

# CS 188: Artificial Intelligence

## Constraint Satisfaction Problems II

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# Today

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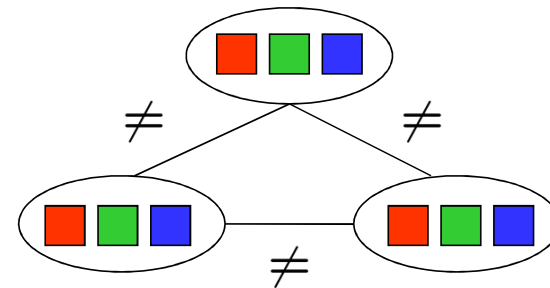
- Efficient Solution of CSPs
- Local Search



# Reminder: CSPs

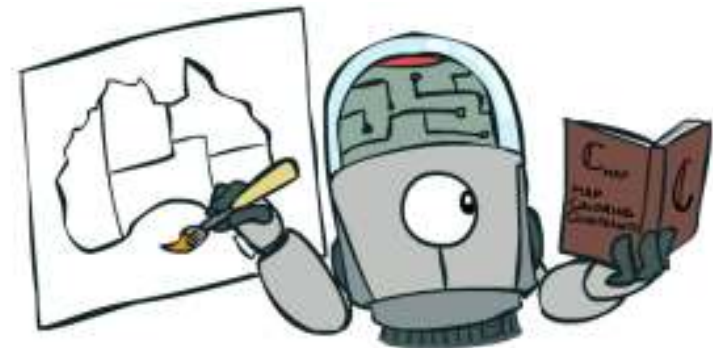
## ■ CSPs:

- Variables
- Domains
- Constraints
  - Implicit (provide code to compute)
  - Explicit (provide a list of the legal tuples)
  - Unary / Binary / N-ary



## ■ Goals:

- Here: find any solution
- Also: find all, find best, etc.



# Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
  return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
  if assignment is complete then return assignment
  var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
  for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
    if value is consistent with assignment given CONSTRAINTS[csp] then
      add {var = value} to assignment
      result ← RECURSIVE-BACKTRACKING(assignment, csp)
      if result ≠ failure then return result
      remove {var = value} from assignment
  return failure
```

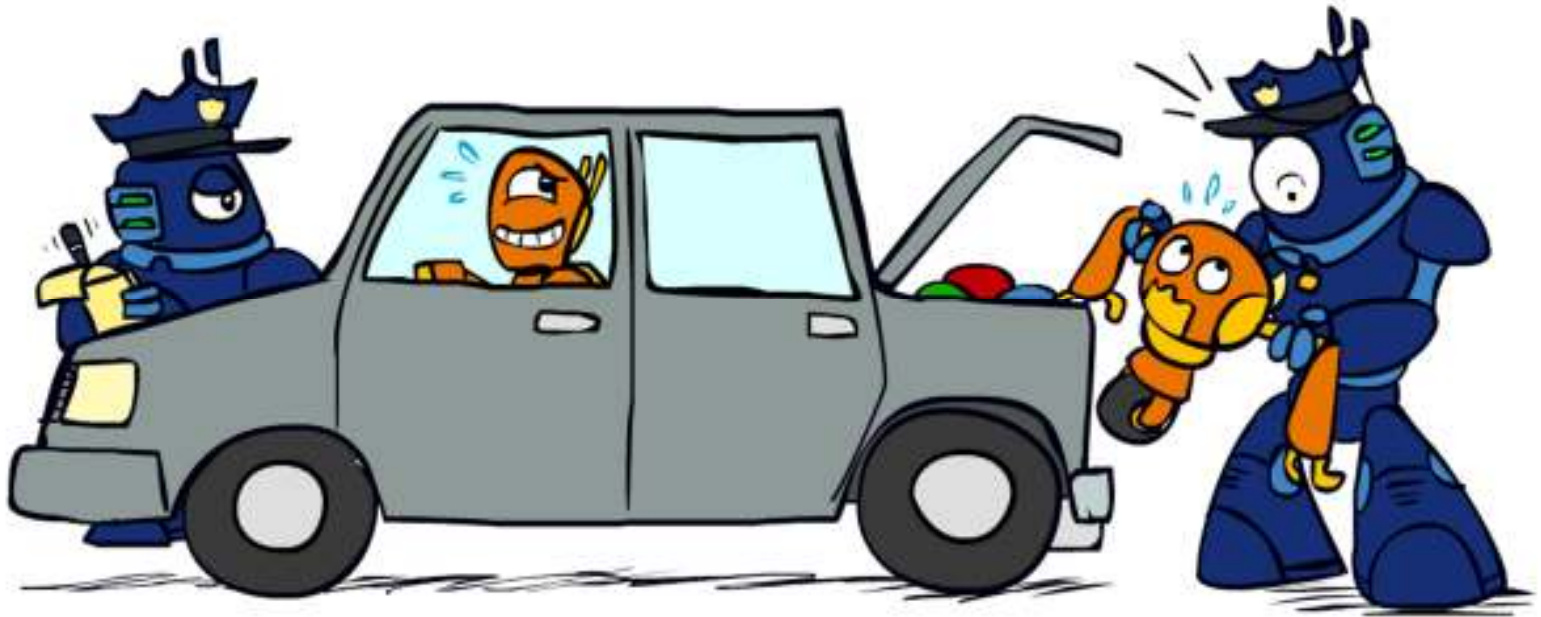
# Improving Backtracking

- General-purpose ideas give huge gains in speed
  - ... but it's all still NP-hard
- Ordering:
  - Which variable should be assigned next? (MRV)
  - In what order should its values be tried? (LCV)
- Filtering: Can we detect inevitable failure early?
