NUMERICAL PERTURBATION ANALYSIS EXAMPLE (30 points possible)

Consider the quadratic program

$$\begin{array}{ll} \text{minimize} & x_1^2 + 2x_2^2 - x_1x_2 - x_1 \\ \text{subject to} & x_1 + 2x_2 \leq u_1 \\ & x_1 - 4x_2 \leq u_2, \\ & x_1 + x_2 \geq -5, \end{array}$$

with variables x_1, x_2, and parameters u_1, u_2.

(a) Solve this QP, for parameter values $u_1 = -2$, $u_2 = -3$, to find optimal primal variable values $x_1^\star$ and $x_2^\star$, and optimal dual variable values $\lambda_1^\star$, $\lambda_2^\star$ and $\lambda_3^\star$. Let $p^\star$ denote the optimal objective value. Verify that the KKT conditions hold for the optimal primal and dual variables you found (within reasonable numerical accuracy).

Hint: See §4.7 of the CVX users' guide to find out how to retrieve optimal dual variables. To specify the quadratic objective, use `quad_form()`.

What is $x_2^\star$? Enter your result rounded to two decimal places.

What is $\lambda_3^\star$? Enter your result rounded to two decimal places.

(b) We will now solve some perturbed versions of the QP, with

$$u_1 = -2 + \delta_1, \qquad u_2 = -3 + \delta_2,$$

where δ_1 and δ_2 each take values from $\{-0.1, 0, 0.1\}$. (There are a total of nine such combinations, including the original problem with $\delta_1 = \delta_2 = 0$.) For each combination of δ_1 and δ_2, make a prediction $p^\star_{\text{pred}}$ of the optimal value of the perturbed QP, and compare it to $p^\star_{\text{exact}}$, the exact optimal value of the perturbed QP (obtained by solving the perturbed QP). Find the values that belong in the two righthand columns in a table with the form shown below. Check that the inequality $p^\star_{\text{pred}} \leq p^\star_{\text{exact}}$ holds.

#	δ_1	δ_2	$p^\star_{\text{pred}}$	$p^\star_{\text{exact}}$
1	0	0		
2	0	-0.1		
3	0	0.1		
4	-0.1	0		
5	-0.1	-0.1		
6	-0.1	0.1		
7	0.1	0		
8	0.1	-0.1		
9	0.1	0.1		

For which perturbations (other than number 1) is $p^\star_{\text{exact}} - p^\star_{\text{pred}}$ the smallest?

- [ ] 2
- [ ] 3
- [ ] 4
- [ ] 5
- [ ] 6
- [ ] 7
- [ ] 8
- [ ] 9

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A SIMPLE EXAMPLE (40 points possible)

Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & x^2+1 \\ \text{subject to} & (x-2)(x-4) \leq 0, \end{array}$$

with variable $x \in \mathbf{R}$.

(a) *Analysis of primal problem.* What is the optimal value?

(b) *Lagrangian and dual function.* Plot the objective x^2+1 versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x,\lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^\star \geq \inf_x L(x,\lambda)$ for $\lambda \geq 0$). Derive and sketch the Lagrange dual function g.

(c) *Lagrange dual problem.* The Lagrange dual problem is

$$\begin{array}{ll} \text{maximize} & \underline{\qquad} \\ \text{subject to} & \lambda \geq 0. \end{array}$$

Help

The correct equation that fills the blank spot is

- $-9\lambda^2/(1+\lambda)+1+8\lambda$
- $-9\lambda^2/(1+\lambda)+2+8\lambda$
- $-9\lambda^2/(1+\lambda)+1+5\lambda$
- $-9\lambda^2/(1+\lambda)+2+5\lambda$
- $-4\lambda^2/(1+\lambda)+1+5\lambda$
- $-4\lambda^2/(1+\lambda)+2+8\lambda$.

Note that the dual problem is convex. What is the dual optimal solution $\lambda^\star$?

- $\lambda^\star = 1$
- $\lambda^\star = 2$
- $\lambda^\star = 3$
- $\lambda^\star = 4$

Note that strong duality holds.

(d) *Sensitivity analysis.* Let $p^\star(u)$ denote the optimal value of the problem

$$\begin{array}{ll} \text{minimize} & x^2+1 \\ \text{subject to} & (x-2)(x-4) \le u, \end{array}$$

as a function of the parameter u.

We can show that

$$p^\star(u) = \begin{cases} \infty & u < -1 \\ \_\_\_\_\_\_ & -1 \le u \le 8 \\ 1 & u \ge 8. \end{cases}$$

The correct equation that fills the blank spot is

- $11+u-6\sqrt{1+u}$
- $3u-6\sqrt{1+u}$
- $7+3(u-4)-6\sqrt{1+u}$.

