

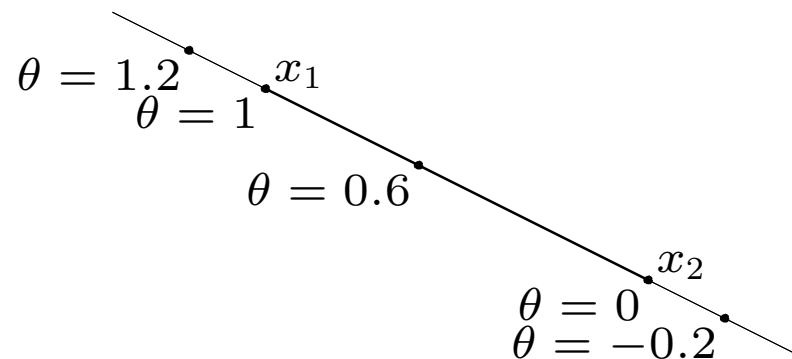
## 2. Convex sets

- affine and convex sets
- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities

# Affine set

**line** through  $x_1, x_2$ : all points

$$x = \theta x_1 + (1 - \theta)x_2 \quad (\theta \in \mathbf{R})$$



**affine set**: contains the line through any two distinct points in the set

**example**: solution set of linear equations  $\{x \mid Ax = b\}$

(conversely, every affine set can be expressed as solution set of system of linear equations)

# Convex set

**line segment** between  $x_1$  and  $x_2$ : all points

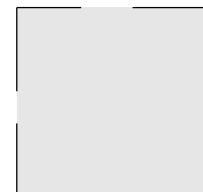
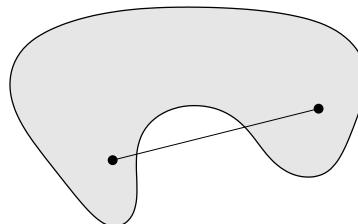
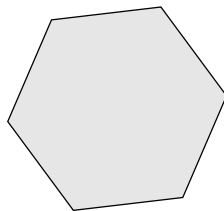
$$x = \theta x_1 + (1 - \theta)x_2$$

with  $0 \leq \theta \leq 1$

**convex set**: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \implies \quad \theta x_1 + (1 - \theta)x_2 \in C$$

**examples** (one convex, two nonconvex sets)



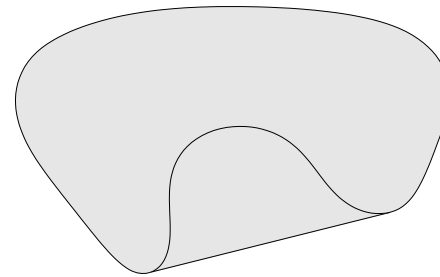
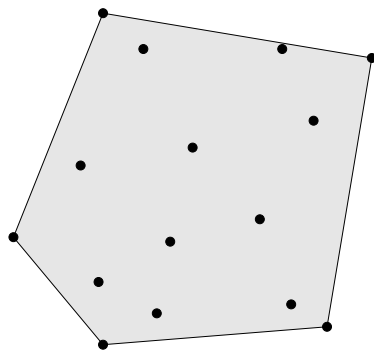
# Convex combination and convex hull

**convex combination** of  $x_1, \dots, x_k$ : any point  $x$  of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k$$

