DUAL FOR GENERAL FORM PROBLEMS

General Form LP

$$\begin{aligned} \max \quad & \mathbf{c}^\intercal \mathbf{x} \\ & P\mathbf{x} = \mathbf{q} \\ & A\mathbf{x} \leq \mathbf{b} \end{aligned}$$

$$\begin{aligned} \max \quad & \mathbf{c}^\intercal \mathbf{x} \\ & P\mathbf{x} \leq \mathbf{q} \leftarrow \mathbf{r} \\ & -P\mathbf{x} \leq -\mathbf{q} \leftarrow \mathbf{s} \\ & A\mathbf{x} \leq \mathbf{b} \leftarrow \mathbf{y} \end{aligned}$$

Dual Derivation

$$
\begin{array}{lrcll}
\max & \mathbf{c}^\intercal\ \mathbf{x} & & & \\
 & P\mathbf{x} & \leq & \mathbf{q} & \leftarrow\ \mathbf{r} \\
 & -P\mathbf{x} & \leq & -\mathbf{q} & \leftarrow\ \mathbf{s} \\
 & A\mathbf{x} & \leq & \mathbf{b} & \leftarrow\ \mathbf{y}
\end{array}
$$

Dual

$$
\begin{array}{lrcl}
\min & \mathbf{q}^\intercal\mathbf{r} - \mathbf{q}^\intercal\mathbf{s} + \mathbf{b}^\intercal\mathbf{y} & & \\
\text{s.t.} & P^\intercal\mathbf{r} - P^\intercal\mathbf{s} + A^\intercal\mathbf{y} = \mathbf{c} & & \\
 & \mathbf{r}, \mathbf{s}, \mathbf{y} & \geq & 0
\end{array}
$$

General Form Dual

$$\begin{aligned} \max \quad & \mathbf{c}^\intercal \mathbf{x} \\ & P\mathbf{x} = \mathbf{q} \quad \leftarrow \mathbf{w} \\ & A\mathbf{x} \leq \mathbf{b} \quad \leftarrow \mathbf{y} \end{aligned}$$

$$\begin{aligned} \min \quad & \mathbf{q}^\intercal \mathbf{w} + \mathbf{b}^\intercal \mathbf{y} \\ \text{s.t.} \quad & P^\intercal \mathbf{w} + A^\intercal \mathbf{y} = \mathbf{c} \\ & \mathbf{y} \geq 0 \end{aligned}$$

Example

$$
\begin{array}{lrrrrrl}
\max & 2x_1 & -3x_2 & +x_3 & & & \\
\text{s.t.} & x_1 & -x_2 & & = & 5 & \leftarrow w_1 \\
 & x_1 & -2x_2 & +x_3 & = & 3 & \leftarrow w_2 \\
 & x_1 & & & \leq & 6 & \leftarrow y_1 \\
 & & & x_3 & \leq & 5 & \leftarrow y_2
\end{array}
$$

$$
\begin{array}{lrrrrr}
\min & 5w_1 & +3w_2 & +6y_1 & +5y_2 & \\
\text{s.t.} & w_1 & +w_2 & +y_1 & = & 2 \\
 & -w_1 & -2w_2 & & = & -3 \\
 & & w_2 & & +y_2 = & 1 \\
 & & & y_1, \quad y_2 & \geq & 0
\end{array}
$$

Complementary Pairs

Dual	Primal
y_1	$(6 - x_1)$
y_2	$(5 - x_3)$

KKT conditions

$$
\begin{array}{lll}
\max & \mathbf{c}^\intercal \mathbf{x} & \\
& P\mathbf{x} = \mathbf{q} & \leftarrow \mathbf{w} \\
& A\mathbf{x} \leq \mathbf{b} & \leftarrow \mathbf{y} \\
\min & \mathbf{q}^\intercal \mathbf{w} + \mathbf{b}^\intercal \mathbf{y} & \\
\text{s.t.} & P^\intercal \mathbf{w} + A^\intercal \mathbf{y} = \mathbf{c} & \\
& \mathbf{y} \geq 0 &
\end{array}
$$

$$
\begin{aligned}
P\mathbf{x} &= \mathbf{q} \\
A\mathbf{x} &\leq \mathbf{b} && \texttt{Primal Feas.}
\end{aligned}
$$

$$
\begin{aligned}
P^\intercal \mathbf{w} + A^\intercal \mathbf{y} &= \mathbf{c} \\
\mathbf{y} &\geq 0 && \texttt{Dual Feas.}
\end{aligned}
$$

$$
\begin{aligned}
y_j(A_j\mathbf{x} - b_j) &= 0 && j = 1, \ldots, m \\
& && \texttt{Complementarity}
\end{aligned}
$$