- Structure: Can we exploit the problem structure?



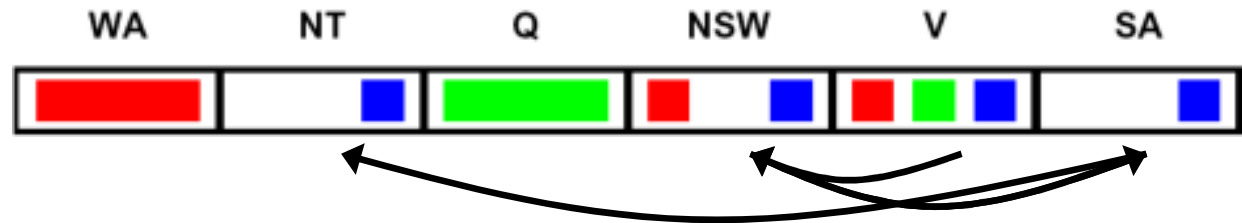
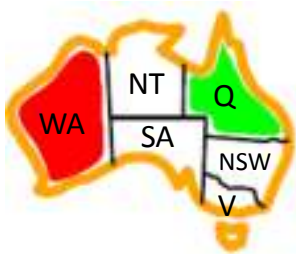
# Arc Consistency and Beyond

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# Arc Consistency of an Entire CSP

- A simple form of propagation makes sure **all** arcs are simultaneously consistent:

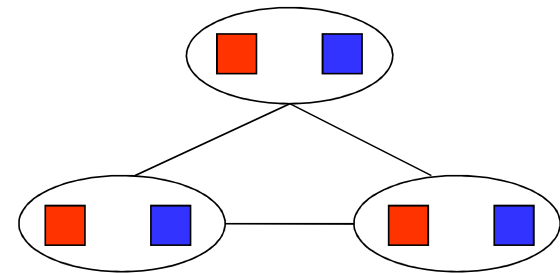
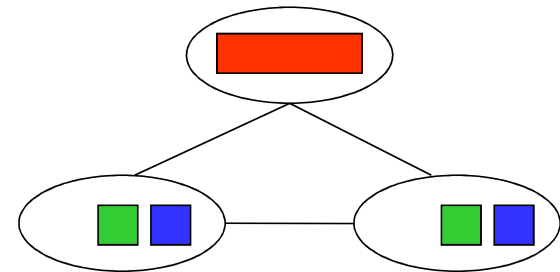


- Arc consistency detects failure earlier than forward checking
- Important: If X loses a value, neighbors of X need to be rechecked!
- Must rerun after each assignment!

*Remember:  
Delete from  
the tail!*

# Limitations of Arc Consistency

- After enforcing arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)
- Arc consistency still runs inside a backtracking search!



*What went wrong here?*



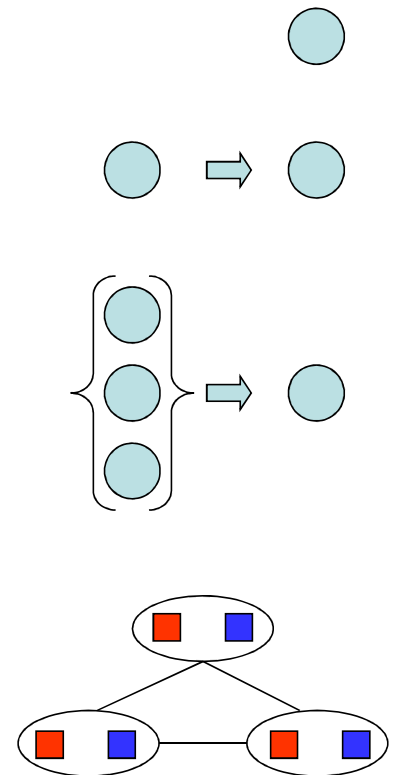
# K-Consistency

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# K-Consistency

- Increasing degrees of consistency
  - 1-Consistency (Node Consistency): Each single node's domain has a value which meets that node's unary constraints
  - 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other
  - K-Consistency: For each k nodes, any consistent assignment to k-1 can be extended to the k<sup>th</sup> node.
- Higher k more expensive to compute
- (You need to know the k=2 case: arc consistency)



# Strong K-Consistency

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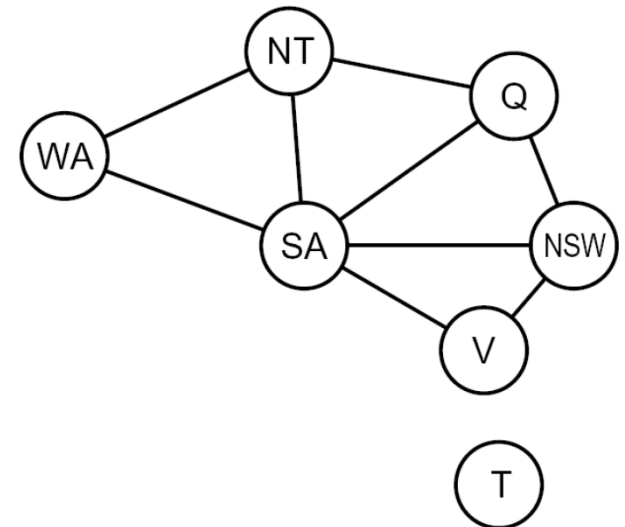
- Strong  $k$ -consistency: also  $k-1, k-2, \dots 1$  consistent
- Claim: strong  $n$ -consistency means we can solve without backtracking!
- Why?
