Linear-Time Selection

Randomized Selection (Algorithm)

Design and Analysis of Algorithms I

Prerequisites

Watch this after:

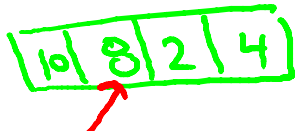
- QuickSort - Partitioning around a pivot
- QuickSort – Choosing a good pivot
- Probability Review, Part I

The Problem

For simplicity

Input : array A with n distinct numbers and a number

Output : i^{th} order statistic (i.e., i^{th} smallest element of input array)



3^{rd} order statistic

Example : median.

(i = (n+1)/2 for n odd,

i = n/2 for n even)

Reduction to Sorting

O(nlog(n)) algorithm

1) Apply MergeSort 2) return i^{th} element of sorted array

Fact : can't sort any faster [see optional video]

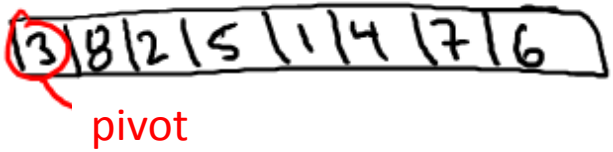
Next : O(n) time (randomized) by modifying Quick Sort.

Optional Video : O(n) time deterministic algorithm.
-- pivot = "median of medians" (warning : not practical)

Partitioning Around a Pivot

Key Idea : partition array around a pivot element.

-Pick element of array



-Rearrange array so that

-Left of pivot => less than pivot

-Right of pivot => greater than pivot

Note : puts pivot in its "rightful position".

Suppose we are looking for the 5th order statistic in an input array of length 10. We partition the array, and the pivot winds up in the third position of the partitioned array. On which side of the pivot do we recurse, and what order statistic should we look for?

- ○ The 3rd order statistic on the left side of the pivot.
- ○ The 2nd order statistic on the right side of the pivot.
- ○ The 5th order statistic on the right side of the pivot.
- ○ Not enough information to answer question – we might need to recurse on the left or the right side of the pivot.

Randomized Selection

Rselect (array A, length n, order statistic i)

0) if n = 1 return A[1]
1) Choose pivot p from A uniformly at random
2) Partition A around p
   let j = new index of p
3) If j = i, return p
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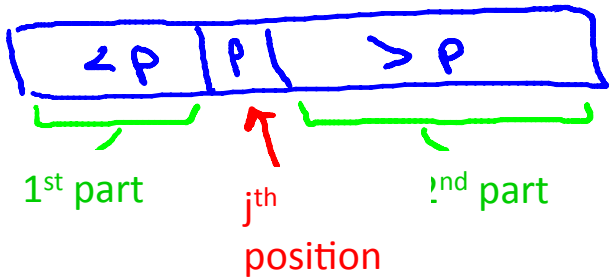
5) [if j<i] return Rselect (2nd part of A, n-j, i-j)



Properties of RSelect

Claim : Rselect is correct (guaranteed to output ith order statistic)

Proof : by induction. [like in optional QuickSort video]

Running Time ? : depends on "quality" of the chosen pivots.

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What is the running time of the RSelect algorithm if pivots are always chosen in the worst possible way?

- $\theta(n)$
- $\theta(n \log n)$
- $\theta(n^2)$
- $\theta(2^n)$

Example :

-- suppose i = n/2
-- suppose choose pivot = minimum every time
=> $\Omega(n)$ time in each of $\Omega(n)$ recursive calls

Running Time of RSelect?

Running Time ? : depends on which pivots get chosen.
(could be as bad as $\theta(n^2)$)

Key : find pivot giving "balanced" split.

Best pivot: the median ! (but this is circular)

⇒Would get recurrence $T(n) \leq T(n/2) + O(n)$
⇒ T(n) = O(n) [case 2 of Master Method]
Hope : random pivot is "pretty good" "often enough"

Running Time of RSelect

Rselect Theorem : for every input array of length n, the average running time of Rselect is O(n)

-- holds for every input [no assumptions on data]

-- "average" is over random pivot choices made by the algorithm

Linear-Time Selection

Randomized Selection (Analysis)

Design and Analysis of Algorithms I

Running Time of RSelect

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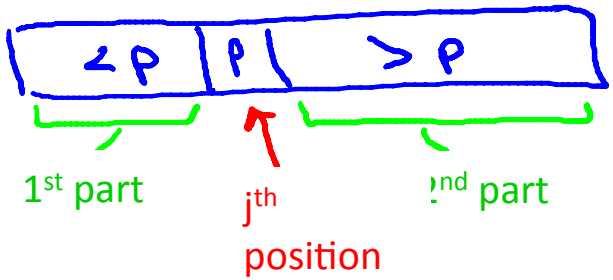


Proof I: Tracking Progress via Phases

Note : Rselect uses <= cn operations outside of recursive call [for some constant c > 0] [from partitioning]

Notation : Rselect is in phase j if current array size between $(\frac{3}{4})^{j+1} \cdot n$ and $(\frac{3}{4})^{j} \cdot n$

-X_j = number of recursive calls during phase j

Note : running time of RSelect $\leq \sum_{phases\ j} X_j \cdot c \cdot (\frac{3}{4})^{j} \cdot n$

# of phase j subproblems

<= array size during phase j

Work per phase j subproblem

Proof II: Reduction to Coin Flipping

X_j = # of recursive calls during phase j → Size between $(\frac{3}{4})^{j+1} \cdot n$ and $(\frac{3}{4})^{j} \cdot n$

Note : if Rselect chooses a pivot giving a 25 – 75 split (or better) then current phase ends !
(new subarray length at most 75 % of old length)

Recall : probability of 25-75 split or better is 50%

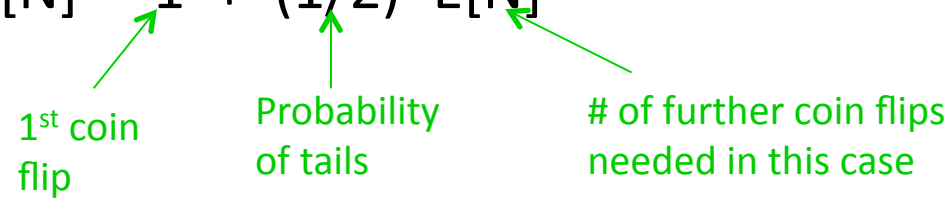
So : $E[X_j]$ <= expected number of times you need to flip a fair coin to get one "heads"
(heads ~ good pivot, tails ~ bad pivot)

Proof III: Coin Flipping Analysis

Let N = number of coin flips until you get heads.

