Measuring complexity

- Goals in designing programs
  1. It returns the correct answer on all legal inputs
  2. It performs the computation efficiently
- Typically (1) is most important, but sometimes (2) is also critical, e.g., programs for collision detection
- Even when (1) is most important, it is valuable to understand and optimize (2)

How do we measure complexity?

- Given a function, would like to answer: "How long will this take to run?"
- Could just run on some input and time it.
- Problem is that this depends on:
  1. Speed of computer
  2. Specifics of Python implementation
  3. Value of input
- Avoid (1) and (2) by measuring time in terms of number of basic steps executed

Computational complexity

- How much time will it take a program to run?
- How much memory will it need to run?

- Need to balance minimizing computational complexity with conceptual complexity
  – Keep code simple and easy to understand, but where possible optimize performance

Measuring basic steps

- Use a **random access machine (RAM)** as model of computation
  – Steps are executed sequentially
  – Step is an operation that takes constant time
    - Assignment
    - Comparison
    - Arithmetic operation
    - Accessing object in memory
- For point (3), measure time in terms of size of input

But complexity might depend on value of input?

```
def linearSearch(L, x):
    for e in L:
        if e == x:
            return True
    return False
```

- If x happens to be near front of L, then returns True almost immediately
- If x not in L, then code will have to examine all elements of L
- Need a general way of measuring

Example

```
def fact(n):
    answer = 1
    while n > 1:
        answer *= n
        n -= 1
    return answer
```

- Number of steps
  - 1 (for assignment)
  - 5*n (1 for test, plus 2 for first assignment, plus 2 for second assignment in while; repeated n times through while)
  - 1 (for return)
- 5*n + 2 steps
- But as n gets large, 2 is irrelevant, so basically 5*n steps

Cases for measuring complexity

- **Best case:** minimum running time over all possible inputs of a given size
  - For linearSearch – constant, i.e. independent of size of inputs
- **Worst case:** maximum running time over all possible inputs of a given size
  - For linearSearch – linear in size of list
- **Average (or expected) case:** average running time over all possible inputs of a given size
- We will focus on worst case – a kind of **upper bound** on running time

Example

- What about the multiplicative constant (5 in this case)?
- We argue that in general, multiplicative constants are not relevant when comparing algorithms

Example

```
def sqrtExhaust(x, eps):
    step = eps**2
    ans = 0.0
    while abs(ans**2 - x) >= eps and ans <= max(x, 1):
        ans += step
    return ans
```

- If we call this on 100 and 0.0001, will take one billion iterations of the loop
  - Have roughly 8 steps within each iteration

Measuring complexity

- Given this difference in iterations through loop, multiplicative factor (number of steps within loop) probably irrelevant
- Thus, we will focus on measuring the complexity as a function of input size
  - Will focus on the largest factor in this expression
  - Will be mostly concerned with the worst case scenario

Example

```
def sqrtBi(x, eps):
    low = 0.0
    high = max(1, x)
    ans = (high + low)/2.0
    while abs(ans**2 - x) >= eps:
        if ans**2 < x:
            low = ans
        else:
            high = ans
        ans = (high + low)/2.0
    return ans
```

- If we call this on 100 and 0.0001, will take thirty iterations of the loop
  - Have roughly 10 steps within each iteration
- 1 billion or 8 billion versus 30 or 300 – it is size of problem that matters

Asymptotic notation

- Need a formal way to talk about relationship between running time and size of inputs
- Mostly interested in what happens as size of inputs gets very large, i.e. approaches infinity

Example

```python
def f(x):
    for i in range(1000):
        ans = i
    for i in range(x):
        ans += 1
    for i in range(x):
        for j in range(x):
            ans += 1
```

Complexity is $1000 + 2x + 2x^2$, if each line takes one step

Example

- So really only need to consider the nested loops (quadratic component)
- Does it matter that this part takes $2x^2$ steps, as opposed to say x^2 steps?
  - For our example, if our computer executes 100 million steps per second, difference is 5.5 hours versus 2.25 hours
  - On the other hand if we can find a linear algorithm, this would run in a fraction of a second
  - So multiplicative factors probably not crucial, but order of growth is crucial

Example

- $1000 + 2x + 2x^2$
- If x is small, constant term dominates
  - E.g., x = 10 then 1000 of 1220 steps are in first loop
- If x is large, quadratic term dominates
  - E.g. x = 1,000,000, then first loop takes 0.000000005% of time, second loop takes 0.0001% of time (out of 2,000,002,001,000 steps)!

