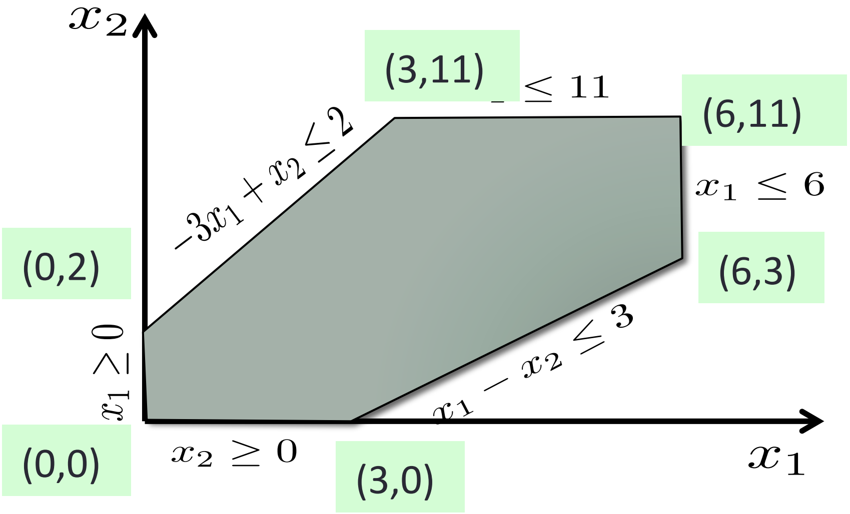
POLYHEDRA: VERTICES

Linear Programming Problem

From Two Weeks Ago.

$$
\begin{array}{lrrcr}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$

Note: Not drawn to scale



Goal: Solve LP using Simplex and visualize!

Active Constraints

$$
\begin{array}{llllll}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$

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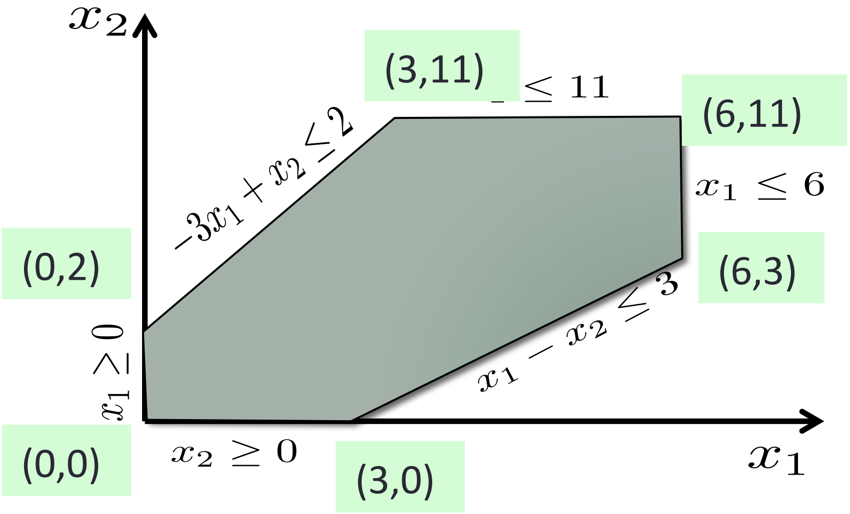
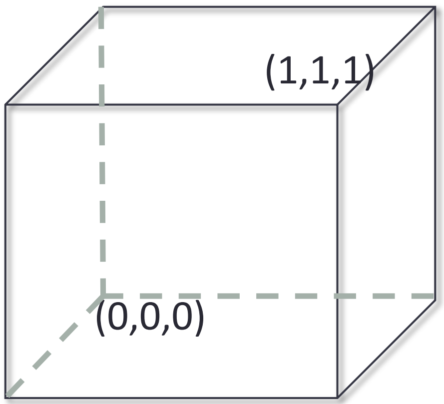


Active Constraints

$$
\begin{aligned}
x_1 &\leq 1 \\
x_2 &\leq 1 \\
x_3 &\leq 1 \\
x_1 &\geq 0 \\
x_2 &\geq 0 \\
x_3 &\geq 0
\end{aligned}
$$



Basic Geometric Facts

- Intersection of 2 lines in 2D yields a point.
  - Lines must be non-parallel.
- Intersection of 3 planes in 3D yields a point.
  - Exclude parallel planes, or other corner cases.
- Intersection of 4 hyper-planes in 4D yields a point.
  - Again, some corner cases.

Intersection of n hyper-planes

$$
\begin{array}{ccccccc}
a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & = & b_1 & \leftarrow \color{red}{\mathcal{H}_1} \\
 & & \ddots & & \vdots & & \\
a_{n1}x_1 & +a_{n2}x_2 & +\cdots+ & a_{nn}x_n & = & b_n & \leftarrow \color{red}{\mathcal{H}_n}
\end{array}
$$

$$
\textbf{rank}\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = n
$$

Vertex (Definition)

A feasible solution **x** to the constraints is a vertex iff

$$\begin{array}{cccccc}
a_{11}x_1 & +a_{12}x_2 & + \cdots + & a_{1n}x_n & \leq & b_1 \\
 & & \ddots & & \vdots & \\
a_{j1}x_1 & +a_{j2}x_2 & + \cdots + & a_{jn}x_n & \leq & b_j \\
 & & \ddots & & \vdots & \\
a_{m1}x_1 & +a_{m2}x_2 & + \cdots + & a_{mn}x_n & \leq & b_m
\end{array}$$

at least n ineqs. are active for x.