(a "geometric random variable")

Note : $E[N] = 1 + (1/2)*E[N]$



Solution : $E[N] = 2$ (Recall $E[X_j] <= E[N]$)

Putting It All Together

Expected running time of RSelect

$$\leq E[cn \sum_{phase\ j} (\frac{3}{4})^j X_j] \qquad (*)$$

$$= cn \sum_{phase\ j} (\frac{3}{4})^j E[X_j] \qquad \text{[LIN EXP]}$$

= E[# of coin flips N] = 2

$$\leq 2cn \sum_{phase\ j} (\frac{3}{4})^j$$

geometric sum,
<= 1/(1-3/4) = 4

$$\leq 8cn = O(n)$$

Q.E.D.

Linear-Time Selection

Deterministic Selection (Algorithm)

Design and Analysis of Algorithms I

The Problem

For simplicity

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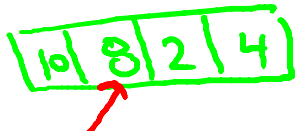
Output : i^{th} order statistic (i.e., i^{th} smallest element of input array)



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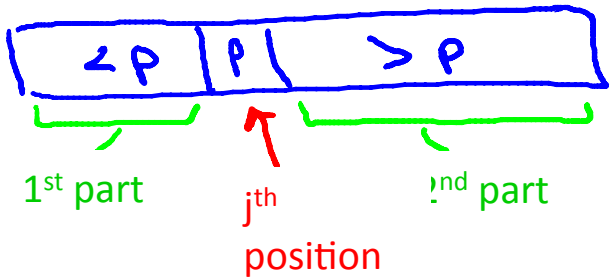


Guaranteeing a Good Pivot

Recall : “best” pivot = the median ! (seems circular!)

Goal : find pivot guaranteed to be pretty good.

Key Idea : use “median of medians”!

A Deterministic ChoosePivot

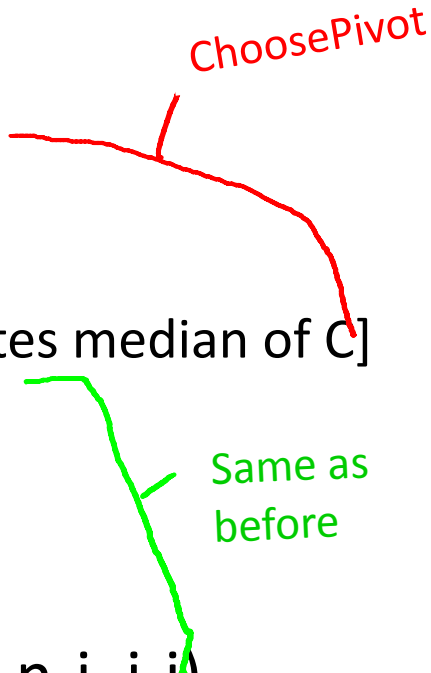
ChoosePivot(A,n)

-- logically break A into n/5 groups of size 5 each
-- sort each group (e.g., using Merge Sort)
-- copy n/5 medians (i.e., middle element of each sorted group)
   into new array C
-- recursively compute median of C (!)
-- return this as pivot

The DSelect Algorithm

DSelect(array A, length n, order statistic i)

1. Break A into groups of 5, sort each group
2. C = the n/5 "middle elements"
3. p = DSelect(C,n/5,n/10) [recursively computes median of C]
4. Partition A around p
5. If j = i return p
6. If j < i return DSelect(1st part of A, j-1, i)
7. [else if j > i] return DSelect(2nd part of A, n-j, i-j)



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How many recursive calls does DSelect make?

- 0
- 1
- 2
- 3

Running Time of DSelect

Dselect Theorem : for every input array of length n, Dselect runs in O(n) time.

Warning : not as good as Rselect in practice

1) Worse constraints 2) not-in-place

History : from 1973

Blum – Floyd – Pratt – Rivest – Tarjan

('95) ('78) ('02) ('86)

Linear-Time Selection

Deterministic Selection (Analysis)

Design and Analysis of Algorithms I

The DSelect Algorithm

DSelect(array A, length n, order statistic i)

1. Break A into groups of 5, sort each group
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6. If j < i return DSelect(1st part of A, j-1, i)
7. [else if j > i] return DSelect(2nd part of A, n-j, i-j)

Choose Pivot

Same as before

What is the asymptotic running time of step 1 of the DSelect algorithm?

