ILP AND COMBINATORIAL OPTIMIZATION

Reducing 3-SAT to ILP

3-SAT Problem

x_1, x_2, x_3, x_4 Boolean Variables

$$(x_1 \text{ OR } x_2 \text{ OR } \neg x_3)$$
$$(\neg x_2 \text{ OR } \neg x_4 \text{ OR } x_1)$$
$$(x_1 \text{ OR } x_2 \text{ OR } \neg x_3)$$

Find values for Boolean variables

such that

All the Clauses are True.

3-SAT Problem (Infeasible/Unsat)

x_1, x_2, x_3, x_4 Boolean Variables

$$(x_1 \text{ OR } \neg x_4 \text{ OR } x_2)$$
$$(\neg x_1 \text{ OR } \neg x_4 \text{ OR } x_2)$$
$$(x_4 \text{ OR } x_2)$$
$$(\neg\ x_2)$$

No Boolean valuation satisfies all 4 clauses.

Reducing 3-SAT to ILP

$x_1, \ldots, x_n$ are Boolean variables.

$$
\begin{aligned}
C_1 : &\quad (\ell_{1,1} \text{ OR } \ell_{1,2} \text{ OR } \ell_{1,3}) \\
\vdots &\quad \ddots \qquad \text{m Clauses.} \\
C_m : &\quad (\ell_{m,1} \text{ OR } \ell_{m,2} \text{ OR } \ell_{m,3})
\end{aligned}
$$

$\ell_{i,j}$ stands for a variable x_k or its negation $\neg x_k$

ILP reduction.

$$x_j \rightarrow y_j \in \{0, 1\}$$

False = 0
True = 1

$$\neg x_j \equiv (1 - y_j)$$

Clauses

Inequalities

$$(x_1 \text{ OR } x_2 \text{ OR } \neg x_5) \rightarrow y_1 + y_2 + (1 - y_5) \geq 1$$

Example-1

$$(x_1 \text{ OR } x_2 \text{ OR } \neg x_3)$$

$$(\neg x_2 \text{ OR } \neg x_4 \text{ OR } x_1)$$

$$(x_1 \text{ OR } x_2 \text{ OR } \neg x_3)$$

Example-2

$$(x_1 \text{ OR } \neg x_4 \text{ OR } x_2)$$
$$(\neg x_1 \text{ OR } \neg x_4 \text{ OR } x_2)$$
$$(x_4 \text{ OR } x_2)$$
$$(\neg\ x_2)$$