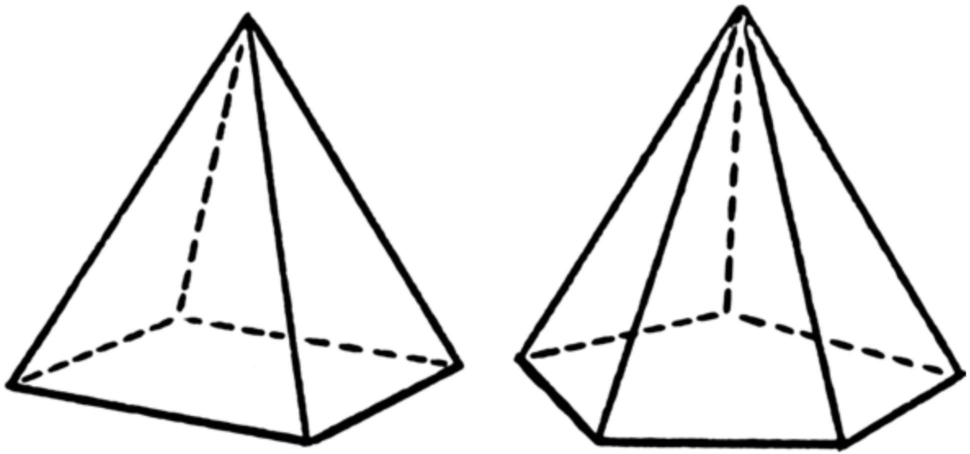
rank of the active constraints for **x** is n

Vertex Issue #1

- Does every point **x** that activates n constraints form a vertex?

Vertex Issue #2

- Can a vertex activate more than n constraints?



Vertex Issue #3

- What if there are more variables than constraints?

Number of Vertices

- n-dimensional hyper cube has 2^n vertices.

- In general, combinatorial explosion of vertices.
  - m constraints, n variables: $\binom{m}{n}$ upper bound on vertices

DICTIONARIES AND VERTICES

Main Message

Dictionaries of Simplex = Vertices of the feasible region.

Linear Programming Problem

From Two Weeks Ago.

$$
\begin{array}{lrrcr}
\text{max.} & x_1 & +2x_2 & & \\
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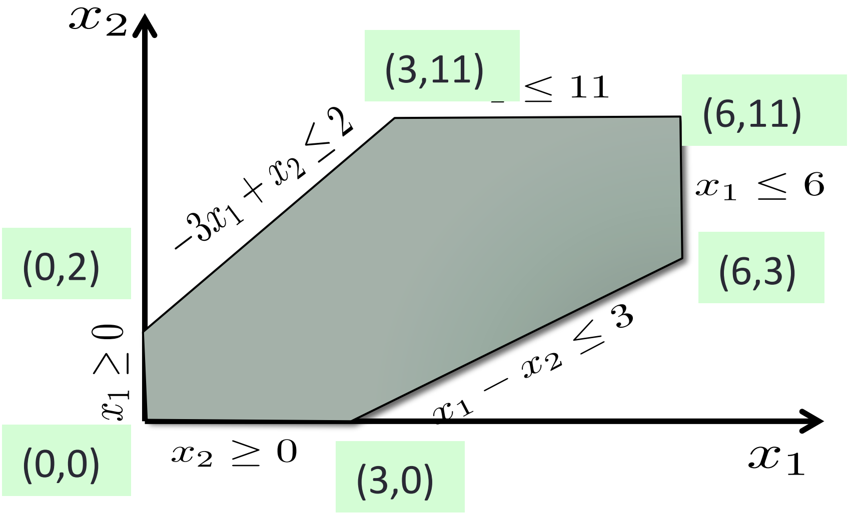
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$

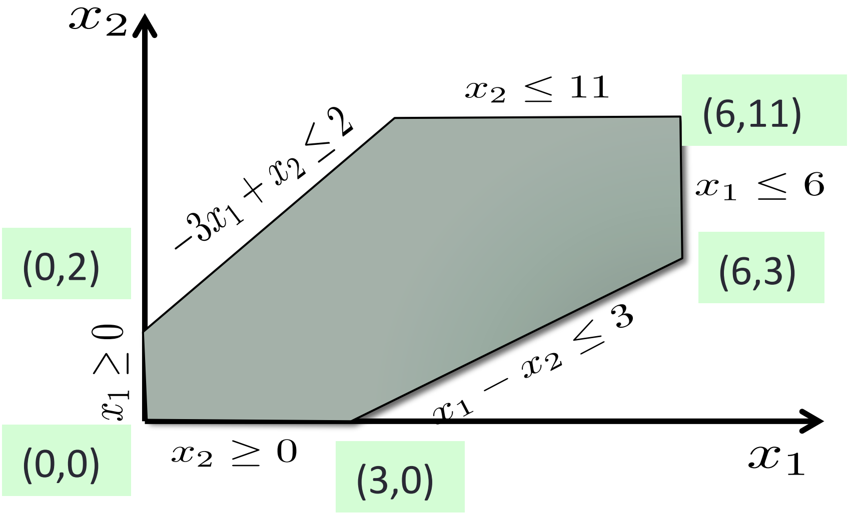
Note: Not drawn to scale



Goal: Solve LP using Simplex and visualize!