- $\theta(1)$
- $\theta(\log n)$
- $\theta(n)$
- $\theta(n \log n)$

Note : sorting an array with 5 elements takes <= 120 operations

[why 120 ? Take m = 5 in our $6m(\log_2 m+1)$ bound for Merge Sort]

$6*5*(\log_2 5+1) <= 120$

<=3

# of gaps

ops per group

So : <= (n/5)*120 = 24n = O(n) for all groups

The DSelect Algorithm

DSelect(array A, length n, order statistic i)

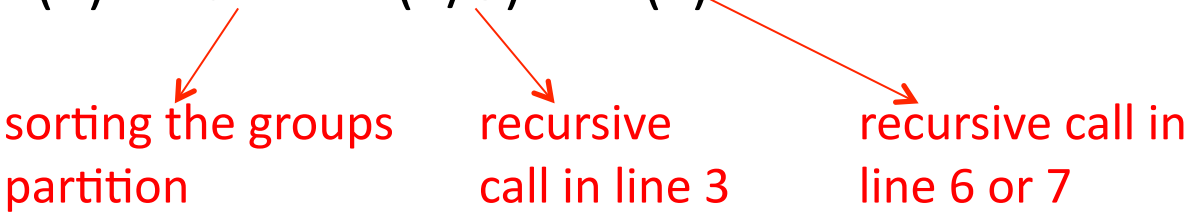
1. Break A into groups of 5, sort each group $\theta(n)$
2. C = the n/5 "middle elements" $\theta(n)$
3. p = DSelect(C,n/5,n/10) [recursively computes median of C] $T(\frac{n}{5})$
4. Partition A around p $\theta(n)$
5. If j = i return p
6. If j < i return DSelect(1st part of A, j-1, i) $T(?)$
7. [else if j > i] return DSelect(2nd part of A, n-j, i-j) $T(?)$

Rough Recurrence

Let T(n) = maximum running time of Dselect on an input array of length n.

There is a constant c >= 1 such that :

1. T(1) = 1
2. T(n) <= c*n + T(n/5) + T(?)



The Key Lemma

Key Lemma : 2^{nd} recursive call (in line 6 or 7) guaranteed to be on an array of size <= 7n/10 (roughly)

Upshot : can replace "?" by "7n/10"

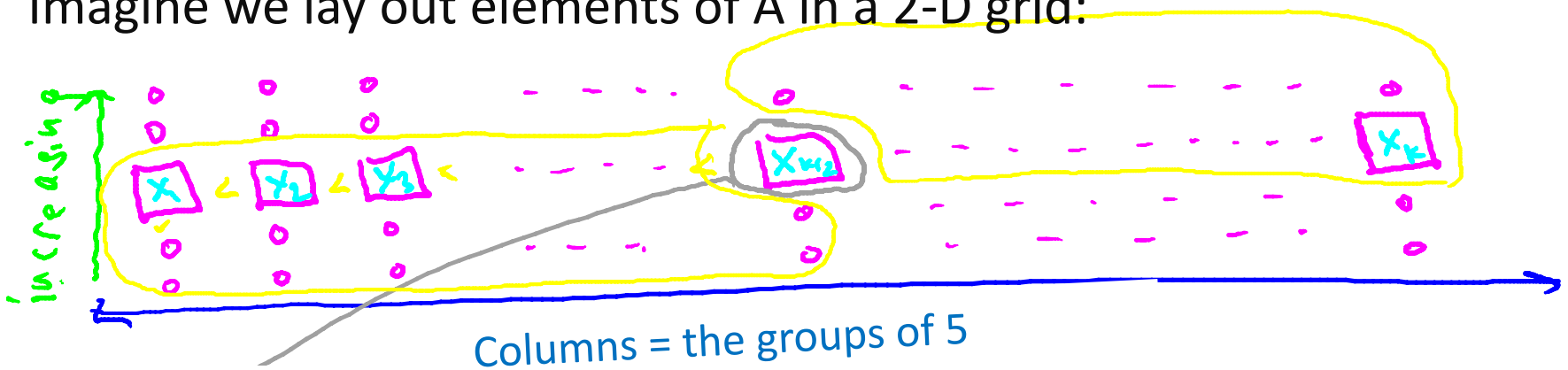
Rough Proof : Let k = n/5 = # of groups
Let x_i = i^{th} smallest of the k "middle elements"
[So pivot = $x_{k/2}$]

Goal : >= 30% of input array smaller than $x_{k/2}$,
>= 30% is bigger

Rough Proof of Key Lemma

Thought Experiment :

Imagine we lay out elements of A in a 2-D grid:

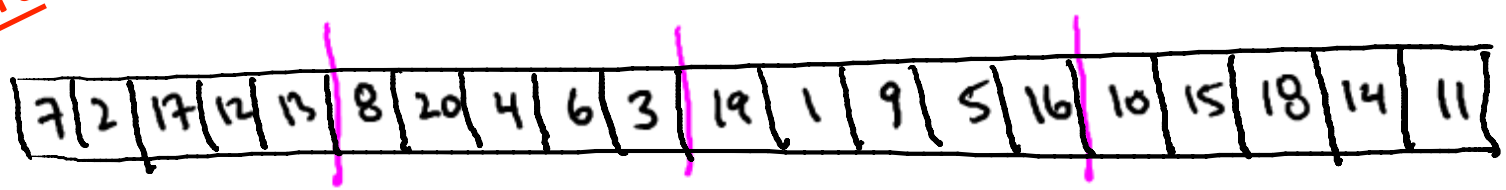


Key point : $x_{k/2}$ bigger than 3 out of 5 (60%) of the elements in ~ 50% of the groups

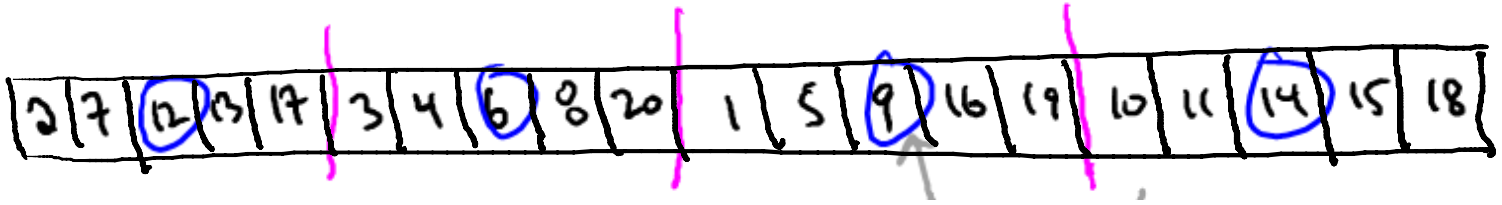
=> bigger than 30% of A (similarly, smaller than 30% of A)

