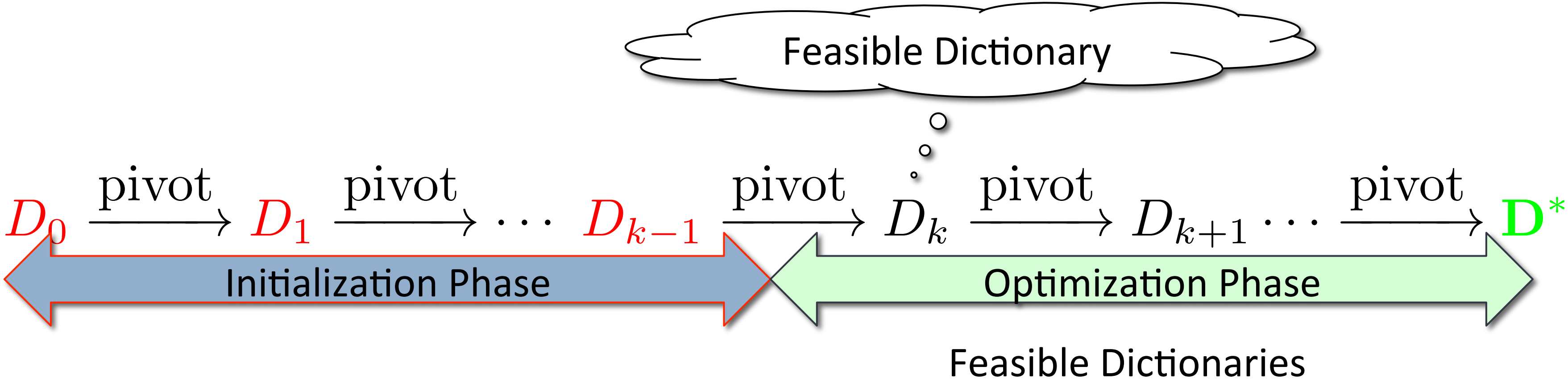
INITIALIZATION USING THE DUAL

Simplex Algorithm



Initialization

- Goal: Find a feasible dictionary for the problem.

$$\boxed{\begin{array}{llll} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ & A\,\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}}$$

Observation:
Objective does not matter.

Initialization

$$
\begin{aligned}
\max \quad & \mathbf{c}^{\mathsf{T}}\mathbf{x} & & \\
& A\,\mathbf{x} & \leq & \; \mathbf{b} \\
& \mathbf{x} & \geq & \; 0
\end{aligned}
$$

Original Problem

If all entries in **b ≥ 0**

- Initialization not needed.

If any entry in **b <0**

- initialization needed.

Linear Program (Dual)

$$
\begin{array}{llll}
\max & \mathbf{c}^\intercal \mathbf{x} & & \\
 & A\,\mathbf{x} & \leq & \mathbf{b} \\
 & \mathbf{x} & \geq & 0
\end{array}
$$

Dual Problem

$$
\begin{array}{llll}
\min & \mathbf{b}^\intercal \mathbf{y} & & \\
 & A^\intercal\,\mathbf{y} & \geq & \mathbf{c} \\
 & \mathbf{y} & \geq & 0
\end{array}
$$

$$
\begin{array}{llll}
\max & -\mathbf{b}^\intercal \mathbf{y} & & \\
 & -A^\intercal\,\mathbf{y} & \leq & -\mathbf{c} \\
 & \mathbf{y} & \geq & 0
\end{array}
$$

Standard Form Converted Dual

Initialization Using Dual: Basic Idea

1. Change the objective function

$$
\begin{aligned}
\max \mathbf{d}^T \; & \mathbf{x} & & \\
A\,\mathbf{x} & \leq & \mathbf{b} \\
\mathbf{x} & \geq & 0
\end{aligned}
$$

Dual Problem

$$
\begin{aligned}
\max \quad & -\mathbf{b}^\intercal \mathbf{y} & & \\
& -A^\intercal\,\mathbf{y} & \leq & -\mathbf{d} \\
& \mathbf{y} & \geq & 0
\end{aligned}
$$

All
Positive

Idea: choose **d** to be all negative entries.

