Dynamic Programming

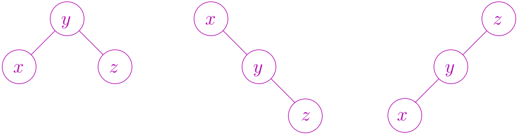
Algorithms: Design and Analysis, Part II

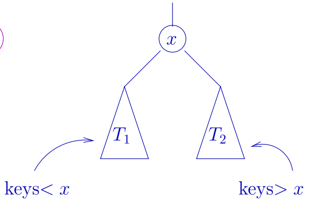
Optimal Binary Search Trees: Problem Definition

A Multiplicity of Search Trees

Recall: For a given set of keys, there are lots of valid search trees.

[say $x < y < z$]





the search
tree property

Question: What is the "best" search tree for a given set of keys?

A good answer: A balanced search tree, like a red-black tree.
(Recall Part I)
$\Rightarrow$ Worst-case search time $= \Theta(\text{height}) = \Theta(\log n)$

Exploiting Non-Uniformity

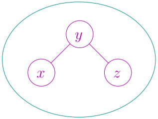
Question: Suppose we have keys $x < y < z$ and we know that:

80% of searches are for x

10% of searches are for y

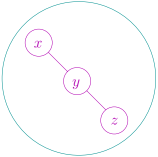
10% of searches are for z

What is the average search time (i.e., number of nodes looked at) in the trees:



$0.8 \cdot 2 + 0.1 \cdot 1 + 0.1 \cdot 2 = 1.9$

and



$0.8 \cdot 1 + 0.1 \cdot 2 + 0.1 \cdot 3 = 1.3$

respectively?

A) 2 and 3

B) 2 and 1

C) 1.9 and 1.2

D) 1.9 and 1.3

Problem Definition

Input: Frequencies $p_1, p_2, \ldots, p_n$ for items $1, 2, \ldots, n$.
[Assume items in sorted order, $1 < 2 < \ldots < n$]

Goal: Compute a valid search tree that minimizes the weighted (average) search time.

$C(T) = \sum_{\text{items } i} p_i$ [search time for i in T]

Depth of i in $T + 1$

Example: If T is a red-black tree, then $C(T) = O(\log n)$.
(Assuming $\sum_i p_i = 1$.)

Comparison with Huffman Codes

Similarities:

- Output = a binary tree
- Goal is (essentially) to minimize average depth with respect to given probabilities

Differences:

- With Huffman codes, constraint was prefix-freeness [i.e., symbols only at leaves]
- Here, constraint = search tree property [seems harder to deal with]

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Optimal BSTs: Optimal Substructure

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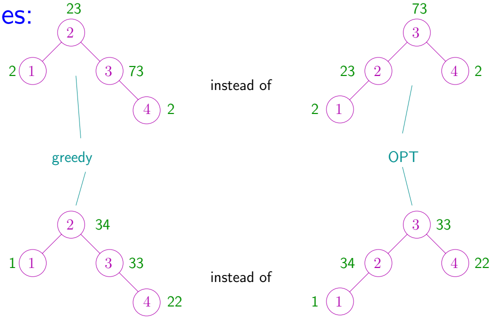
Greedy Doesn't Work

Intuition: Want the most (respectively, least) frequently accessed items closest (respectively, furthest) from the root.

Ideas for greedy algorithms:

- Bottom-up [populate lowest level with least frequently accessed keys]
- Top-down [put most frequently accessed item at root, recurse]

Counter examples:



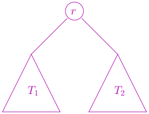
Choosing the Root

Issue: With the top-down approach, the choice of root has hard-to-predict repercussions further down the tree.
[stymies both greedy and naive divide + conquer approaches]

Idea: What if we knew the root?
(i.e., maybe can try all possibilities within a dynamic programming algorithm!)

Optimal Substructure

Question: Suppose an optimal BST for keys $\{1, 2, \ldots, n\}$ has root r, left subtree T_1, right subtree T_2. Pick the strongest statement that you suspect is true.



A) Neither T_1 nor T_2 need be optimal for the items it contains.

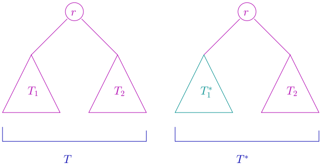
B) At least one of T_1, T_2 is optimal for the items it contains.

C) Each of T_1, T_2 is optimal for the items it contains.

D) T_1 is optimal for the keys $\{1, 2, \ldots, r-1\}$ and T_2 for the keys $\{r+1, r+2, \ldots, n\}$

Proof of Optimal Substructure

Let T be an optimal BST for keys $\{1, 2, \ldots, n\}$ with frequencies $p_1, \ldots, p_n$. Suppose T has root r. Suppose for contradiction that T_1 is not optimal for $\{1, 2, \ldots, r-1\}$ [other case is similar] with $C(T_1^*) < C(T_1)$. Obtain T^* from T by "cutting+pasting" T_1^* in for T_1.



Note: To complete contradiction + proof, only need to show that $C(T^*) < C(T)$.