Linear Programming Problem

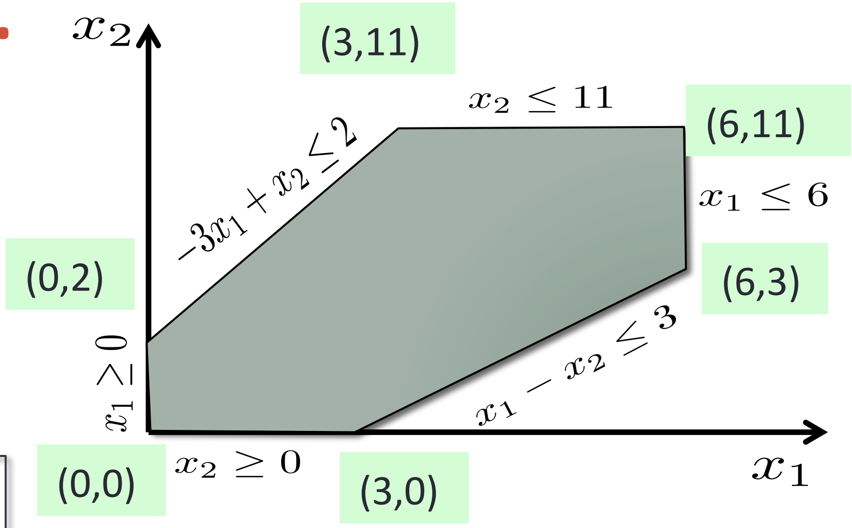
$$
\begin{array}{lrrcr}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$



Dictionary Vertex Corr.

$$
\begin{array}{llllll}
\max & x_1 & +2x_2 & & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 & \leftarrow x_3 \\
 & & x_2 & \leq & 11 & \leftarrow x_4 \\
 & x_1 & -x_2 & \leq & 3 & \leftarrow x_5 \\
 & x_1 & & \leq & 6 & \leftarrow x_6 \\
 & x_1, & x_2 & \geq & 0 &
\end{array}
$$

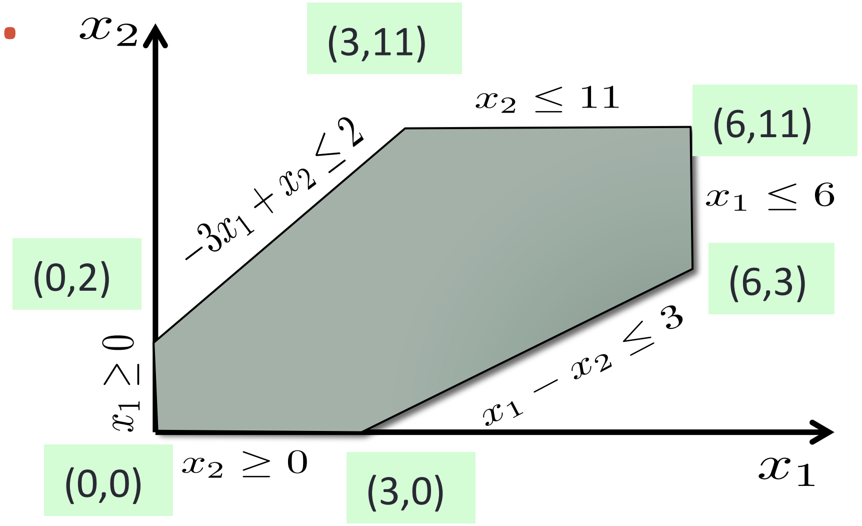
$$
\begin{array}{rcl}
x_1 & = & \\
x_2 & = & \\
x_5 & = & \\
x_6 & = & \\
\hline
z & = & \qquad x_4 \qquad x_3
\end{array}
$$



Dictionary Vertex Corr.

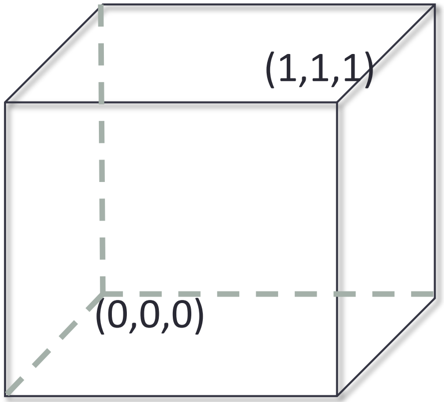
$$
\begin{array}{llllllll}
\max & x_1 & +2x_2 & & & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 & \leftarrow & \color{red}{x_3} \\
 & & x_2 & \leq & 11 & \leftarrow & \color{red}{x_4} \\
 & x_1 & -x_2 & \leq & 3 & \leftarrow & \color{red}{x_5} \\
 & x_1 & & \leq & 6 & \leftarrow & \color{red}{x_6} \\
 & x_1, & x_2 & \geq & 0 & &
\end{array}
$$

$$
\boxed{
\begin{array}{c|lll}
x_1 & \cdots & & \\
x_3 & \cdots & & \\
x_4 & \cdots & & \\
x_6 & \cdots & & \\
\hline
z & ? & ?x_2 & ?x_5
\end{array}
}
$$



