)\text{col } 2$  dominates col 3. Removing column 3 leaves a 3 by 2 game whose payoffs for a given  $q$  are displayed in the graph on the right. The upper envelope takes on its minimum value at the intersection of row 1 and row 2. Solving the 2 by 2 game in the upper left corner of the original matrix gives the solution. Player I's optimal strategy is  $(1/3, 2/3, 0)$ , Player II's optimal strategy is  $(1/3, 2/3, 0)$ , and the value is  $16/3$ .

6. (a) The first row is dominated by the third; the seventh is dominated by the fifth. Then the third column is dominated by the first; the fourth is dominated by the second; the fifth column is dominated by the seventh. Then the middle row is dominated. When these three rows and columns are removed, the resulting matrix is the 4 by 4 identity matrix with value  $v = 1/4$  and optimal strategies giving equal weight  $1/4$  to each choice. This results in the optimal strategies  $\mathbf{p} = (0, .25, .25, 0, .25, .25, 0)$  for I, and  $\mathbf{q} = (.25, .25, 0, 0, 0, .25, .25)$  for II.

(b) For all  $n$ , domination reduces the game matrix to the identity matrix. We find the value for arbitrary  $n \geq 2$  to be  $v_n = 1/(2k)$  for  $n = 4k - 2, 4k - 1, 4k$ , and  $v_n = 1/(2k + 1)$  for  $n = 4k + 1$ . For  $n$  equal to 2 or 3, the optimal strategies are simple special cases. For  $n \geq 4$ , an optimal strategy for Player I is  $\mathbf{p} = (p_1, p_2, \dots, p_n)$ , symmetric about its midpoint and such that for  $i \leq (n + 1)/2$ ,

$$p_i = \begin{cases} v_n & \text{if } i = 2 \text{ or } 3 \text{ mod } 4 \\ 0 & \text{if } i = 0 \text{ or } 1 \text{ mod } 4. \end{cases}$$

Similarly, an optimal strategy for Player II is  $\mathbf{q} = (q_1, \dots, q_n)$ , symmetric about its midpoint and such that for  $j \leq (n + 1)/2$ ,

$$q_j = \begin{cases} v_n & \text{if } j = 1 \text{ or } 2 \text{ mod } 4 \\ 0 & \text{if } j = 0 \text{ or } 3 \text{ mod } 4. \end{cases}$$

7. If Player I uses  $\mathbf{p}$  and Player II uses column 1, the average payoff is  $(6/37)5 + (20/37)4 + (11/37) = 121/37$ . Similarly for columns 2, 3, 4 and 5, the average payoffs are

121/37, 160/37, 121/37, and 161/37. So Player I can guarantee an average payoff of at least 121/37 by using  $\mathbf{p}$ . Similarly, if Player II uses  $\mathbf{q}$  and Player I uses rows 1, 2, 3, or 4, the average payoffs are 121/37, 121/37, 120/37, and 121/27 respectively. By using  $\mathbf{q}$ , Player II can keep the average payoff to at most 121/37. Thus, 121/37 is the value of the game and  $\mathbf{p}$  and  $\mathbf{q}$  are optimal strategies.

8. If  $(52/143, 50/143, 41/143)$  is optimal, then the value is the minimum of the inner product of this vector with the three columns of the matrix. This inner product with each of the three columns gives the same number, namely 96/143, which is then the value.

9. The matrix is

$$\begin{array}{c} 1 \quad 2 \quad 3 \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \end{array}$$

$(1/2)\text{row } 1 + (1/2)\text{row } 3$  dominates row 2; then  $(1/2)\text{col } 1 + (1/2)\text{col } 3$  dominates row 2. Solving the resulting 2 by 2 game and moving back to the original game, we find the value is 1 and an optimal strategy for I is  $(1/2, 0, 1/2)$ , and an optimal strategy for II is  $(1/2, 0, 1/2)$ . However, the pure strategy of choosing col 2 is also optimal. In fact it is better than the mixed strategy  $(1/2, 0, 1/2)$  whenever Player I makes the mistake of playing row 2.

10. In an  $n \times n$  magic square,  $\mathbf{A} = (a_{ij})$ , there is a number  $s$  such that  $\sum_i a_{ij} = s$  for all  $j$ , and  $\sum_j a_{ij} = s$  for all  $i$ . If Player I uses the mixed strategy  $\mathbf{p} = (1/n, 1/n, \dots, 1/n)$  his average payoff is  $V = s/n$  no matter what Player II does. The same goes for player II, so the value is  $s/n$  and  $\mathbf{p}$  is optimal for both players. In the example,  $n = 4$  and  $s = 34$ , so the value of the game is  $17/2$  and the optimal strategy is  $(1/4, 1/4, 1/4, 1/4)$ .

11. (a) First, 6 dominates 4 and 5. With 4 and 5 removed,  $C$  dominates  $D$  and  $F$ ;  $A$  dominates  $E$ . Also, the mixture  $(3/4)A + (1/4)C$  dominates  $B$ . Then with  $B$ ,  $D$ ,  $E$  and  $F$  removed, 3 dominates 2 and 1. The resulting 2 by 2 game

$$\begin{array}{c} A \quad C \\ \begin{array}{c} 3 \\ 6 \end{array} \begin{pmatrix} 18 & 31 \\ 23 & 19 \end{pmatrix} \end{array}$$

is easily solved. The value is  $21 + \frac{14}{17}$ . Optimal for Player I is  $(0, 0, 4/17, 0, 0, 13/17)$  and optimal for Player II is  $(12/17, 0, 5/17, 0, 0, 0)$ .

(b) Neither player was using an optimal strategy. The German choice was very poor, and the Allies were lucky. (Or did they have inside information?)

### Solutions to Exercises II.3.

1.(a) There is a saddle at row 2, column 3. The value is 1.

(b) The inverse is  $\mathbf{A}^{-1} = \frac{1}{5} \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 1 \\ -3 & 6 & -4 \end{pmatrix}$ .

(c) The mixed strategy  $(1/4, 1/2, 1/4)$ , for example, is optimal for II.

(d) Equation (16) gives  $\mathbf{q} = (2/5, 4/5, -1/5)$ . Equations (16) are valid when  $\mathbf{A}$  is nonsingular and Player I has an optimal strategy giving positive weight to each strategy. That is not the case here.

2.(a) If  $d_i = 0$  for some  $i$ , then (row  $i$ , col  $i$ ) is a saddlepoint of value zero. And row  $i$  and col  $i$  are optimal pure strategies for the players.

(b) If  $d_i > 0$  and  $d_j < 0$ , then (row  $i$ , col  $j$ ) is a saddlepoint of value zero. And row  $i$  and col  $j$  are optimal pure strategies for the players.

(c) If all  $d_i < 0$ , then the same analysis as in Section 3.3 holds. The value is  $V = \sum_1^m 1/d_i$ , and the players have the same optimal strategy,  $(V/d_1, \dots, V/d_m)$ .

3. The matrix is

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix}.$$

This is a diagonal game of value  $V = (1/2 + 1/4 + 1/8 + 1/16)^{-1} = 16/15$ . The optimal strategy for both players is  $(8/15, 4/15, 2/15, 1/15)$ .

4. The matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 1/2 & 1 & 0 \\ 1/2 & 1/2 & 1/2 & 1 \end{pmatrix}.$$

This is a triangular game. If  $V$  is the value and if  $(p_1, p_2, p_3, p_4)$  is Player I's optimal strategy, then equations (12) become  $V = p_4 = p_3 + (1/2)p_4 = p_2 + (1/2)(p_3 + p_4) = p_1 + (1/2)(p_2 + p_3 + p_4)$ . We may solve the equations one at a time to find  $p_4 = V$ ,  $p_3 = (1/2)V$ ,  $p_2 = (1/4)V$  and  $p_1 = (1/8)V$ . Since the sum of the  $p$ 's is one, we find  $(\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1)V = 1$ , so that  $V = 8/15$ . This is the value and  $\mathbf{p} = (1/15, 2/15, 4/15, 8/15)$  is optimal for Player I and  $\mathbf{q} = (8/15, 4/15, 2/15, 1/15)$  is optimal for II.

5. This is similar to Exercise 4. Equations (12) become:

$$\begin{aligned} p_n &= V \\ p_{n-1} - p_n &= V \\ p_{n-2} - p_{n-1} - p_n &= V \\ &\vdots \\ p_1 - p_2 - \cdots - p_{n-2} - p_{n-1} - p_n &= V \end{aligned}$$

The solution is  $p_n = V$ ,  $p_{n-1} = 2V$ , ...,  $p_1 = 2^{n-1}V$ . Since  $1 = p_1 + p_2 + \cdots + p_n = [2^{n-1} + 2^{n-2} + \cdots + 1]V = [2^n - 1]V$ , we find that the value is  $V = 1/(2^n - 1)$ . The optimal strategy for I (and for II also) is  $(2^{n-1}, 2^{n-2}, \dots, 2, 1)/(2^n - 1)$ .

6. The matrix  $\mathbf{A}$  has components  $a_{ij} = 0$  for  $i < j$  and  $a_{ij} = b^{i-j}$  for  $i \geq j$ . It is easy to show that

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ b & 1 & 0 & & 0 \\ b^2 & b & 1 & & 0 \\ \vdots & & \ddots & \ddots & \\ b^{n-1} & b^{n-2} & \cdots & b & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -b & 1 & 0 & & 0 \\ 0 & -b & 1 & & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & \cdots & -b & 1 \end{pmatrix} = I$$

Therefore, we may use Theorem 3.2 to find the value  $V$  as the reciprocal of the sum of all the elements of  $\mathbf{A}^{-1}$ ,  $V = 1/(n - (n-1)b)$ , and I's optimal strategy is proportional to the sums of the columns of  $\mathbf{A}^{-1}$ ,  $\mathbf{p} = (1 - b, 1 - b, \dots, 1 - b, 1)/(n - (n-1)b)$ , and II's optimal strategy is  $\mathbf{q} = (1, 1 - b, 1 - b, \dots, 1 - b)/(n - (n-1)b)$ , proportional to the sums of the rows of  $\mathbf{A}^{-1}$ .

7. We may use Theorem 3.2 with  $\mathbf{A}_n^{-1}$  replaced by  $\mathbf{B}_n$ . Since the sum of the  $i$ th row of  $\mathbf{B}_n$  is  $2^{i-1}$  (the binomial theorem), we have  $\mathbf{B}_n \mathbf{1} = (1, 2, 4, \dots, 2^{n-1})^T$ , and so  $\mathbf{1}^T \mathbf{B}_n \mathbf{1} = 2^n - 1$ . Similarly, the sum of column  $j$  is  $\sum_{k=j}^n \binom{k-1}{j-1} = \binom{n}{j}$  (easily proved by induction). So that  $\mathbf{1}^T \mathbf{B} = (\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n})$ . From this we may conclude that the value is  $V = 1/(2^n - 1)$ , the optimal strategy of Player I is  $\mathbf{p} = (1, 2, \dots, 2^{n-1})/(2^n - 1)$ , and the optimal strategy of Player II is  $\mathbf{q} = (\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n})/(2^n - 1)$ .