  - Choose any assignment to any variable
  - Choose a new variable
  - By 2-consistency, there is a choice consistent with the first
  - Choose a new variable
  - By 3-consistency, there is a choice consistent with the first 2
  - ...
- Lots of middle ground between arc consistency and  $n$ -consistency! (e.g.  $k=3$ , called path consistency)

# Structure



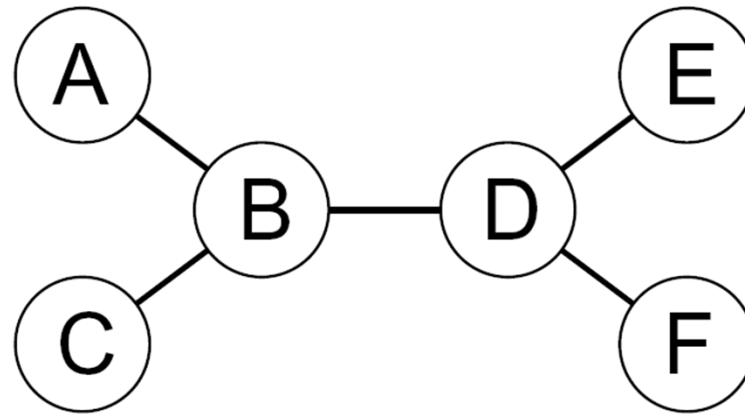
# Problem Structure

- Extreme case: independent subproblems
  - Example: Tasmania and mainland do not interact
- Independent subproblems are identifiable as connected components of constraint graph
- Suppose a graph of  $n$  variables can be broken into subproblems of only  $c$  variables:
  - Worst-case solution cost is  $O((n/c)(d^c))$ , linear in  $n$
  - E.g.,  $n = 80$ ,  $d = 2$ ,  $c = 20$
  - $2^{80} = 4$  billion years at 10 million nodes/sec
  - $(4)(2^{20}) = 0.4$  seconds at 10 million nodes/sec



# Tree-Structured CSPs

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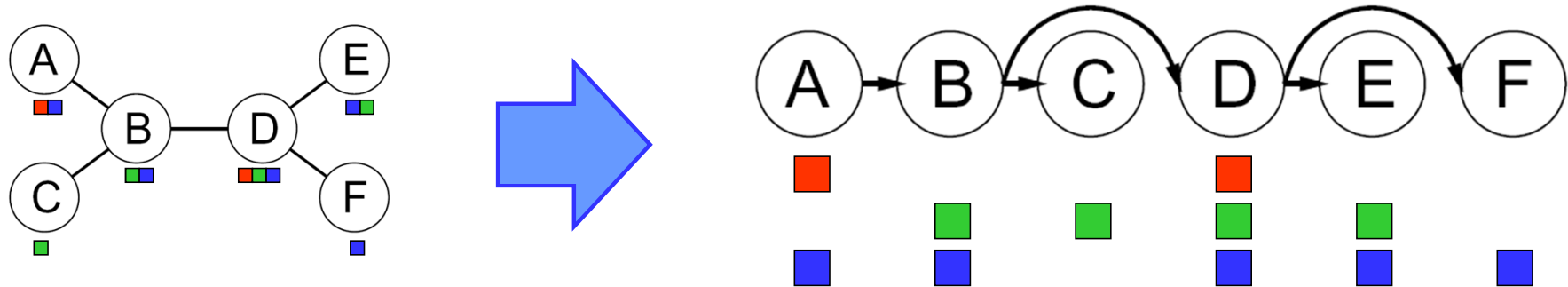


- Theorem: if the constraint graph has no loops, the CSP can be solved in  $O(n d^2)$  time
  - Compare to general CSPs, where worst-case time is  $O(d^n)$
- This property also applies to probabilistic reasoning (later): an example of the relation between syntactic restrictions and the complexity of reasoning

# Tree-Structured CSPs

- Algorithm for tree-structured CSPs:

- Order: Choose a root variable, order variables so that parents precede children



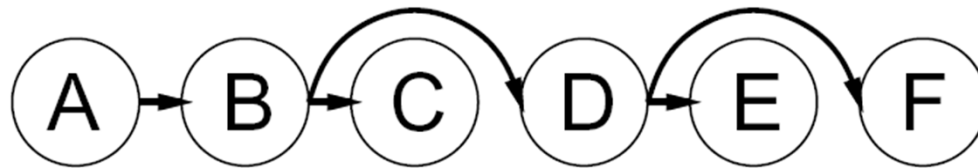
- Remove backward: For  $i = n : 2$ , apply  $\text{RemoveInconsistent}(\text{Parent}(X_i), X_i)$
- Assign forward: For  $i = 1 : n$ , assign  $X_i$  consistently with  $\text{Parent}(X_i)$

- Runtime:  $O(n d^2)$  (why?)