Initialization Using Dual

$$\begin{aligned} \max\ & \mathbf{d}^T \mathbf{x} \\ & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

d has all negative entries

Dualize

$$\begin{aligned} \max\ & -\mathbf{b}^\intercal \mathbf{y} \\ & -A^\intercal \mathbf{y} \leq -\mathbf{d} \\ & \mathbf{y} \geq 0 \end{aligned}$$

Optimization Phase Simplex

$$\begin{array}{c|cc} \mathbf{y}_b & \mathbf{q} & +P\,\mathbf{y}_n \\ \hline w & w_0 & +\mathbf{r}^T\,\mathbf{y}_n \end{array}$$

Final Dual Dictionary

Dualize

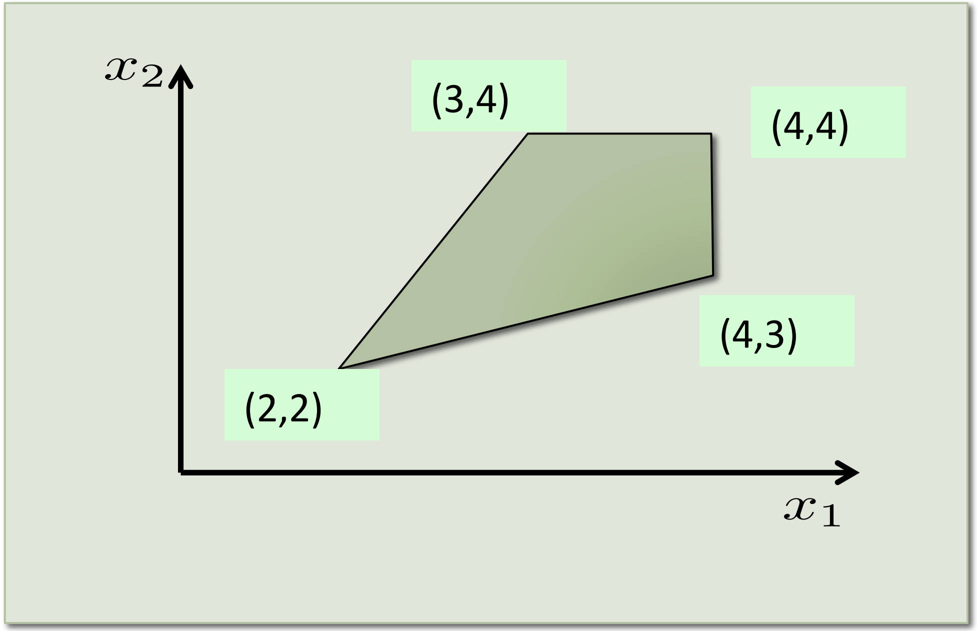
$$\begin{array}{c|cc} \mathbf{y}^c_n & -\mathbf{r} & -P^T\,\mathbf{y}^c_b \\ \hline \end{array}$$

Restore Original Obj.

Feasible Primal Dictionary

Example

$$
\begin{array}{rccc}
\text{max.} & x_1 + 2x_2 & & \\
\text{s.t.} & -2x_1 + x_2 & \leq & -2 \\
 & x_2 & \leq & 4 \\
 & x_1 - 2x_2 & \leq & -2 \\
 & x_1 & \leq & 4 \\
 & x_1, x_2 & \geq & 0
\end{array}
$$



Example #1: Problem Transformation

$$\begin{array}{llcl} \max & -x_1 - x_2 & & \\ \text{s.t.} & -2x_1 + x_2 & \leq & -2 \\ & x_2 & \leq & 4 \\ & x_1 - 2x_2 & \leq & -2 \\ & x_1 & \leq & 4 \\ & x_1, x_2 & \geq & 0 \end{array}$$

$$\begin{array}{llcl} \max & -x_1 - x_2 & & \\ \text{s.t.} & -2x_1 + x_2 + x_3 & = & -2 \\ & x_2 + x_4 & = & 4 \\ & x_1 - 2x_2 + x_5 & = & -2 \\ & x_1 + x_6 & = & 4 \\ & x_1, x_2, x_3, \ldots, x_6 & \geq & 0 \end{array}$$

$$\begin{array}{llllllcl} \max & 2y_1 & -4y_2 & +2y_3 & -4y_4 & & & \\ \text{s.t.} & 2y_1 & & -y_3 & -y_4 & & \leq & 1 \\ & -y_1 & -y_2 & +2y_3 & & & \leq & 1 \\ & & & & y_1, \ldots, y_4 & & \geq & 0 \end{array}$$

$$\begin{array}{llllllcl} \max & 2y_1 & -4y_2 & +2y_3 & -4y_4 & & & \\ \text{s.t.} & 2y_1 & & -y_3 & -y_4 & +y_5 & = & 1 \\ & -y_1 & -y_2 & +2y_3 & & +y_6 & = & 1 \\ & & & & & y_1, \ldots, y_4, y_5, y_6 & \geq & 0 \end{array}$$