Example

Input

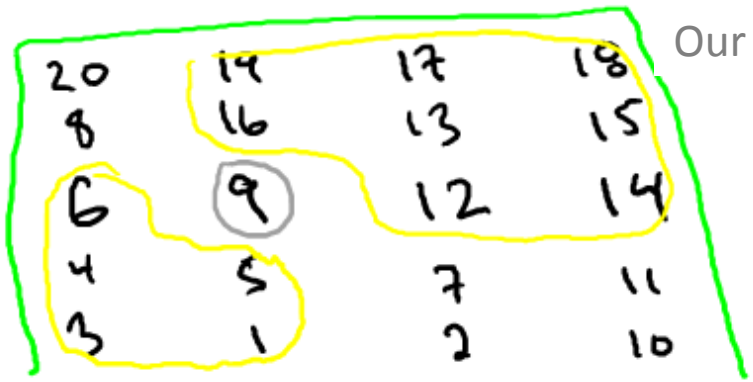


After sorting groups of 5



Our pivot !

The grid :



Design and Analysis
of Algorithms I

Linear-Time Selection

Deterministic Selection (Analysis II)

Rough Recurrence (Revisited)

Let T(n) = maximum running time of Dselect on an input array of length n.

There is a constant c >= 1 such that :

1. T(1) = 1
2. T(n) <= c*n + T(n/5) + T(?)

<= 7n/10 by Key Lemma

sorting the groups partition

recursive call in line 3

recursive call in line 6 or 7

Rough Recurrence (Revisited)

$T(1) = 1$, $T(n) <= cn + T(n/5) + T(7n/10)$

Constant c>=1

Note : different-sized subproblems => can't use Master Method!

Strategy : "hope and check"

Hope : there is some constant a [independent of n]
Such that $T(n) <= an$ for all $n >=1$
[if true, then $T(n) = O(n)$ and algorithm is linear time]

Analysis of Rough Recurrence

T(1) = 1 ; T(n) <= cn + T(n/5) + T(7n/10)

Constant c>=1

Claim : Let a = 10c

Then T(n) <= an for all n >= 1

=> Dselect runs in O(n) time

Proof : by induction on n

Base case : T(1) = 1 <= a*1 (since a >= 1)

Inductive Step : [n > 1]

Inductive Hypothesis : $T(k) \leq ak \; \forall \; k < n$

We have $T(n) \leq cn + T(n/5) + T(7n/10)$

GIVEN

$\leq cn + a(n/5) + a(7n/10)$

IND HYP

$= n(c + 9a/10) = an$

Choice of a

Q.E.D.

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Design and Analysis
of Algorithms I

Linear-Time Selection

An $\Omega(n \log n)$ Sorting Lower Bound

A Sorting Lower Bound

Theorem : every "comparison-based" sorting algorithm has worst-case running time $\Omega(n \log n)$

[assume deterministic, but lower bound extends to randomized]

Comparison-Based Sort : accesses input array elements only via comparisons ~ "general purpose sorting method"

Examples : Merge Sort, Quick Sort, Heap Sort

Non Examples : Bucket Sort, Counting Sort, Radix Sort

Good for data from distributions

good for small integers

good for medium-size integers

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Proof Idea

Fix a comparison-based sorting method and an array length n

⇒Consider input arrays containing {1,2,3,…,n} in some order.

⇒ n! such inputs

Suppose algorithm always makes <= k comparisons to correctly sort these n! inputs.

=> Across all n! possible inputs, algorithm exhibits <= 2^k distinct executions

i.e., resolution of the comparisons

Proof Idea (con'd)

By the Pigeonhole Principle : if $2^k < n!$, execute identically on two distinct inputs => must get one of them incorrect.

So : Since method is correct,

$$2^k \geq n!$$

$$\geq \left(\frac{n}{2}\right)^{\frac{n}{2}}$$

$$=> k \geq \frac{n}{2} \cdot \log_2 \frac{n}{2} = \Omega(n \log n)$$

Contraction Algorithm

Overview

Design and Analysis of Algorithms I

Goals for These Lectures

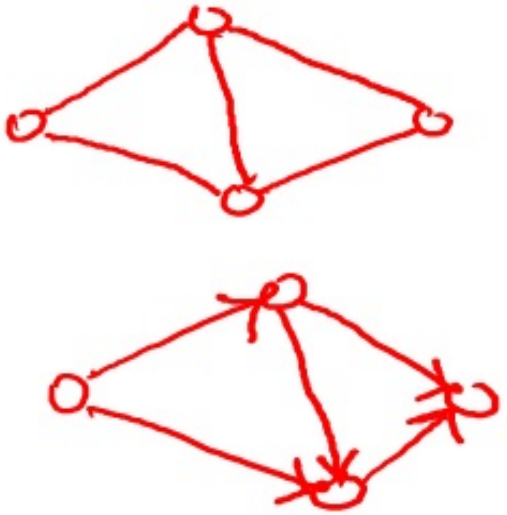
- Further practice with randomized algorithms
  - In a new application domain (graphs)
- Introduction to graphs and graph algorithms

Also: "only" 20 years ago!

Graphs

Two ingredients

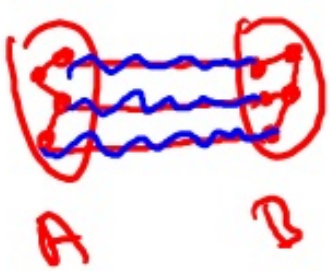
- Vertices aka nodes (V)
- Edges (E) = pairs of vertices
  - can be undirected [unordered pair] or directed [ordered pair] (aka arcs)



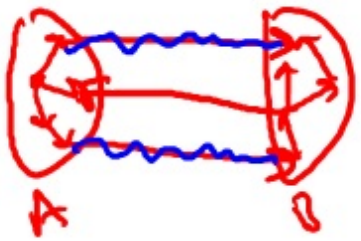
Examples: road networks, the Web, social networks, precedence constraints, etc.

