CSCI5654 (Linear Programming, Fall 2013)
Lecture-5

Today's Lecture

1. Go over Simplex steps at a higher level.
2. Unbounded problem instances.
3. Initialization.

Simplex

1. Convert to standard form.
2. Introduce Slack Variables.
3. Find initial feasible solution and corr. dictionary.
4. While dictionary is not final:
    4.1 Find a non-basic variable x_i to improve.
    4.2 Find a basic variable x_j that constrains x_i the most.
    4.3 Insert x_i into basis and remove x_j from basis.
5. Output final result.

Example-2

$$
\begin{array}{rrcl}
\text{maximize} & 2x_1 + 3x_2 - 5x_3 & & \\
\text{s.t.} & x_1 - x_2 & \leq & 5 \\
& -x_1 + \quad x_3 & \leq & 6 \\
& -2x_1 + \quad x_3 & \leq & 2 \\
& -x_1 + x_2 & \leq & 4 \\
& x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Initial Dictionary

$$\begin{array}{rcl} x_4 & = & 5 - x_1 + x_2 \\ x_5 & = & 6 + x_1 \qquad\quad - x_3 \\ x_6 & = & 2 + 2x_1 \qquad - x_3 \\ x_7 & = & 4 + x_1 - x_2 \\ \hline z & = & 2x_1 + 3\boxed{x_2} - 5x_3 \end{array}$$

Q: Choose x_2 to enter the basis. What variable should leave?

Finding Leaving Variable

$$
\begin{array}{rcccccccll}
x_4 & = & 5 & - & x_1 & + & \color{red}{1}x_2 & + & 0x_3 & \leftarrow x_4 \text{ does not constrain increase of } x_2 \\
x_5 & = & 6 & + & x_1 & + & \color{red}{0}x_2 & - & x_3 & \leftarrow x_5 \text{ does not constrain on increase of } x_2 \\
x_6 & = & 2 & + & 2x_1 & + & \color{red}{0}x_2 & - & x_3 & \leftarrow x_6 \text{ does not constrain on increase of } x_2 \\
x_7 & = & 4 & + & x_1 & + & \color{red}{-1}x_2 & + & 0x_3 & \leftarrow x_7 \ \underline{\text{constrains the increase of}} \ x_2 \\
\hline
z & = & 0 & + & 2x_1 & + & 3\boxed{x_2} & - & 5x_3 &
\end{array}
$$

Calculation:

$$
\begin{array}{llllll}
x_4 \geq 0 & \Rightarrow & 5 + x_2 \geq 0 & \Rightarrow & x_2 \geq -5 & \leftarrow \text{ but } x_2 \geq 0 \text{ anyways!} \\
x_7 \geq 0 & \Rightarrow & 4 - x_2 \geq 0 & \Rightarrow & x_2 \leq 4 & \leftarrow \text{ constrains } x_2 \text{ increase.}
\end{array}
$$

Second Dictionary

Note: x_7 leaves the initial dictionary.

$$
\begin{array}{rcrcrcrcr}
x_2 & = & 4 & + & x_1 & - & x_7 & & \\
x_4 & = & 9 & & & - & x_7 & & \\
x_5 & = & 6 & + & x_1 & & & - & x_3 \\
x_6 & = & 2 & + & 2x_1 & & & - & x_3 \\
\hline
z & = & 12 & + & 5x_1 & - & 3x_7 & - & 5x_3
\end{array}
$$

Note: what variables enters? what variable can leave?

Unbounded LP

$$
\begin{array}{rcrcrlcccc}
x_2 & = & 4 & + & 1 & x_1 & - & x_7 & & \\
x_4 & = & 9 & + & 0 & x_1 & - & x_7 & & \\
x_5 & = & 6 & + & 1 & x_1 & & & - & x_3 \\
x_6 & = & 2 & + & 2 & x_1 & & & - & x_3 \\
\hline
z & = & 12 & + & 5 & x_1 & - & 3x_7 & - & 5x_3
\end{array}
$$

Unbounded:
For some entering variable x_j , s.t.
<u>no</u> row upper bounds the value of x_j.

Alternatively, the entries in some column are all non-negative.

Simplex

1. Convert to standard form.
2. Introduce Slack Variables.
3. Find initial feasible solution and corr. dictionary.
4. While dictionary is <u>not final</u>:
   (<u>some positive coeff. in objective row</u>)
   4.1 Find a entering non-basic variable x_i to improve.
       - Coefficient of x_i in the objective row is positive.
   4.2 Find a basic variable x_j that constrains x_i the most.
       - Look at the column corr. to entering variable.
       - For all negative entries $c_{i,j}$ compute upper bound on x_i increase.
   4.3 Insert x_i into basis and remove x_j from basis.
       - Rewrite x_i in terms of x_j and substitute.
5. Output final result.