8.(a) Assuming all strategies active, the optimal strategy for I satisfies,  $p_1 = V$ ,  $-p_1 + 2p_2 = V$  and  $-p_1 + p_2 + 3p_3 = V$ , from which we find  $p_1 = V$ ,  $p_2 = V$  and  $p_3 = V/3$ . Since  $1 = p_1 + p_2 + p_3 = V + V + (1/3)V = (7/3)V$ , we have  $V = 3/7$  and  $\mathbf{p} = (p_1, p_2, p_3) = (3/7, 3/7, 1/7)$ . A similar analysis for Player II gives  $\mathbf{q} = (q_1, q_2, q_3) = (5/7, 1/7, 1/7)$ . Since  $\mathbf{p}$  and  $\mathbf{q}$  are nonnegative, these are the optimal strategies, and  $V$  is the value.

(b) If we subtract 1 from all entries of the matrix, we end up with a diagonal game with 1, 1/2, 1/3 and 1/4 along the diagonal. The value of that game is 1/10, and the optimal strategies for both players is (1/10, 2/10, 3/10, 4/10). The original game has value 11/10 and the same optimal strategies.

(c) The last column is dominated by the first, and the bottom row is dominated by the mixture of row 1 and row 2 with probability 1/2 each. The resulting three by three

matrix is a diagonal game with value  $1/[(1/2) + (1/3) + (1/4)] = 12/13$ . The optimal strategy for both players is  $(6/13, 4/13, 3/13, 0)$ .

9.(a) The matrix is

$$\begin{pmatrix} 0 & -2 & 1 & 1 & 1 & \dots \\ 2 & 0 & -2 & 1 & 1 & \dots \\ -1 & 2 & 0 & -2 & 1 & \\ -1 & -1 & 2 & 0 & -1 & \\ \vdots & \vdots & & & & \ddots \end{pmatrix}$$

(b) The game is symmetric and has value zero (if it exists). If the first five rows and columns are the active ones, the equations become

$$\begin{aligned} 2p_2 - p_3 - p_4 - p_5 &= 0 \\ -2p_1 + 2p_3 - p_4 - p_5 &= 0 \\ p_1 - 2p_2 + 2p_4 - p_5 &= 0 \\ p_1 + p_2 - 2p_3 + 2p_5 &= 0 \\ p_1 + p_2 + p_3 - 2p_4 &= 0 \end{aligned}$$

If we interchange  $(p_1, p_2)$  with  $(p_5, p_4)$  in these equations, we get the same set of equations. So in the solution, we must have  $p_1 = p_5$  and  $p_2 = p_4$ . Using this, the top two equations become  $p_2 = p_1 + p_3$  and  $2p_3 = 3p_2 + p_1$ , which together with  $2p_1 + 2p_2 + p_3 = 1$  gives  $p_1 = p_5 = 1/16$ ,  $p_2 = p_4 = 5/16$  and  $p_3 = 4/16$ . If Player I uses  $\mathbf{p} = (1/16, 5/16, 4/16, 5/16, 1/16, 0, 0, \dots)$  on the game with general  $n$ , then Player II will never use columns 6 or greater because the average payoff to Player I would be positive. Thus, the value is zero and  $\mathbf{p}$  is optimal for both players.

10. (a) The matrix is

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ \vdots \end{matrix} & \begin{pmatrix} 0 & -1 & 2 & 2 & 2 & 2 & \dots \\ 1 & 0 & -1 & -1 & -1 & 2 & \dots \\ -2 & 1 & 0 & -1 & -1 & -1 & \dots \\ -2 & 1 & 1 & 0 & -1 & -1 & \dots \\ -2 & 1 & 1 & 1 & 0 & -1 & \dots \\ -2 & -2 & 1 & 1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{matrix}$$

One can see that columns 6, 7,  $\dots$  are all dominated by column 1. Similarly for rows. This reduces the game to a 5 by 5 matrix. Columns 3 and 4 are dominated by column 5. This reduces the game to 3 by 3.

(b) The game restricted to rows and columns 1, 2 and 5 has matrix  $\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$ ,

whose solution (see (25)) is  $(1/4, 1/2, 1/4)$ . It is easy to check that the mixed strategy

$(1/4, 1/2, 0, 0, 1/4, 0, \dots)$  gives Player I an average payoff of at least 0 for every pure strategy of Player II. So this strategy is optimal for Player I, and by symmetry Player II as well.

11.(a) The matrix is skew-symmetric so this is a symmetric game. So the value is 0. To find an optimal strategy for I, we try  $(p_1, p_2, p_3)$  against the columns. The first column gives  $-p_2 + 2p_3 = 0$  (since 0 is the value), and the second gives  $p_1 - 3p_3 = 0$ . We have  $p_1 = 3p_3$  and  $p_2 = 2p_3$ . Then since the probabilities sum to 1, we have  $3p_3 + 2p_3 + p_3 = 1$  or  $p_3 = 1/6$ . Then,  $p_1 = 1/2$  and  $p_2 = 1/3$ . The optimal strategy for both players is  $(1/2, 1/3, 1/6)$ .

(b) This is a Latin square game, so  $(1/3, 1/3, 1/3)$  is optimal for both players and the value is  $v = (0 + 1 - 2)/3 = -1/3$ .

(c)  $(1/4)\text{row } 1 + (1/4)\text{row } 2 + (1/4)\text{row } 3 + (1/4)\text{row } 4$  dominates row 5. After removing row 5, the matrix is a Latin square. So  $(1/4, 1/4, 1/4, 1/4)$  is optimal for II, and  $(1/4, 1/4, 1/4, 1/4, 0)$  is optimal for I. The value is  $v = (1 + 4 - 1 + 5)/4 = 9/4$ .

12. The answer given by the Matrix Game Solver gives the same value and optimal strategy for Player I as in the text, but gives the optimal strategy for Player II as  $(7/90, 32/90, 48/90, 3/90)$ . This shows that although there may be a unique invariant optimal strategy, there may be other noninvariant optimal strategies as well. The simplex method only finds basic feasible solutions, and so will not find the invariant optimal solution  $(1/18, 4/9, 4/9, 1/18)$ , because it is not basic.

In (15), the middle row is strictly dominated by  $(3/4)$  the top row plus  $(1/4)$  the bottom row. Our solution and the one found by the Matrix Game Solver both give zero weight to  $(3,1)$  and  $(1,3)$ .

13. (a) The reduced matrix has a saddle point.

$$\begin{array}{c} (1,0)^* \\ (2,0)^* \\ (1,1) \end{array} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

So the value is 1,  $(1,1)$  (or  $(2,0)^*$ ) is optimal for Player I and  $(1,0)^*$  is optimal for Player II.

(b) The reduced matrix is

$$\begin{array}{c} (2,0)^* \\ (3,0)^* \\ (2,1)^* \end{array} \begin{pmatrix} (1,1) & (1,1) \\ 3/2 & 1/2 \\ 0 & 2 \end{pmatrix}$$

The value is 1. An optimal strategy for Player I is to use  $(3,0)^*$  with probability  $2/3$ , and  $(2,1)^*$  with probability  $1/3$ . This corresponds to playing  $(3,0)$  and  $(0,3)$  with probability  $1/3$  each, and  $(2,1)$  and  $(1,2)$  with probability  $1/6$  each. An optimal strategy for Player II is to use  $(1,1)$  with probability  $1/2$ , and  $(2,0)$  and  $(0,2)$  with probability  $1/4$  each.

14. (a) In the matrix below, row 4 is dominated by  $(1/2)\text{row } 2 + (1/2)\text{row } 3$ . Then col 2 is dominated by  $(1/2)\text{col } 1 + (1/2)\text{col } 3$ . Then row 2 is dominated by  $(2/3)\text{row } 1 +$



(1/3)row 3.

$$\begin{array}{c} (3,0,0)^* \quad (2,1,0)^* \quad (1,1,1) \\ \begin{array}{c} (4,0,0)^* \\ (3,1,0)^* \\ (2,2,0)^* \\ (2,1,1)^* \end{array} \left( \begin{array}{ccc} 4/3 & 2/3 & 0 \\ 1/3 & 1 & 1 \\ -1 & 4/3 & 3 \\ -1/3 & 1/3 & 2 \end{array} \right) \end{array}$$

We find that  $(3/4, 0, 1/4, 0)$  is optimal for Player I,  $(9/16, 0, 7/16)$  is optimal for Player II, and the value is  $3/4$ .

(b) In the matrix below, row 4 is dominated by  $(1/2)\text{row 3} + (1/2)\text{row 5}$ . But we might as well use the Matrix Game Solver directly.

$$\begin{array}{c} (3,0,0,0)^* \quad (2,1,0,0)^* \quad (1,1,1,0) \\ \begin{array}{c} (4,0,0,0)^* \\ (3,1,0,0)^* \\ (2,2,0,0)^* \\ (2,2,1,1)^* \\ (1,1,1,1) \end{array} \left( \begin{array}{ccc} 1 & 1/4 & -1/2 \\ 1/2 & 3/4 & 1/2 \\ -1/2 & 1 & 2 \\ 1/4 & 1/2 & 3/2 \\ 1 & 0 & 1 \end{array} \right) \end{array}$$

The value is  $3/5$ ,  $(0, 4/5, 0, 0, 1/5)$  is optimal for Player I, and  $(8/15, 2/5, 1/15)$  is optimal for Player II.

15. Consider the following strategies for Player II.

A: Start at the center square; if this is a hit continue with a 2, 4, 6, or 8 in random order each order equally likely; if this is a miss, shoot at the corners 1,3,7,9 in a random, equally likely order, and when a hit occurs, choose one of the two possible middle edge squares at random, then the other.

B: Start at the four middle edge squares, 2,4,6,8 in some random order; when a hit occurs, try the center next, then the possible corner squares.

C: Start at the four middle edge squares, 2,4,6,8 in some random order; when a hit occurs, try the possible corners next, then the center.

There are many other strategies for Player II, but they should be dominated by some mixture of these. In particular, starting at a corner square should be dominated by starting at a middle edge.

Using invariance, Player I has the two strategies,  $[1, 2]^*$  and  $[2, 5]^*$ . Suppose Player I uses  $[1, 2]^*$  and Player II uses C. Then the first hit will occur on shot 1, 2, 3, or 4 with probability  $1/4$  each. After the first hit it takes on the average 1.5 more shots to get the other hit. The average number of shots then is

$$(1/4)(2.5) + (1/4)(3.5) + (1/4)(4.5) + (1/4)(5.5) = 4.$$

But if Player II starts off by shooting in the center before trying the corners, it will take one more shot on the average, namely 5. This gives the top row of the matrix below. The whole matrix turns out to be

$$\begin{array}{c} A \quad B \quad C \\ \begin{array}{c} [1, 2]^* \\ [2, 5]^* \end{array} \left( \begin{array}{ccc} 5 & 5 & 4 \\ 3.5 & 3.5 & 5.5 \end{array} \right) \end{array}$$

The first two columns are equivalent. Player I's optimal strategy is  $(2/3, 1/3)$ . This translates into choosing one of the 12 positions at random with probability  $1/12$  each. One optimal strategy for Player II is to randomize with equal probability between B and C. The value is 4.5.