with  $\theta_1 + \dots + \theta_k = 1$ ,  $\theta_i \geq 0$

**convex hull**  $\text{conv } S$ : set of all convex combinations of points in  $S$

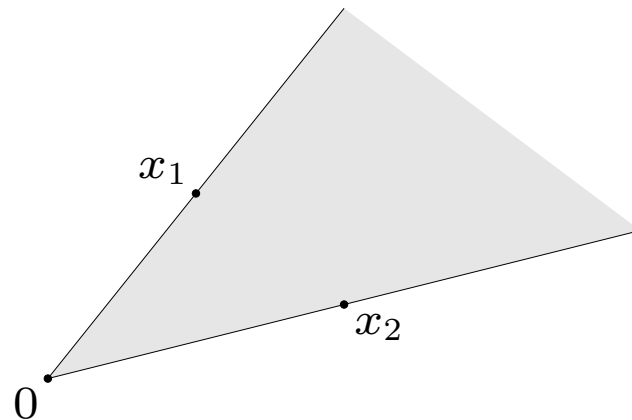


# Convex cone

**conic (nonnegative) combination** of  $x_1$  and  $x_2$ : any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

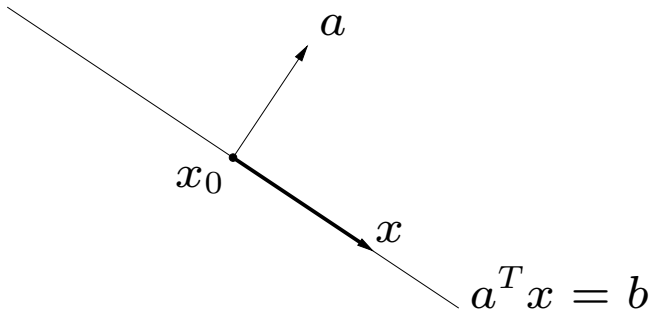
with  $\theta_1 \geq 0$ ,  $\theta_2 \geq 0$



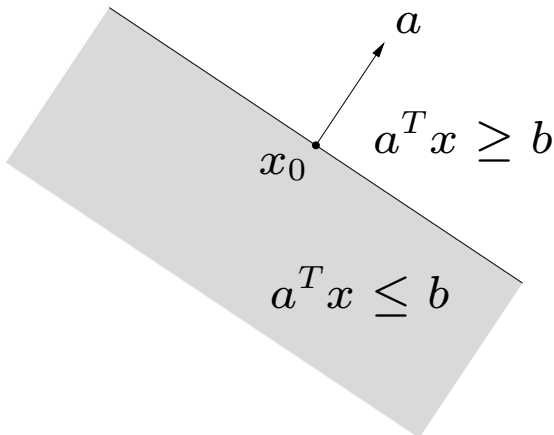
**convex cone**: set that contains all conic combinations of points in the set

# Hyperplanes and halfspaces

**hyperplane:** set of the form  $\{x \mid a^T x = b\}$  ( $a \neq 0$ )



**halfspace:** set of the form  $\{x \mid a^T x \leq b\}$  ( $a \neq 0$ )



- $a$  is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

# Euclidean balls and ellipsoids

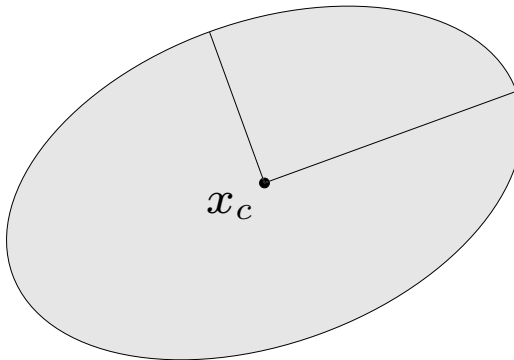
**(Euclidean) ball** with center  $x_c$  and radius  $r$ :

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + ru \mid \|u\|_2 \leq 1\}$$

**ellipsoid:** set of the form

$$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$$

with  $P \in \mathbf{S}_{++}^n$  (*i.e.*,  $P$  symmetric positive definite)



other representation:  $\{x_c + Au \mid \|u\|_2 \leq 1\}$  with  $A$  square and nonsingular

# Norm balls and norm cones

**norm:** a function  $\|\cdot\|$  that satisfies

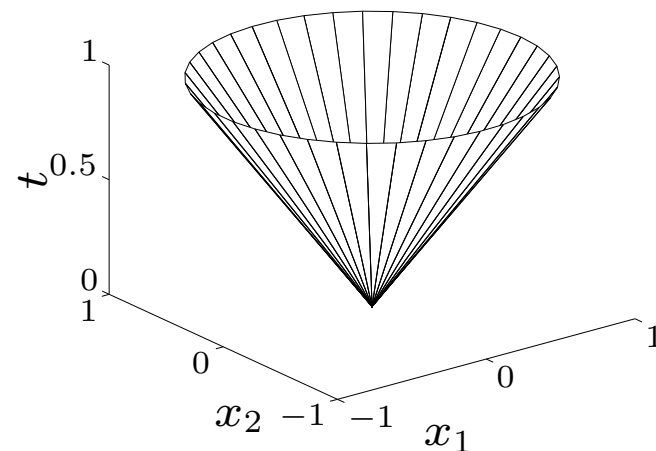
- $\|x\| \geq 0$ ;  $\|x\| = 0$  if and only if  $x = 0$
- $\|tx\| = |t| \|x\|$  for  $t \in \mathbf{R}$
- $\|x + y\| \leq \|x\| + \|y\|$

notation:  $\|\cdot\|$  is general (unspecified) norm;  $\|\cdot\|_{\text{symb}}$  is particular norm