# Tree-Structured CSPs

- Claim 1: After backward pass, all root-to-leaf arcs are consistent
- Proof: Each  $X \rightarrow Y$  was made consistent at one point and  $Y$ 's domain could not have been reduced thereafter (because  $Y$ 's children were processed before  $Y$ )



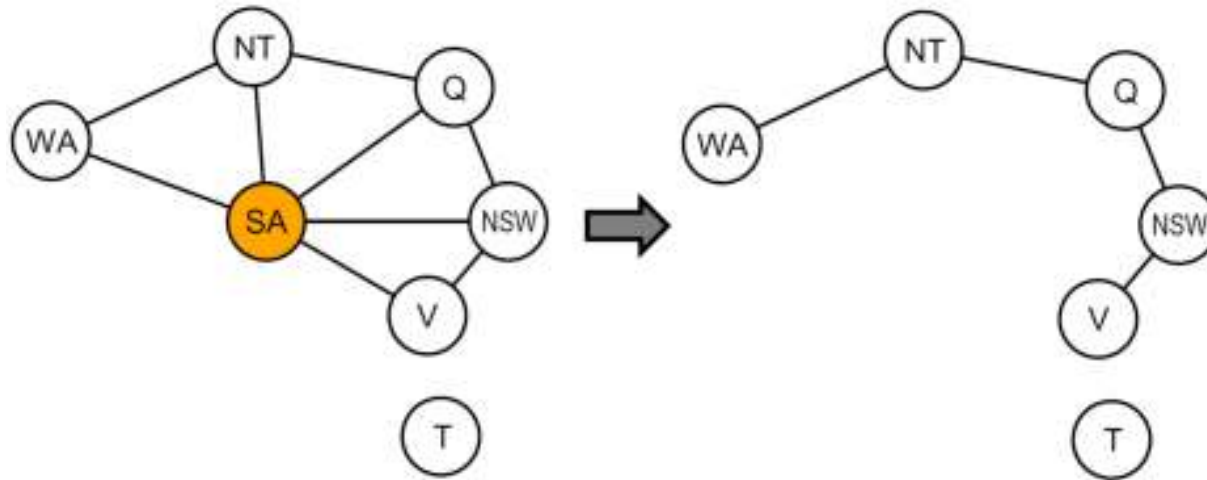
- Claim 2: If root-to-leaf arcs are consistent, forward assignment will not backtrack
- Proof: Induction on position
- Why doesn't this algorithm work with cycles in the constraint graph?
- Note: we'll see this basic idea again with Bayes' nets



# Improving Structure



# Nearly Tree-Structured CSPs



- Conditioning: instantiate a variable, prune its neighbors' domains
- Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree
- Cutset size  $c$  gives runtime  $O(d^c (n-c) d^2)$ , very fast for small  $c$