Rules of thumb for complexity

- Asymptotic complexity
  - Describe running time in terms of number of basic steps
  - If running time is sum of multiple terms, keep one with the largest growth rate
  - If remaining term is a product, drop any multiplicative constants
- Use “Big O” notation (aka Omicron)
  - Gives an upper bound on asymptotic growth of a function

Complexity classes

- *O(1)* denotes constant running time
- *O(log n)* denotes logarithmic running time
- *O(n)* denotes linear running time
- *O(n log n)* denotes log-linear running time
- $O(n^c)$ denotes polynomial running time (c is a constant)
- $O(c^n)$ denotes exponential running time (c is a constant being raised to a power based on size of input)

Logarithmic complexity

- Complexity grows as log of size of one of its inputs
- Example:
  - Bisection search
  - Binary search of a list

Constant complexity

- Complexity independent of inputs
- Very few interesting algorithms in this class, but can often have pieces that fit this class
- Can have loops or recursive calls, but number of iterations or calls independent of size of input

Logarithmic complexity

```
def intToStr(i):
    digits = '0123456789'
    if i == 0:
        return '0'
    result = ''
    while i > 0:
        result = digits[i%10] + result
        i = i/10
    return result
```

Logarithmic complexity

```
def intToStr(i):
    digits = '0123456789'
    if i == 0:
        return '0'
    result = ''
    while i > 0:
        result = digits[i%10]
                 + result
        i = i/10
    return result
```

- Only have to look at loop as no function calls
- Within while loop constant number of steps
- How many times through loop?
  - How many times can one divide i by 10?
  - *O(log(i))*

Linear complexity

- Complexity can depend on number of recursive calls

```
def fact(n):
    if n == 1:
        return 1
    else:
        return n*fact(n-1)
```

- Number of recursive calls?
  - Fact(n), then fact(n-1), etc. until get to fact(1)
  - Complexity of each call is constant
  - *O(n)*

Linear complexity

- Searching a list in order to see if an element is present
- Add characters of a string, assumed to be composed of decimal digits

```
def addDigits(s):
    val = 0
    for c in s:
        val += int(c)
    return val
```

- *O(len(s))*

Log-linear complexity

- Many practical algorithms are log-linear
- Very commonly used log-linear algorithm is merge sort
- Will return to this

Polynomial complexity

- Most common polynomial algorithms are quadratic, i.e., complexity grows with square of size of input
- Commonly occurs when we have nested loops or recursive function calls

Quadratic complexity

```
def isSubset(L1, L2):
    for e1 in L1:
        matched = False
        for e2 in L2:
            if e1 == e2:
                matched = True
                break
        if not matched:
            return False
    return True
```

- Outer loop executed len(L1) times
- Each iteration will execute inner loop up to len(L2) times
- *O(len(L1)*len(L2))*
- Worst case when L1 and L2 same length, none of elements of L1 in L2
- *O(len(L1)²)*

Quadratic complexity

```
def isSubset(L1, L2):
    for e1 in L1:
        matched = False
        for e2 in L2:
            if e1 == e2:
                matched = True
                break
        if not matched:
            return False
    return True
```

Quadratic complexity

Find intersection of two lists, return a list with each element appearing only once

```
def intersect(L1, L2):
    tmp = []
    for e1 in L1:
        for e2 in L2:
            if e1 == e2:
                tmp.append(e1)
    res = []
    for e in tmp:
        if not(e in res):
            res.append(e)
    return res
```