Solving the modified dual

$$
\begin{array}{llllllll}
\text{max} & 2y_1 & -4y_2 & +2y_3 & -4y_4 & & & \\
\text{s.t.} & 2y_1 & & -y_3 & -y_4 & +{\color{red}y_5} & = & {\color{blue}1} \\
 & -y_1 & -y_2 & +2y_3 & & +{\color{red}y_6} & = & {\color{blue}1} \\
 & & & & \multicolumn{2}{r}{y_1, \ldots, y_4, {\color{red}y_5, y_6}} & \geq & 0
\end{array}
$$

$$
\begin{array}{c|cllll}
y_5 & 1 & -2y_1 & & +y_3 & +y_4 \\
y_6 & 1 & +y_1 & +y_2 & -2y_3 & \\
\hline
w & 0 & +2y_1 & -4y_2 & +2y_3 & -4y_4
\end{array}
$$

y_1 enters and y_5 leaves

Solving the modified dual (step 2)

$$
\begin{array}{c|llllll}
y_1 & 0.5 & -0.5y_5 & & +0.5y_3 & +0.5y_4 \\
y_6 & 1.5 & -0.5y_5 & +1y_2 & -1.5y_3 & +0.5y_4 \\
\hline
w & 1 & -1y_5 & -4y_2 & +3y_3 & -3y_4
\end{array}
$$

y_3 enters and y_6 leaves

Solving the modified dual (step 3)

$$
\begin{array}{c|cl}
y_1 & 1 & -\frac{2}{3}y_5 + \frac{1}{3}y_2 - \frac{1}{3}y_6 + \frac{2}{3}y_4 \\
y_3 & 1 & -\frac{1}{3}y_5 + \frac{2}{3}y_2 - \frac{2}{3}y_6 + \frac{1}{3}y_4 \\
\hline
w & 4 & -2y_5 \; -2y_2 \; -2y_6 \; -2y_4
\end{array}
$$

Final Dual Dictionary

Convert Dual back to Primal

$$\begin{array}{lrcl} \max & -x_1 - x_2 & & \\ \text{s.t.} & -2x_1 + x_2 + x_3 & = & -2 \\ & x_2 + x_4 & = & 4 \\ & x_1 - 2x_2 + x_5 & = & -2 \\ & x_1 + x_6 & = & 4 \\ & x_1, x_2, x_3, \ldots, x_6 & \geq & 0 \end{array}$$

$$\begin{array}{llllllcl} \max & 2y_1 & -4y_2 & +2y_3 & -4y_4 & & & \\ \text{s.t.} & 2y_1 & & -y_3 & -y_4 & +y_5 & = & 1 \\ & -y_1 & -y_2 & +2y_3 & & +y_6 & = & 1 \\ & & & & & y_1, \ldots, y_4, y_5, y_6 & \geq & 0 \end{array}$$

x_1	y_5
x_2	y_6
x_3	y_1
x_4	y_2
x_5	y_3
x_6	y_4

Conversion to Primal Dictionary

$$
\begin{array}{c|l}
y_1 & 1 \quad -\frac{2}{3}y_5 + \frac{1}{3}y_2 - \frac{1}{3}y_6 + \frac{2}{3}y_4 \\
y_3 & 1 \quad -\frac{1}{3}y_5 + \frac{2}{3}y_2 - \frac{2}{3}y_6 + \frac{1}{3}y_4 \\
\hline
w & 4 \quad -2y_5 \; -2y_2 \; -2y_6 \; -2y_4
\end{array}
$$

$$
\begin{array}{c|lll}
x_1 & 2 & +\frac{2}{3}x_3 & +\frac{1}{3}x_5 \\
x_4 & 2 & -\frac{1}{3}x_3 & -\frac{2}{3}x_5 \\
x_2 & 2 & +\frac{1}{3}x_3 & +\frac{2}{3}x_5 \\
x_6 & 2 & -\frac{2}{3}x_3 & -\frac{1}{3}x_5 \\
\hline
w & -4 & -x_3 & -x_5
\end{array}
$$

x_1	y_5
x_2	y_6
x_3	y_1
x_4	y_2
x_5	y_3
x_6	y_4

Initialization: Restoring Objective

$$
\begin{array}{c|cll}
x_1 & 2 & +\frac{2}{3}x_3 & +\frac{1}{3}x_5 \\
x_4 & 2 & -\frac{1}{3}x_3 & -\frac{2}{3}x_5 \\
x_2 & 2 & +\frac{1}{3}x_3 & +\frac{2}{3}x_5 \\
x_6 & 2 & -\frac{2}{3}x_3 & -\frac{1}{3}x_5 \\
\hline
\boxed{\mathrm{z}} & \multicolumn{3}{c}{\boxed{6 + \frac{4}{3}x_3 + \frac{5}{3}x_5}}
\end{array}
$$

