Advanced Union-Find

Path Compression: The Hopcroft-Ullman Analysis

Algorithms: Design and Analysis, Part II

Hopcroft-Ullman Theorem

Theorem: [Hopcroft-Ullman 73] With Union by Rank and path compression, m UNION+FIND operations take $O(m \log^* n)$ time, where $\log^* n =$ the number of times you need to apply log to n before the result is ≤ 1.

[Will focus on interesting case where $m = \Omega(n)$]

Measuring Progress

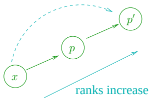
Intuition: Installing shortcuts should significantly speed up subsequent FINDs+UNIONs.

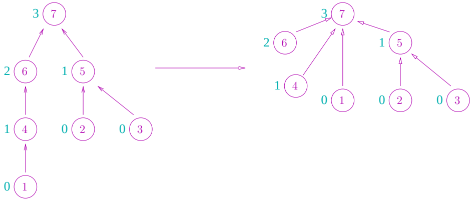
Question: How to track this progress and quantify the benefit?

Idea: Consider a non-root object x. $\longrightarrow$ Recall: rank[x] frozen

Progress measure: rank[parent[x]] - rank[x]

Path compression increases this progress measure: If x has old parent p, new parent $p' \neq p$, then rank[p']>rank[p].





Proof Setup

Rank blocks: $\{0\}, \{1\}, \{2, 3, 4\}, \{5, \ldots, \overset{16}{2^4}\}, \{17, 18, \ldots, \overset{65536}{2^{16}}\}, \{65537, \ldots, 2^{65536}\}, \ldots, \{\ldots, n\}$

Note: There are $O(\log^* n)$ different rank blocks.

Semantics: Traversal $x \to$ parent(x) is "fast progress" $\iff$ rank[parent[x]] in larger block than rank[x]

Definition: At a given point in time, call object x good if

(1) x or x's parent is a root OR

(2) rank[parent[x]] in larger block than rank[x]

x is bad otherwise.

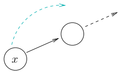
Proof of Hopcroft-Ullman

Point: Every FIND visits only $O(\log^* n)$ good nodes [$2 + \#$ of rank blocks $= O(\log^* n)$]

Upshot: Total work done during m operations $= O(m \log^* n)$ (visits to good objects) $+$ total $\#$ of visits to bad nodes (need to bound globally by separate argument)

Consider: A rank block $\{k+1, k+2, \ldots, 2^k\}$.

Note: When a bad node is visited



its parent is changed to one with strictly larger rank $\Rightarrow$ Can only happen 2^k times before x becomes good (forevermore).

Proof of Hopcroft-Ullman II

Total work: $O(m \log^* n) + O(\boxed{\# \text{ visits to bad nodes}})$.

$\leq n$ for each of $O(\log^* n)$ rank blocks

Consider: A rank block $\{k+1, k+2, \ldots, 2^k\}$.

Last slide: For each object x with final rank in this block, # visits to x while x is bad is $\boxed{\leq 2^k.}$

Rank Lemma: Total number of objects x with final rank in this rank block is $\sum_{i=k+1}^{2^k} n/2^i \boxed{\leq n/2^k.}$

$\leq n$ visits to bad objects in this rank block.

Recall: Only $O(\log^* n)$ rank blocks.

Total work: $O((m+n)\log^* n)$.