Quadratic complexity

```
def intersect(L1, L2):
    tmp = []
    for e1 in L1:
        for e2 in L2:
            if e1 ==
  e2:

  tmp.append(e1)
    res = []
    for e in tmp:
        if not(e in
  res):

  res.append(e)
    return res
```

- First nested loop takes *len(L1)*len(L2)* steps
- Second loop takes at most *len(L1)* steps
- Latter term overwhelmed by former term
- *O(len(L1)*len(L2))*

Exponential complexity

```
def genSubsets(L):
    res = []
    if len(L) == 0:
        return [[]] #list of empty list
    smaller = genSubsets(L[:-1])
    # get all subsets without last element
    extra = L[-1:]
    # create a list of just last element
    new = []
    for small in smaller:
        new.append(small+extra)
    # for all smaller solutions, add one with last
  element
    return smaller+new
    # combine those with last element and those
  without
```

Exponential complexity

- Recursive functions where more than one recursive call for each size of problem
  – Towers of Hanoi
- Many important problems are inherently exponential
  – Unfortunate, as cost can be high
  – Will lead us to consider approximate solutions more quickly

Exponential complexity

```
def genSubsets(L):
    res = []
    if len(L) == 0:
        return [[]]
    smaller = genSubsets(L[:-1])
    extra = L[-1:]
    new = []
    for small in smaller:
        new.append(small+extra)
    return smaller+new
```

- Assuming append is constant time
- Time includes time to solve smaller problem, plus time needed to make a copy of all elements in smaller problem

Exponential complexity

```
def genSubsets(L):
    res = []
    if len(L) == 0:
        return [[]]
    smaller = genSubsets(L[:-1])
    extra = L[-1:]
    new = []
    for small in smaller:
        new.append(small+extra)
    return smaller+new
```

- But important to think about size of smaller
- Know that for a set of size k there are 2^k cases
- So to solve need $2^{n-1} + 2^{n-2} + \ldots + 2^0$ steps
- Math tells us this is $O(2^n)$

Comparing complexities

- So does it really matter if our code is of a particular class of complexity?
- Depends on size of problem, but for large scale problems, complexity of worst case makes a difference

Complexity classes

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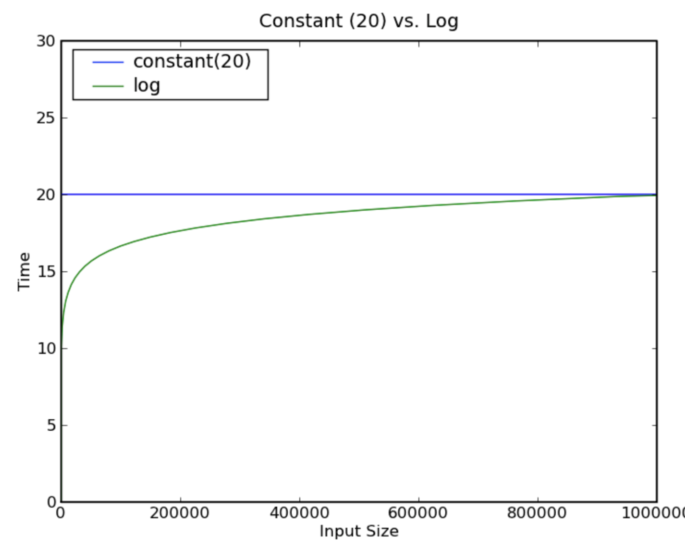
Constant versus logarithmic