16. Invariance reduces Player I to two strategies; choose 1 and 3 with probability  $1/2$  each, denoted by  $1^*$ , and choose 3. Similarly, invariance and dominance reduces Player II to two strategies, we call A and B. For A, start with 2. For B, with probability  $1/2$ , start with 1 and if it's not successful follow it with 3, and with probability  $1/2$  start with 3 if it's not successful and follow it with 1. This leads to a 2 by 2 game with matrix

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} 1^* \\ 2 \end{array} & \begin{pmatrix} 2 & 3/2 \\ 1 & 3 \end{pmatrix} \end{array}$$

The value is  $9/5$ . An optimal strategy for I is to choose 2 with probability  $1/5$ , and 1 or 3 equally likely with probability  $2/5$  each. An optimal strategy for II is guess 2 first with probability  $3/5$ , and otherwise to guess 1 then 3, or 3 then 1 with probability  $1/5$  each; that is, II never guesses 1 then 2 then 3 and never guesses 3 then 2 then 1.

17. To make Player I indifferent in choosing among rows 1 through  $k$ , Player II will choose  $\mathbf{q} = (q_1, \dots, q_k, 0, \dots, 0)$  so that  $u_i \sum_{j \neq i} q_j = V_k$  for  $i = 1, \dots, k$  for some constant  $V_k$ . Using  $\sum_1^k q_j = 1$ , this reduces to  $(1 - q_i) = V_k/u_i$ . Since  $\sum_1^k (1 - q_i) = k - 1$ , we have

$$V_k = \frac{k-1}{\sum_1^k 1/u_i} \quad \text{and} \quad q_i = \begin{cases} 1 - V_k/u_i & \text{for } i = 1, \dots, k \\ 0 & \text{for } i = k+1, \dots, m \end{cases} \quad (1)$$

The  $q_i$  are nondecreasing but we must have  $q_i \geq 0$ , which reduces to  $V_k \leq u_k$ . If  $k < m$ , we also require that Player I will not choose rows  $k+1$  to  $m$ . This reduces to  $u_{k+1} \leq V_k$ . Therefore, if  $u_{k+1} \leq V_k \leq u_k$ , Player II can achieve  $V_k$  by using  $\mathbf{q}$ . (It is easy to show that  $V_2 < u_2$  and that  $V_k \leq u_k$  implies that  $V_{k-1} \leq u_{k-1}$ . This shows that such a  $k$  exists and is unique.)

To make Player II indifferent in choosing among columns 1 through  $k$ , Player I will choose  $\mathbf{p} = (p_1, \dots, p_k, 0, \dots, 0)$  so that  $\sum_1^k p_i u_i - p_j u_j = V_k$  for some constant  $V_k$  and  $j = 1, \dots, k$ . This shows that  $p_j$  is equal to some constant over  $u_j$  for  $j = 1, \dots, k$ . Using  $\sum_1^k p_j = 1$ , we find

$$p_j = \begin{cases} \frac{1/u_j}{\sum_1^k 1/u_i} & \text{for } j = 1, \dots, k \\ 0 & \text{for } j = k+1, \dots, m \end{cases} \quad (2)$$

Solving for  $V_k$  shows it indeed has the same value as above. All the  $p_j$  are nonnegative, so we only have to show that Player II will not want to choose columns  $k+1, \dots, m$ . The expected payoff is the same for each of these columns, namely,  $\sum_1^k p_i u_i$  which is clearly greater than  $V_k$ , so Player I can achieve at least  $V_k$ .

In summary, find the largest  $k$  in  $\{2, \dots, m\}$  such that  $(k-1)/u_k \leq \sum_1^k 1/u_i$ . Then the value and the optimal strategies are given by (1) and (2).

18. The payoff matrix is  $\mathbf{A}_n = (a_{ij})$ , where

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i - j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that  $\mathbf{A}_n \mathbf{B}_n = \mathbf{I}_n$ , so that  $\mathbf{A}_n^{-1} = \mathbf{B}_n$ . The sum of the  $i$ th row of  $\mathbf{A}_n^{-1}$  is  $i(n+1-i)/2$ . By symmetry these are also the column sums. Since they are all positive the game is completely mixed, and the optimal strategy, the same for both players, is proportional to  $(n, 2(n-1), 3(n-2), \dots, n)$ , namely,  $p(i) = 6i(n+1-i)/(n(n+1)(n+2))$ . The sum of all numbers in  $\mathbf{A}_n^{-1}$  is  $n(n+1)(n+2)/12$ , so the value is its reciprocal,  $v_n = 12/(n(n+1)(n+2))$ .

19. (a)  $n$  odd: Let  $\mathbf{x} = (1, 0, 1, 0, \dots, 1)^T$ . Then  $\mathbf{x}^T \mathbf{A} = (2, 2, \dots, 2)$ . There are  $(n+1)/2$  1's in  $\mathbf{x}$ , so  $\mathbf{p} = 2\mathbf{x}/(n+1)$  is a mixed strategy for I that guarantees I will win  $4/(n+1)$  no matter what column II chooses. The matrix is symmetric, so the same strategy guarantees II will lose  $4/(n+1)$  no matter what I does. Thus,  $\mathbf{p}$  is an optimal strategy for both I and II, and  $4/(n+2)$  is the value.

(b)  $n$  even: Let  $k = n/2$  and  $\mathbf{x} = (k, 1, k-1, 2, \dots, 1, k)^T$ . Then  $\mathbf{x}^t \mathbf{A} = (2k+1, 2k+1, \dots, 2k+1)$ . The sum of the elements of  $\mathbf{x}$  is  $k(k+1)$  so  $\mathbf{p} = \mathbf{x}/(k(k+1))$  is a mixed strategy for I that guarantees I will win  $(2k+1)/(k(k+1))$  no matter what column II chooses. The same strategy guarantees II will lose  $(2k+1)/(k(k+1))$  no matter what I does. Thus,  $\mathbf{p}$  is an optimal strategy for both I and II, and the value is  $v = (2k+1)/(k(k+1)) = 4(n+1)/(n(n+2))$ .

## Solutions to Exercises of Section II.4.

1.(a) If Player II uses the mixed strategy,  $(1/5, 1/5, 1/5, 2/5)$ , I's expected payoff from rows 1, 2, and 3 are  $17/5$ ,  $17/5$ , and  $23/5$  respectively. So I's Bayes strategy is row 3, giving expected payoff  $23/5$ .

(b) If II guesses correctly that I will use the Bayes strategy against  $(1/5, 1/5, 1/5, 2/5)$ , she should choose column 3, giving Player I a payoff of  $-1$ .

2.(a) We have  $b_{ij} = 5 + 2a_{ij}$  for all  $i$  and  $j$ . Hence,  $A$  and  $B$  have the same optimal strategies for the players, and the value of  $B$  is  $\text{Val}(B) = 5 + 2\text{Val}(A) = 5$ . The optimal strategy for I is  $(6/11, 3/11, 2/11)$ .

(b) Since we are given that  $\text{Val}(A) = 0$ , we may solve for the optimal  $\mathbf{q}$  for II using the equations,  $-q_2 + q_3 = 0$ , and  $2q_1 - 2q_3 = 0$ . So  $q_1 = q_3$  and  $q_2 = q_3$ . Since the probabilities sum to 1, all three must be equal to  $1/3$ . So  $(1/3, 1/3, 1/3)$  is optimal for Player II for both matrices  $A$  and  $B$ .

3.(a) Let  $\epsilon > 0$  and let  $\mathbf{q}$  be any element of  $Y^*$ . Then since  $\sum_{j=1}^n q_j \rightarrow 1$  as  $n \rightarrow \infty$ , we have  $\sum_{j=n}^{\infty} q_j \rightarrow 0$ , so that there is an integer  $N$  such that  $\sum_{j=N}^{\infty} q_j < \epsilon$ . If Player I uses  $i = N$ , the expected payoff is  $\sum_{j=1}^{\infty} L(N, j)q_j = \sum_{j=1}^{N-1} q_j - \sum_{j=N+1}^{\infty} q_j < 1 - 2\epsilon$ . Thus for every  $\mathbf{q} \in Y^*$ , we have  $\sup_{1 \leq i < \infty} \sum_{j=1}^{\infty} L(i, j)q_j \geq 1 - 2\epsilon$ . Since this is true for all  $\epsilon > 0$ , it is also true for  $\epsilon = 0$ .

(b) Since (a) is true for all  $\mathbf{q} \in Y^*$ , we have  $\bar{V} = \inf_{\mathbf{q} \in Y^*} \sup_{1 \leq i < \infty} \sum_{j=1}^{\infty} L(i, j)q_j \geq 1$ . Since no payoff is greater than 1, we have  $\bar{V} = 1$ .

(c) The game is symmetric, so  $\underline{V} = -\bar{V}$ . Hence,  $\underline{V} = -1$ .

(d) Any strategy is minimax for Player I since any strategy guarantees an expected payoff of at least  $\underline{V} = -1$ .

4. The value of  $\mathbf{A}$  is positive since the simple strategy  $(2/3)\text{row } 1 + (1/3)\text{row } 2$  guarantees a positive return for Player I. But let's add 1 to  $\mathbf{A}$  anyway to get  $\mathbf{B}$ :

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \\ 4 & -2 & 1 \end{pmatrix}.$$

The simplex tableau is displayed below on the left. We are asked to pivot in the second column. But there is only one positive number there, so we must pivot on the first row second column. We arrive at:

	$y_1$	$y_2$	$y_3$		$\longrightarrow$		$y_1$	$x_1$	$y_3$	
$x_1$	1	②	3	1		$y_2$	1/2	1/2	3/2	1/2
$x_2$	3	0	-1	1		$x_2$	3	0	-1	1
$x_3$	4	-2	1	1		$x_3$	5	1	4	2
	-1	-1	-1	0			-1/2	1/2	1/2	1/2

There is still a negative element on the bottom edge so we continue. It is unique and in the first column, so we pivot in the first column. the smallest of the ratios is  $1/3$  occurring in the second row. So we pivot on the second row first column to find:

$$\begin{array}{c|ccc|c} & y_1 & x_1 & y_3 & \\ \hline y_2 & 1/2 & 1/2 & 3/2 & 1/2 \\ x_2 & \textcircled{3} & 0 & -1 & 1 \\ x_3 & 5 & 1 & 4 & 2 \\ \hline & -1/2 & 1/2 & 1/2 & 1/2 \end{array} \longrightarrow \begin{array}{c|ccc|c} & x_2 & x_1 & y_3 & \\ \hline y_2 & -1/6 & 1/2 & 5/3 & 1/3 \\ y_1 & 1/3 & 0 & -1/3 & 1/3 \\ x_3 & -5/3 & 1 & 17/3 & 1/3 \\ \hline & 1/6 & 1/2 & 1/3 & 2/3 \end{array}$$

From this we see that  $\text{Val}(\mathbf{B}) = 3/2$ , so that  $\text{Val}(\mathbf{A}) = 1/2$ . For either game  $(p_1, p_2, p_3) = (3/4, 1/4, 0)$  is optimal for Player I and  $(q_1, q_2, q_3) = (1/2, 1/2, 0)$  is optimal for Player II. You may use the sure-fire test to see that this is correct.

