DUAL LINEAR PROGRAM: WEAK/STRONG DUALITY

Linear Program (Dual)

$$\begin{array}{llcl} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ & A\,\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}$$

Dual Problem

$$\begin{array}{llcl} \min & \mathbf{b}^\intercal \mathbf{y} & & \\ & A^\intercal\,\mathbf{y} & \geq & \mathbf{c} \\ & \mathbf{y} & \geq & 0 \end{array}$$

$$\begin{array}{llcl} \max & -\mathbf{b}^\intercal \mathbf{y} & & \\ & -A^\intercal\,\mathbf{y} & \leq & -\mathbf{c} \\ & \mathbf{y} & \geq & 0 \end{array}$$

Standard Form Converted Dual

Primal vs. Dual

$$\boxed{\begin{array}{llll} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ & A\,\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}}$$

- n variables
- m constraints
- Problem Matrix: A
- Objective vector: c (max)
- RHS of inequality: b (<=)

$$\boxed{\begin{array}{llll} \min & \mathbf{b}^\intercal \mathbf{y} & & \\ & A^\intercal\,\mathbf{y} & \geq & \mathbf{c} \\ & \mathbf{y} & \geq 0 & \end{array}}$$

- m variables
- n constraints
- Problem Matrix: A^T
- Objective Vector: b (min)
- RHS of inequality: c (>=)

Weak Duality Theorem

$$\begin{array}{llll} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ & A\,\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}$$

$$\boxed{\begin{array}{llll} \min & \mathbf{b}^\intercal \mathbf{y} & & \\ & A^\intercal\,\mathbf{y} & \geq & \mathbf{c} \\ & \mathbf{y} & \geq 0 & \end{array}}$$

Let **y** be any dual feasible solution and let **x** be any primal feasible solution.

$$\mathbf{b}^\intercal\,\mathbf{y} \;\geq\; \mathbf{c}^\intercal\mathbf{x}$$

Weak Duality Theorem

Let $\mathbf{y}^*$ be an optimal solution for the dual and $\mathbf{x}^*$ be an optimal solution to the primal.

$$\mathbf{b}^\intercal \, \mathbf{y}^* \;\geq\; \mathbf{c}^\intercal \mathbf{x}^*$$

Another Useful Result

- Let **x** be any primal feasible solution and **y** be any dual feasible solution such that

$$\mathbf{b}^\intercal\ \mathbf{y} = \ \mathbf{c}^\intercal\mathbf{x}$$

  - **x** is primal optimal, and
  - **y** is dual optimal.

Strong Duality

- Let $\mathbf{x}^*$ be a primal optimal solution for an LP.
  - The dual problem has an optimal solution, and
  - Any dual optimal solution $\mathbf{y}^*$ satisfies

$$\mathbf{b}^\intercal \mathbf{y}^* = \mathbf{c}^\intercal \mathbf{x}^*$$

Unboundedness of Primal/Dual

- If Primal unbounded then dual is infeasible.
- If Dual unbounded then primal is infeasible.

Relation between Primal and Dual Problems

	Infeasible	Unbounded	Optimal
Infeasible	Possible	Possible	
Unbounded	Possible		
Optimal			Possible

If Primal is unbounded then dual is infeasible.
If Dual is unbounded then primal is infeasible.

Both problems can be infeasible

Both Primal and Dual Infeasible.

$$
\begin{array}{lcccc}
\max & & x_2 & & \\
& x_1 & & \leq & -1 \\
& & -x_2 & \leq & -1 \\
& x_1 , & x_2 & \geq & 0
\end{array}
$$

$$
\begin{array}{lcccc}
\min & -y_1 & -y_2 & & \\
& y_1 & & \geq & 0 \\
& & -y_2 & \geq & 1 \\
& y_1 , & y_2 & \geq & 0
\end{array}
$$