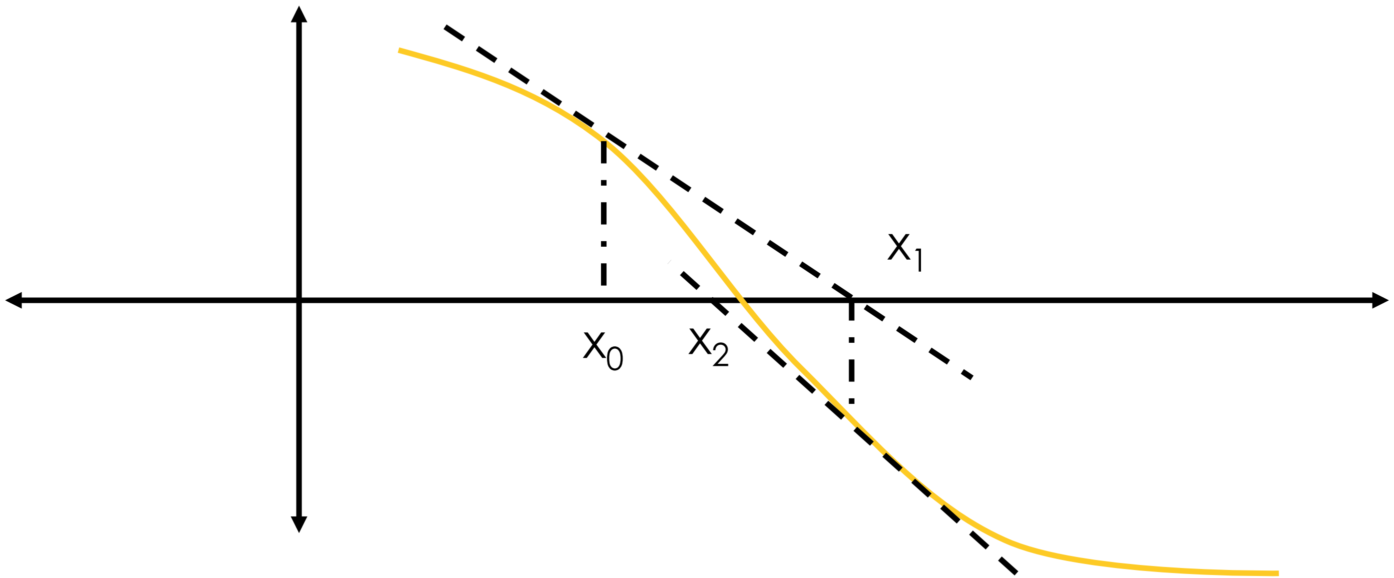
NEWTON'S METHOD

Basic Goal

Solve the (system) of equations: $F(\mathbf{x}) = 0$

$F(\mathbf{x})$ assumed continuous and differentiable (smooth)



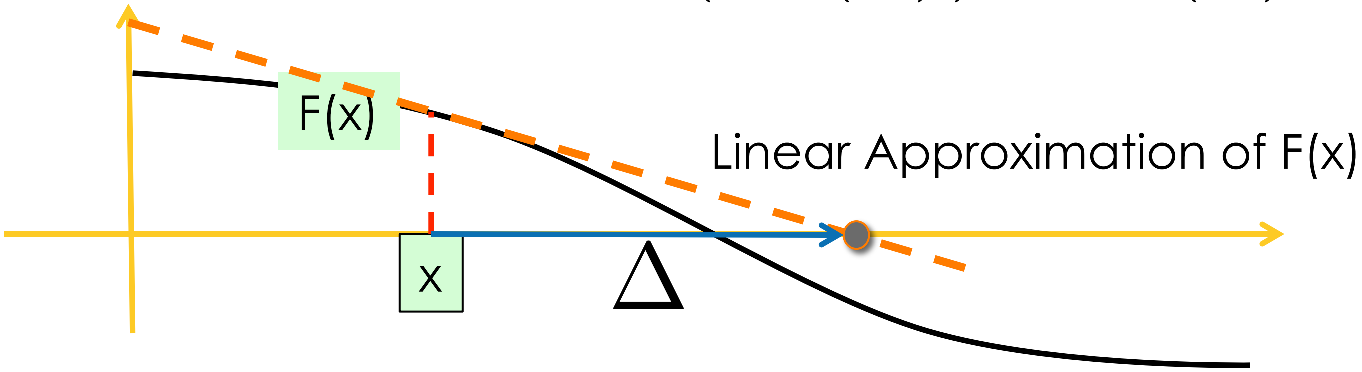
Newton Step

Currently, $F(\mathbf{x}) \neq 0$

Linear Approximation: $F(\mathbf{x}+\Delta) \simeq F(\mathbf{x}) + F'(\mathbf{x})\Delta$