5. (a) For all  $j = 0, 1, 2, \dots$ ,

$$\begin{aligned} \sum_{i=0}^{\infty} p_i A(i, j) &= \sum_{i=0}^{j-1} \frac{1}{2^{(i+1)}} (-4^i) + \sum_{i=j+1}^{\infty} \frac{1}{2^{(i+1)}} (4^j) \\ &= -\frac{1}{2} (1 + 2 + \dots + 2^{j-1}) + 4^j \frac{1}{2^{(j+2)}} (1 + \frac{1}{2} + \frac{1}{4} + \dots) \\ &= -\frac{1}{2} (2^j - 1) + \frac{1}{2} 2^j = \frac{1}{2}. \end{aligned}$$

(b) If both players use the mixed strategy,  $\mathbf{p}$ , the payoff is  $\sum \sum p_i A(i, j) p_j$ . The trouble is that the answer depends on the order of summation. If we sum over  $i$  first, we get  $+1/2$ , and if we sum over  $j$  first we get  $-1/2$ . In other words, Fubini's Theorem does not hold here. For Fubini's Theorem, we need  $\sum \sum p_i |A(i, j)| p_j < \infty$ , which is not the case here. The whole theory of using mixed strategies in games depends heavily on Utility Theory. In Utility Theory, at least as presented in Appendix 1, the utility functions are bounded. So it would seem most logical to restrict attention to games in which the payoff function,  $A$ , is bounded. That is certainly one way to avoid such examples. However, in many important problems the payoff function is unbounded, at least on one side, so one usually assumes that the payoff function is bounded below, say.

There is another way of dealing with the problem that is more germane to the example above, and that is by restricting the notion of a mixed strategy to be a probability distribution that gives weight to only a finite number of pure strategies. (Then Fubini's theorem holds because the summations are finite.) If this is done in the example, then one can easily see that the value of the game does not exist. This seems to be the "proper" solution of the game because it is just a blown-up version of the game, "the-player-that-chooses-the-larger-integer-wins".

6. We take the initial  $i_1 = 1$ , and find

$k$	$i_k$	$s_k(1)$	$s_k(2)$	$\underline{V}_k$	$j_k$	$t_k(1)$	$t_k(2)$	$\bar{V}_k$
1	1	1	-1	-1	2	-1	<b>2</b>	2
2	2	<b>1</b>	1	.5	1	0	<b>2</b>	1
3	2	<b>1</b>	3	.3333	1	1	<b>2</b>	0.6667
4	2	<b>1</b>	5	0.25	1	<b>2</b>	2	.5

The largest of the  $\underline{V}_k$ , namely .5, is equal to the smallest of the  $\bar{V}_k$ . So the value of the game is  $1/2$ . An optimal strategy for Player I is found at  $k = 2$  to be  $\mathbf{p} = (.5, .5)$ . An optimal strategy for Player II is  $\mathbf{q} = (.75, .25)$  found at  $k = 4$ . Don't expect to find the value of a game by this method again!

7. (a) The upper left payoff of  $\sqrt{2}$  was chosen so that there would be no ties in the fictitious play. So Player II knows exactly what Player I will do and will be able to guarantee a zero payoff at each future stage. If Player II's relative frequency,  $q_k$ , of column 1 by stage  $k$  goes above  $1/(\sqrt{2} + 1)$ , Player I will play row 1, causing Player II to play column 2, thus causing  $q_k$  to decrease. Thus  $q_k$  converges to  $1/(\sqrt{2} + 1)$ , which in fact is Player II's optimal strategy for the game. Similarly, Player I's relative frequency,  $p_k$ , of row 1 converges to  $1/(\sqrt{2} + 1)$ , which is his optimal strategy.

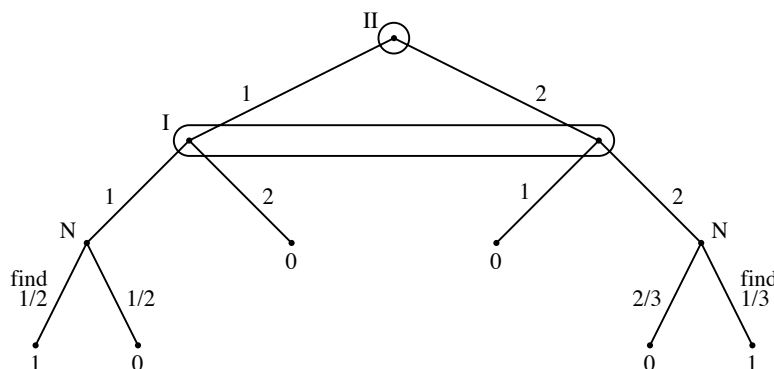
(b) Player I should play the same pure strategy as his opponent at each stage, gaining either  $\sqrt{2}$  or 1 at each stage. The same argument as in (a) shows that Player II's average relative frequency of column 2 converges to  $1/(\sqrt{2} + 1)$ , so Player I's limiting average payoff is

$$\frac{1}{\sqrt{2} + 1} \cdot \sqrt{2} + \frac{\sqrt{2}}{\sqrt{2} + 1} = 2(2 - \sqrt{2}),$$

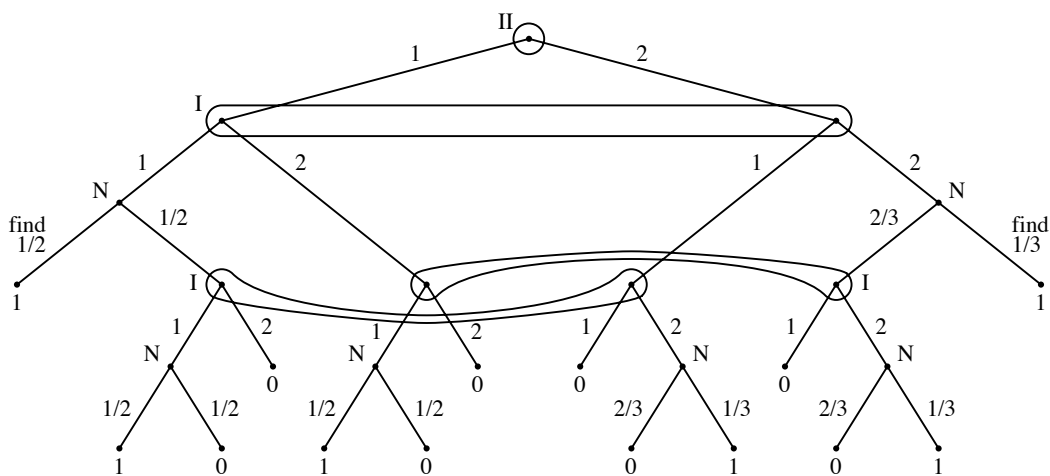
twice the value of the game.

## Solutions to Exercises of Section II.5.

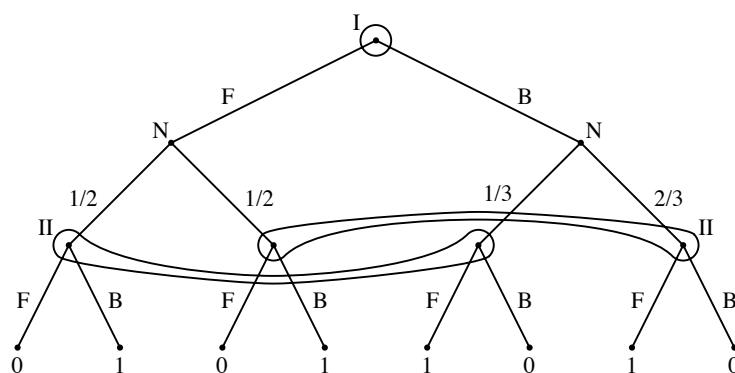
1. **The Silver Dollar.** I hides the dollar, II searches for it with random success.



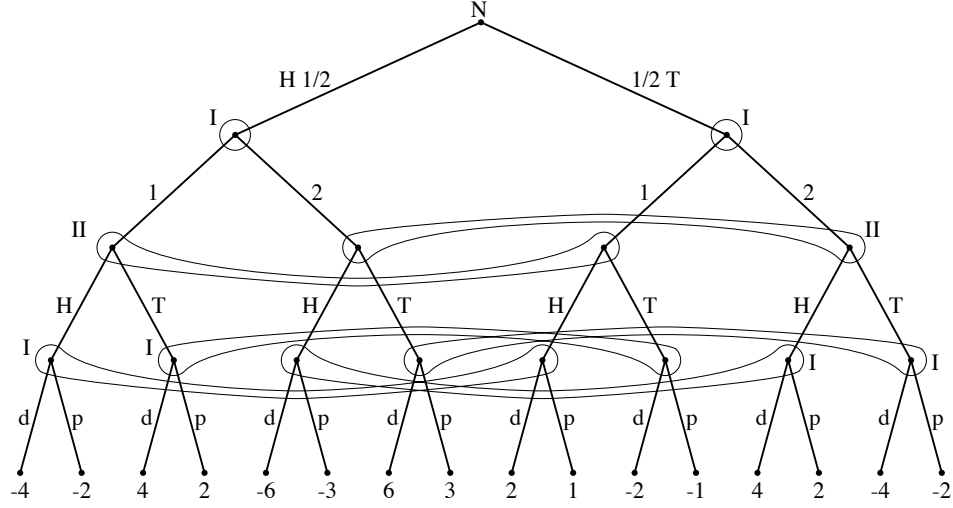
2. **Two Guesses for the Silver Dollar.** I hides the dollar, II searches for it twice with random success.



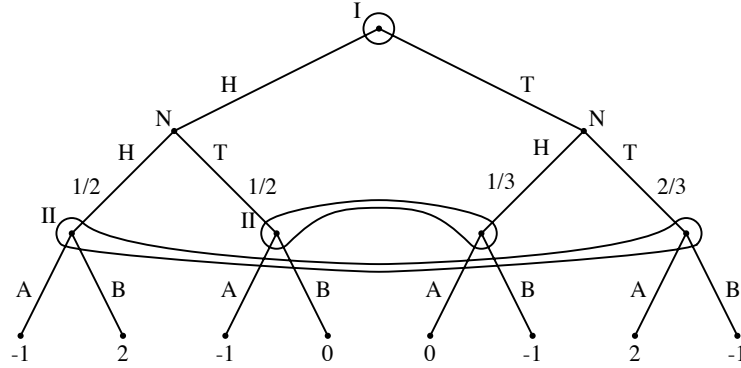
3. **Guessing the Probability of a Coin.** I chooses the fair (F) or biased (B) coin. II observes H or T on one toss of the coin and must guess which coin.



4. **A Forgetful Player.** A fair coin is tossed. I hears the outcome and bets 1 or 2. II guesses H or T. I forgets the toss and doubles or passes.



#### 5. A One-Shot Game of Incomplete Information.



There is an alternate Kuhn tree where the simultaneous move is replaced by Player II moving first and then Player I moving, not knowing what move Player II made.

6. In Figure 2, replace  $1/4$  by  $p$ ,  $3/4$  by  $1 - p$ , and  $\pm 3$  by  $\pm(1 + b)$ . Again one can argue that Player I should bet with a winning card; he wins 1 if he checks, and wins at least 1 if he bets. In other words, as in the analysis in the text of the resulting  $4 \times 2$  matrix, the first row dominates the third row and the second row dominates the fourth row. The top two rows of the matrix are

$$\begin{matrix} & c & f \\ (b, b) & \left( (1 + b)(2p - 1) & 1 \right) \\ (b, c) & \left( p(2 + b) - 1 & 2p - 1 \right) \end{matrix}$$

If  $(2p - 1)(1 + b) \geq 1$ , (that is, if  $p \geq (2 + b)/(2 + 2b)$ ), there is a saddle-point in the upper right corner. The value of the game is 1, Player I should always bet, and Player II should always fold.