Example #3

$$
\begin{array}{lcll}
\max \; x_1 + x_2 - x_3 & & & \\
x_1 & \leq & 1 & \leftarrow x_4 \\
\quad x_2 & \leq & 1 & \leftarrow x_5 \\
\qquad x_3 & \leq & 1 & \leftarrow x_6 \\
x_1 & \geq & 0 & \\
\quad x_2 & \geq & 0 & \\
\qquad x_3 & \geq & 0 &
\end{array}
$$



Linear Programming Problem (Standard Form)

$$\begin{array}{rrcl} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}$$

Feasible Dictionary

$$\begin{array}{ccccccccc} x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} \\ & \vdots & & & & & & \\ x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} \\ \hline z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In} \end{array}$$

(1) Solution associated will make at least n constraints active.

(2) Rank of active constraints is n.

Summary

- Vertex (definition).
  - A feasible point that makes at least n inequalities active.
  - The rank of active inequalities equals n.
- Feasible Dictionaries in Simplex:
  - Solution associated must be a vertex of the feasible region.
- What does pivoting do?

PIVOTING AND VERTICES

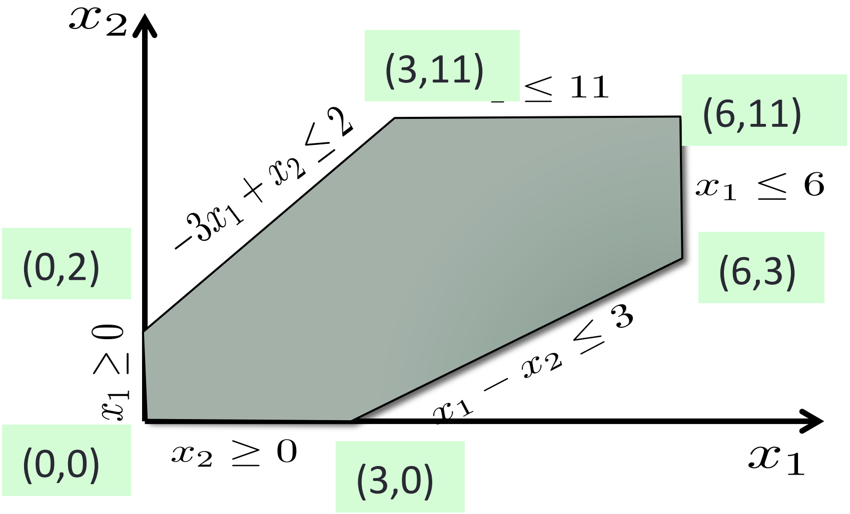
What happens when we pivot?

- Entering variable leaves non-basic set.
  - Leaving variable becomes non-basic.

Adjacent Vertices

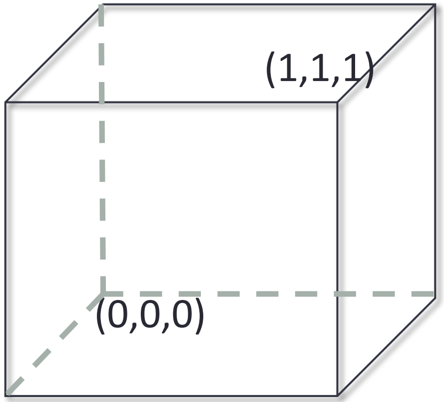
$$
\begin{array}{llllll}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$

Note: Not drawn to scale



Example #2: Adjacent Vertices

$$
\begin{array}{ccccc}
x_1 & & & \leq & 1 \\
& x_2 & & \leq & 1 \\
& & x_3 & \leq & 1 \\
x_1 & & & \geq & 0 \\
& x_2 & & \geq & 0 \\
& & x_3 & \geq & 0
\end{array}
$$



Adjacent Vertices

Definition: Two vertices are adjacent if and only if

- At least (n-1) active constraints are common.
- Rank of common active constraints is (n-1).

$$
\begin{array}{cccccc}
a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 \\
 & & \ddots & & \vdots & \\
a_{j1}x_1 & +a_{j2}x_2 & +\cdots+ & a_{jn}x_n & \leq & b_j \\
 & & \ddots & & \vdots & \\
a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m
\end{array}
$$

Active for
both vertices

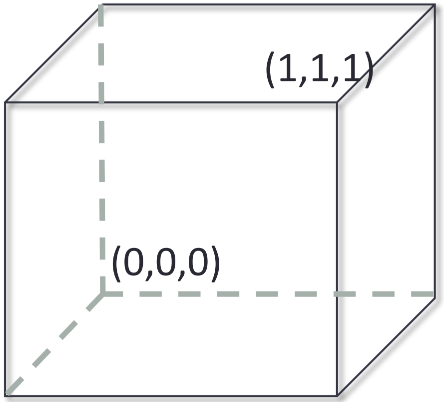
Claim

For non-degenerate/non-final dictionary D_1 if D_2 is obtained on pivot, then the vertices corr. to D_1 and D_2 are adjacent.

