ACT II: THE DICTIONARY

A convenient data structure for Linear Programs and representation of solutions.

Standard Form (Matrix Notation)

$$
\begin{array}{lrcl}
\text{maximize} & c_1x_1 + \ldots + c_nx_n & & \\
\text{subj.to.} & a_{11}x_1 + \ldots + a_{1n}x_n & \leq & b_1 \\
& a_{21}x_1 + \ldots + a_{2n}x_n & \leq & b_2 \\
& \ddots & \vdots & \\
& a_{m1}x_1 + \ldots + a_{mn}x_n & \leq & b_m \\
& x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

$$
\begin{array}{lr}
\text{maximize} & \mathbf{c}^\intercal \mathbf{x} \\
\text{subj.to.} & \mathbf{A}\ \mathbf{x} \leq \mathbf{b} \\
& \mathbf{x} \geq 0
\end{array}
$$

Slack Form

Trick: Change inequality constraint to an equality.

$$\mathbf{a}^\intercal \mathbf{x} \leq b \xrightarrow{\text{add slack}} \mathbf{a}^\intercal \mathbf{x} + x_s = b$$

maximize	$\mathbf{c}^\intercal \mathbf{x}$	
subj.to.	$\mathbf{A}\ \mathbf{x} \leq \mathbf{b}$	
	$\mathbf{x} \geq 0$	

maximize	$\mathbf{c}^\intercal \mathbf{x}$		
s.t.	$\mathbf{A}\mathbf{x} + \mathbf{x}_{\text{slack}}$	$=$	$\mathbf{b}$
	$\mathbf{x}$	$\geq$	$\mathbf{0}$
	$\mathbf{x}_{\text{slack}}$	$\geq$	$\mathbf{0}$

Example (Slack Variable Addition)

$$
\begin{aligned}
\text{max.} \quad & 5x_1 + 4x_2 + 3x_3 & & \\
\text{s.t.} \quad & 2x_1 + 3x_2 + x_3 & \leq & \quad 5 \\
& 4x_1 + x_2 + 2x_3 & \leq & \quad 11 \\
& 3x_1 + 4x_2 + 2x_3 & \leq & \quad 8 \\
& x_1, x_2, x_3 & \geq & \quad 0
\end{aligned}
$$

Example (Slack Variable Addition)

$$
\begin{aligned}
\text{max.} \quad & 5x_1 + 4x_2 + 3x_3 & & \\
\text{s.t.} \quad & 2x_1 + 3x_2 + x_3 + x_4 & = & \ 5 \\
& 4x_1 + x_2 + 2x_3 + x_5 & = & \ 11 \\
& 3x_1 + 4x_2 + 2x_3 + x_6 & = & \ 8 \\
& x_1, x_2, x_3, x_4, x_5, x_6 & \geq & \ 0
\end{aligned}
$$

Dictionary

$$
\begin{array}{lll}
\text{max.} & 5x_1 + 4x_2 + 3x_3 & \\
\text{s.t.} & 2x_1 + 3x_2 + x_3 + x_4 = 5 & \\
& 4x_1 + x_2 + 2x_3 + x_5 = 11 & \\
& 3x_1 + 4x_2 + 2x_3 + x_6 = 8 & \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 &
\end{array}
$$

$$
\begin{array}{rcrrrr}
x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\
x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\
x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\
\hline
z & = & 0 & +5x_1 & +4x_2 & +3x_3
\end{array}
$$

Dictionary

$$\begin{aligned} \mathbf{x}_b &= \mathbf{b} - \mathbf{A}\mathbf{x}_i \\ \hline z &= c_0 + \mathbf{c}^\intercal \mathbf{x}_i \end{aligned}$$

Summary So Far

Transform Problem with Slack Variables

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^\intercal \mathbf{x} \\ \text{subj.to.} & \mathbf{A}\,\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}$$

$\Leftrightarrow$

$$\begin{array}{lrcl} \text{maximize} & \mathbf{c}^\intercal \mathbf{x} & & \\ \text{s.t.} & \mathbf{A}\mathbf{x} + \mathbf{x}_{\text{slack}} & = & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \\ & \mathbf{x}_{\text{slack}} & \geq & \mathbf{0} \end{array}$$