**norm ball** with center  $x_c$  and radius  $r$ :  $\{x \mid \|x - x_c\| \leq r\}$

**norm cone:**  $\{(x, t) \mid \|x\| \leq t\}$

Euclidean norm cone is called second-order cone



norm balls and cones are convex

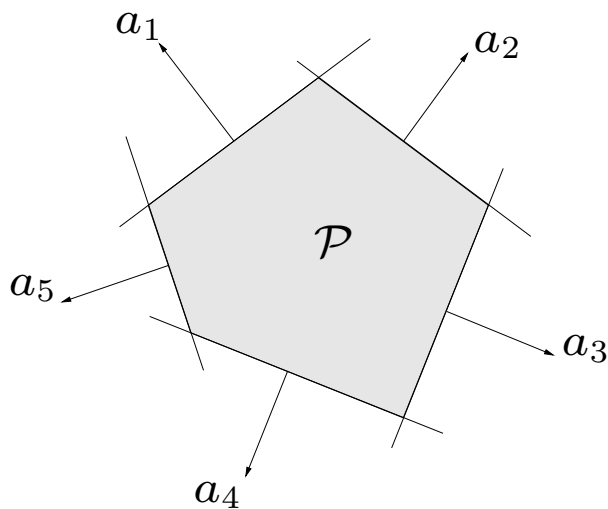


# Polyhedra

solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \quad Cx = d$$

( $A \in \mathbf{R}^{m \times n}$ ,  $C \in \mathbf{R}^{p \times n}$ ,  $\preceq$  is componentwise inequality)



polyhedron is intersection of finite number of halfspaces and hyperplanes

# Positive semidefinite cone

notation:

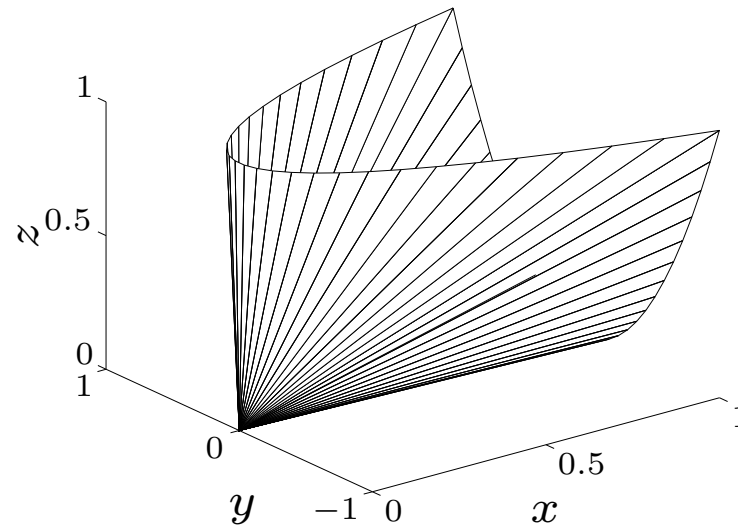
- $\mathbf{S}^n$  is set of symmetric  $n \times n$  matrices
- $\mathbf{S}_+^n = \{X \in \mathbf{S}^n \mid X \succeq 0\}$ : positive semidefinite  $n \times n$  matrices

$$X \in \mathbf{S}_+^n \iff z^T X z \geq 0 \text{ for all } z$$

$\mathbf{S}_+^n$  is a convex cone

- $\mathbf{S}_{++}^n = \{X \in \mathbf{S}^n \mid X \succ 0\}$ : positive definite  $n \times n$  matrices

example:  $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}_+^2$



# Operations that preserve convexity

practical methods for establishing convexity of a set  $C$

1. apply definition

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \implies \quad \theta x_1 + (1 - \theta)x_2 \in C$$