Cuts of Graphs

Definition: a cut of a graph (V, E) is a partition of V into two non-empty sets A and B.



[undirected]



[directed]

Definition: the crossing edges of a cut (A, B) are those with:

- the one endpoint in each of (A, B) [undirected]
- tail in A, head in B [directed]

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Roughly how many cuts does a graph with n vertices have?

- n
- n^2
- 2^n
- n^n

The Minimum Cut Problem

- INPUT: An undirected graph G = (V, E).

[Parallel  edges allowed]

[See other video for representation of the input]

- GOAL: Compute a cut with fewest number of crossing edges. (a min cut)

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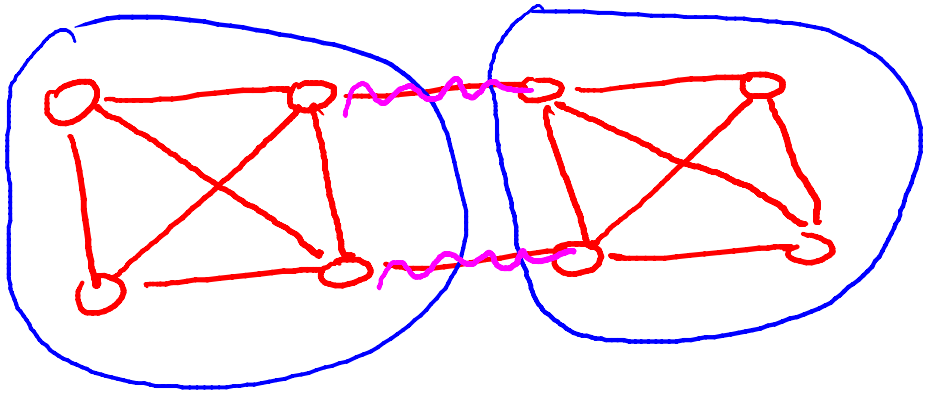
What is the number of edges crossing a minimum cut in the graph shown below?

- 1
- 2
- 3
- 4



A Few Applications

- indentify network bottlenecks / weaknesses
- community detection in social networks
- image segmentation
  - input = graph of pixels
  - use edge weights

    [(u,v) has large weight ⇔ "expect" u,v to come from some object]

  hope: repeated min cuts identifies the primary objects in picture.

Design and Analysis
of Algorithms I

Graph Algorithms

Representing Graphs

Graphs

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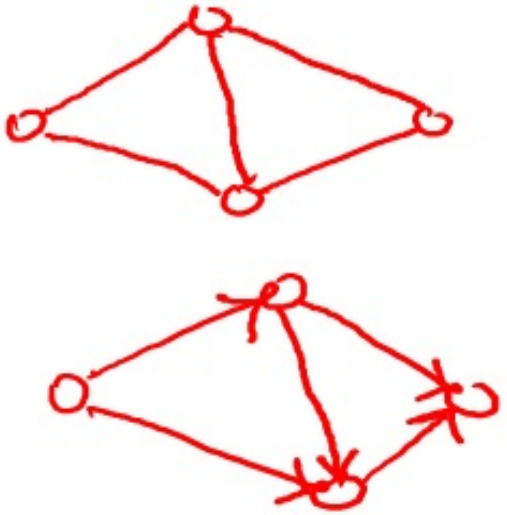
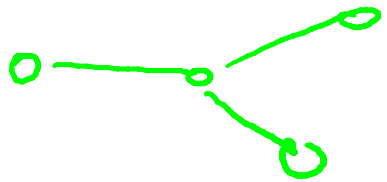
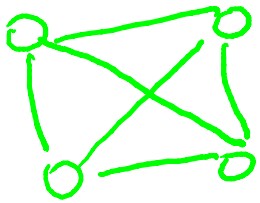


Examples: road networks, the Web, social networks, precedence constraints, etc.

Consider an undirected graph that has n vertices, no parallel edges, and is connected (i.e., “in one piece”). What is the minimum and maximum number of edges that the graph could have, respectively ?

- ○ $n-1$ and $n(n-1)/2$
- ○ $n-1$ and n^2
- ○ n and 2^n
- ○ n and n^n





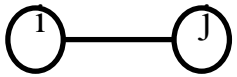
Sparse vs. Dense Graphs

Let n = # of vertices, m = # of edges.

In most (but not all) applications, m is $\Omega(n)$ and $O(n^2)$

- in a "sparse" graph, m is or is close to $O(n)$
- in a "dense" graph, m is closer to $\theta(n^2)$

The Adjacency Matrix

Represent G by a n x n 0-1 matrix A where
$A_{ij} = 1 \Leftrightarrow$ G has an i-j edge 

<u>Variants</u>

- A_{ij} = # of i-j edges (if parallel edges)
- A_{ij} = weight of i-j edge (if any)
- $A_{ij} = \begin{cases} +1 \text{ if } \circ\!\!\rightarrow\!\!\circ \\ -1 \text{ if } \circ\!\!\leftarrow\!\!\circ \end{cases}$

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How much space does an adjacency matrix require, as a function of the number n of vertices and the number m of edges?

- $\theta(n)$
- $\theta(m)$
- $\theta(m+n)$
- $\theta(n^2)$

Adjacency Lists

Ingredients

- array (or list) of vertices
- array (or list) of edges
- each edge points to its endpoints
- each vertex points to edges incident on it

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ver

How much space does an adjacency list representation require, as a function of the number n of vertices and the number m of edges?

- $\theta(n)$
- $\theta(m)$
- $\theta(m+n)$
- $\theta(n^2)$

Adjacency Lists

Ingredients

- array (or list) of vertices
- array (or list) of edges
- each edge points to its endpoints
- each vertex points to edges incident on it

Space

$\theta(n)$

$\theta(m)$

$\theta(m)$

$\theta(m)$

$\theta(m+n)$

$[or \ \theta(max\{m,n\})]$

Question: which is better?