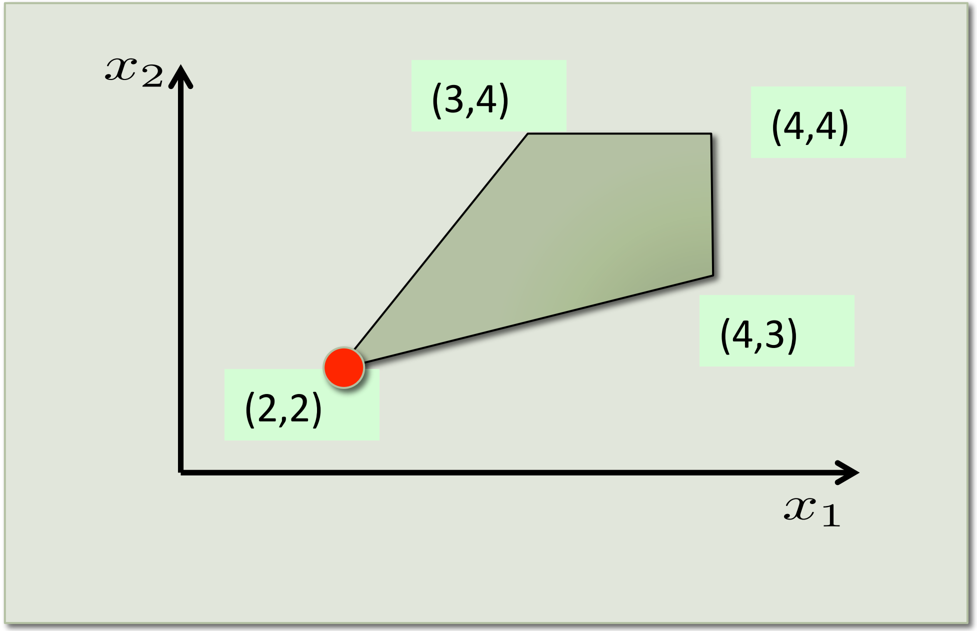
$$
\begin{aligned}
z &= x_1 + 2x_2 \\
&= 2 + \tfrac{2}{3}x_3 + \tfrac{1}{3}x_5 + 2(2 + \tfrac{1}{3}x_3 + \tfrac{2}{3}x_5) \\
&= 6 + \tfrac{4}{3}x_3 + \tfrac{5}{3}x_5
\end{aligned}
$$

Initial Feasible Dictionary Found.
We can now proceed to optimize!!

Example

$$\begin{array}{rlcl} \text{max.} & x_1 + 2x_2 & & \\ \text{s.t.} & -2x_1 + x_2 & \leq & -2 \\ & x_2 & \leq & 4 \\ & x_1 - 2x_2 & \leq & -2 \\ & x_1 & \leq & 4 \\ & x_1, x_2 & \geq & 0 \end{array}$$

$$\begin{array}{c|llll} x_1 & 2 & +\frac{2}{3}x_3 & +\frac{1}{3}x_5 \\ x_4 & 2 & -\frac{1}{3}x_3 & -\frac{2}{3}x_5 \\ x_2 & 2 & +\frac{1}{3}x_3 & +\frac{2}{3}x_5 \\ x_6 & 2 & -\frac{2}{3}x_3 & -\frac{1}{3}x_5 \\ \hline \boxed{z} & \multicolumn{3}{l}{\boxed{6 + \frac{4}{3}x_3 + \frac{5}{3}x_5}} \end{array}$$



Initialization Using Dual

$$\begin{array}{lrcl} \max & \mathbf{d}^T\mathbf{x} & & \\ & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}$$

d has all negative entries

Dualize

$$\begin{array}{lrcl} \max & -\mathbf{b}^\intercal\mathbf{y} & & \\ & -A^\intercal\mathbf{y} & \leq & -\mathbf{d} \\ & \mathbf{y} & \geq & 0 \end{array}$$

Unbounded!

Original primal problem is infeasible.

Optimization Phase Simplex

$$\begin{array}{c|c} \mathbf{y}_b & \mathbf{q} + P\,\mathbf{y}_n \\ \hline w & w_0 + \mathbf{r}^T\mathbf{y}_n \end{array}$$

Final Dual Dictionary