$$\begin{array}{c|cc} x_{B1} & \mathbf{b}_1 & \cdots \\ \hline z & c_0 & +c_{N1}x_{N1} \end{array} \qquad \begin{array}{c|cc} x_{B2} & \mathbf{b}_2 & \cdots \\ \hline z & c_2 & +c_{N2}x_{N1} \end{array}$$

Simplex Pivoting Visualization

$$
\begin{aligned}
\max \quad x_1 + x_2 - x_3 & \\
x_1 \qquad\qquad & \leq 1 \\
x_2 \qquad & \leq 1 \\
x_3 & \leq 1 \\
x_1 \qquad\qquad & \geq 0 \\
x_2 \qquad & \geq 0 \\
x_3 & \geq 0
\end{aligned}
$$



Pivoting Issues

- Does pivoting always move to an adjacent vertex?
  - Yes, if the current dictionary is non-degenerate.
- What happens in the degenerate case?
  - Case-1: Move to an adjacent vertex.
  - Case-2: Remain in the same vertex (?)
- What happens if a dictionary is unbounded?

DEGENERATE POLYHEDRA

Degenerate Dictionaries

$$
\begin{array}{rcrrr}
x_1 & = & 3 & -\frac{1}{3}x_4 & +\frac{1}{3}x_3 \\
x_2 & = & 11 & -x_4 & +0x_3 \\
x_5 & = & 11 & -\frac{2}{3}x_4 & -\frac{1}{3}x_3 \\
x_6 & = & 0 & +\frac{1}{3}x_4 & -\frac{1}{3}x_3 \\
\hline
z & = & 25 & -\frac{7}{3}x_4 & +\frac{1}{3}x_3
\end{array}
$$

1. Understand geometry of degeneracy
2. Highly degenerate polyhedra.

Vertex (Definition)

A feasible solution **x** to the constraints is a vertex iff

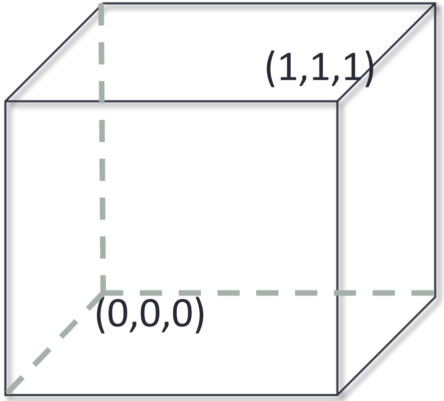
$$
\begin{array}{cccccc}
a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 \\
 & & \ddots & & \vdots & \\
a_{j1}x_1 & +a_{j2}x_2 & +\cdots+ & a_{jn}x_n & \leq & b_j \\
 & & \ddots & & \vdots & \\
a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m
\end{array}
$$

at least n ineqs.
are active for x.

rank of the active constraints for **x** is n

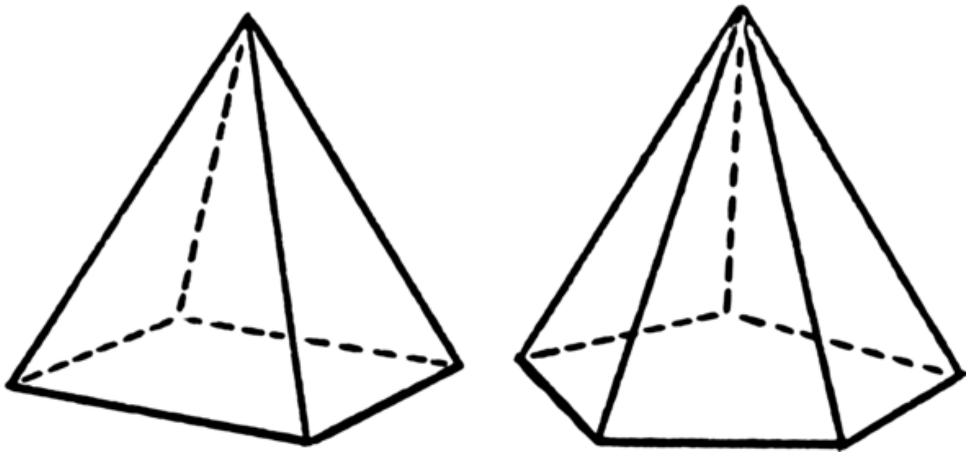
Vertices and Active Constraints

$$\begin{array}{rcl} x_1 & \le & 1 \\ x_2 & \le & 1 \\ x_3 & \le & 1 \\ x_1 & \ge & 0 \\ x_2 & \ge & 0 \\ x_3 & \ge & 0 \end{array}$$

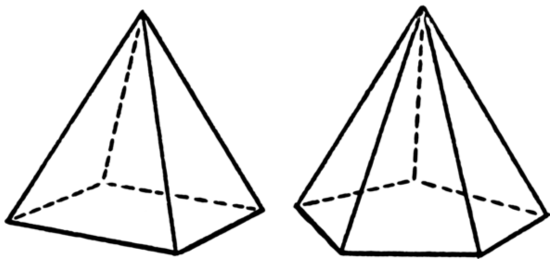


Vertex Issue #2

- Can a vertex activate more than n constraints?