Initialization

Example-3

$$
\begin{array}{rrcl}
\text{max.} & x_1 - x_2 + x_3 & & \\
\text{s.t.} & 2x_1 - x_2 + x_3 & \leq & 4 \\
 & 2x_1 - 3x_2 + x_3 & \leq & -5 \\
 & -x_1 + x_2 - 2x_3 & \leq & -1 \\
 & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Example-3

$$\begin{array}{rcl} x_4 & = & 4 - 2x_1 + x_2 - 2x_3 \\ x_5 & = & -5 - 2x_1 + 3x_2 - x_3 \\ x_6 & = & -1 + x_1 - x_2 + 2x_3 \\ \hline z & = & x_1 - x_2 + x_3 \end{array}$$

Problem: After adding slack variables, initial dictionary is not feasible.

Auxilliary Problem

Idea: Add new variable x_0.
Modified Problem:

$$
\begin{array}{rrcl}
\text{maximize} & -x_0 & & \\
\text{s.t.} & a_{11}x_1 + \cdots + a_{1n}x_n - x_0 & \leq & b_1 \\
& a_{21}x_1 + \cdots + a_{2n}x_n - x_0 & \leq & b_2 \\
& & \vdots & \\
& a_{m1}x_1 + \cdots + a_{mn}x_n - x_0 & \leq & b_m \\
& x_0, x_1, \ldots, x_n & \geq & 0
\end{array}
$$

Auxilliary Problem

Original Problem:

$$\begin{array}{rrcl} \text{max.} & \vec{c} \cdot \vec{x} & & \\ \text{s.t.} & A\vec{x} & \leq & \vec{b} \\ & \vec{x} & \geq & \vec{0} \end{array}$$

Auxilliary Problem:

$$\begin{array}{rrcl} \text{max.} & -x_0 & & \\ \text{s.t.} & A\vec{x} - x_0\vec{1} & \leq & \vec{b} \\ & \left(\begin{array}{c} \vec{x} \\ x_0 \end{array}\right) & \geq & \vec{0} \end{array}$$

Auxilliary Problem: Example -3

$$\begin{array}{rrcl}
\text{max.} & -x_0 & & \\
\text{s.t.} & 2x_1 - x_2 + x_3 - x_0 & \leq & 4 \\
 & 2x_1 - 3x_2 + x_3 - x_0 & \leq & -5 \\
 & -x_1 + x_2 - 2x_3 - x_0 & \leq & -1 \\
 & x_0, x_1, x_2, x_3 & \geq & 0
\end{array}$$

Auxilliary Problem: Property

Theorem: For any given LP,

1. Auxilliary LP is always feasible and bounded.
2. If orig. LP is feasible then optimal value of auxilliary LP is 0, OR
3. If orig. LP is infeasible then optimal value of auxilliary LP is < 0.

Proof:

1. Auxilliary LP is always feasible.

$$\begin{array}{rrcl} \text{max.} & -x_0 & & \\ \text{s.t.} & A\vec{x} - x_0\vec{1} & \leq & \vec{b} \\ & \left(\begin{array}{c} \vec{x} \\ x_0 \end{array}\right) & \geq & \vec{0} \end{array}$$

Idea: Set $\vec{x} = \vec{0}$ and x_0 to absolute value of smallest (negative) coefficient in $\vec{b}$. (??)

1.1 Aux. LP is always bounded. We are maximizing $z = -x_0$ while we have $x_0 \geq 0$. This gives us $z \leq 0$. Therefore, z is bounded from above by 0.

Auxilliary Problem: Example -3

$$\begin{array}{rrcl} \text{max.} & -x_0 & & \\ \text{s.t.} & 2x_1 - x_2 + x_3 - x_0 & \leq & 4 \\ & 2x_1 - 3x_2 + x_3 - x_0 & \leq & \boxed{-5} \leftarrow \text{ smallest coeff.} \\ & -x_1 + x_2 - 2x_3 - x_0 & \leq & -1 \\ & x_0, x_1, x_2, x_3 & \geq & 0 \end{array}$$

Feasible solution: $x_1 = x_2 = x_3 = 0$, $x_0 = 5$.

Auxilliary Problem: Example -3

$$
\begin{array}{rcl}
x_4 & = & 4 + x_0 - 2x_1 + x_2 - x_3 \\
x_5 & = & -5 + x_0 - 2x_1 - 3x_2 - x_3 \\
x_6 & = & -1 + x_0 + x_1 - x_2 + 2x_3 \\
\hline
w & = & -x_0
\end{array}
$$

Note: This is still infeasible.