Otherwise, (if  $p < (2 + b)/(2 + 2b)$ ), the game does not have a saddle-point and we can use the straightforward method for solving two by two games. It is optimal for Player I to choose row 1 with probability  $pb/((2 + b)(1 - p))$ , and row 2 otherwise. It is optimal for Player II to choose column 1 with probability  $2/(2 + b)$ , and column 2 otherwise. The value is  $(4p(1 + b) - (2 + b))/(2 + b)$ .

7.(a) The strategic (normal) form is

$$\begin{array}{c} \begin{array}{cccc} & (d, f) & (d, g) & (e, f) & (e, g) \\ a & \begin{pmatrix} -1 & -1 & 1 & 3 \end{pmatrix} \\ b & \begin{pmatrix} 1 & 0 & 1 & 2 \end{pmatrix} \\ c & \begin{pmatrix} 1 & 1 & -1 & 1 \end{pmatrix} \end{array} \end{array}$$

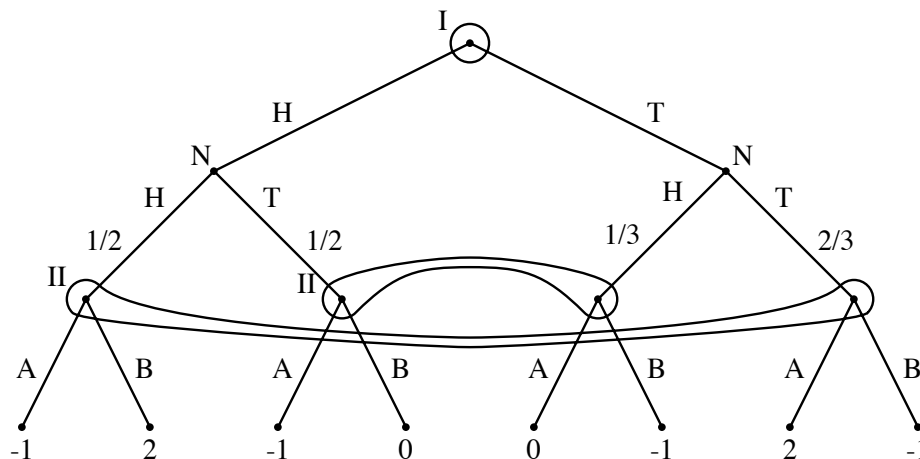
(b) Column 2 dominates columns 1 and 4. Then row 2 dominates row 1. The resulting two by two matrix is  $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ , with value  $1/3$ . The optimal mixed strategy for Player I is  $(0, 2/3, 1/3)$ . The optimal mixed strategy for Player 2 is  $(0, 2/3, 1/3, 0)$ .

8.(a) The strategic (normal) form is

$$\begin{array}{c} \begin{array}{cccc} & (a, c) & (a, d) & (b, c) & (b, d) \\ (A, C) & \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \\ (A, D) & \begin{pmatrix} 0 & 2 & 1 & 3 \end{pmatrix} \\ (B, C) & \begin{pmatrix} 3/2 & 1/2 & -1/2 & -3/2 \end{pmatrix} \\ (B, D) & \begin{pmatrix} 1/2 & 3/2 & 1/2 & 3/2 \end{pmatrix} \end{array} \end{array}$$

(b) Column 3 dominates column 2. The mixture,  $(2/3)\text{row } 2 + (1/3)\text{row } 3$ , dominates row 4. The mixture,  $(1/3)\text{row } 2 + (2/3)\text{row } 3$ , dominates row 1. The resulting two by three matrix is  $\begin{pmatrix} 0 & 1 & 3 \\ 3/2 & -1/2 & -3/2 \end{pmatrix}$ . The first two columns of this matrix are active. The value is  $1/2$ . An optimal mixed strategy for Player I in the original game is  $(0, 2/3, 1/3, 0)$ . An optimal mixed strategy for Player II is  $(1/2, 0, 1/2, 0)$ . It is interesting to note that Player I also has an optimal pure strategy, namely row 4.

9. (a)



(b) The matrix is

		AA	AB	BA	BB
H		-1	-1/2	1/2	1
T		4/3	1	-2/3	-1

(c) Optimal for I =  $(4/7, 3/7)$ . Optimal for II =  $(1/3, 0, 2/3, 0)$ . Value = 0.

10. (a) The matrix is

		1	2
1		1/2	0
2		0	1/3

The mixed strategy  $(2/5, 3/5)$  is optimal for both I and II. The value is  $1/5$ .

(b) The matrix is

		1	2
11		3/4	0
12		1/2	1/3
21		1/2	1/3
22		0	5/9

An optimal strategy for I is  $(0, 0, 10/13, 3/13)$ .

The optimal strategy for II is  $(4/13, 9/13)$ . The value is  $5/13$ .

(c) The matrix is

		FF	FB	BF	BB
F		0	1/2	1/2	1
B		1	1/3	2/3	0

Optimal for I is  $(4/7, 3/7)$ . Optimal for II is  $(1/7, 6/7, 0, 0)$ . The value is  $3/7$ .

(d) The matrix is 64 by 4, much too large write out by hand. However, simple arguments show that most of Player I's pure strategies are dominated. First some notation. We denote Player I's pure strategies by a six-tuple,  $ab; wxyz$ , where  $a$  and  $b$  are 1 or 2 (the amount bet) for information sets  $I_1$  and  $I_2$  respectively, and each of  $w$ ,  $x$ ,  $y$  and  $z$  are  $p$  or  $d$  (pass or double) for information sets  $I_3$ ,  $I_4$ ,  $I_5$  and  $I_6$  respectively. Thus, for example,  $12; pdpp$  represents the strategy: Bet 1 with heads and 2 with tails; his partner passes unless 1 is bet and Player II guesses heads, in which case he doubles.

If Player I uses a strategy starting 12, then his partner upon hearing a bet of 1 and a guess of heads should pass rather than double since that means a loss of 2 rather than 4.

Similarly, on hearing a bet of 1 and a guess of tails, his partner should double. Continuing in this way, we see that the strategy 12;*pddp* dominates all strategies beginning 12.

Similarly, we may see that the strategy 21;*dppd* dominates all strategies beginning 21, the strategy 11;*pdx* (where *x* stands for “any”) dominates all strategies beginning 11, and 22;*xcpd* dominates all strategies beginning 22. Thus, dominance reduces the game to the following 4 by 4 matrix.

$$\begin{array}{c}
 \begin{array}{cccc}
 & HH & HT & TH & TT \\
 11; pdx & \left( \begin{array}{cccc}
 -1/2 & -1/2 & 1 & 1 \\
 12; pddp & 1 & -2 & 4 & 1 \\
 21; dppd & -1/2 & 4 & -2 & 5/2 \\
 22; xcpd & -1/2 & 1 & -1/2 & 1
 \end{array} \right)
 \end{array}
 \end{array}$$

Now, we can see that col 1 dominates col 4. Moreover an equiprobable mixture of rows 2 and 3 dominate rows 1 and 4. This reduces the game to a 2 by 3 matrix which is easily solvable. For the above matrix, (0,3/5,2/5,0) is optimal for I, (4/5,1/5,0,0) is optimal for II and the value is 2/5.

We can describe Player I's strategy as follows. 60% of the time, Player I bets low on heads and high on tails, and his partner doubles when Player II is wrong. The other 40% of the time, Player I bets high on heads and low on tails, and his partner doubles when Player II is wrong.

$$\begin{array}{c}
 \begin{array}{cc}
 & 1 & 2 \\
 11 & \left( \begin{array}{cc}
 4 & 0 \\
 12 & 2 & 4 \\
 12 & 2 & 0 \\
 22 & 0 & 4
 \end{array} \right)
 \end{array}
 \end{array}$$

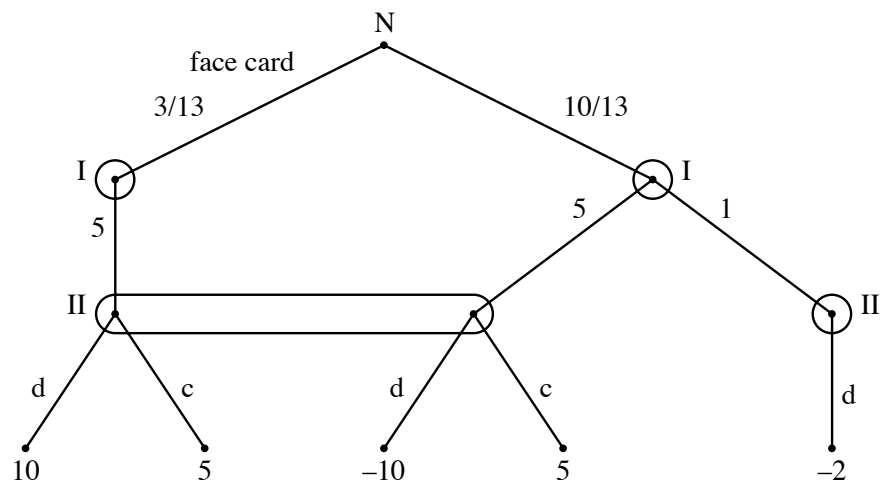
(e) The matrix is . An optimal strategy for I is (1/3,2/3,0,0). An optimal strategy for II is (2/3,1/3). The value is 8/3.

11. Suppose Player I uses *f* with probability  $p_1$  and *c* with probability  $p_2$  (and so *g* with probability  $1 - p_1$  and *d* with probability  $1 - p_2$ ). Suppose Player II uses *a* with probability  $q$  (and *b* with probability  $1 - q$ ). The average payoff is then

$$\begin{aligned}
 v &= q(p_1 - (1 - p_1)(1 - p_2)) + (1 - q)(-p_1p_2 + 2(1 - p_1)) \\
 &= q(4p_1 + p_2 - 3) + 2 - 2p_1 - p_1p_2.
 \end{aligned}$$

Therefore, Player II will choose  $q = 0$  if  $4p_1 + p_2 \geq 3$ , and  $q = 1$  if  $4p_1 + p_2 < 3$ . Against this, the best Player I can do is to use  $p_1 = 1/2$  and  $p_2 = 1$  (or  $p_1 = 3/4$  and  $p_2 = 0$ ) giving an average payoff of 1/2. If  $q$  is announced, then Player I will use  $p_1 = 1$  and  $p_2 = 0$  if  $q \geq 2/3$ , and  $p_1 = 0$  and  $p_2 = 1$  if  $q < 2/3$ . Against this, the best Player II can do is  $q = 2/3$ , which gives Player I an average payoff of 2/3. Therefore, the value of the game does not exist if behavioral strategies must be used.

12. (a)



(b) The matrix is 
$$\begin{matrix} & \begin{matrix} c & d \end{matrix} \\ \begin{matrix} 5 \\ 1 \end{matrix} & \begin{pmatrix} 5 & -70/13 \\ -5/13 & 10/13 \end{pmatrix} \end{matrix}.$$

(c) The strategy  $(1/10, 9/10)$  is optimal for I,  $(8/15, 7/15)$  is optimal for II and the value is  $2/13$ .

## Solutions to Exercises of Section II.6.

1. (a)  $G_1$  has a saddle point. The value is 3, the pure strategy  $(1, 0)$  is optimal for I, and  $(0, 1)$  is optimal for II.

The value of  $G_2$  is 3, the strategy  $(.4, .6)$  is optimal for I, and  $(.5, .5)$  is optimal for II.

The value of  $G_3$  is  $-1$ , the strategy  $(.5, .5)$  is optimal for I and for II.