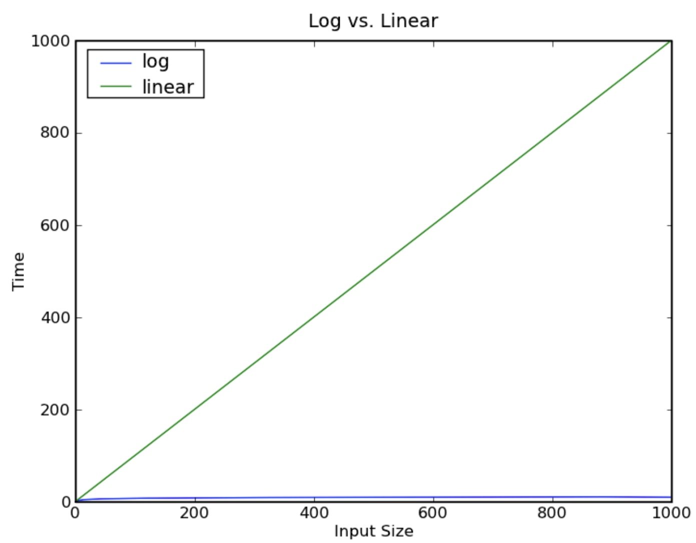
Observations

- A logarithmic algorithm is often almost as good as a constant time algorithm
- Logarithmic costs grow very slowly

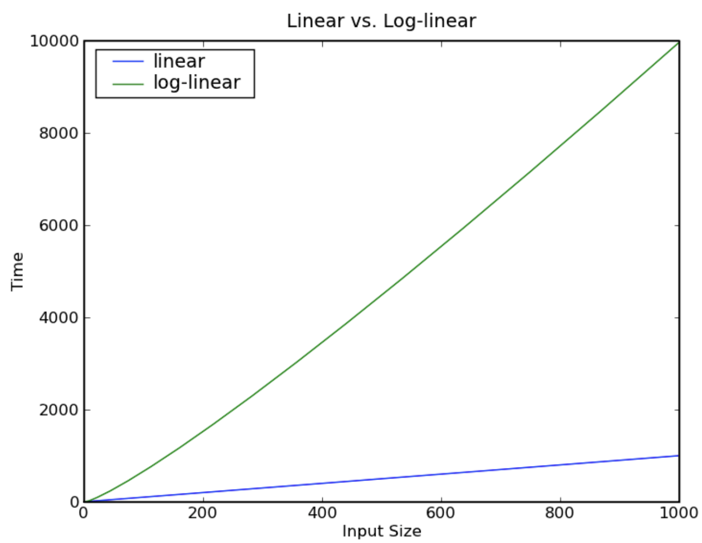
Observations

- Logarithmic clearly better for large scale problems than linear
- Does not imply linear is bad, however

Logarithmic versus Linear



Linear versus Log-linear



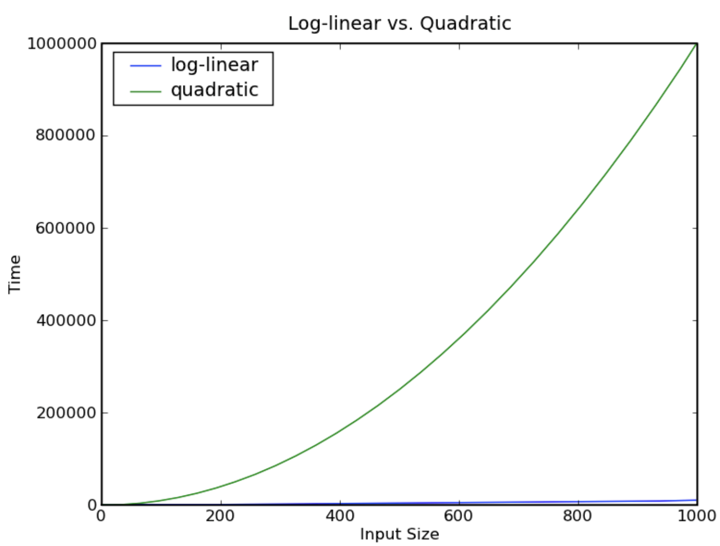
Observations

- While *log(n)* may grow slowly, when multiplied by a linear factor, growth is much more rapid than pure linear
- *O(n log n)* algorithms are still very valuable

Observations

- Quadratic is often a problem, however.
- Some problems inherently quadratic but if possible always better to look for more efficient solutions

Log-linear versus Quadratic



Quadratic versus Exponential

- Exponential algorithms very expensive
  - Right plot is on a log scale, since left plot almost invisible given how rapidly exponential grows
- Exponential generally not of use except for small problems