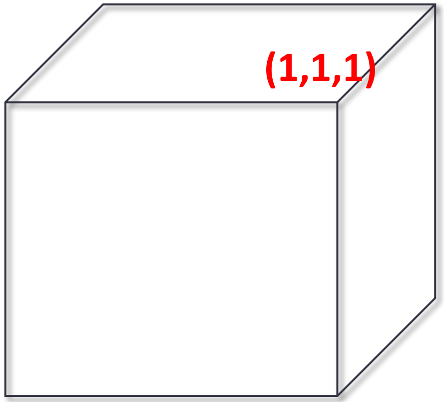
Degenerate Vertex (Definition)



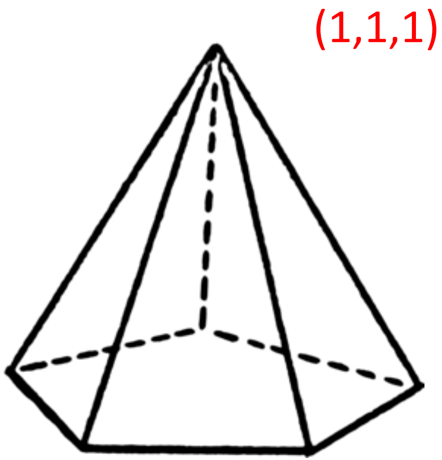
Vertex **x** is degenerate iff

it activates n+k constraints for k > 0

Degenerate vs. Non-degenerate vertex



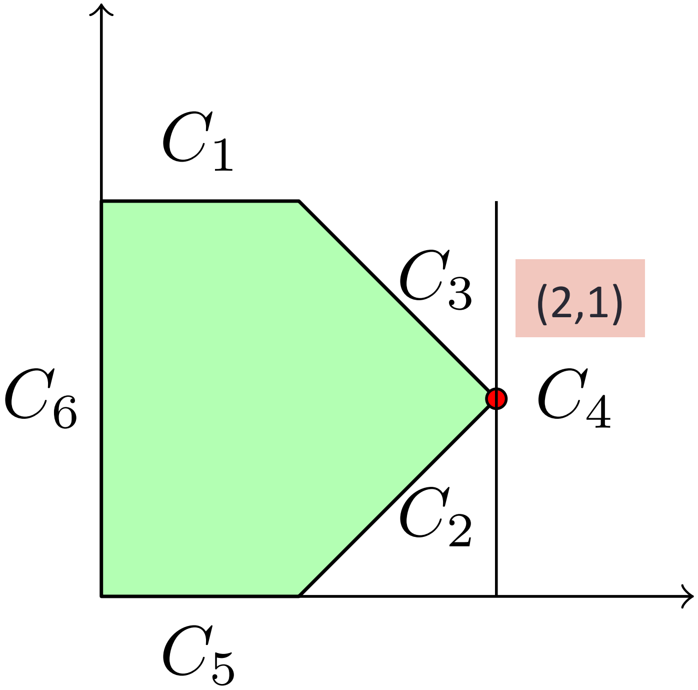
Non-degenerate:
- Activates exactly n constraints.
- Exactly n faces meet at the vertex.
- Unique dictionary associated with vertex.



Degenerate:
- Activates n + k constraints (k > 0)
- More than n faces meet at the vertex.
- Multiple dictionaries associated with vertex.

Degeneracy due redundancy

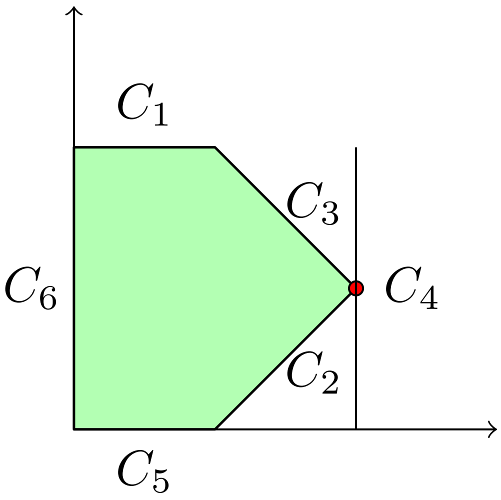
$$
\begin{array}{lrrcl}
\max & x & & & \\
\text{s.t.} & & & & \\
C_1: & & y & \leq & 2 \\
C_2: & x & -y & \leq & 1 \\
C_3: & x & +y & \leq & 3 \\
C_4: & x & & \leq & 2 \\
C_5: & -x & & \leq & 0 \\
C_6: & & -y & \leq & 0
\end{array}
$$