The game  $G$  is thus equivalent to a game with matrix  $\begin{pmatrix} 0 & 3 \\ 3 & -1 \end{pmatrix}$ .

Hence, the value of  $G$  is  $9/7$ , and the strategy  $(4/7, 3/7)$  is optimal for both players.

(b1) Interchanging the second and third columns, we have a matrix of the form  $G = \begin{pmatrix} 0 & G_1 \\ G_2 & 0 \end{pmatrix}$ , where  $G_1 = \begin{pmatrix} 6 & 2 \\ 3 & 5 \end{pmatrix}$  and  $G_2 = \begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}$ . Since  $\text{Val}(G_1) = 4$  and  $\text{Val}(G_2) = 3$ , we find  $\text{Val}(G) = 12/7$ .

(b2) The matrix has the form,  $G = \begin{pmatrix} G_1 & 2 \\ 1 & G_2 \end{pmatrix}$ , where  $G_2 = \begin{pmatrix} 6 & 3 \\ 4 & 7 \end{pmatrix}$  and  $G_1 = \begin{pmatrix} 3 & 1 & 5 \\ 1 & 3 & 5 \\ 4 & 4 & 1 \end{pmatrix} = \begin{pmatrix} G_3 & 5 \\ 4 & 1 \end{pmatrix}$ , where  $G_3 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ . Since  $\text{Val}(G_3) = 2$ , we have  $\text{Val}(G_1) = 3$ , and then since  $\text{Val}(G_2) = 5$ , we have  $\text{Val}(G) = 13/6$ .

2. The games  $G_{m,n}$  are defined by the induction

$$G_{m,n} = \begin{matrix} & \begin{matrix} \text{act} & \text{wait} \end{matrix} \\ \begin{matrix} \text{inspect} \\ \text{wait} \end{matrix} & \begin{pmatrix} 1 & G_{m-1,n-1} \\ 0 & G_{m,n-1} \end{pmatrix} \end{matrix} \quad \text{for } n = m+1, m+2, \dots \quad \text{and } m = 1, 2, \dots$$

with boundary conditions  $G_{0,n} = (0)$ . Let  $V_{m,n} = \text{Value}(G_{m,n})$ . We have  $V_{m,n} = 1$  for  $m \geq n$  and  $V_{0,n} = 0$  for  $n \geq 1$ . Also from the Example given in the text, we have  $V_{1,n} = 1/n$ . We compute the next few values,

$$V_{2,3} = \text{Value} \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} = \frac{2}{3} \quad V_{2,4} = \text{Value} \begin{pmatrix} 1 & 1/3 \\ 0 & 2/3 \end{pmatrix} = \frac{2}{4}$$

and perhaps a few more, and then we conjecture  $V_{m,n} = m/n$  for  $0 \leq m \leq n$ . Let us check this conjecture by induction. It is true for  $m = 0$  and for  $m = n$ . Suppose that  $0 < m < n$  and suppose the conjecture is true for all smaller values. Then,

$$\begin{aligned} V_{m,n} &= \text{Value} \begin{pmatrix} 1 & V_{m-1,n-1} \\ 0 & V_{m,n-1} \end{pmatrix} \\ &= \text{Value} \begin{pmatrix} 1 & (m-1)/(n-1) \\ 0 & m/(n-1) \end{pmatrix} \\ &= \frac{m}{n}. \end{aligned}$$

and the conjecture is verified. The optimal strategy for I in  $G_{m,n}$  is the mixed strategy  $(m/n, (n-m)/n)$ , and the optimal strategy for II is  $(1/n, (n-1/n))$ . It is interesting to

note that to play optimally player II does not need to keep track of how many times I has searched for him.

3. The induction is

$$G_n = \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} G_{n-1} & 0 \\ 0 & G_{n-2} \end{pmatrix} \end{matrix} \quad \text{for } n = 2, 3, \dots$$

and boundary conditions  $G_0 = (1)$  and  $G_1 = (1)$ . Let  $V_n = \text{Value}(G_n)$ . The recursion for the  $V_n$  becomes

$$V_n = \text{Value} \begin{pmatrix} V_{n-1} & 0 \\ 0 & V_{n-2} \end{pmatrix} = \frac{V_{n-1}V_{n-2}}{V_{n-1} + V_{n-2}} \quad \text{for } n = 2, 3, \dots$$

with boundary conditions  $V_0 = 1$  and  $V_1 = 1$ . Taking reciprocals of this equation, we find

$$\frac{1}{V_n} = \frac{1}{V_{n-1}} + \frac{1}{V_{n-2}} \quad \text{for } n = 2, 3, \dots$$

This is the recursion for the Fibonacci sequence. Since  $1/V_0 = F_0$  and  $1/V_1 = F_1$ , we must have  $1/V_n = F_n$ . Hence we have  $V_n = 1/F_n$  for all  $n$  and we may compute the optimal strategy for I and II to be  $(F_{n-1}/F_n, F_{n-2}/F_n)$  for  $n = 2, 3, \dots$

4. Use of the first row implies  $v_n \geq n + 2$ . Use of the first column implies  $v_n \leq n + 3$ . Since  $n + 2 \leq v_n \leq n + 3$ , none of the games have saddle points. So for  $n = 0, 1, \dots$ ,

$$v_n = \text{Val} \begin{pmatrix} n+3 & n+2 \\ n+1 & v_{n+1} \end{pmatrix} = n+3 - \frac{2}{v_{n+1} - n}.$$

Let  $w_n = v_n - n + 1$  for  $n = 0, 1, \dots$ . Then the  $w_n$  satisfy

$$w_n = 4 - \frac{2}{w_{n+1}}.$$

In fact, the  $w_n$  are the values of the games  $G'_n$  where

$$G'_n = 1 + \begin{pmatrix} 3 & 2 \\ 1 & G'_{n+1} \end{pmatrix} \quad \text{for } n = 0, 1, \dots$$

In game  $G'_n$ , I receives 1 from II and then the players choose row and column; if the players choose the second row second column, then I receives 1 from II and they next play  $G'_{n+1}$ . It may be seen that each of the games  $G'_n$  has the same structure. It is as if the players were playing the recursive game  $G'$  where

$$G' = 1 + \begin{pmatrix} 3 & 2 \\ 1 & G' \end{pmatrix}.$$

So all the games  $G'_n$  should have the same values. If so, denoting the common value by  $w$ , we would have  $w = 4 - (2/w)$ , or  $w^2 - 4w + 2 = 0$ . This has a unique solution in the interval  $3 \leq w \leq 4$ , namely  $w = 2 + \sqrt{2}$ . From this we have  $v_n = n + 1 + \sqrt{2}$ . The optimal strategies are the same for all games, namely,

$$\begin{aligned} \left( \frac{1 + \sqrt{2}}{2 + \sqrt{2}}, \frac{1}{2 + \sqrt{2}} \right) &= (.707 \dots, .293 \dots) \quad \text{is optimal for I for all games } G_n \\ \left( \frac{\sqrt{2}}{2 + \sqrt{2}}, \frac{2}{2 + \sqrt{2}} \right) &= (.414 \dots, .586 \dots) \quad \text{is optimal for II for all games } G_n. \end{aligned}$$

5. The game matrix of  $G_{1,n}$  reduces to

$$\begin{pmatrix} 1 - \frac{n}{n+1}V_{n-1,1} & \frac{n}{n+1} - \frac{n}{n+1}V_{n-1,1} \\ 0 & 1 \end{pmatrix}.$$

(a) So Player I's optimal strategy uses odds  $1 : 1/(n+1) = n+1 : 1$ ; i.e, he should bluff with probability  $1/(n+2)$ .

(b) Player II's optimal odds are  $\frac{1}{n+1} + \frac{n}{n+1}V_{n-1,1} : 1 - \frac{n}{n+1}V_{n-1,1} = 1 + nV_{n-1,1} : n+1 - nV_{n-1,1}$ ; i.e., she should call with probability  $(n+1 - nV_{n-1,1})/(n+2) = V_{1,n}$ .

6. (a) If  $Q \geq 2$ , the top row forever is optimal for I, the second column is optimal for II, and the value is  $v = 2$ . If  $0 \leq Q \leq 2$ , the top row forever is optimal for I, the first column forever is optimal for II, and the value is  $v = Q$ . If  $Q \leq 0$ , the bottom row is optimal for I, the first column forever is optimal for II, and the value is  $v = 0$ .

(b) If  $Q \geq 1$ , the value is  $v = 1$ ,  $(1, 0, 0)^\infty$  (i.e. the top row forever) is optimal for I, and  $(0, 1/2, 1/2)^\infty$  is optimal for II. If  $Q \leq 1$ , the value is still  $v = 1$ ,  $(1, 0, 0)^\infty$  is optimal for II, and  $(1 - \epsilon, \epsilon/2, \epsilon/2)^\infty$  is  $\epsilon$ -optimal for I (actually  $\epsilon/(2 - \epsilon)$ -optimal)

7. Since in game  $G_3$ , I can choose the second row and II can choose the second column, we have  $0 \leq v_3 \leq 1$ . But since  $v_3 \leq 1$ , the most that I can hope to achieve in  $G_2$  is  $\text{Val} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = 2/3$ , so we have  $0 \leq v_2 \leq 2/3$ . Similarly,  $1/2 \leq v_1 \leq 1$ . So none of the game matrices have saddle points and we can write

$$v_1 = \frac{1}{2 - v_2} \quad v_2 = \frac{2v_3}{2 + v_3} \quad v_3 = \frac{1}{2 - v_1}.$$

Substitution of the first equation into the third and then the result into the second yields the quadratic equation,  $v_2 = (4 - 2v_2)/(8 - 5v_2)$ , or  $5v_2^2 - 10v_2 + 4 = 0$ . Solving this gives  $v_2 = (5 - \sqrt{5})/5$  as the root less than 1. From this we can find  $v_1 = (5 - \sqrt{5})/4$  and  $v_3 = 3 - \sqrt{5}$ . The complete solution is

$$G_1 : \quad \text{Val}(G_1) = v_1 = (5 - \sqrt{5})/5 = .691 \dots$$

$$\text{Optimal for I} = \text{Optimal for II} = (v_1, 1 - v_1) = (.691 \dots, .309 \dots)$$

$$G_2 : \quad \text{Val}(G_2) = v_2 = (5 - \sqrt{5})/5 = .553 \dots$$

$$\text{Optimal for I} = \text{Optimal for II} = \left( \frac{5 + \sqrt{5}}{10}, \frac{5 - \sqrt{5}}{10} \right) = (.724 \dots, .276 \dots)$$

$$G_3 : \quad \text{Val}(G_3) = v_3 = 3 - \sqrt{5} = .764 \dots$$

$$\text{Optimal for I} = \text{Optimal for II} = (v_3, 1 - v_3) = (.764 \dots, .236 \dots)$$

independent of  $Q$ .

