Least-Norm

- Problem Description

- Underdetermined Problem

"fall" vs "fat"

- Which solution is "best"?

- Matrix Form Solution

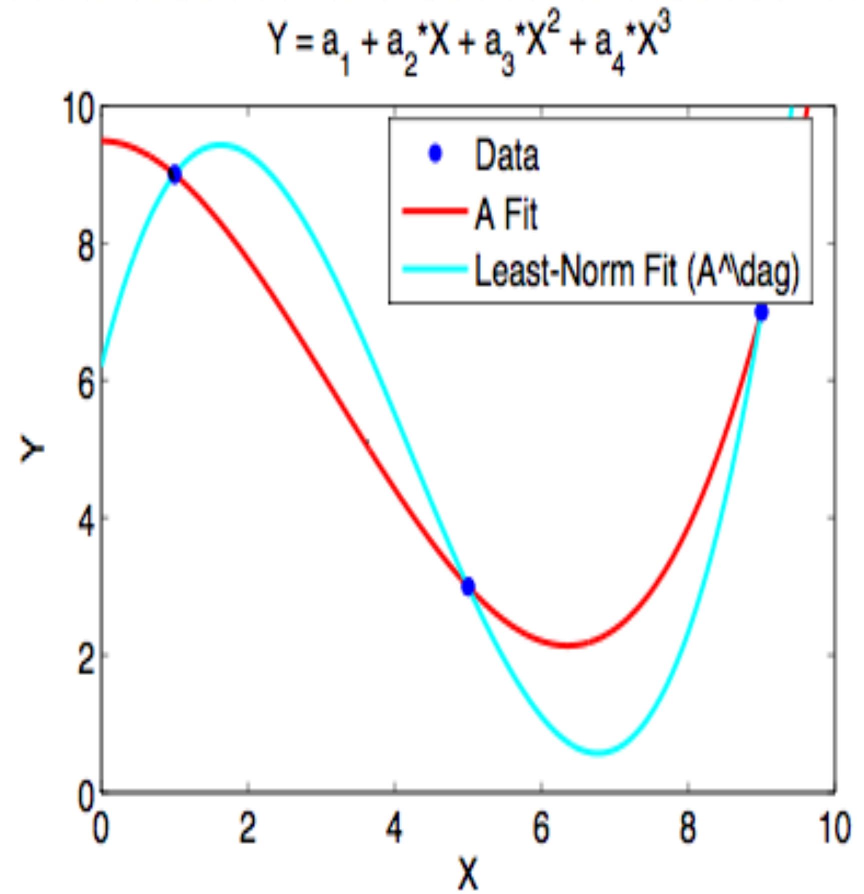
- Minimum Power Application

Problem Description

- m data points (x_i, y_i)

- Questions: Can you fit a polynomial ($y_i = \underbrace{a_0}_{} + \underbrace{a_1}_{} x_i + \underbrace{a_2}_{} x_i^2 \cdots + a_n x_i^n$), with n coefficients, through all the data points?

$m < n$



Underdetermined Problem

- More than enough unknowns to solve the fewer equations ($n > m$)

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ & & \vdots & & \\ 1 & x_m & x_m^2 & \cdots & x_m^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

- Infinite Solutions!

$$\boldsymbol{Ax = b}$$

$$a_0 + a_1 x_i + a_2 x_i^2 + a_3 x_i^3 - y_i = 9$$

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Which Solution is "Best"

- All Residuals are zero

$$r = Ax - b = 0$$

- Choose the solutions with the "smallest" coefficients (i.e. smallest elements of the solution)

$$\begin{array}{ll} \text{minimize} & \|x\|^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2 \\ \text{subject to} & \boxed{Ax = b} \end{array}$$

A closed-form solution!