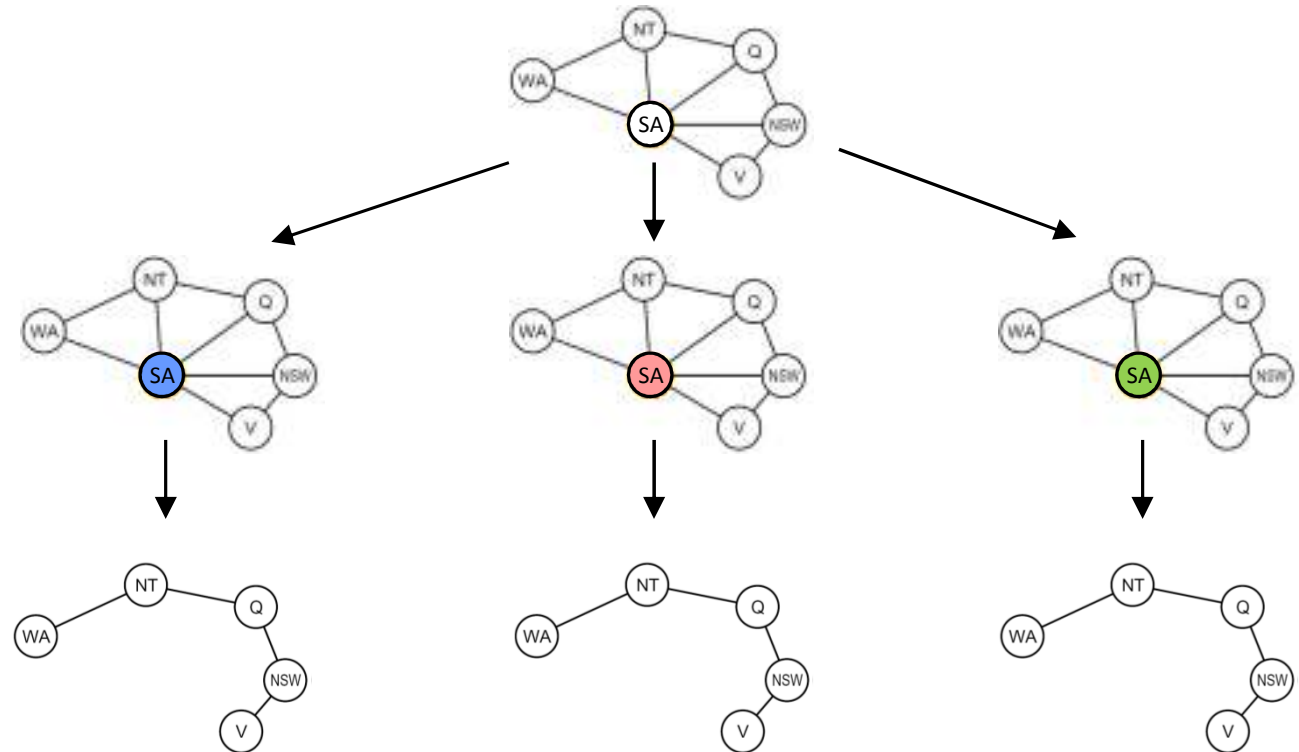
# Cutset Conditioning

Choose a cutset

Instantiate the cutset  
(all possible ways)

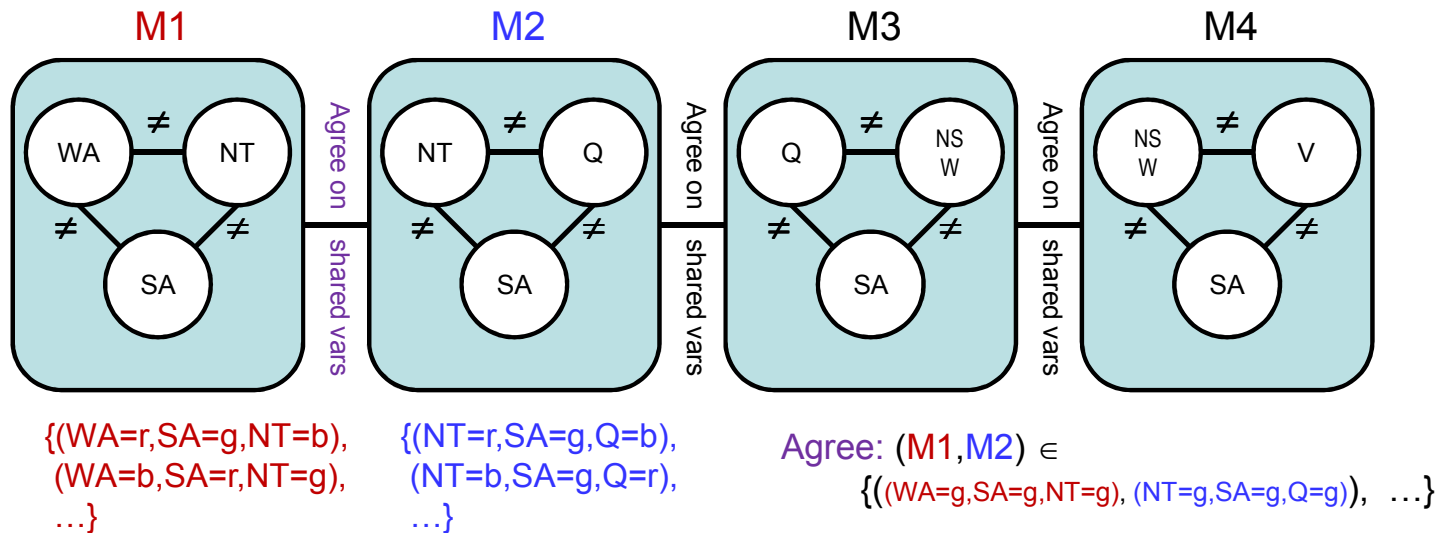
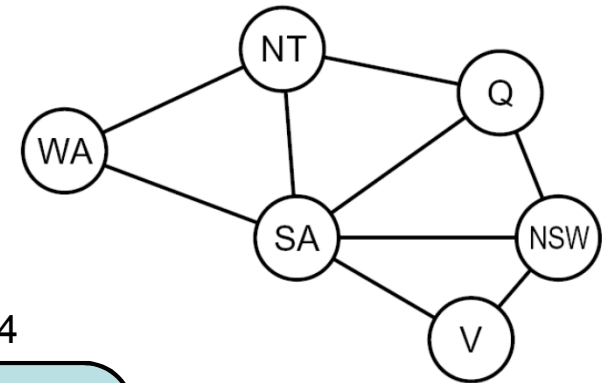
Compute residual CSP  
for each assignment

Solve the residual CSPs  
(tree structured)



# Tree Decomposition\*

- Idea: create a tree-structured graph of mega-variables
- Each mega-variable encodes part of the original CSP
- Subproblems overlap to ensure consistent solutions

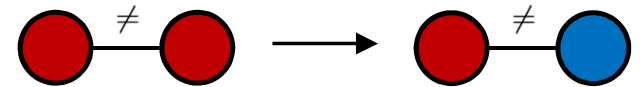


# Iterative Improvement

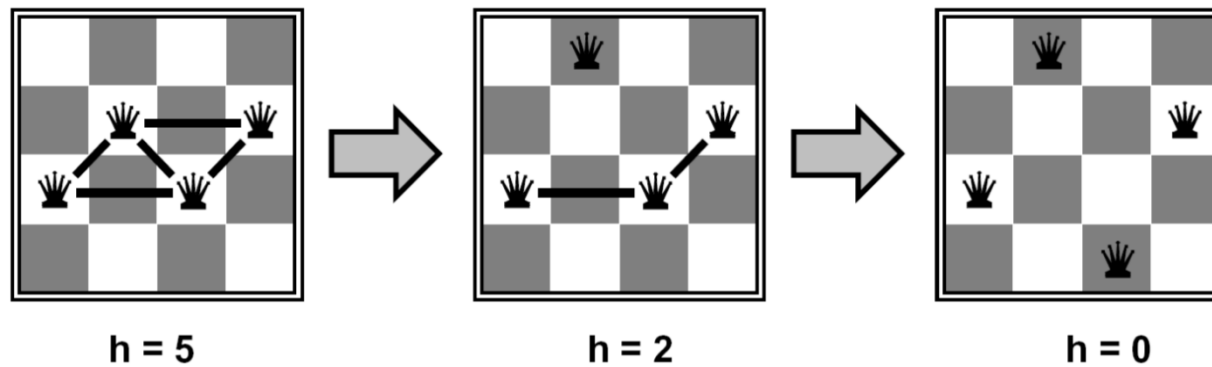


# Iterative Algorithms for CSPs

- Local search methods typically work with “complete” states, i.e., all variables assigned
- To apply to CSPs:
  - Take an assignment with unsatisfied constraints
  - Operators *reassign* variable values
  - No fringe! Live on the edge.
