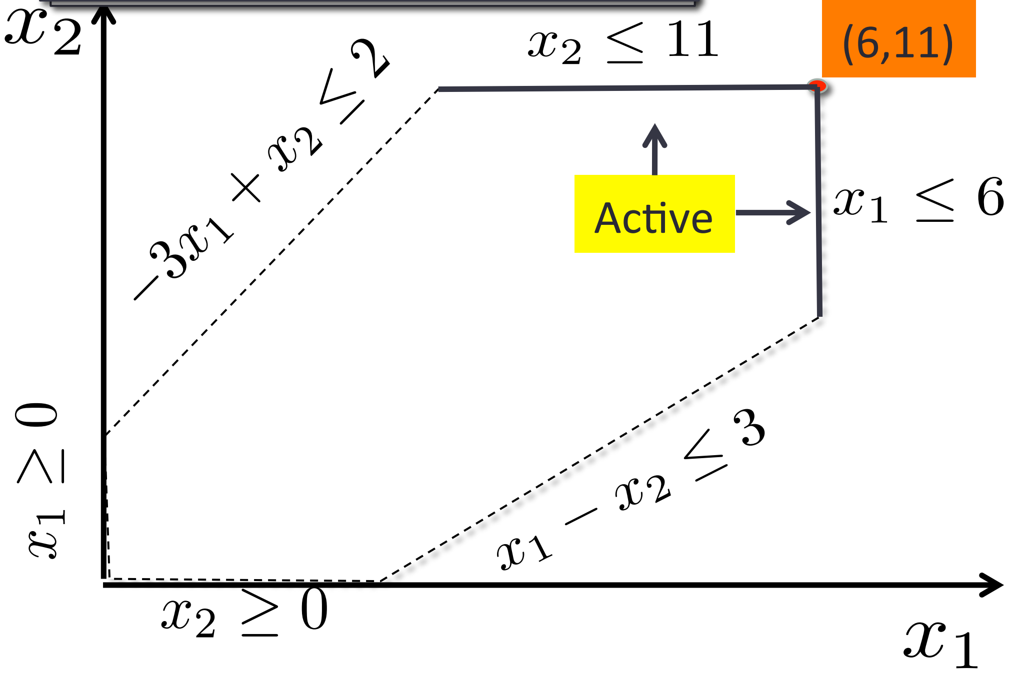
COMPLEMENTARY SLACKNESS THEOREM

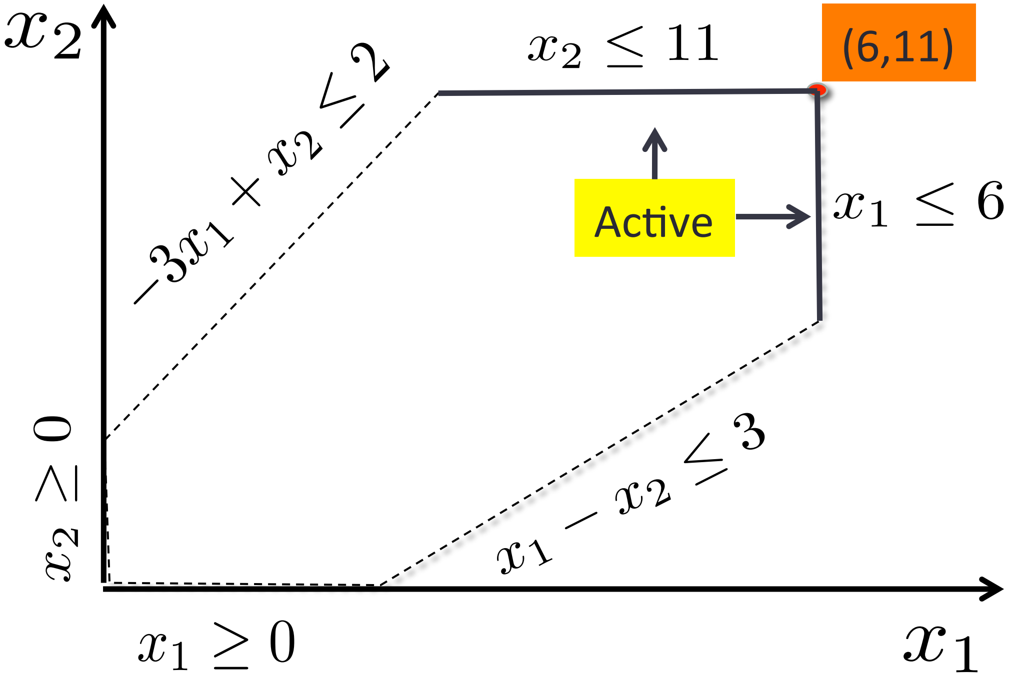
$$\begin{aligned} x_3 &= 2 + 3x_1 - x_2 \\ &= 2 + 18 - 11 \\ &= 9 \end{aligned}$$

$$\begin{array}{lllllll}
\text{max.} & & x_1 & +2x_2 & & & \\
{\color{red}x_3} & \rightarrow & -3x_1 & +x_2 & \leq & 2 \leftarrow & {\color{red}y_1} \\
{\color{red}x_4} & \rightarrow & & +x_2 & \leq & 11 \leftarrow & {\color{red}y_2} \\
{\color{red}x_5} & \rightarrow & x_1 & -x_2 & \leq & 3 \leftarrow & {\color{red}y_3} \\
{\color{red}x_6} & \rightarrow & x_1 & & \leq & 6 \leftarrow & {\color{red}y_4} \\
& & x_1, & x_2 & \geq & 0 \leftarrow & {\color{red}y_5, y_6}
\end{array}$$



Primal	Dual
x_1 : 6	y_5 : 0
x_2 : 11	y_6 : 0
x_3 : 9	y_1 : 0
x_4 : 0	y_2 : 2
x_5 : 8	y_3 : 0
x_6 : 0	y_4 : 1

Active vs. Inactive Constraints



Primal	Dual
$x_1 : 6$	$y_5 : 0$
$x_2 : 11$	$y_6 : 0$
$x_3 : 9$	$y_1 : 0$
$x_4 : 0$	$y_2 : 2$
$x_5 : 8$	$y_3 : 0$
$x_6 : 0$	$y_4 : 1$

x_2 <= 11 is active.