Proof of Optimal Substructure (con'd)

A Calculation:

$=1+$search time for i in T_1 $\qquad$ $=1+$search time for i in T_2

$$
\begin{aligned}
C(T) &= \sum_{i=1}^{n} p_i\,[\text{search time for } i \text{ in } T] \\
&= p_r \cdot 1 + \sum_{i=1}^{r-1} p_i\,\boxed{[\text{search time for } i \text{ in } T]} \\
&\quad + \sum_{i=r+1}^{n} p_i\,\boxed{[\text{search time for } i \text{ in } T]} \\
&= \boxed{\sum_{i=1}^{n} p_i} + \boxed{\sum_{i=1}^{r-1} p_i\,[\text{search time for } i \text{ in } T_1]} \\
&\quad + \boxed{\sum_{i=r+1}^{n} p_i\,[\text{search time for } i \text{ in } T_2]}
\end{aligned}
$$

a constant (independent of T) $\qquad$ $= C(T_1)$ $\qquad$ $= C(T_2)$

Similarly: $C(T^*) = \sum_{i=1}^{n} p_i + C(T_1^*) + C(T_2)$

Upshot: $C(T_1^*) < C(T_1)$ implies $C(T^*) < C(T)$, contradicting optimality of T. QED!

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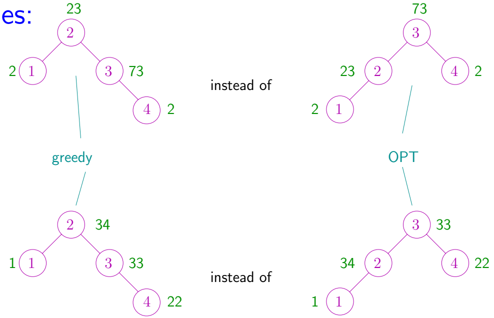

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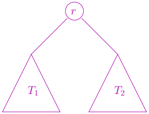

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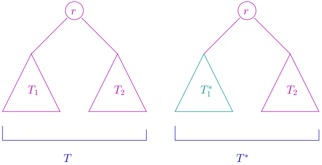

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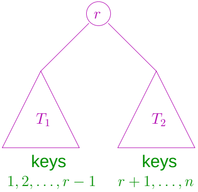


Note: Items in a subproblem are either a prefix or a suffix of the original problem.

Relevant Subproblems

Question: Let $\{1, 2, \ldots, n\}$ = original items. For which subsets $S \subseteq \{1, 2, \ldots, n\}$ might we need to compute the optimal BST for S?

A) Prefixes ($S = \{1, 2, \ldots, i\}$ for every i)

B) Prefixes and suffixes ($S = \{1, \ldots, i\}$ and $\{i, \ldots, n\}$ for every i)

C) <u>Contiguous intervals ($S = \{i, i+1, \ldots, j-1, j\}$ for every $i \leq j$)</u>

D) All subsets S

The Recurrence

Notation: For $1 \leq i \leq j \leq n$, let C_{ij} = weighted search cost of an optimal BST for the items $\{i, i+1, \ldots, j-1, j\}$ [with probabilities $p_i, p_{i+1}, \ldots, p_j$]

Recurrence: For every $1 \leq i \leq j \leq n$:

$$C_{ij} = \min_{r=i,\ldots,j} \left\{ \sum_{k=i}^{j} p_k + C_{i,r-1} + C_{r+1,j} \right\}$$

(Recall formula $C(T) = \sum_k p_k + C(T_1) + C(T_2)$ from last video)

Interpret $C_{xy} = 0$ if $x > y$

Correctness: Optimal substructure narrows candidates down to $(j - i + 1)$ possibilities, recurrence picks the best by brute force.

The Algorithm

Important: Solve smallest subproblems (with fewest number $(j - i + 1)$ of items) first.

Let A = 2-D array. [$A[i, j]$ represents opt BST value of items $\{1, \ldots, j\}$]

For $s = 0$ to $n - 1$ [s represents $j - i$]

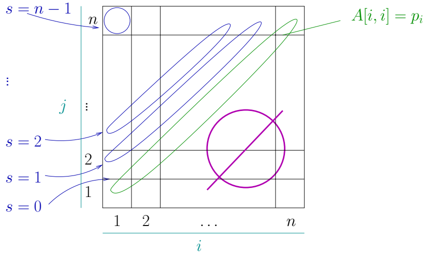
  For $i = 1$ to n [so $i + s$ plays role of j]

    $A[i, i+s] = \min_{r=1,\ldots,i+s}\{\sum_{k=1}^{i+s} p_k + \boxed{A[i, r-1]} + \boxed{A[r+1, i+s]}\ \}$

Return $A[1, n]$

Interpret as 0 if 1st index > 2nd index. Available for $O(1)$-time lookup

Pictorially:



Running Time

- $\Theta(n^2)$ subproblems
- $\Theta(j-i)$ time to compute $A[i,j]$

$\Rightarrow \Theta(n^3)$ time overall

Fun fact: [Knuth '71, Yoo '80] Optimized version of this DP algorithm correctly fills up entire table in only $\Theta(n^2)$ time [$\Theta(1)$ on average per subproblem]

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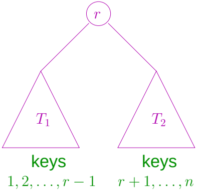

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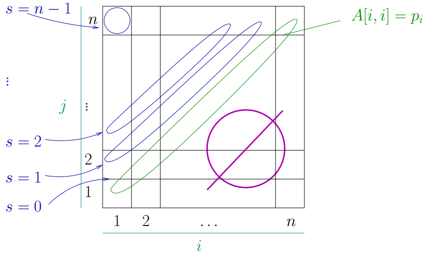

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