A SIMPLE PROBLEM (50 points possible)

Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{array}$$

For each of the following objective functions, select a solution of the problem.

$f_0(x_1, x_2) = x_1 + x_2$

- (2/5, 1/5)
- (0, 1)
- (1/2, 1/6)
- (1/3, 1/3)
- Unbounded below

$f_0(x_1, x_2) = -x_1 - x_2$

- (2/5, 1/5)
- (0, 1)
- (1/2, 1/6)
- (1/3, 1/3)
- Unbounded below

$f_0(x_1, x_2) = x_1$

- (2/5, 1/5)
- (0, 1)
- (1/2, 1/6)
- (1/3, 1/3)
- Unbounded below

$f_0(x_1, x_2) = \max\{x_1, x_2\}$

- (2/5, 1/5)
- (0, 1)

- (1/2, 1/6)
- (1/3, 1/3)
- Unbounded below

$f_0(x_1, x_2) = x_1^2 + 9x_2^2$

- (2/5, 1/5)
- (0, 1)
- (1/2, 1/6)
- (1/3, 1/3)
- Unbounded below

Show Answer

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"HELLO WORLD" IN CVX (20 points possible)

Use CVX to verify the optimal values in the two cases given in the exercise above titled "A simple problem".

When $f_0(x_1, x_2) = x_1 + x_2$:

When $f_0(x_1, x_2) = x_1^2 + 9x_2^2$:

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RELAXATION OF BOOLEAN LP (30 points possible)

In a *Boolean linear program*, the variable x is constrained to have components equal to zero or one:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i \in \{0, 1\}, \quad i = 1, \ldots, n. \end{array}$$

In general, such problems are very difficult to solve, even though the feasible set is finite (containing at most 2^n points).

In a general method called *relaxation*, the constraint that x_i be zero or one is replaced with the linear inequalities $0 \leq x_i \leq 1$:

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & 0 \le x_i \le 1, \quad i = 1, \ldots, n.\end{array}$$

We refer to this problem as the *LP relaxation* of the Boolean LP. The LP relaxation is far easier to solve than the original Boolean LP.

What can you say about the optimal value of the LP relaxation?

- It is a lower bound on the optimal value of the Boolean LP
- It is a upper bound on the optimal value of the Boolean LP
- It is equal to the optimal value of the Boolean LP
- Nothing can be concluded

What can you say about the Boolean LP if the LP relaxation is infeasible?

- The Boolean LP is feasible if the relaxation is infeasible
- The Boolean LP is unbounded below if the relaxation is infeasible
- The Boolean LP is infeasible if the relaxation is infeasible
- Nothing can be concluded

It sometimes happens that the LP relaxation has a solution with $x_i \in \{0,1\}$. What can you say in this case?

- The optimal solution of the relaxation is also optimal for the Boolean LP
- Nothing can be concluded

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HEURISTIC SUBOPTIMAL SOLUTION FOR BOOLEAN LP (20 points possible)

This exercise builds on exercises 4.15 and 5.13 in *Convex Optimization*, which involve the Boolean LP

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i \in \{0,1\}, \quad i = 1, \ldots, n,\end{array}$$

with optimal value $p^\star$. Let x^{rlx} be a solution of the LP relaxation

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & 0 \preceq x \preceq \mathbf{1},\end{array}$$

so $L = c^T x^{\text{rlx}}$ is a lower bound on $p^\star$. The relaxed solution x^{rlx} can also be used to guess a Boolean point $\hat{x}$, by rounding its entries, based on a threshold $t \in [0,1]$:

$$\hat{x}_i = \begin{cases} 1 & x_i^{\text{rlx}} \ge t \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, \ldots, n$. Evidently $\hat{x}$ is Boolean (i.e., has entries in $\{0, 1\}$). If it is feasible for the Boolean LP, i.e., if $A\hat{x} \preceq b$, then it can be considered a guess at a good, if not optimal, point for the Boolean LP. Its objective value, $U = c^T\hat{x}$, is an upper bound on $p^\star$. If U and L are close, then $\hat{x}$ is nearly optimal; specifically, $\hat{x}$ cannot be more than $(U - L)$-suboptimal for the Boolean LP.

This rounding need not work; indeed, it can happen that for all threshold values, $\hat{x}$ is infeasible. But for some problem instances, it can work well.

Of course, there are many variations on this simple scheme for (possibly) constructing a feasible, good point from $x^{\rm rlx}$.

Finally, we get to the problem. Generate problem data using

```
rng(0,'v5uniform');
n=100;
m=300;
A=rand(m,n);
b=A*ones(n,1)/2;
c=-rand(n,1);
```

Note: in older versions of Matlab, you can initialize the random number generator with

```
rand('state',0);
```

You can think of x_i as a job we either accept or decline, and $-c_i$ as the (positive) revenue we generate if we accept job i. We can think of $Ax \preceq b$ as a set of limits on m resources. A_{ij}, which is positive, is the amount of resource i consumed if we accept job j; b_i, which is positive, is the amount of resource i available.

Find a solution of the relaxed LP and examine its entries.

What is the associated lower bound L?

Carry out threshold rounding for (say) 100 values of t, uniformly spaced over $[0, 1]$. For each value of t, note the objective value $c^T\hat{x}$ and the maximum constraint violation $\max_i(A\hat{x} - b)_i$. Plot the objective value and the maximum violation versus t.

Find a value of t for which $\hat{x}$ is feasible, and gives minimum objective value, and note the associated upper bound U. What is the gap $U - L$ between the upper bound on $p^\star$ and the lower bound on $p^\star$?

If you define vectors `obj` and `maxviol`, you can find the upper bound as `U=min(obj(find(maxviol<=0)))`.

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