1. y_2 (dual) is non-zero.
2. x_4 (slack) is zero

$-3x_1 + x_2$ <= 2 is inactive.

1. y_1 (dual) is zero.
2. x_3 (slack) is non-zero

Complementary Slackness (Main Idea)

- Let **x** be a primal feasible solution
- Let **y** be a dual feasible solution.
- *Complementarity Condition:* Product of complementary pairs are all zero.

$$x_i \times y_j = 0$$

Complementary Pairs

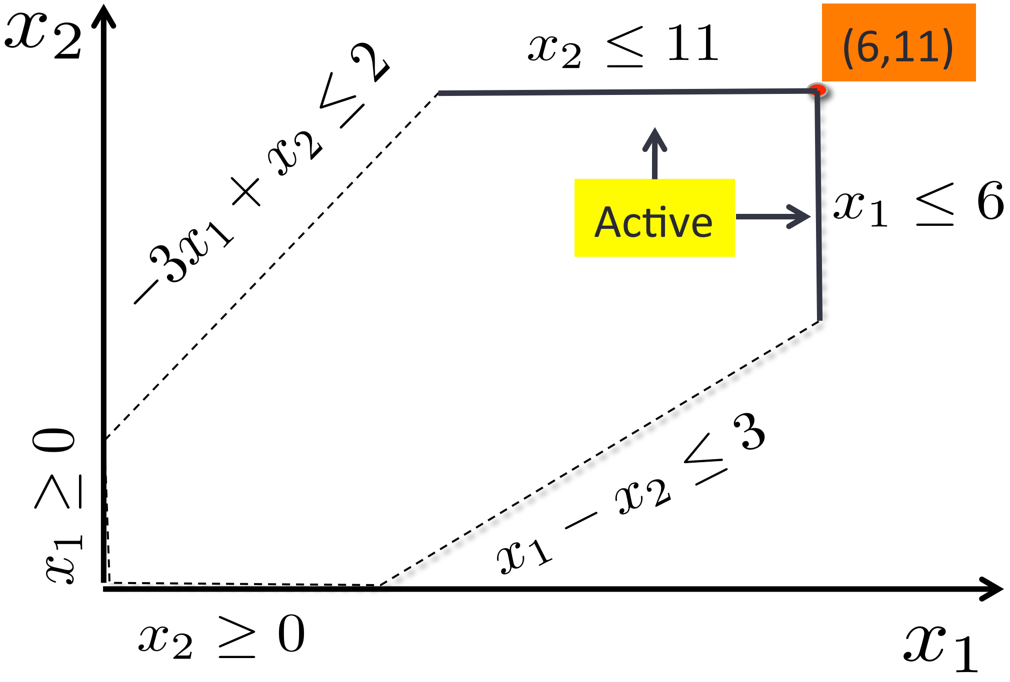
- Theorem: **x, y** are primal and dual optimal respectively.

Proof

Will be discussed separately on a discussion forum.

An Example

$$
\begin{array}{rrrrrl}
\text{max.} & x_1 & +2x_2 & & & \\
{\color{red}x_3 \rightarrow} & -3x_1 & +x_2 & \leq & 2 & {\color{red}\leftarrow y_1} \\
{\color{red}x_4 \rightarrow} & & +x_2 & \leq & 11 & {\color{red}\leftarrow y_2} \\
{\color{red}x_5 \rightarrow} & x_1 & -x_2 & \leq & 3 & {\color{red}\leftarrow y_3} \\
{\color{red}x_6 \rightarrow} & x_1 & & \leq & 6 & {\color{red}\leftarrow y_4} \\
 & x_1, & x_2 & \geq & 0 & {\color{red}\leftarrow y_5, y_6}
\end{array}
$$



Primal	Dual
$x_1 : 6$	$y_5 : 0$
$x_2 : 11$	$y_6 : 0$
$x_3 : 9$	$y_1 : 0$
$x_4 : 0$	$y_2 : 2$
$x_5 : 8$	$y_3 : 0$
$x_6 : 0$	$y_4 : 1$

Complementary Slackness

$$
\begin{array}{c|cc}
\mathbf{x_B} & \mathbf{b} & +A\mathbf{x_I} \\
\hline
z & z_0 & +\mathbf{c}^\intercal \mathbf{x_I}
\end{array}
\qquad
\begin{array}{c|cc}
\mathbf{x_I}^c & -\mathbf{c} & -A^\intercal \mathbf{x_B}^c \\
\hline
d & -z_0 & -\mathbf{b}^\intercal \mathbf{x_B}^c
\end{array}
$$

$$x_I = \mathbf{0}, x_B = \mathbf{b} \qquad x_B^c = \mathbf{0}, x_I^c = -\mathbf{c}$$

Claim: The solutions represented by primal and dual dictionaries are complementary pairs.