Matrix Form Solution

$\neq (A^T A)^{-1} A^T$ – least-squares

- Matrix Solution:

$$x^\star = \underbrace{A^T(AA^T)^{-1}}_{A^+}b$$

A^+ – psuedo-inverse

- In Matlab

fast

```
>> x_star = pinv(A)*b;
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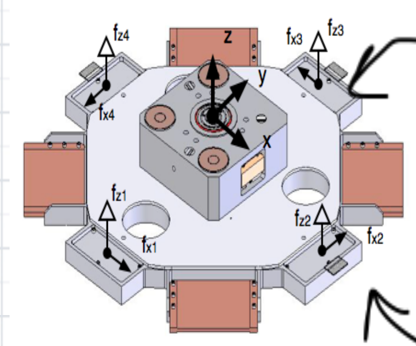
$x^* \neq A \backslash b$

x_star does not equal $\boxed{\texttt{A\textbackslash b}}$ (in general);

Minimum Power Application

Stage



motor

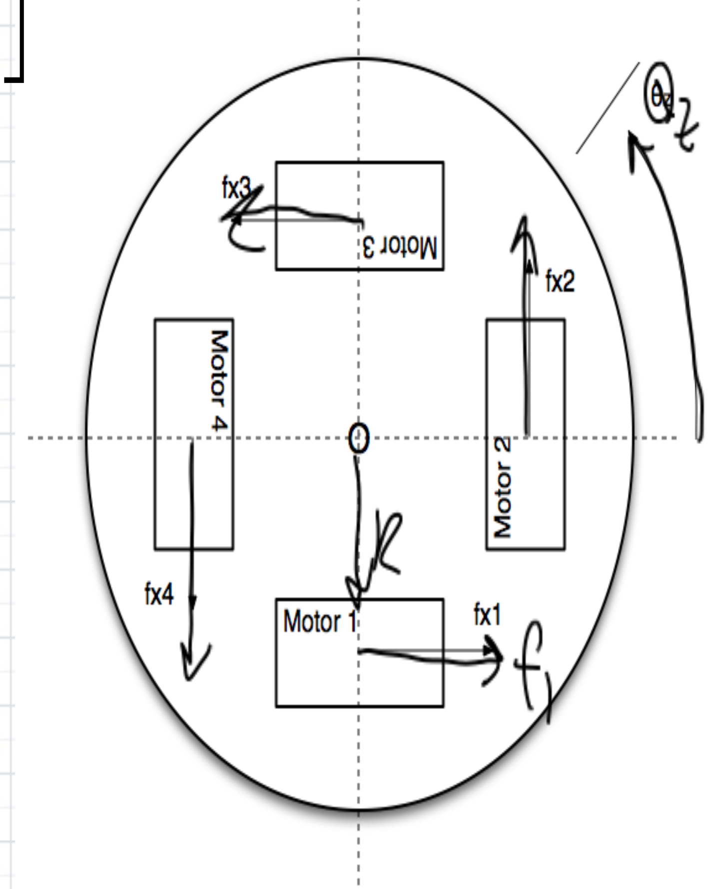
$$\begin{bmatrix} F_x \\ F_y \\ \tau_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ R & R & R & R \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

Global forces

$$\boxed{\boldsymbol{F}^{3\times 1} = A\boldsymbol{f}^{4\times 1}}$$

$$F_x = f_1 - f_3$$

TOP VIEW



y

x

★ Assuming θ is small

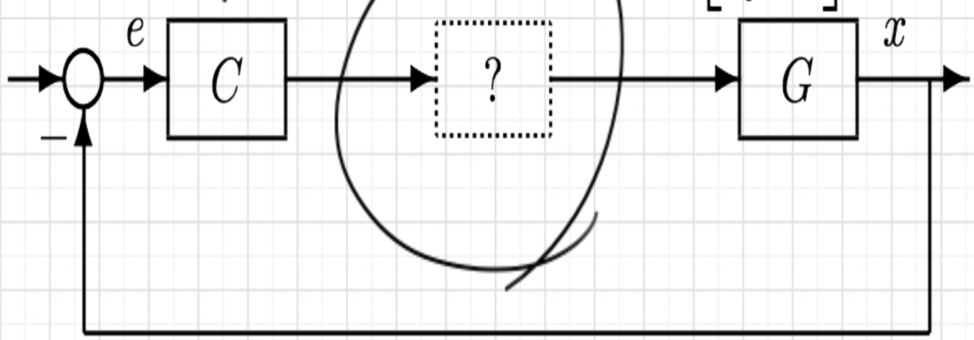
(small angle approximation)

Control Loop

desired

$$r = \begin{bmatrix} x_d \\ y_d \\ \theta_d \end{bmatrix} \qquad \boldsymbol{F}_d = \begin{bmatrix} F_x \\ F_y \\ T \end{bmatrix} \qquad \boldsymbol{f} = \begin{bmatrix} f_{x1} \\ f_{x2} \\ f_{x3} \\ f_{x4} \end{bmatrix}$$

control



3 eqns $\qquad$ 4 unknowns