8. Whatever the values of  $G_1$ ,  $G_2$  and  $G_3$ , the game  $G_1$  is a latin square game and so has optimal strategies  $(1/3, 1/3, 1/3)$  for both players. So we must have  $v_1 =$

$(v_1 + v_2 + v_3)/3$ , or equivalently  $2v_1 = v_2 + v_3$ . From the form of  $G_2$  and  $G_3$ , we see that  $0 \leq v_2 \leq 2$  and  $0 \leq v_3 \leq 1$ . Hence,  $0 \leq v_1 \leq 3/2$ . We now see that  $G_2$  does not have a saddle point, and that  $0 \leq v_2 < 1$  so that  $G_3$  does not have a saddle point either. We arrive at the three equations,

$$2v_1 = v_2 + v_3 \quad v_2 = \frac{2v_1}{2 + v_1} \quad v_3 = \frac{1}{2 - v_2}.$$

Eliminating  $v_1$  and  $v_3$  leads to a quadratic equation for  $v_2$ , namely  $v_2^2 + 2v_2 - 1 = 0$ . This has one positive root, namely  $v_2 = \sqrt{2} - 1$ . From this we can find  $v_1 = (4\sqrt{2} - 2)/7$  and  $v_3 = (3 + \sqrt{2})/7$ . The complete solution is

$$G_1 : \quad \text{Val}(G_1) = v_1 = (4\sqrt{2} - 2)/7 = .522 \dots$$

$$\text{Optimal for I} = \text{Optimal for II} = (1/3, 1/3, 1/3)$$

$$G_2 : \quad \text{Val}(G_2) = v_2 = \sqrt{2} - 1 = .414 \dots$$

$$\text{Optimal for I} = \text{Optimal for II} = (.793 \dots, .207 \dots)$$

$$G_3 : \quad \text{Val}(G_3) = v_3 = (3 + \sqrt{2})/7 = .631 \dots$$

$$\text{Optimal for I} = \text{Optimal for II} = (v_3, 1 - v_3) = (.631 \dots, .369 \dots)$$

independent of  $Q$ .

9. This is a recursive game of the form

$$G = \begin{pmatrix} .8 + .2(-G^T) & .5 + .5(-G^T) \\ .6 + .4(-G^T) & .7 + .3(-G^T) \end{pmatrix}$$

and the value,  $v$ , of the game satisfies

$$v = \text{Val} \begin{pmatrix} .8 - .2v & .5 - .5v \\ .6 - .4v & .7 - .3v \end{pmatrix}$$

The game is in favor of the server, so the value is between zero and one and the game does not have a saddle point. The optimal strategy for the server is to serve (high,low) with probabilities proportional to  $(.1 + .1v, .3 + .3v)$ , namely  $(1/4, 3/4)$ . The optimal strategy for the receiver is to receive (near,far) with probabilities proportional to  $(.2 + .2v, .2 + .2v)$ , namely,  $(1/2, 1/2)$ . Using the second of these equalizing strategies, the value may be found to be  $v = (1/2)(.8 - .2v) + (1/2)(.5 - .5v) = .65 - .35v$ . Solving for  $v$  gives  $v = 13/27 = .481 \dots$ .

10. (a) We may think of the basic game,  $G$ , as the one in which player I chooses a number  $k$  to be the number of times he tosses the coin before challenging II. In this, player II has no choice and the matrix  $G$  is an  $\infty \times 1$  matrix, which is to say an infinite dimensional column vector. The probability of tossing  $k$  heads in a row is  $p^k$ . Counting 1 for a win and  $-1$  for a loss, the expected payoff given that I tosses  $k$  heads in a row is  $p^k(-1) + (1 - p^k) = 1 - 2p^k$ . Thus the  $k$ th component of  $G$  is  $p^k(1 - 2p^k) + (1 - p^k)(-G^T)$ .



Whatever the value  $v = \text{Val}(G)$ , Player I will choose  $k$  to maximize this. We have the equation

$$v = \max_k (p^k(1 - 2p^k) + (1 - p^k)(-v)).$$

Clearly  $v > 0$ , so there is a finite integer  $k$  at which the maximum is taken on, call it  $k_0$ . Then  $v = p^{k_0}(1 - 2p^{k_0})/(2 - p^{k_0})$  and since  $v$  takes on its maximum value at  $k_0$ , we have

$$v = \max_k \left( \frac{p^k(1 - 2p^k)}{2 - p^k} \right).$$

When  $p = .5$ , evaluating  $p^k(1 - 2p^k)/(2 - p^k)$  at  $k = 1, 2, 3, 4, \dots$  gives  $0, 1/14 = .0714\dots, 1/20 = .05, 7/248 = .0282\dots$ , and so on, with a clear maximum at  $k = 2$ .

(b) For arbitrary  $p$ , there is a maximum value attainable by  $v$ . Replace  $p^k$  by  $y$  in the formula for  $v$  and write it as  $f(y) = y(1 - 2y)/(2 - y)$ . Calculus gives  $f'(y) = (2y^2 - 8y + 2)/(2 - y)^2$ , so the function  $f(y)$  has a unique maximum on the interval  $(0,1)$  attained when  $y^2 - 4y + 1 = 0$ . The root of this equation in the interval  $(0,1)$  is  $y = 2 - \sqrt{3}$ , and the value attained there is  $V^* = f(2 - \sqrt{3}) = 7 - 4\sqrt{3} = .0718\dots$ . This is quite close to  $.0714\dots$  attainable when  $p = .5$ . If  $p = 2 - \sqrt{3}$ , then  $V^*$  is attainable with  $k = 1$ . As  $p \rightarrow \infty$ , it becomes easier and easier to choose  $k$  so that  $p^k$  is close to  $2 - \sqrt{3}$  so the value converges to  $V^*$ .

11. The first row shows the value is at least 1, and the first column shows the value is at most 4. So  $1 \leq v \leq 4$ . Then we see by “down-up-down-up” that the game does not have a saddle-point, so

$$v = \text{Val} \begin{pmatrix} 4 & 1 + (v/3) \\ 0 & 1 + (2v/3) \end{pmatrix} = \frac{4 + (8v/3)}{4 + (v/3)}.$$

This leads to the quadratic equation,  $v^2 + 4v - 12 = 0$ , which has solutions  $v = -2 \pm 4$ . Since  $v$  is positive, we have  $v = 2$  as the value. The matrix becomes  $\begin{pmatrix} 4 & 1 + (2/3) \\ 0 & 1 + (4/3) \end{pmatrix}$ . Player I's stationary optimal strategy is  $(1/2, 1/2)$ , and Player II's stationary optimal strategy is  $(1/7, 6/7)$ .

12. We have

$$v(1) = \text{Val} \begin{pmatrix} 2 & 2 + (v(2)/2) \\ 0 & 4 + (v(2)/2) \end{pmatrix} \quad v(2) = \text{Val} \begin{pmatrix} -4 & 0 \\ -2 + (v(1)/2) & -4 + (v(1)/2) \end{pmatrix}.$$

It may be difficult to guess that the matrices do not have saddle-points, so let us assume they do not and check later to see if this assumption is correct. If neither matrix has a saddle-point, then the equations become,

$$v(1) = \frac{8 + v(2)}{4} \quad v(2) = \frac{16 - 2v(1)}{-6}.$$

Solving these equations simultaneously, we find  $v(1) = 16/11$  and  $v(2) = -24/11$ . With these values the matrices above become

$$\begin{pmatrix} 2 & 2 - (12/11) \\ 0 & 4 - (12/11) \end{pmatrix} \quad \begin{pmatrix} -4 & 0 \\ -2 + (8/11) & -4 + (8/11) \end{pmatrix}.$$

Since these do not have saddle-points, our assumption is valid and  $v(1) = 16/11$  and  $v(2) = -24/11$  are the values. In  $G^{(1)}$ , the optimal stationary strategies are  $(8/11, 3/11)$  for I and  $(1/2, 1/2)$  for II. In  $G^{(2)}$ , the optimal stationary strategies are  $(1/3, 2/3)$  for I and  $(6/11, 5/11)$  for II.

## Solutions to Exercises of Section II.7.

1. (a) The value is 0. Player II has a pure optimal strategy, namely  $y = 0$ , since  $A(x, 0) = 0$  for all  $x \in X$ . Player I's optimal strategy is 1 and  $-1$  with equal probability  $1/2$ , since if  $\mathbf{q} \in Y_F^*$ ,

$$A(1/2, \mathbf{q}) = (1/2) \sum_j j q_j - (1/2) \sum_j j q_j = 0$$

because the sums are finite.

$$(b) \text{ If } q_j = \begin{cases} (1/4)(1/|j|) & \text{for } |j| = 2^{-k} \text{ for some } k \geq 2, \\ 0 & \text{otherwise} \end{cases}, \text{ then } \sum_j q_j = 1 \text{ and}$$

$$A(x, \mathbf{q}) = (\cdots - \frac{1}{4} - \frac{1}{4} - \frac{1}{4}) + (\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots) = -\infty + \infty.$$

2. It is easier to use Method 2. The value occurs at the intersection of the curve  $y_2 = e^{-y_1}$  and the line  $y_1 = y_2$ , namely, it is the solution of the equation  $v = e^{-v}$ . This is about  $v = .5671$ . The optimal strategy for II is  $y = v$ . The slope of the tangent line to the curve  $y_2 = e^{-y_1}$  at the point  $y_1 = v$  is  $-e^{-v} = -v$ . The normal to this is the negative of the reciprocal, namely  $1/v$ . The optimal strategy of Player I takes  $x_1$  and  $x_2$  in proportions  $v : 1$ . This is the mixed strategy  $(v/(1+v), 1/(1+v)) = (.3619, .6381)$ .

3. The value occurs where the curve  $(y_1 - 3)^2 + 4(y_2 - 2)^2 = 4$  is first hit by the line  $y_1 = y_2$ . Thus it satisfies the equation  $(v - 3)^2 + 4(v - 2)^2 = 4$  with  $v < 2$ . This leads to the equation,  $v^5 - 22v + 21 = 0$ , or  $(5v - 7)(v - 3) = 0$ , so  $v = 7/5$  is the value. Player II's optimal pure strategy is  $(v, v)$ . The slope of the curve  $(y_1 - 3)^2 + 4(y_2 - 2)^2 = 4$  at the point  $(v, v)$  is  $y'_2 = \frac{-(y_1 - 3)}{4(y_2 - 2)} = \frac{-(v - 3)}{4(v - 2)}$ . The normal is  $\frac{4(2 - v)}{(3 - v)} = \frac{3}{2}$ . So Player I's optimal strategy is  $(2/5, 3/5)$ .

4. (a) For the upper semi-continuous payoff, the value is  $1/2$ . An optimal strategy for Player I is to choose 0 and 1 with probability  $1/2$  each. For any  $0 < \epsilon < 1/6$ , an optimal strategy for Player II is choose  $1/3 - \epsilon$  and  $2/3 + \epsilon$  with probability  $1/2$  each.

(b) For the lower semi-continuous payoff, the value is  $1/2$ . An optimal strategy for Player II is to choose  $1/3$  and  $2/3$  with probability  $1/2$  each. Player I also has an optimal strategy. It is the same as above, namely, to choose 0 and 1 with probability  $1/2$  each.

5. This is essentially the game, He-who-chooses-the-smaller-positive-number-wins.

6. The game is symmetric, so if the value, if it exists, is zero and the players have identical optimal strategies. The pure strategy  $x = 1$  for Player I dominates the pure strategies  $x$  such that  $0 \leq x \leq 1 - 2b$ . Similarly for Player II. The game, played on the remaining square,  $[1 - 2b, 1]^2$ , is a latin square game of the form of Example 1 of Section

7.2. So the optimal strategy for both players is the uniform distribution on the interval  $[1 - 2b, 1]$ , and the value is zero. If  $b = 0$ , the optimal strategies are the pure strategies,  $x = 1$  and  $y = 1$ .