Answer: depends on graph density and operations needed.

This course: focus on adjacency lists.

Contraction Algorithm

The Algorithm

Design and Analysis of Algorithms I

The Minimum Cut Problem

- INPUT: An undirected graph G = (V, E).

[Parallel  edges allowed]

[See other video for representation of the input]

- GOAL: Compute a cut with fewest number of crossing edges. (a min cut)

Tim Roughgarden

Random Contraction Algorithm

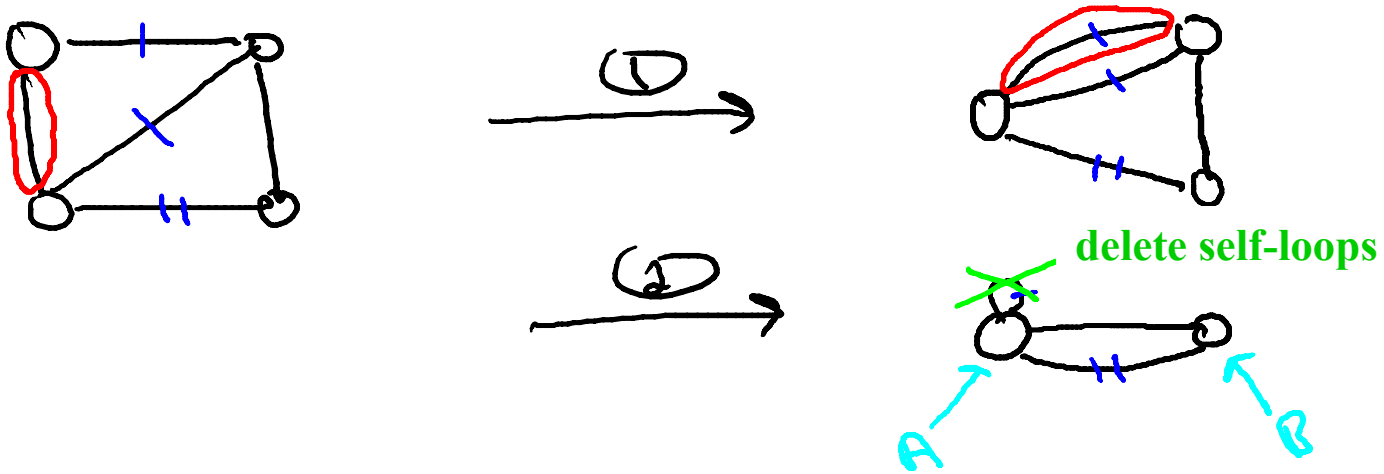
[due to Karger, early 90s]

While there are more than 2 vertices:

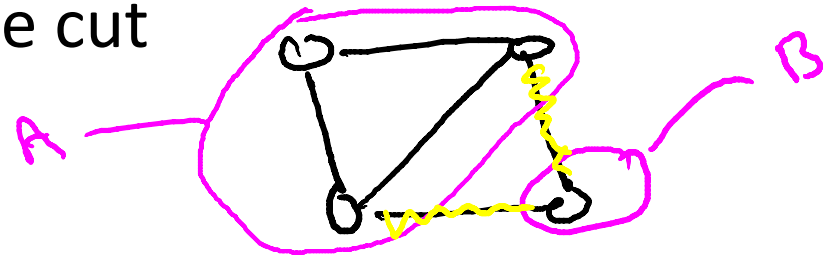
- pick a remaining edge (u,v) uniformly at random
- merge (or "contract") u and v into a single vertex
- remove self-loops

return cut represented by final 2 vertices.