2. show that  $C$  is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, . . . ) by operations that preserve convexity

- intersection
- affine functions
- perspective function
- linear-fractional functions

# Intersection

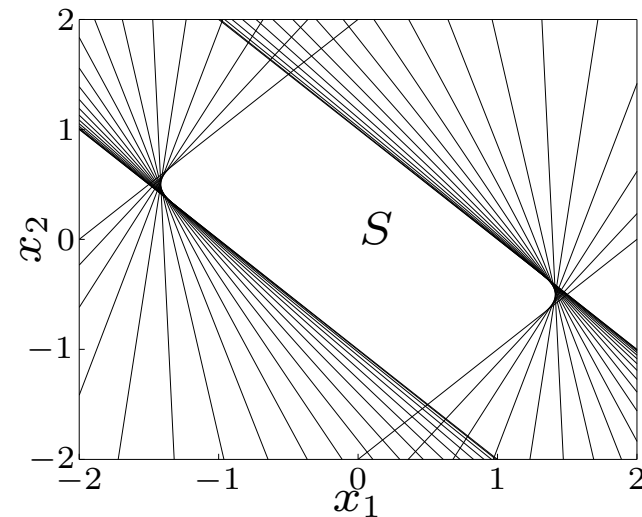
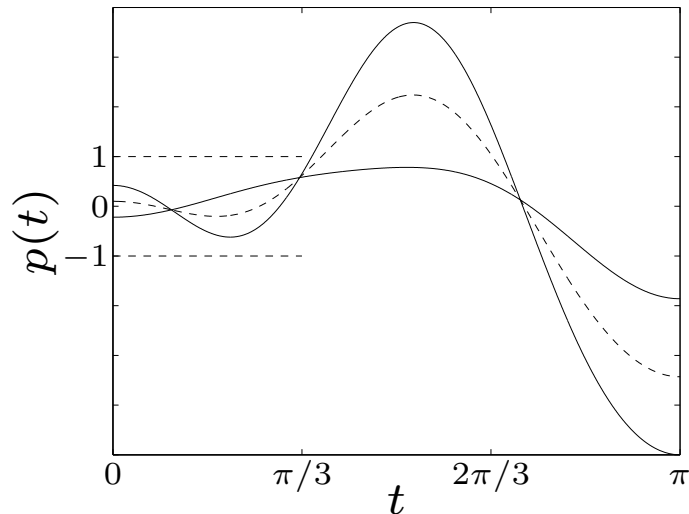
the intersection of (any number of) convex sets is convex

**example:**

$$S = \{x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$

where  $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$

for  $m = 2$ :



# Affine function

suppose  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$  is affine ( $f(x) = Ax + b$  with  $A \in \mathbf{R}^{m \times n}$ ,  $b \in \mathbf{R}^m$ )

- the image of a convex set under  $f$  is convex

$$S \subseteq \mathbf{R}^n \text{ convex} \implies f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

- the inverse image  $f^{-1}(C)$  of a convex set under  $f$  is convex

$$C \subseteq \mathbf{R}^m \text{ convex} \implies f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\} \text{ convex}$$

## examples

- scaling, translation, projection
- solution set of linear matrix inequality  $\{x \mid x_1 A_1 + \cdots + x_m A_m \preceq B\}$   
(with  $A_i, B \in \mathbf{S}^p$ )
- hyperbolic cone  $\{x \mid x^T P x \leq (c^T x)^2, c^T x \geq 0\}$  (with  $P \in \mathbf{S}_+^n$ )