- Algorithm: While not solved,
  - Variable selection: randomly select any conflicted variable
  - Value selection: min-conflicts heuristic:
    - Choose a value that violates the fewest constraints
    - I.e., hill climb with  $h(n)$  = total number of violated constraints



# Example: 4-Queens



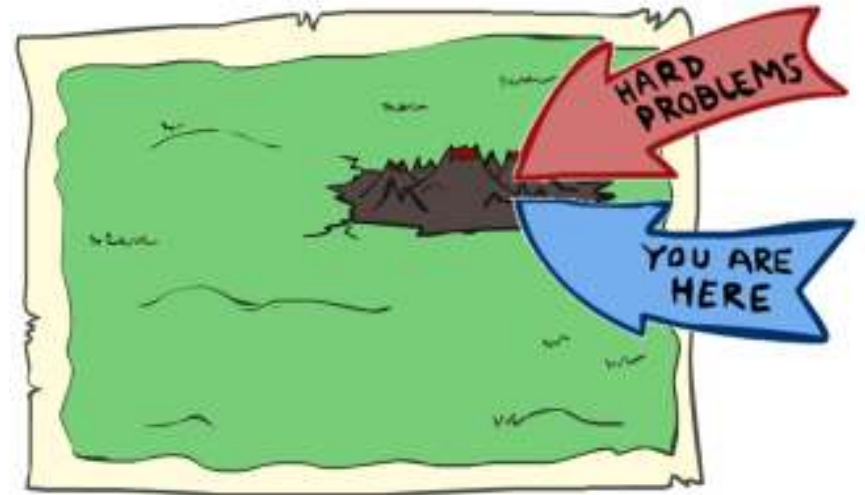
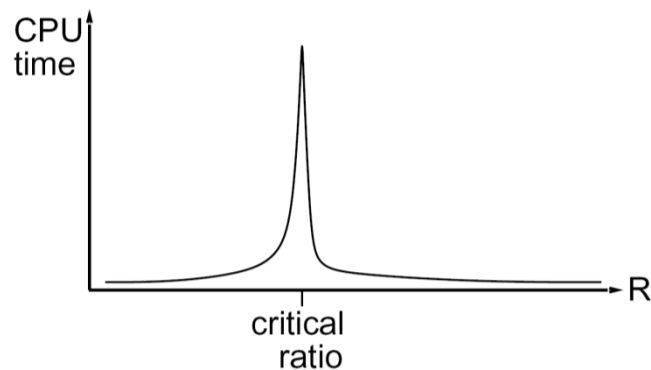
- States: 4 queens in 4 columns ( $4^4 = 256$  states)
- Operators: move queen in column
- Goal test: no attacks
- Evaluation:  $c(n)$  = number of attacks

[demos: iterative n-queens, map coloring]

# Performance of Min-Conflicts

- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)!
- The same appears to be true for any randomly-generated CSP *except* in a narrow range of the ratio

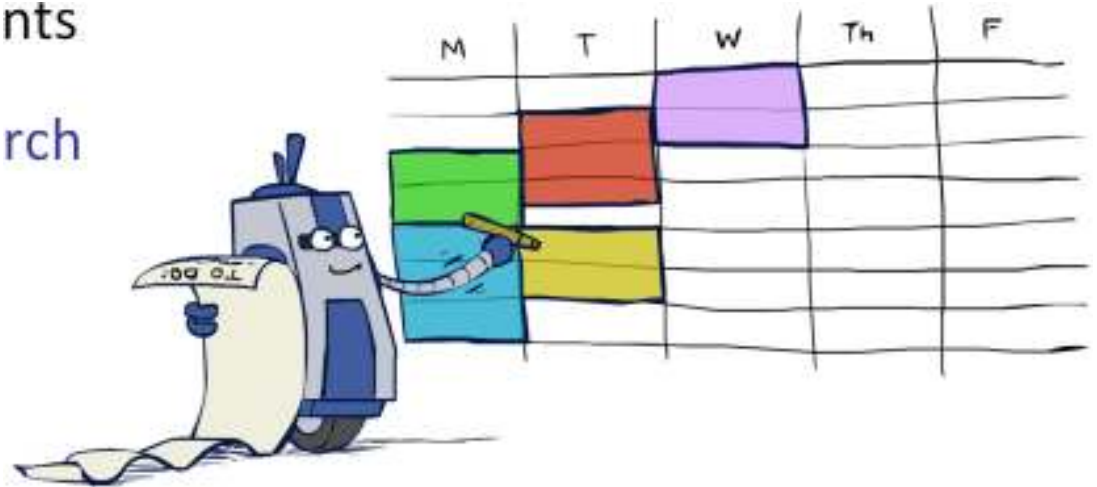
$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$





# Summary: CSPs

- CSPs are a special kind of search problem:
  - States are partial assignments
  - Goal test defined by constraints
- Basic solution: backtracking search
- Speed-ups:
  - Ordering
  - Filtering
  - Structure
- Iterative min-conflicts is often effective in practice



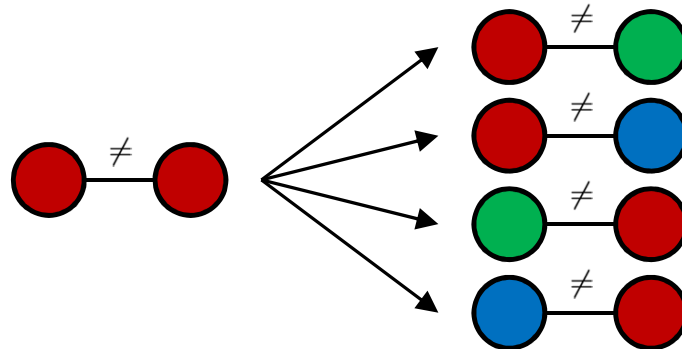
# Local Search

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# Local Search

- Tree search keeps unexplored alternatives on the fringe (ensures completeness)
- Local search: improve a single option until you can't make it better (no fringe!)