Example

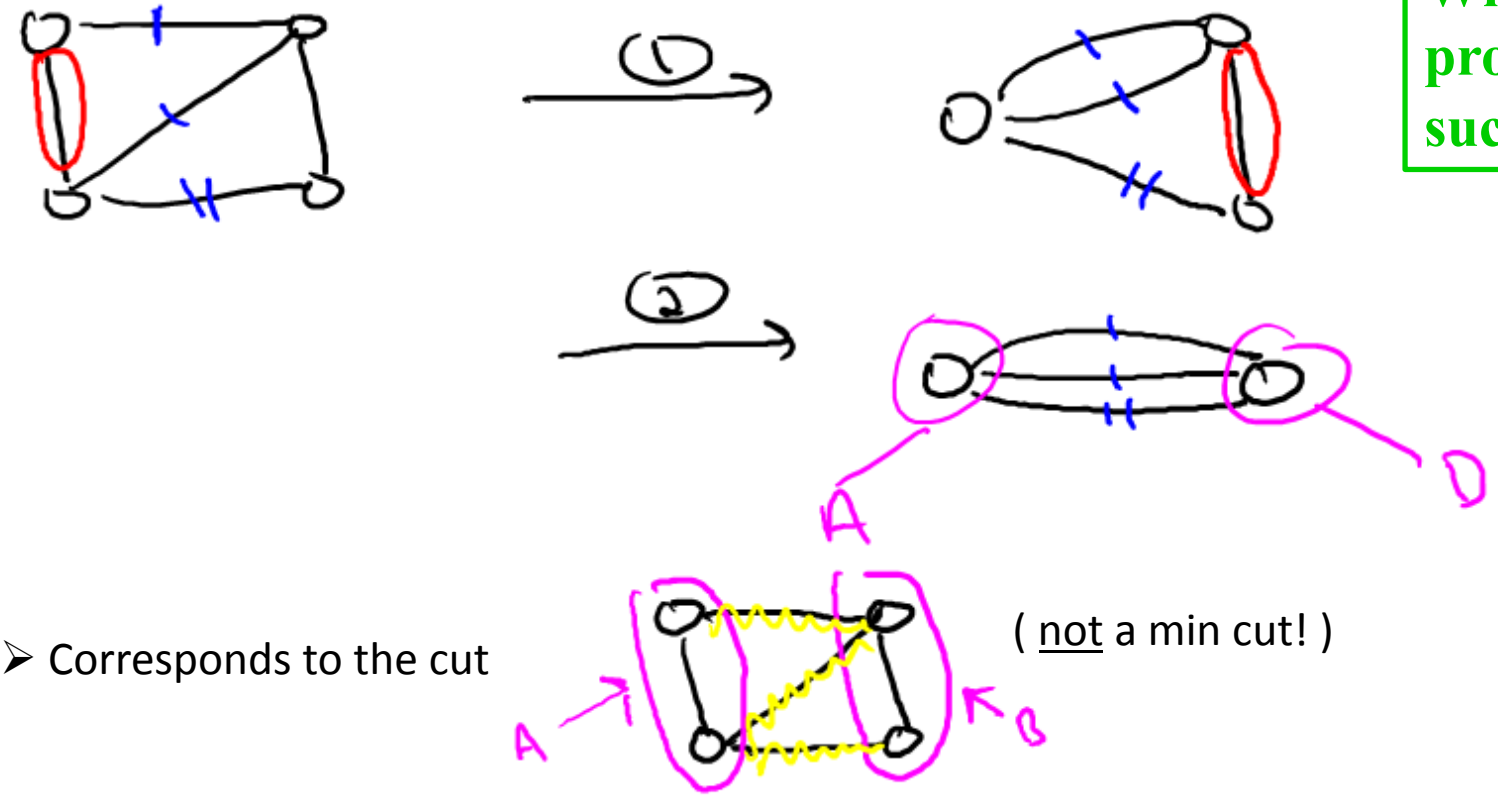


=> Corresponds to the cut



(a min cut!)

Example (con'd)



KEY QUESTION: What is the probability of success?

- Corresponds to the cut (not a min cut!)

Contraction Algorithm

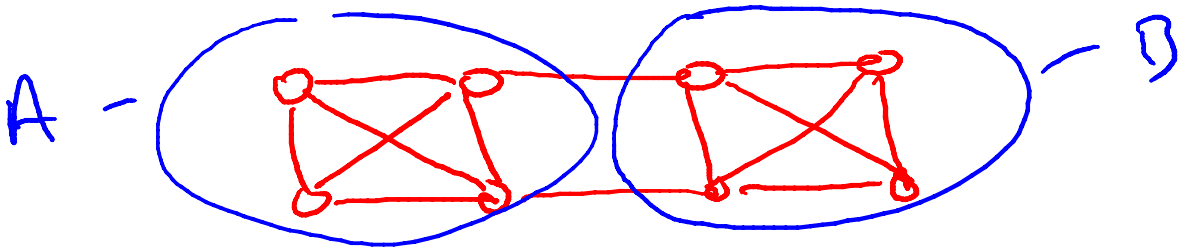
The Analysis

Design and Analysis of Algorithms I

The Minimum Cut Problem

Input: An undirected graph G = (V, E).
[parallel edges  allowed]
[See other video for representation of input]

Goal: Compute a cut with fewest number of crossing edges.
(a min cut)



Random Contraction Algorithm

[due to Karger, early 90s]

While there are more than 2 vertices:

- pick a remaining edge (u,v) uniformly at random
- merge (or "contract") u and v into a single vertex
- remove self-loops

return cut represented by final 2 vertices.

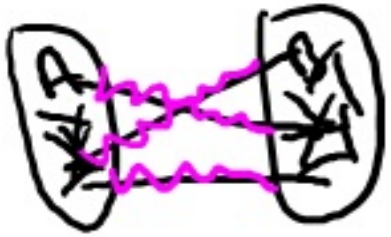
The Setup

Question: what is the probability of success?

Fix a graph G = (V, E) with n vertices, m edges.

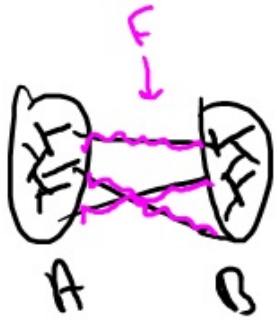
Fix a minimum cut (A, B).

Let k = # of edges crossing (A, B). (Call these edges F)



What Could Go Wrong?

1. Suppose an edge of F is contracted at some point
   ⇨ algorithm will not output (A,B).



2. Suppose only edges inside A or inside B get contracted ⇨ algorithm will output (A, B).

Thus: Pr [output is (A, B)] = Pr [never contracts an edge of F]

Let S_i = event that an edge of F contracted in iteration i.

Goal: Compute $\Pr[\neg S_1 \wedge \neg S_2 \wedge \neg S_3 \wedge \ldots\ldots \wedge \neg S_{n-2}]$

What is the probability that an edge crossing the minimum cut (A, B) is chosen in the first iteration (as a function of the number of vertices n, the number of edges m, and the number k of crossing edges)?

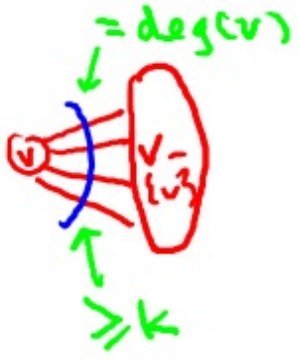
- k/n
- k/m
- k/n^2
- n/m

$$\Pr[S_1] = \frac{\#\text{ of crossing edges}}{\#\text{ of edges}} = \frac{k}{m}$$

The First Iteration

Key Observation: degree of each vertex is at least k

# of incident edges



Reason: each vertex v defines a cut ({v}. V-{v}).