Degeneracy due to redundancy

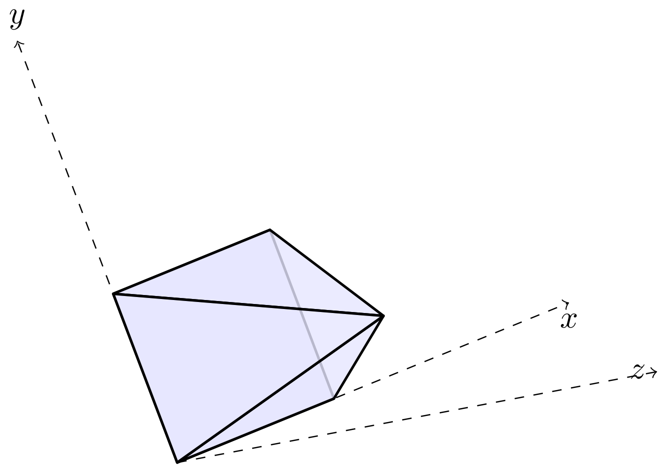
$$\begin{array}{c|cll}
w_1 & 1 & -\frac{1}{2}w_2 & +\frac{1}{2}w_3 \\
x & 2 & -\frac{1}{2}w_2 & -\frac{1}{2}w_3 \\
y & 1 & +\frac{1}{2}w_2 & -\frac{1}{2}w_3 \\
w_4 & 0 & +\frac{1}{2}w_2 & +\frac{1}{2}w_3 \\
\hline
z & 2 & -\frac{1}{2}w_2 & -\frac{1}{2}w_3
\end{array}$$

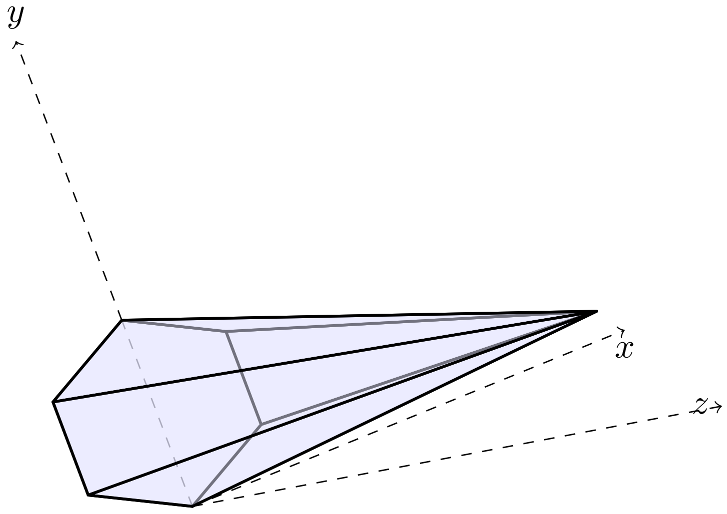
$$\begin{array}{c|cll}
w_1 & 1 & -w_2 & +w_4 \\
x & 2 & & -w_4 \\
y & 1 & +w_2 & -w_4 \\
w_3 & 0 & -w_2 & +2w_4 \\
\hline
z & 2 & & -w_4
\end{array}$$



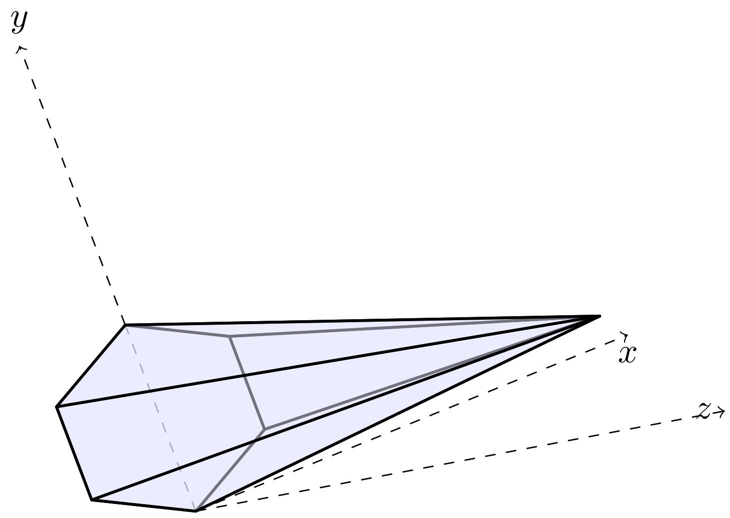
Degeneracy without redundancy

Removing any of the constraints changes feasible region.





Simplex over Degenerate Polyhedra



UNBOUNDED POLYHEDRA: RAYS

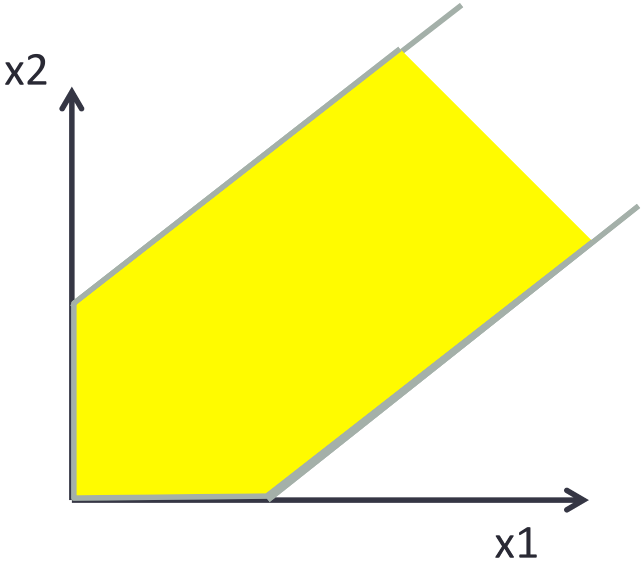
Thus far...