1. Force x_0 to enter the basis.
2. Force variable with smallest negative constant coeff. to leave.

Example-3: Initialization

$$
\begin{array}{rcl}
x_4 & = & 4 + x_0 - 2x_1 + x_2 - x_3 \\
x_5 & = & -5 + x_0 - 2x_1 - 3x_2 - x_3 \\
x_6 & = & -1 + x_0 + x_1 - x_2 + 2x_3 \\
\hline
w & = & -x_0
\end{array}
$$

Note: According to our initialization rules, x_0 enters and x_5 leaves.

Example-3: Dictionary # 1

$$
\begin{array}{rcl}
x_0 & = & 5 + 2x_1 - 3x_2 + x_3 + x_5 \\
x_4 & = & 9 - 2x_2 - x_3 + x_5 \\
x_6 & = & 4 + 3x_1 - 4x_2 + 3x_3 + x_5 \\
\hline
w & = & -5 - 2x_1 + 3x_2 - x_3 - x_5
\end{array}
$$

Note: This is a feasible dictionary. We proceed according to basic Simplex.

Rule #3: Whenever x_0 can leave the basis, we let x_0 leave
in fact, the resulting dictionary will be final.

Example-3: Dictionary # 1

$$\begin{array}{rcl} x_0 & = & 5 + 2x_1 - 3x_2 + x_3 + x_5 \\ x_4 & = & 9 - 2x_2 - x_3 + x_5 \\ x_6 & = & 4 + 3x_1 - 4x_2 + 3x_3 + x_5 \\ \hline w & = & -5 - 2x_1 + 3x_2 - x_3 - x_5 \end{array}$$

x_2 is entering variable.
Calculation for leaving variable:

$$\begin{array}{lcl} x_0 \geq 0 & \Rightarrow & x_2 \leq \frac{5}{3} (\sim 2.6666...) \\ x_4 \geq 0 & \Rightarrow & x_2 \leq 9 \\ x_6 \geq 0 & \Rightarrow & x_2 \leq 1 \end{array}$$

Leaving variable is ??

Example-3: Dictionary # 2

$$
\begin{array}{rcl}
x_0 & = & 2 - .25x_1 - .75x_6 - 1.25x_3 + .25x_5 \\
x_2 & = & 1 + .75x_1 - .25x_6 + .75x_3 + .25x_5 \\
x_4 & = & 7 - 1.5x_1 - .5x_6 - 2.5x_3 + .5x_5 \\
\hline
w & = & -2 + .25x_1 - .75x_6 + 1.25x_3 - .25x_5
\end{array}
$$

x_3 is entering.
What variable leaves?

Example-3: Initialization (final dictionary)

$$
\begin{array}{rcl}
x_3 & = & 1.6 - .2x_1 + .2x_5 + .6x_6 - .8x_0 \\
x_2 & = & 2.2 + .6x_1 + .4x_5 + .2x_6 - .6x_0 \\
x_4 & = & 3 - x_1 - x_6 - 2x_0 \\
\hline
w & = & -x_0
\end{array}
$$

Note: This is optimal with optimal value 0.
Rule #4: Remove x_0 from this dictionary and replace with original objective function.

Initial Basis for Example-3

$$
\begin{array}{rclcccccc}
x_3 & = & 1.6 & - & .2x_1 & + & .2x_5 & + & .6x_6 \\
x_2 & = & 2.2 & + & .6x_1 & + & .4x_5 & + & .2x_6 \\
x_4 & = & 3 & - & x_1 & & & - & x_6 \\
\hline
z & = & -.6 & + & .2x_1 & - & .2x_5 & + & .4x_6
\end{array}
$$

We can start rest of Simplex from this basis.

Initialization

1. Formulate auxilliary problem.
2. Force x_0 to enter and variable with least -ve constant coeff. to leave.
3. Iterate according to basic simplex.
   - Whenever x_0 can leave the basis, force it leave.
4. If optimal value is 0:
   - Remove x_0 from the dictionary.
   - Replace original objective function.
5. If optimal value is < 0, declare solution INFEASIBLE.

Initialization: Properties

Theorem: For the first step of initialization:

1. after inserting x_0 into basis, and
2. forcing variable with least negative constant term to leave

we will always obtain a <u>feasible basis</u>.

Initialization: Properties

Theorem: If optimal value of initialization simplex is 0, then final dictionary will have x_0 as its only non-basic variable.

$$
\begin{array}{rcll}
x_{i_1} & = & \cdots & i_1 \neq 0 \\
 & \vdots & & \\
x_{i_m} & = & \cdots & i_m \neq 0 \\
\hline
w & = & -x_0 &
\end{array}
$$

Next Lecture

- Degeneracy
- Cycling (termination of Simplex)
- Lexicographic rule & Bland's Rule.
- Heuristics for choosing entering/leaving variable.
- Klee-Minty Examples.