$$F(\mathbf{x}) + F'(\mathbf{x})\Delta = \mathbf{0}$$

$$\Delta = -(F'(\mathbf{x}))^{-1}F(\mathbf{x})$$



Newton Step (n-dimensions)

$$F : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

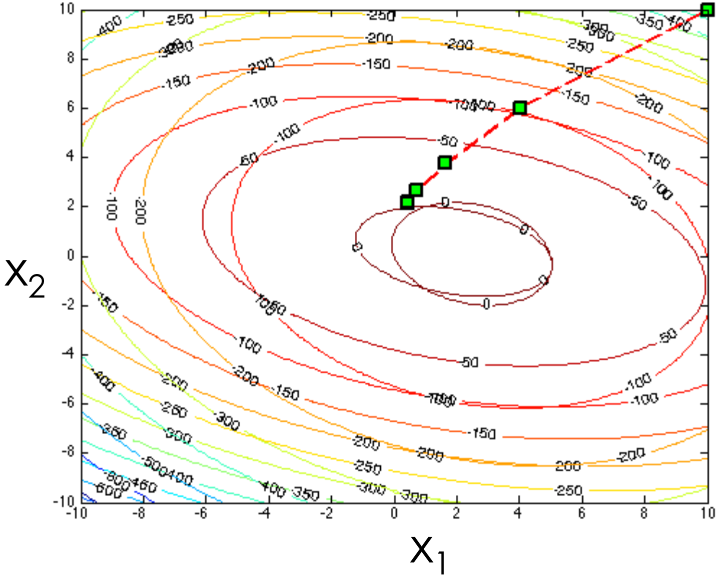
$$F(x) = \begin{bmatrix} F_1(\mathbf{x}) \\ \vdots \\ F_n(\mathbf{x}) \end{bmatrix}$$

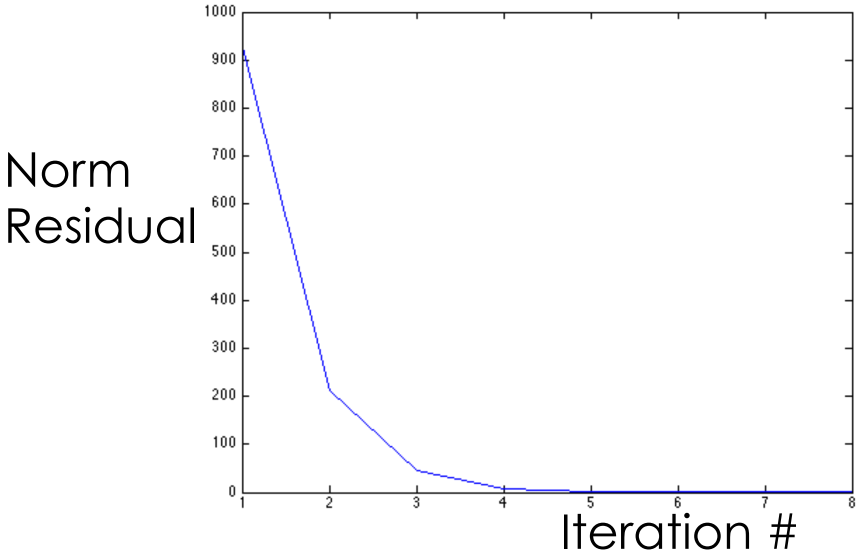
$$F'(\mathbf{x}) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$$

$$\Delta = -(F'(\mathbf{x}))^{-1} F(\mathbf{x})$$

Newton's method example

$$f(x_1, x_2) = \begin{pmatrix} -x_1^2 - 3x_2^2 - x_1 x_2 + 3x_2 + 4x_1 + 5 \\ -2x_1^2 - 3x_2^2 - x_1 x_2 + 10x_1 + 3x_2 \end{pmatrix} = 0$$





Newton Method Convergence

- If method converges, it does so to a root of F(x).
- Convergence is not guaranteed.
  - Only if starting point in the "basin of attraction" of the root.
  - F'(x) should not vanish at the root ("simple" root).
- Convergence is quadratic.
  - Often very fast when it does converge.

Complexity of each newton step.

- Goal: Compute root of F(x) using Newton’s method.
- Compute the Jacobian F’(x)
- Newton Step:
  - Invert the Jacobian and multiply with value of function.

Newton Method for Optimization

- Goal: minimize function F(x) for all x.
- Unconstrained minimization problem.