- New successor function: local changes



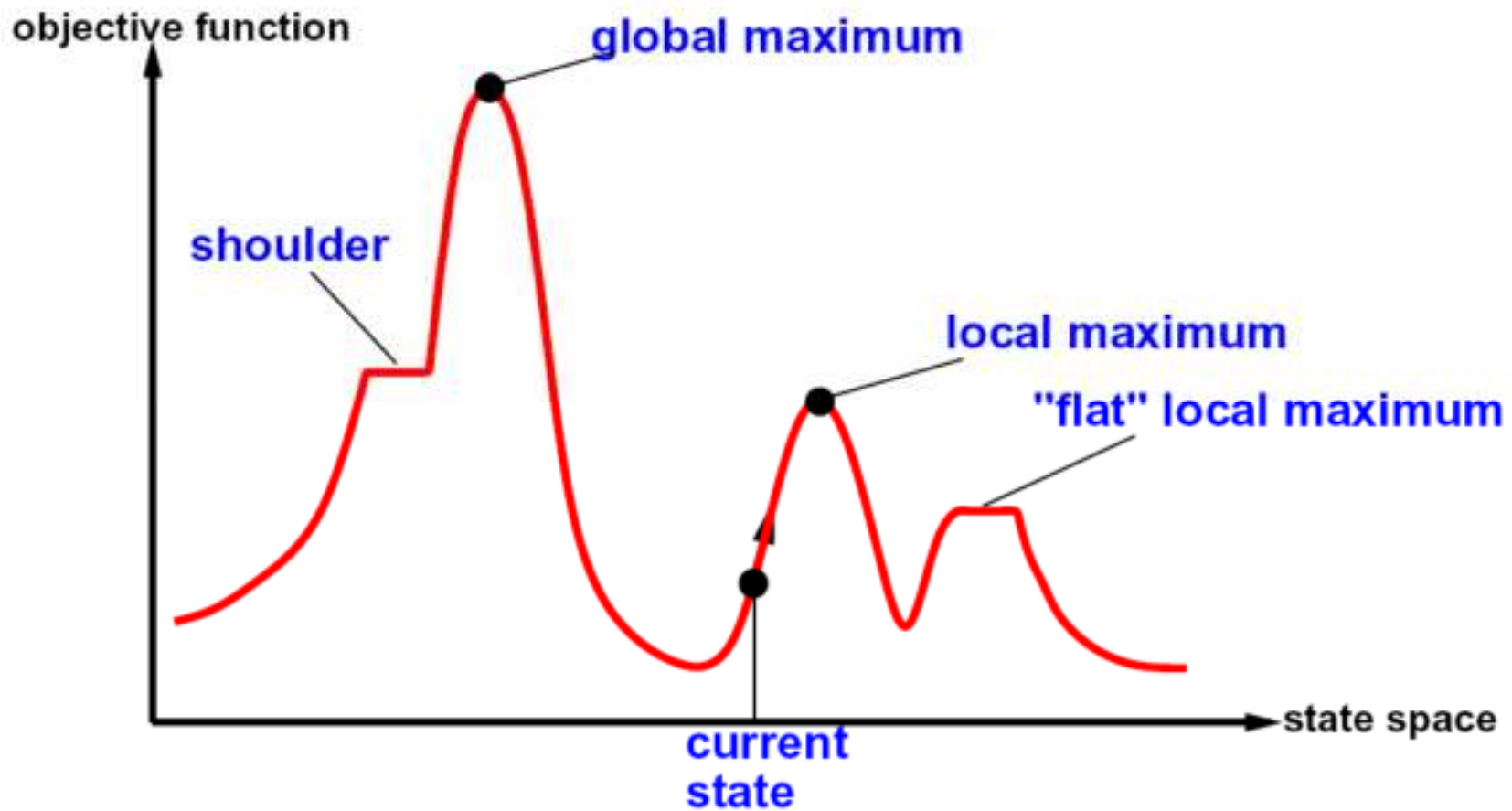
- Generally much faster and more memory efficient (but incomplete and suboptimal)

# Hill Climbing

- Simple, general idea:
  - Start wherever
  - Repeat: move to the best neighboring state
  - If no neighbors better than current, quit
- What's bad about this approach?
  - Complete?
  - Optimal?
- What's good about it?



# Hill Climbing Diagram



# Simulated Annealing

- Idea: Escape local maxima by allowing downhill moves
  - But make them rarer as time goes on

```
function SIMULATED-ANNEALING( problem, schedule ) returns a solution state
  inputs: problem, a problem
           schedule, a mapping from time to "temperature"
  local variables: current, a node
                   next, a node
                   T, a "temperature" controlling prob. of downward steps

  current ← MAKE-NODE( INITIAL-STATE[ problem ] )
  for t ← 1 to ∞ do
    T ← schedule[ t ]
    if T = 0 then return current
    next ← a randomly selected successor of current
     $\Delta E \leftarrow \text{VALUE}[\textit{next}] - \text{VALUE}[\textit{current}]$ 
    if  $\Delta E > 0$  then current ← next
    else current ← next only with probability  $e^{\Delta E / T}$ 
```



# Simulated Annealing

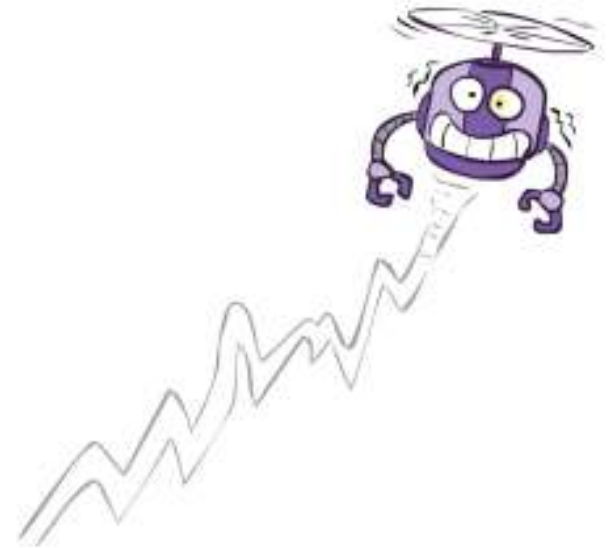
- Theoretical guarantee:

- Stationary distribution:  $p(x) \propto e^{-\frac{E(x)}{kT}}$
- If T decreased slowly enough, will converge to optimal state!