# Perspective and linear-fractional function

**perspective function**  $P : \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n$ :

$$P(x, t) = x/t, \quad \text{dom } P = \{(x, t) \mid t > 0\}$$

images and inverse images of convex sets under perspective are convex

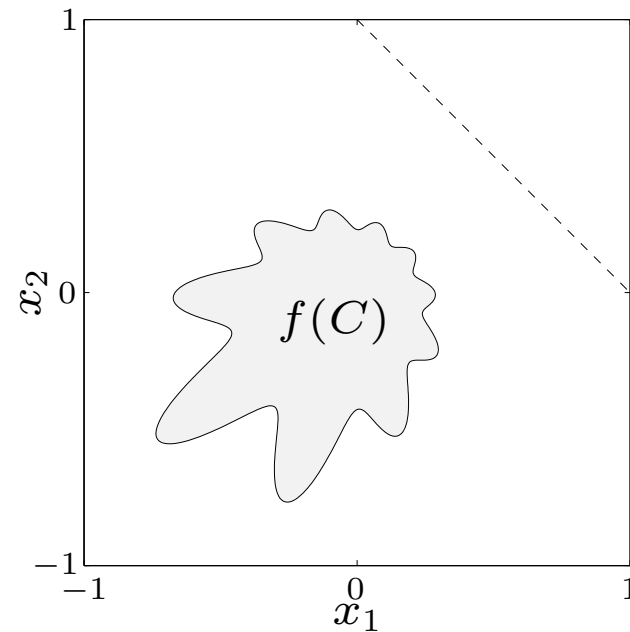
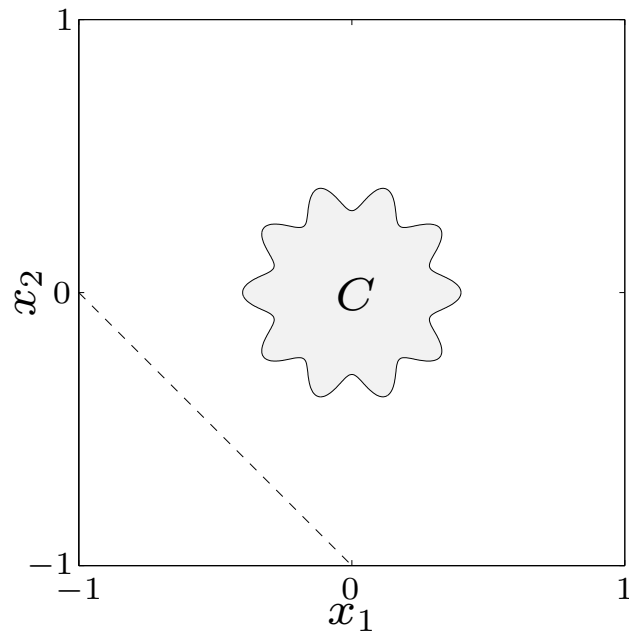
**linear-fractional function**  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ :

$$f(x) = \frac{Ax + b}{c^T x + d}, \quad \text{dom } f = \{x \mid c^T x + d > 0\}$$

images and inverse images of convex sets under linear-fractional functions are convex

**example** of a linear-fractional function

$$f(x) = \frac{1}{x_1 + x_2 + 1}x$$



# Generalized inequalities

a convex cone  $K \subseteq \mathbf{R}^n$  is a **proper cone** if

- $K$  is closed (contains its boundary)
- $K$  is solid (has nonempty interior)
- $K$  is pointed (contains no line)

## examples

- nonnegative orthant  $K = \mathbf{R}_+^n = \{x \in \mathbf{R}^n \mid x_i \geq 0, i = 1, \dots, n\}$
- positive semidefinite cone  $K = \mathbf{S}_+^n$
- nonnegative polynomials on  $[0, 1]$ :

$$K = \{x \in \mathbf{R}^n \mid x_1 + x_2 t + x_3 t^2 + \dots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$$



**generalized inequality** defined by a proper cone  $K$ :

$$x \preceq_K y \iff y - x \in K, \quad x \prec_K y \iff y - x \in \mathbf{int} K$$

### examples

- componentwise inequality ( $K = \mathbf{R}_+^n$ )

$$x \preceq_{\mathbf{R}_+^n} y \iff x_i \leq y_i, \quad i = 1, \dots, n$$

- matrix inequality ( $K = \mathbf{S}_+^n$ )

$$X \preceq_{\mathbf{S}_+^n} Y \iff Y - X \text{ positive semidefinite}$$

these two types are so common that we drop the subscript in  $\preceq_K$

**properties:** many properties of  $\preceq_K$  are similar to  $\leq$  on  $\mathbf{R}$ , *e.g.*,

$$x \preceq_K y, \quad u \preceq_K v \implies x + u \preceq_K y + v$$

# Minimum and minimal elements

$\preceq_K$  is not in general a *linear ordering*: we can have  $x \not\preceq_K y$  and  $y \not\preceq_K x$