7. (a) The upper envelope is  $\max\{A(0, y), A(1, y)\} = \max\{y^2, 2(1 - y)^2\}$ . This has a minimum when  $y^2 = 2(1 - y)^2$ . This reduces to  $y^2 - 4y + 2 = 0$  whose solution in  $[0, 1]$  is  $y_0 = 2 - \sqrt{2} = .586 \dots$ . The slope of  $A(0, y)$  and that of  $A(1, y)$  at  $y = y_0$  is proportional to  $2y_0 : -4 + 2y_0$  which reduces to  $2 - \sqrt{2} : \sqrt{2}$ . So Player I's optimal strategy is mix  $x = 0$  and  $x = 1$  with probabilities  $(2 - \sqrt{2})/2$  and  $\sqrt{2}/2$ , respectively. Numerically this is  $(.293 \dots, .707 \dots)$ .

(b) This is a convex-concave game so both player have optimal pure strategies. If  $y_0$  is an optimal pure strategy for Player II, then  $x_0$  must maximize  $A(x, y_0)$ . As a function of  $x$  this is a line of slope  $e^{-y_0} - y_0$ . So

$$x_0 = \begin{cases} 0 & \text{if } e^{-y_0} < y_0 \\ \text{any} & \text{if } e^{-y_0} = y_0 \\ 1 & \text{if } e^{-y_0} > y_0 \end{cases}$$

We are bound to have a solution to this equation if  $e^{-y_0} = y_0$ . So  $y_0 = .5671 \dots$ . But  $y$  must minimize  $A(x_0, y)$ , whose derivative,  $-x_0 e^{-y} + 1 - x_0$  must be zero at  $y_0$ . This gives  $x_0(e^{-y_0} + 1) = 1$ . Since  $e^{-y_0} = y_0$ , we have  $x_0 = 1/(1 + y_0) = .6381 \dots$ .

8. (a) The payoff is convex in  $\mathbf{y}$  for every  $\mathbf{y}$ , so Player II has an optimal pure strategy. Player I will hide in one of the three corners to make it as hard as possible for Player II to come close. Therefore Player II will choose the point equidistant from  $(-1, 0)$ ,  $(1, 0)$ , and  $(0, 2)$ . This will be the point  $(0, a)$  such that the distance from  $(0, a)$  to  $(1, 0)$  is equal to the distance from  $(0, a)$  to  $(0, 2)$ . This gives  $1 + a^2 = (2 - a)^2$ , or  $a = 3/4$ . So II's optimal strategy is  $\mathbf{y} = (0, 3/4)$ , and the value of the game is  $(2 - a)^2 = 25/26$ . By symmetry, I's optimal strategy gives equal probability, say  $p$ , to  $(-1, 0)$  and  $(1, 0)$  and probability  $1 - 2p$  to  $(0, 2)$ , such that  $p(-1, 0) + p(1, 0) + (1 - 2p)(0, 2) = (0, 3/4)$ . This gives  $p = 5/16$ . Player I gives probability  $5/16$  to each of  $(-1, 0)$  and  $(1, 0)$  and probability  $6/16$  to  $(0, 2)$ .

(b) Find the smallest sphere,  $\{\mathbf{y} : \|\mathbf{y} - \mathbf{y}_0\|^2 \leq r^2\}$  containing  $S$ . Then,  $r^2$  is the value, and Player II has the optimal pure strategy  $\mathbf{y}_0$ . Let  $X_0 = \{\mathbf{x} \in S : \|\mathbf{x} - \mathbf{y}_0\|^2 = r^2\}$ . Then, there exist  $\mathbf{x}_1, \dots, \mathbf{x}_k$  in  $X_0$  (for some  $k \leq n+1$ ) and probabilities  $p_1, \dots, p_k$  adding to one, such that  $\sum_1^k p_i \mathbf{x}_i = \mathbf{y}_0$ . An optimal mixed strategy for Player I is to choose  $\mathbf{x}_i$  with probability  $p_i$ .

9. Let us assume that the "good" strategies are those that give all mass to some interval,  $[a, b]$ , with  $0 < a < 1 < b$ , and let us search for a strategy  $F$  with a density  $f(x)$  for which  $A(F, G)$  is constant over good strategies,  $G$ . Let  $\Phi(y) = \int_a^y x f(x) dx$ . Then

$$A(F, G) = \int_a^b \int_a^y (y + x) f(x) dx dG(y) - 1 = \int_a^b [yF(y) + \Phi(y)] dG(y) - 1 \quad (2)$$

Since  $\int_a^b dG(y) = 1$  and  $\int_a^b y dG(y) = 1$ , the payoff (2) will be constant for all good  $G$ , provided  $yF(y) + \Phi(y)$  is linear in  $y \in (a, b)$ , for some  $\alpha$ . This means the second derivative

of  $yF(y) + \Phi(y)$  must be zero:

$$3yf(y) + 2f'(y) = 0 \quad (3)$$

whose solution is

$$f(y) = cy^{-3/2} \quad \text{for } y \in (a, b), \quad (4)$$

for some constant  $c > 0$ . We have

$$F(y) = 2c[a^{-1/2} - y^{-1/2}] \quad \text{and} \quad \Phi(y) = 2c[y^{1/2} - a^{1/2}] \quad \text{for } a < y < b \quad (5)$$

For  $f$  to be a density, the constant  $c$  must satisfy

$$2c\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right) = 1. \quad (6)$$

For  $E(X) = 1$ , we must have

$$2c(\sqrt{b} - \sqrt{a}) = 1. \quad (7)$$

We see from (6) and (7) that  $b = 1/a$ , and  $c = \sqrt{b}/(2(b-1))$ . Therefore from (5),

$$yF(y) + \Phi(y) = \frac{2c}{\sqrt{a}}(y - a) = \frac{b}{b-1}\left(y - \frac{1}{b}\right) \quad (8)$$

The value of this game is zero. The distribution with density

$$f(y) = \sqrt{b}/(2(b-1))y^{-3/2} \quad \text{for } 1/b < y < b \quad (9)$$

is an optimal pure strategy for both players.

**Proof.** Suppose Player I uses  $f(x)$  of (9). Since there can be no ties, we must show that if Player II uses any distribution function  $G(y)$  on  $[0, b]$  having mean 1, then  $E((Y+X)I(X < Y)) \geq 1$ . But,

$$\begin{aligned} E((Y+X)I(X < Y)) &= \int_{1/b}^b \int_{1/b}^y (y+x)f(x) dx dG(y) = \int_{1/b}^b [yF(y) + \Phi(y)] dG(y) \\ &= \int_{1/b}^b \frac{b}{b-1}\left(y - \frac{1}{b}\right) dG(y) \\ &\geq \int_0^b \frac{b}{b-1}\left(y - \frac{1}{b}\right) dG(y) = \frac{b}{b-1}\left(1 - \frac{1}{b}\right) = 1 \end{aligned}$$

This also shows that if Player II puts any mass below  $1/b$ , then Player I's expected payoff is positive.

10. The payoff function,  $B(u, v)$ , is displayed in Figure 7.7.

This is a Latin Square type game with  $\int_0^1 B(u, v) du$  equal to the constant  $c$  for all  $v$ , and similarly  $\int_0^1 B(u, v) dv = c$  for all  $u$ . Thus, the optimal mixed strategy for both players is the uniform distribution on  $(0,1)$ , and the value of the game is  $c = .60206\dots$

This implies that in the original game the optimal strategy for I is to choose  $U$  from a uniform distribution on  $(0,1)$  and let  $X = 10^U$ , and similarly for II. The value is  $c$ .

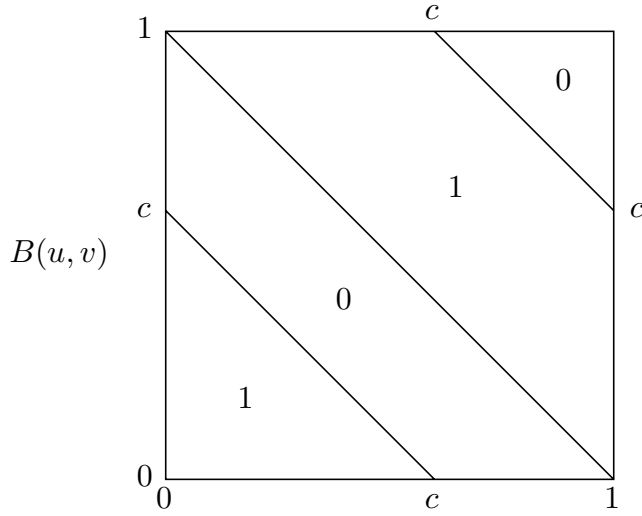
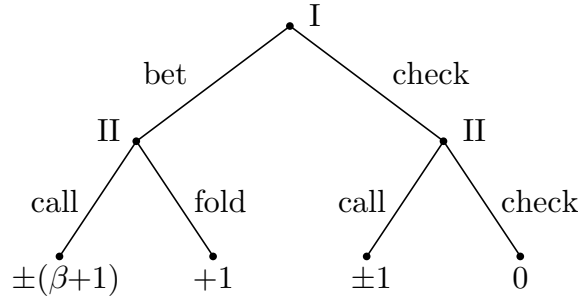


Figure 7.7

It is interesting to note that if the payoff is changed so that Player I wins 1 if and only if the first significant digit is in some set, such as  $\{2, 4, 7, 8\}$ , the optimal strategies of the players remain the same. Only the value changes.

11. (a)



(b) Indifference at  $a$ :  $c - (\beta + 1)(1 - c) = 0 \cdot d - (1 - d)$ .

Indifference at  $b$ :  $c + (\beta + 1)(b - c - (1 - b)) = 0 \cdot d + (b - c - (1 - b))$ .

Indifference at  $c$ :  $(\beta + 1)(-a + 1 - b) = a + 1 - b$ .

Indifference at  $d$ :  $d - a - (b - d) = 0$ .

Simplifying gives

$$d = (\beta + 2)c - \beta$$

$$d = \beta(1 + c - 2b)$$

$$\beta(1 - b) = (\beta + 2)a$$

$$d = (a + b)/2$$

(c) At  $\beta = 2$ , I get  $a = 5/33$ ,  $b = 23/33$ ,  $c = 20/33$ , and  $d = 14/33$ .

The value is:  $v = (\beta + 1)[(1 - b)(b - c) - a(1 - c)] + ac + (1 - b)c - (1 - b)(b - a) - (d - a)^2$ . This turns out to be negative for all  $\beta > 0$ . In particular for  $\beta = 2$ ,  $v = -2/33$ ; the game favors Player II.

12. It is optimal for Player I to play the points  $x_0 = 0$  and  $x_1 = 0.7$ , and for Player II to play the points  $y_0 = 0$  and  $y_1 = 0.5$  (but any  $0.4 < y_1 < 0.6$  works as well). This leads to the following 2 by 2 game with matrix

$$\begin{array}{cc} & \begin{array}{cc} y_0 & y_1 \end{array} \\ \begin{array}{c} x_0 \\ x_1 \end{array} & \begin{pmatrix} .00 & .40 \\ .60 & .24 \end{pmatrix} \end{array}$$

Methods of Chapter 2.2 may be used to solve this game. The value is  $v = 6/19$ . Player I bets nothing with probability  $9/19$ , and bets \$70 with probability  $10/19$ . Player II bets nothing with probability  $4/19$ , and \$50 with probability  $15/19$ . Player I wins with probability  $v = .316\dots$