$F \longrightarrow f$

★ Assume that force ≈ current (electromagnetic)

$\text{Current}^2 \approx \text{Power} \implies \text{force}^2 \approx \text{Power}$

$$\min \; f_1^2 + f_2^2 + f_3^2 + f_4^2 = \|f\|_2^2 = \min \text{Power}$$

$$\text{s.t.} \quad F = \underline{A} f$$

Least-Norm Solution

min $\|f\|_2^2$? min Power

s.t $F_d = Af$

Given

motor 1 $f_1 = \frac{Fx}{2} + \frac{\tau_z}{4R}$

$f_2 = \frac{Fy}{2} + \frac{\tau_z}{4R}$

motor 3 $f_3 = -\frac{Fx}{2} + \frac{\tau_z}{4R}$

$f_4 = -\frac{Fy}{2} + \frac{\tau_z}{4R}$

$f^* = A^+ F_d = \begin{bmatrix} f_1^+ \\ f_2^+ \\ f_3^+ \\ f_3^+ \end{bmatrix} ?$

Null Space of A

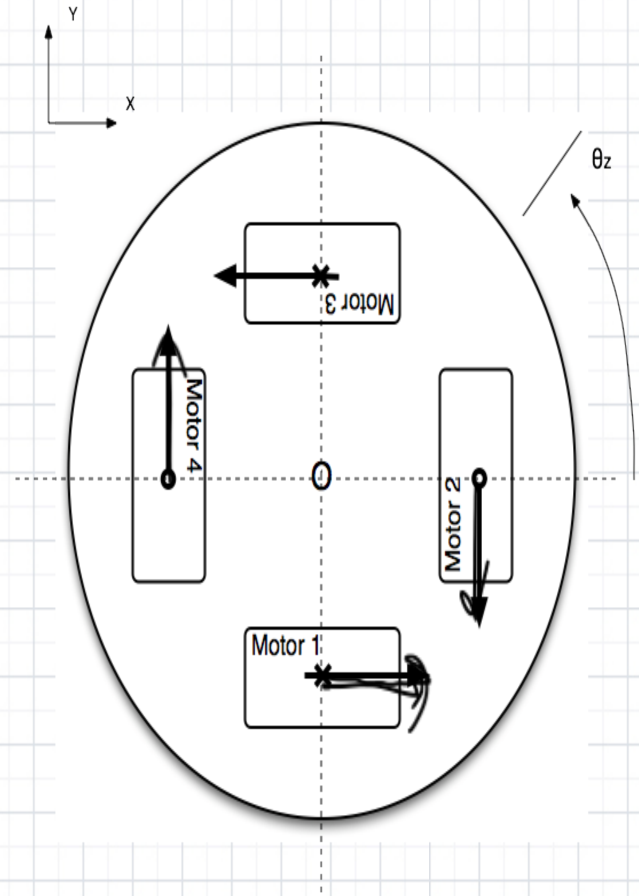
* Why are there an infinite set of solutions!

if $x_n \in N(A)$

$\Rightarrow \quad A x_n = 0 = \text{zero}!$

$$F = Af = A(\overbrace{f + f_N}^{\hat{f} \neq f}) \quad \text{where } f_N \in N(A)$$

$$= Af + \cancelto{0}{Af_N}$$



$f_1 = c$

$f_2 = -c$

$f_3 = c$

$f_4 = -c$

$$Af = 0$$

where $f \neq 0$

$$F_d = Af = A\left(f + \underbrace{c\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}}_{\hat{f}}\right) = Af$$

$$\hat{f} \neq f$$