Huffman Codes

Introduction and Motivation

Algorithms: Design and Analysis, Part II

Binary Codes

Binary code: Maps each character of an alphabet Σ to a binary string.

Example: $\Sigma =$ a-z and various punctuation (size 32 overall, say)

Obvious encoding: Use the 32 5-bit binary strings to encode this Σ (a fixed-length code)

Can we do better? Yes, if some characters of Σ are much more frequent than others, using a variable-length code.

Ambiguity

Example: Suppose $\Sigma = \{$A,B,C,D$\}$. Fixed-length encoding would be $\{$00,01,10,11$\}$.

Suppose instead we use the encoding $\{$0,01,10,1$\}$. What is 001 an encoding of?

A) AB $\rightarrow$ Leads to 001

B) CD

C) AAD $\rightarrow$ Also leads to 001

D) Not enough info to answer question

Prefix-Free Codes

Problem: With variable-length codes, not clear where one character ends + the next one begins.

Solution: Prefix-free codes - make sure that for every pair $i, j \in \Sigma$, neither of the encodings $f(i), f(j)$ is a prefix of the other.

Example: $\{0,10,110,111\}$

Why useful? Can give shorter encodings with non-uniform character frequencies.

Example

Example:

A	60%	00	0
B	25%	01	10
C	10%	10	110
D	5%	11	111
Σ	frequencies	fixed-length	variable-length (prefix free)

Fixed-length encoding: 2 bits/character

Variable-length encoding: How many bits needed on average?

A) 1.5 B) 1.55 C) 2 D) 2.5

$0.6 \cdot 1 + 0.25 \cdot 2 + (0.1 + 0.05) \cdot 3 = 1.55$

Huffman Codes

Problem Definition

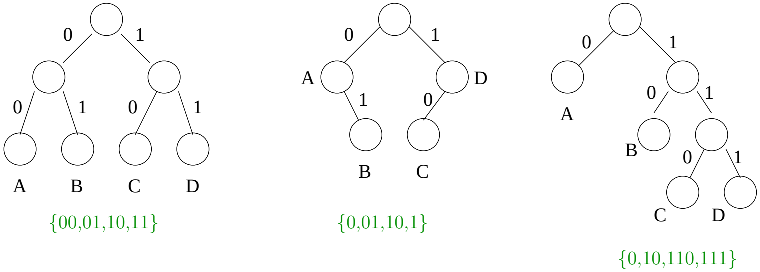
Algorithms: Design and Analysis, Part II

Codes as Trees

Goal: Best binary prefix-free encoding for a given set of character frequencies.

Useful fact: Binary codes $\leftrightarrow$ Binary trees

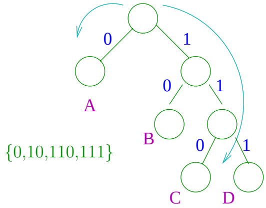
Examples: ($\Sigma = \{$A,B,C,D$\}$)



Prefix-Free Codes as Trees

In general: - Left child edges $\leftrightarrow$ "0", right child edges $\leftrightarrow$ "1"
- For each $i \in \Sigma$, exactly one node labeled "i"
- Encoding of $i \in \Sigma \leftrightarrow$ Bits along path from node to the node "i"
- Prefix-free $\leftrightarrow$ Labelled nodes = the leaves
  [since prefixes $\leftrightarrow$ one node an ancestor of another]

To decode: Repeadetly follow path from root until you hit a leaf.
[ex. 0110111 $\mapsto$ ACD] (unambiguous since only leaves are labelled)



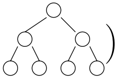
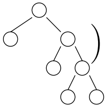
Note: Encoding length of $i \in \Sigma$ = depth of i in tree.

Problem Definition

Input: Probability p_i for each character $i \in \Sigma$.

Notation: If T = tree with leaves $\leftrightarrow$ symbols of Σ, then average encoding length $L(T) = \sum_{i \in \Sigma} p_i \cdot [\text{depth of } i \text{ in } T]$

Example: If $p_A = 60\%, p_B = 25\%, p_C = 10\%, p_D = 5\%$, then

$L($  $) = 2$ while $L($  $) = 1.55$

Output: A binary tree T minimizing the average encoding length $L(\cdot)$.

Huffman Codes

A Greedy Algorithm

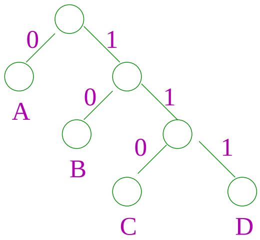
Algorithms: Design and Analysis, Part II

Codes as Trees

Input: Probability p_i for each character $i \in \Sigma$.