Since $\underbrace{\sum_{v} degree(v)}_{\geq kn} = 2m,$ we have $m \geq \frac{kn}{2}$

Since $\Pr[S_1] = \frac{k}{m}, \Pr[S_1] \leq \frac{2}{n}$

The Second Iteration

Recall: $\Pr[\neg S_1 \wedge \neg S_2] = \Pr[\neg S_2 | \neg S_1] . \Pr[\neg S_1]$

$= 1 - \frac{k}{\#\text{ of remaining edge}}$ $\geq (1 - \frac{2}{n})$

what is this?

Note: all nodes in contracted graph define cuts in G (with at least k crossing edges).

- all degrees in contracted graph are at least k

So: # of remaining ec $\geq \frac{1}{2}k(n-1)$

So $\Pr[\neg S_2 | \neg S_1] \geq 1 - \frac{2}{(n-1)}$

All Iterations

In general:

$$\Pr[\neg S_1 \wedge \neg S_2 \wedge \neg S_3 \wedge .... \wedge \neg S_{n-2}]$$

$$= \Pr[\neg S_1] \Pr[\neg S_2 | \neg S_1] \Pr[\neg S_3 | \neg S_2 \wedge \neg S_1] ..... \Pr[\neg S_{n-2} | \neg S_1 \wedge ... \wedge \neg S_{n-3}]$$

$$\geq (1 - \tfrac{2}{n})(1 - \tfrac{2}{n-1})(1 - \tfrac{2}{n-2}).....(1 - \tfrac{2}{n-(n-4)})(1 - \tfrac{2}{n-(n-3)})$$

$$= \tfrac{n-2}{n} \cdot \tfrac{n-3}{n-1} \cdot \tfrac{n-4}{n-2} ..... \tfrac{2}{4} \cdot \tfrac{1}{3} = \tfrac{2}{n(n-1)} \geq \tfrac{1}{n^2}$$

Problem: low success probability! (But: non trivial)

recall $\simeq 2^n$ cuts !

Repeated Trials

Solution: run the basic algorithm a large number N times, remember the smallest cut found.

Question: how many trials needed?

Let T_i = event that the cut (A, B) is found on the i^{th} try.

- by definition, different T_i's are independent

So: $\Pr[\text{all N trails fail}] = \Pr[\neg T_1 \wedge \neg T_2 \wedge \ldots \wedge \neg T_N]$

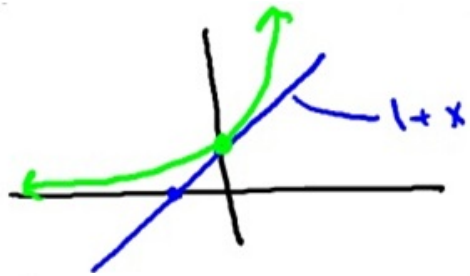
$$= \prod_{i=1}^{N} \Pr[\neg T_i] \leq (1 - \frac{1}{n^2})^N$$

By independence !

Repeated Trials (con'd)

Calculus fact: $\forall$real numbers x, $1+x \leq e^x$

Pr[all trials fail] $\leq (1-\frac{1}{n^2})^N$



So: if we take $N = n^2, \Pr[\text{all fail}] \leq (e^{-\frac{1}{n^2}})^{n^2} = \frac{1}{e}$

If we take $N = n^2 \ln n, \Pr[\text{all fail}] \leq (\frac{1}{e})^{\ln n} = \frac{1}{n}$

Running time: polynomial in n and m but slow $(\Omega(n^2 m))$

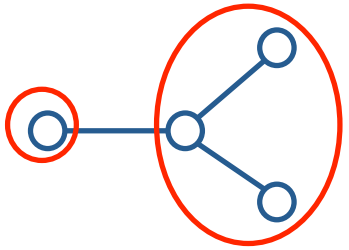
But: can get big speed ups (to roughly $O(n^2)$) with more ideas.

Design and Analysis
of Algorithms I

Contraction Algorithm

Counting Mininum Cuts

The Number of Minimum Cuts



NOTE: A graph can have multiple min cuts.
[e.g., a tree with n vertices has (n-1) minimum cuts]

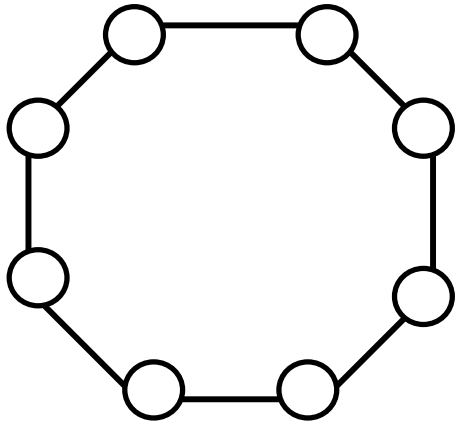
QUESTION: What's the largest number of min cuts that a graph with n vertices can have?

ANSWER: $\binom{n}{2} = \frac{n(n-1)}{2}$

The Lower Bound

Consider the n-cycle.

<u>NOTE</u>: Each pair of the n edges defines a distinct minimum cut (with two crossing edges).



(n = 8)

- has $\geq \binom{n}{2}$ min cuts

The Upper Bound

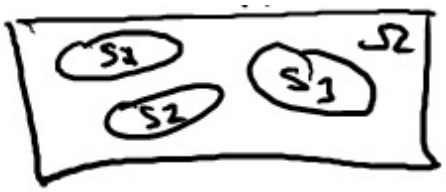
Let (A_1, B_1) , (A_2, B_2) , …, (A_t, B_t) be the min cuts of a graph with n vertices.

By the Contraction Algorithm analysis (without repeated trials):

$$\underbrace{\Pr[output = (A_i, B_i)]}_{S_i} \geq \frac{2}{n(n-1)} = \frac{1}{\binom{n}{2}} \quad \forall i = 1, 2, .., t$$

Note: S_i's are disjoint events (i.e., only one can happen)

➢ their probabilities sum to at most 1



Thus: $\frac{t}{\binom{n}{2}} \leq 1 \Rightarrow t \leq \binom{n}{2}$

QED !

Tim Roughgarden