$x \in S$  is **the minimum element** of  $S$  with respect to  $\preceq_K$  if

$$y \in S \implies x \preceq_K y$$

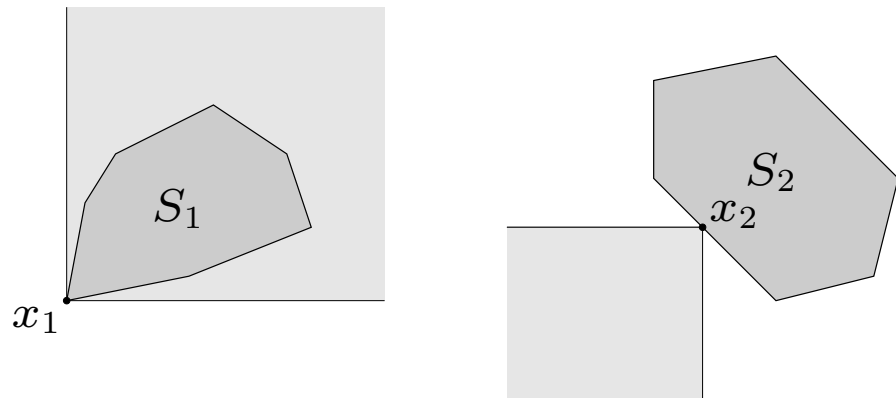
$x \in S$  is **a minimal element** of  $S$  with respect to  $\preceq_K$  if

$$y \in S, \quad y \preceq_K x \implies y = x$$

**example** ( $K = \mathbf{R}_+^2$ )

$x_1$  is the minimum element of  $S_1$

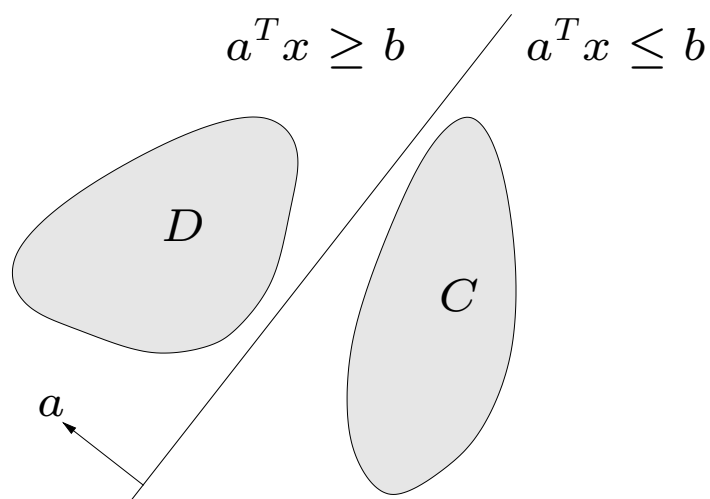
$x_2$  is a minimal element of  $S_2$



# Separating hyperplane theorem

if  $C$  and  $D$  are nonempty disjoint convex sets, there exist  $a \neq 0$ ,  $b$  s.t.

$$a^T x \leq b \text{ for } x \in C, \quad a^T x \geq b \text{ for } x \in D$$



the hyperplane  $\{x \mid a^T x = b\}$  separates  $C$  and  $D$

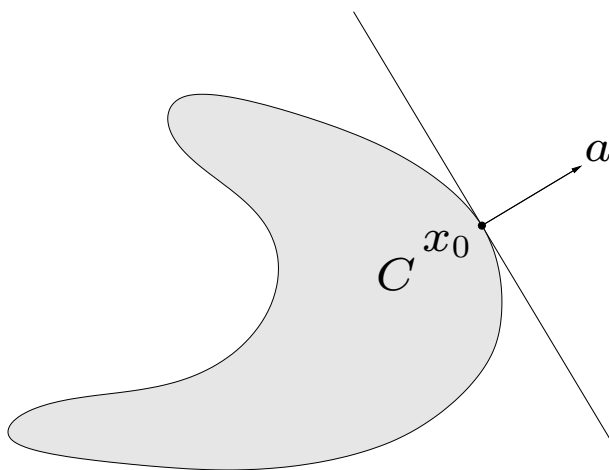
strict separation requires additional assumptions (*e.g.*,  $C$  is closed,  $D$  is a singleton)