Output: Binary tree (with leaves $\leftrightarrow$ symbols of Σ) minimizing the average encoding length:

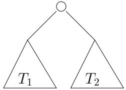
$$L(T) = \sum_{i \in \Sigma} p_i [\text{depth of } i \text{ in } T]$$



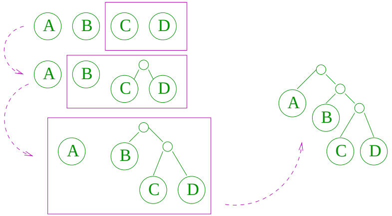
Building a Tree

Question: What's a principled approach for building a tree with leaves $\leftrightarrow$ symbols of Σ?

Natural but suboptimal idea: Top-down/divide+conquer.
- Partition Σ into Σ_1, Σ_2 each with $\approx$50% of total frequency.
- Recursively compute T_1 for Σ_1, T_2 for Σ_2, return:



Huffman's (optimal) idea: Build tree bottom-up using successive mergers.

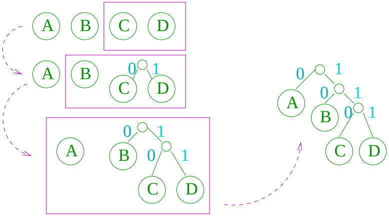


A Greedy Approach

Question: Which pair of symbols is "safe" to merge?

Observation: Final encoding length of $i \in \Sigma = \#$ of mergers its subtree endures.

[Each merger increases encoding length of participating symbols by 1]



Greedy heuristic: In first iteration, merge the two symbols with the smallest frequencies.

How to Recurse?

Suppose: 1st iteration of algorithm merges symbols a & b.

Idea: Replace symbols a, b by a new "meta-symbol" ab.

Question: What should be the frequency p_{ab} if this meta-symbol?

A) $\max\{p_a, p_b\}$

B) $\min\{p_a, p_b\}$

C) $\boxed{p_a + p_b}$ since ab is a proxy for "a or b" (intuitively)

D) $p_a - p_b$

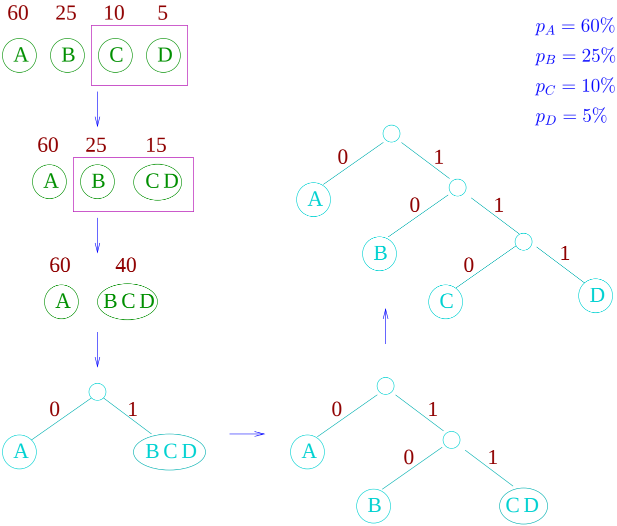
Example

$p_A = 60\%$

$p_B = 25\%$

$p_C = 10\%$

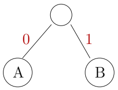
$p_D = 5\%$



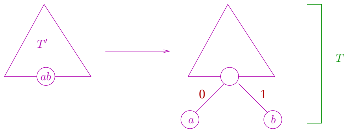
Huffman's Algorithm

(Given frequencies p_i as input)

If $|\Sigma| = 2$ return



Let $a, b \in \Sigma$ have the smallest frequencies.
Let $\Sigma' = \Sigma$ with a, b replaced by new symbol ab.
Define $p_{ab} = p_a + p_b$.
Recursively compute T' (for the alphabet Σ')
Extend T' (with leaves $\leftrightarrow \Sigma'$) to a tree T with leaves $\leftrightarrow \Sigma$ by splitting leaf ab into two leaves a & b.



Return T

Huffman Codes

Huffman's Algorithm: A More Complex Example

Algorithms: Design and Analysis, Part II

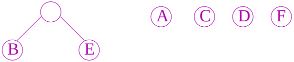
Input and Steps 1 and 2

Input:

Characters	A	B	C	D	E	F
Weights	3	2	6	8	2	6

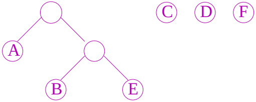
Step 1: Merge B and E:

A	BE	C	D	F
3	4	6	8	6



Step 2: Merge A and BE:

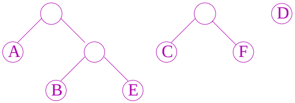
ABE	C	D	F
7	6	8	6



Steps 3 and 4

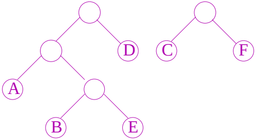
Step 3: Merge C and F:

ABE	CF	D
7	12	8



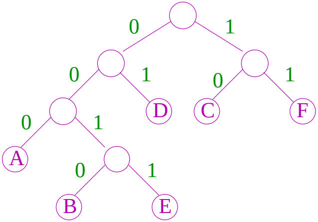
Step 2: Merge ABE and D:

ABDE	CF
15	12



Final Output

Final tree:



Corresponding code:

A 000 D 01
B 0010 E 10
C 0011 F 11

Huffman Codes

Correctness Proof

Algorithms: Design and Analysis, Part II

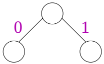
Correctness of Huffman's Algorithm

Theorem: [Huffman 52] Huffman's algorithm computes a binary tree (with leaves $\leftrightarrow$ symbols of Σ) that minimizes the average encoding length

$$L(T) = \sum_{i \in \Sigma} p_i [\text{depth of leaf } i \text{ in } T].$$

Proof: By induction on $n = |\Sigma|$. (Can assume $n \geq 2$.)

Base case: When $n = 2$, algorithm outputs the optimal tree.
(Needs 1 bit per symbol)



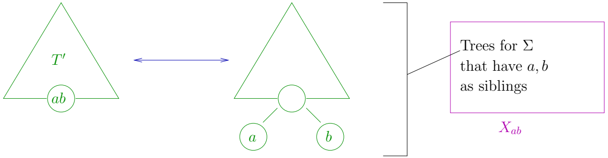
Inductive step: Fix input with $n = |\Sigma| > 2$.

By inductve hypothesis: Algorithm solves smaller subproblems (for Σ') optimally.

Inductive Step

Let $\Sigma' = \Sigma$ with a, b (symbols with smallest frequencies) replaced by meta-symbol ab. Define $p_{ab} = p_a + p_b$.

Recall: Exact correspondence between:



Important: For every such pair T' and T, $L(T) - L(T')$ is (after cancellation)

p_a [a's depth in T] $+ p_b$ [b's depth in T] $- p_{ab}$ [ab's depth in T'] $=$

Each is one more than

$= p_A(d+1) + p_b(d+1) - (p_a + p_b)d = p_a + p_b$, Independent of T, T'!

Proof of Theorem

Inductive hypothesis: Huffman's algorithm computes a tree $\hat{T}'$ that minimizes $L(T')$ for Σ'.

Upshot of last slide: Corresponding tree $\hat{T}$ minimizes $L(T)$ for Σ over all trees in X_{ab} (i.e., where a & b are siblings)

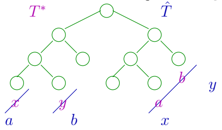
Key lemma: [Completes proof of theorem] There is an optimal tree (for Σ) in X_{ab}. [i.e., a & b were "safe" to merge]

Intuition: Can make an optimal tree better by pushing a & b as deep as possible (since a, b have smallest frequencies).

Proof of Key Lemma

By exchange argument. Let T^* be any tree that minimizes $L(T)$ for Σ. Let x, y be siblings at the deepest level of T^*.

The exchange: Obtain $\hat{T}$ from T^* by swapping $a \leftrightarrow x$, $b \leftrightarrow y$



Note: $\hat{T} \in X_{ab}$ (by choice of x, y).

To finish: Will show that $L(\hat{T}) \leq L(T^*)$
[$\Rightarrow$ $\hat{T}$ also optimal, completes proof]

Reason:

$$
\begin{aligned}
L(T^*) - L(\hat{T}) \quad &= \quad (p_x - p_a) \quad [x\text{'s depth in } T^* \text{ - } a\text{'s depth in } T^*] \\
&+ \quad (p_y - p_b) \quad [y\text{'s depth in } T^* \text{ - } b\text{'s depth in } T^*] \\
&\geq \quad 0 \qquad \text{QED!}
\end{aligned}
$$

≥ 0 since a, b have smallest frequencies $\qquad$ ≥ 0 by choice of x, y

Notes on Running Time

Naive implementation: $O(n^2)$ time, where $n = |\Sigma|$.

Speed ups: - Use a heap! [to perform repeated minimum computations]
- Use keys = frequencies
- After extracting the two smallest-frequency symbols, re-Insert the new meta-symbol [new key = sum of the 2 old ones]

$\Rightarrow$ Iterative, $O(n \log n)$ implementation.

Even faster: (Non-trivial exercise) Sorting + $O(n)$ additional work.
- Manage (meta-)symbols using two queues.

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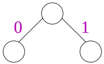

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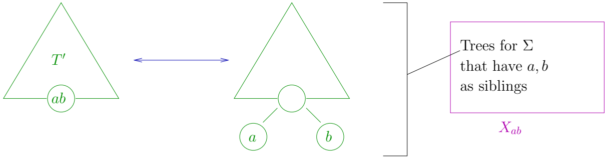

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Tim Roughgarden

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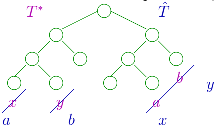

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