Note that $dp^\star(0)/du = -\lambda^\star$.

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LAGRANGIAN RELAXATION OF BOOLEAN LP (20 points possible)

A *Boolean linear program* is an optimization problem of the form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i \in \{0,1\}, \quad i=1,\ldots,n, \end{array}$$

and is, in general, very difficult to solve. In exercise 4.15 we studied the LP relaxation of this problem,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & 0 \leq x_i \leq 1, \quad i = 1, \ldots, n, \end{array}$$

which is far easier to solve, and gives a lower bound on the optimal value of the Boolean LP. In this problem we derive another lower bound for the Boolean LP, and work out the relation between the two lower bounds.

(a) *Lagrangian relaxation.* The Boolean LP can be reformulated as the problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i(1 - x_i) = 0, \quad i = 1, \ldots, n, \end{array}$$

which has quadratic equality constraints. The Lagrange dual of this problem is

$$\begin{array}{ll} \text{maximize} & \underline{\hspace{3em}} \\ \text{subject to} & \mu \succeq 0. \end{array}$$

The correct equation that fills the blank spot is

- ○ $b^T \mu - \sum_{i=1}^n \max\{0, -c_i + a_i^T \mu\}$
- ○ $-(b^T \mu)^2 - \sum_{i=1}^n \max\{0, c_i + a_i^T \mu\}$
- ○ $-b^T \mu + \sum_{i=1}^n \min\{0, c_i + a_i^T \mu\}$.

The optimal value of the dual problem (which is convex) gives a lower bound on the optimal value of the Boolean LP. This method of finding a lower bound on the optimal value is called *Lagrangian relaxation.*

(b) The lower bound obtained via Lagrangian relaxation, and via the LP relaxation are the same. The dual of the LP relaxation is

$$\begin{array}{ll} \text{maximize} & \underline{\hspace{3em}} \\ \text{subject to} & A^T u - v + w + c = 0 \\ & u \succeq 0, v \succeq 0, w \succeq 0. \end{array}$$

The correct equation that fills the blank spot is

- ○ $b^T u - \max(0, \mathbf{1}^T w)$
- ○ $-(b^T u)^2 - \max(0, \mathbf{1}^T w)$
- ○ $-b^T u - \mathbf{1}^T w$.

This is equivalent to the Lagrange relaxation problem derived above. We conclude that the two relaxations give the same value.

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OPTION PRICE BOUNDS (20 points possible)

In this problem we use the methods and results of Example 5.10 to give bounds on the arbitrage-free price of an option. (See Exercise 5.38 for a simple version of option pricing.) We will use all the notation and definitions from Example 5.10.

We consider here options on an underlying asset (such as a stock); these have a payoff or value that depends on S, the value of the underlying asset at the end of the investment period. We will assume that the underying asset can only take on m different values, $S^{(1)}, \ldots, S^{(m)}$. These correspond to the m possible scenarios or outcomes described in Example 5.10.

A risk-free asset has value $r > 1$ in every scenario.

A *put option* at *strike price* K gives the owner the right to sell one unit of the underlying stock at price K. At the end of the investment period, if the stock is trading at a price S, then the put option has payoff $(K - S)_+ = \max\{0, K - S\}$ (since the option is exercised only if $K > S$). Similarly a *call option* at strike price K gives the buyer the right to buy a unit of stock at price K. A call option has payoff $(S - K)_+ = \max\{0, S - K\}$.

A *collar* is an option with payoff

$$\begin{cases} C - S_0 & S > C \\ S - S_0 & F \leq S \leq C \\ F - S_0 & S \leq F \end{cases}$$

where F is the *floor*, C is the *cap* and S_0 is the price of the underlying at the start of the investment period. This option limits both the upside and downside of payoff.

Now we consider a specific problem. The price of the risk-free asset, with $r = 1.05$, is 1. The price of the underlying asset is $S_0 = 1$. We will use $m = 200$ scenarios, with $S^{(i)}$ uniformly spaced from $S^{(1)} = 0.5$ to $S^{(200)} = 2$. The following options are traded on an exchange, with prices listed below.

Type	Strike	Price
Call	1.1	0.06
Call	1.2	0.03
Put	0.8	0.02
Put	0.7	0.01

A collar with floor $F = 0.9$ and cap $C = 1.15$ is not traded on an exchange. Find the range of prices for this collar, consistent with the absence of arbitrage and the prices given above. Enter your results rounded to three decimal places.

Lower bound =

Upper bound =

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