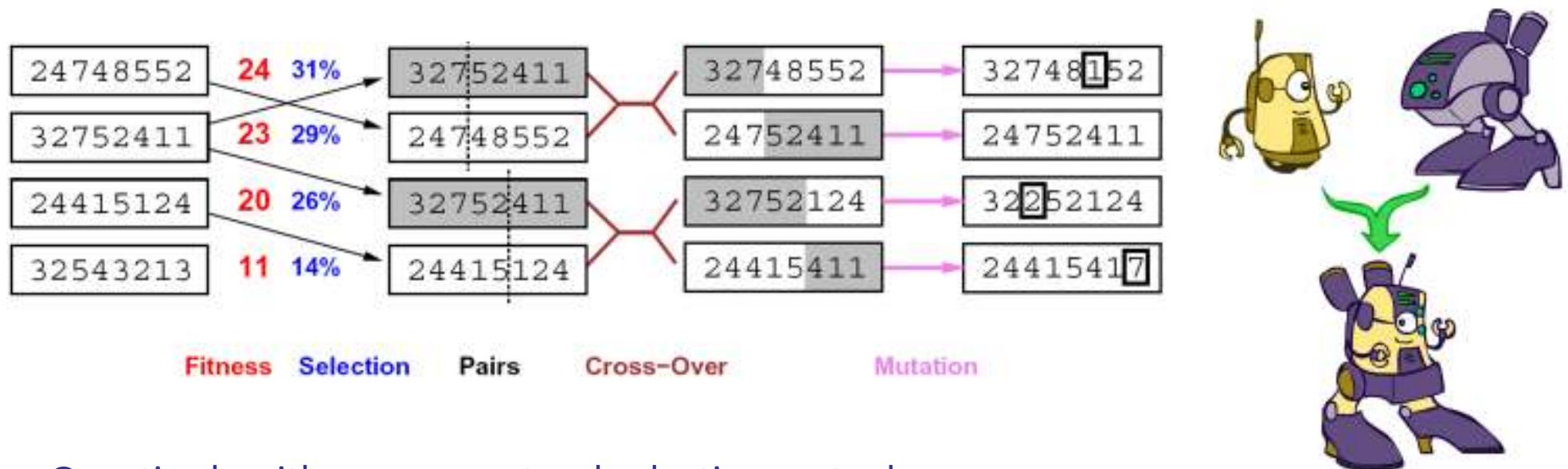
- Is this an interesting guarantee?

- Sounds like magic, but reality is reality:

- The more downhill steps you need to escape a local optimum, the less likely you are to ever make them all in a row
- People think hard about *ridge operators* which let you jump around the space in better ways



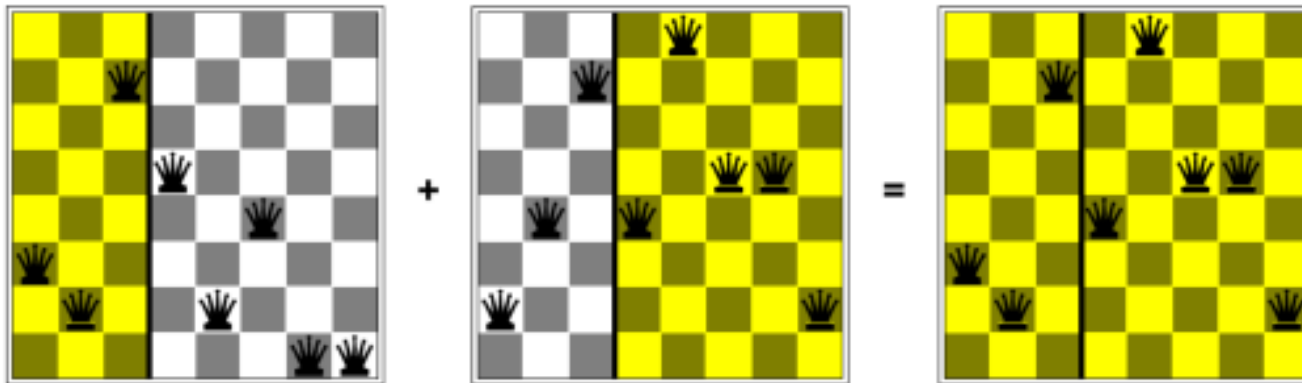
# Genetic Algorithms



- Genetic algorithms use a natural selection metaphor
  - Keep best N hypotheses at each step (selection) based on a fitness function
  - Also have pairwise crossover operators, with optional mutation to give variety
- Possibly the most misunderstood, misapplied (and even maligned) technique around



# Example: N-Queens



- Why does crossover make sense here?
- When wouldn't it make sense?
- What would mutation be?
- What would a good fitness function be?

# Next Time: Adversarial Search!

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