# Supporting hyperplane theorem

**supporting hyperplane** to set  $C$  at boundary point  $x_0$ :

$$\{x \mid a^T x = a^T x_0\}$$

where  $a \neq 0$  and  $a^T x \leq a^T x_0$  for all  $x \in C$



**supporting hyperplane theorem:** if  $C$  is convex, then there exists a supporting hyperplane at every boundary point of  $C$

# Dual cones and generalized inequalities

**dual cone** of a cone  $K$ :

$$K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}$$

examples

- $K = \mathbf{R}_+^n$ :  $K^* = \mathbf{R}_+^n$
- $K = \mathbf{S}_+^n$ :  $K^* = \mathbf{S}_+^n$
- $K = \{(x, t) \mid \|x\|_2 \leq t\}$ :  $K^* = \{(x, t) \mid \|x\|_2 \leq t\}$
- $K = \{(x, t) \mid \|x\|_1 \leq t\}$ :  $K^* = \{(x, t) \mid \|x\|_\infty \leq t\}$

first three examples are **self-dual** cones

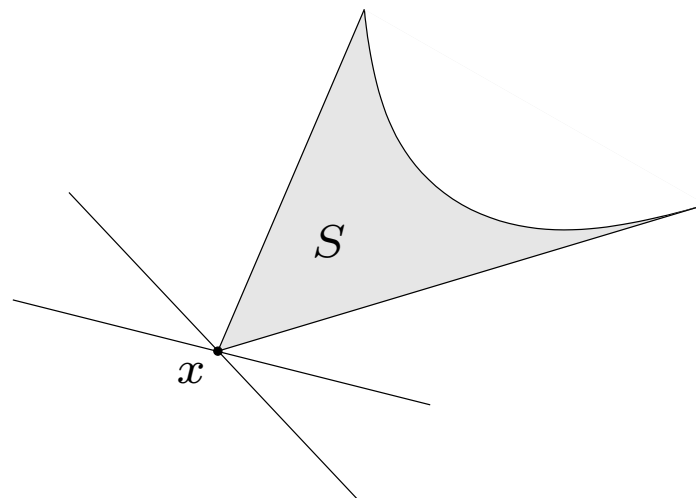
dual cones of proper cones are proper, hence define generalized inequalities:

$$y \succee_{K^*} 0 \iff y^T x \geq 0 \text{ for all } x \succee_K 0$$

# Minimum and minimal elements via dual inequalities

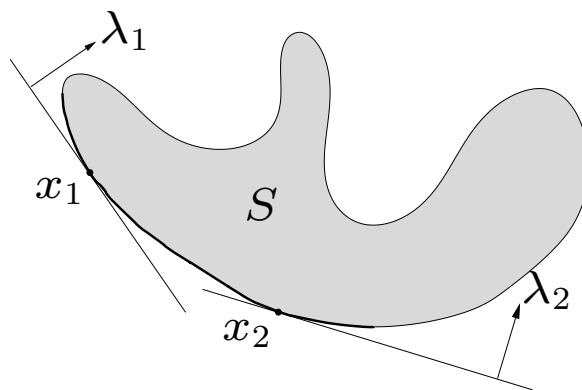
**minimum element** w.r.t.  $\preceq_K$

$x$  is minimum element of  $S$  iff for all  $\lambda \succ_{K^*} 0$ ,  $x$  is the unique minimizer of  $\lambda^T z$  over  $S$



**minimal element** w.r.t.  $\preceq_K$

- if  $x$  minimizes  $\lambda^T z$  over  $S$  for some  $\lambda \succ_{K^*} 0$ , then  $x$  is minimal



- if  $x$  is a minimal element of a *convex* set  $S$ , then there exists a nonzero  $\lambda \succeq_{K^*} 0$  such that  $x$  minimizes  $\lambda^T z$  over  $S$

## optimal production frontier

- different production methods use different amounts of resources  $x \in \mathbf{R}^n$
- production set  $P$ : resource vectors  $x$  for all possible production methods
- efficient (Pareto optimal) methods correspond to resource vectors  $x$  that are minimal w.r.t.  $\mathbf{R}_+^n$

### example ( $n = 2$ )

$x_1, x_2, x_3$  are efficient;  $x_4, x_5$  are not