- Feasible Region: Polyhedra
- Vertices:
  - Activate at l
  - Acti
- Vertice
- Simplex ertex.
  - Degene generate

Unbounded Problems.

Unbounded Linear Programs and Rays

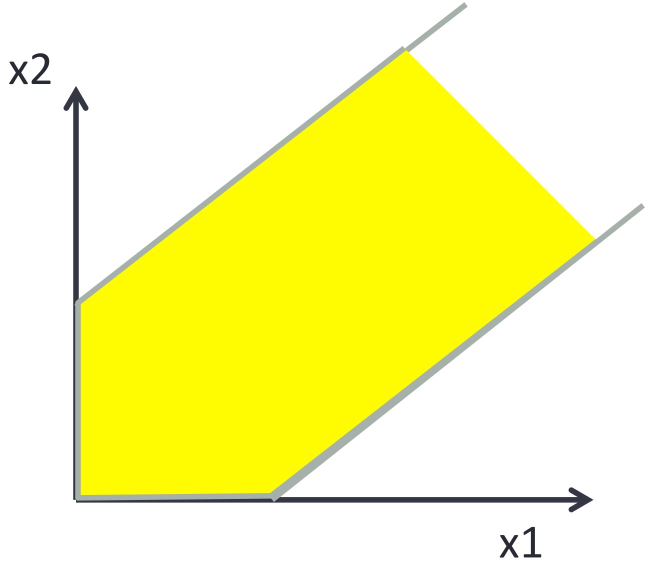
$$\begin{array}{rllll} \max & x_1 & & & \\ \text{s.t.} & x_1 & -x_2 & \leq & 1 \\ & -x_1 & +x_2 & \leq & 1 \\ & x_1 & , x_2 & \geq 0 & \end{array}$$



Ray

Vector $\mathbf{r}$ is a ray of polyhedron P iff for every $\mathbf{x} \in P$ and every $\lambda \geq 0$,

$$\mathbf{x} + \lambda \mathbf{r} \in P$$



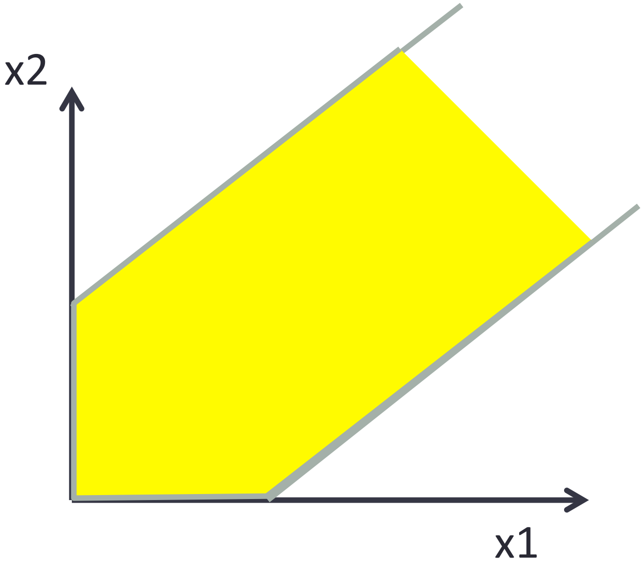
Ray (Fundamental Property)

Polyhedron: $A\mathbf{x} \leq \mathbf{b}$

$\mathbf{r}$ is a ray if and only if $A\mathbf{r} \leq \mathbf{0}$

Ray (Fundamental Property)

$$\begin{array}{llllll} \max & x_1 & & & \\ \text{s.t.} & x_1 & -x_2 & \leq & 1 \\ & -x_1 & +x_2 & \leq & 1 \\ & x_1 & , x_2 & \geq 0 & \end{array}$$



Is (1,1) a ray of this polyhedron?

Example

$$
\begin{array}{rrcl}
\text{maximize} & 2x_1 + 3x_2 - 5x_3 & & \\
\text{s.t.} & x_1 - x_2 & \leq & 5 \\
& -x_1 + \quad\quad x_3 & \leq & 6 \\
& -2x_1 + \quad\quad x_3 & \leq & 2 \\
& -x_1 + x_2 & \leq & 4 \\
& x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Second Dictionary

$$
\begin{array}{rcrcrcrcr}
x_2 & = & 4 & + & x_1 & - & x_7 & & \\
x_4 & = & 9 & & & - & x_7 & & \\
x_5 & = & 6 & + & x_1 & & & - & x_3 \\
x_6 & = & 2 & + & 2x_1 & & & - & x_3 \\
\hline
z & = & 12 & + & 5x_1 & - & 3x_7 & - & 5x_3
\end{array}
$$

Unbounded Dictionary

- No leaving variables.
- Alternatively: all entries in the column corr. to entering variables are non-negative.

$$
\begin{array}{cccccccc}
x_{B1} & = & b_1 & +a_{11}x_{I1} & \cdots & +{\color{red}a_{1j}}x_{Ij} & \cdots \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & \cdots & +{\color{red}a_{2j}}x_{Ij} & \cdots \\
 & \vdots & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & \cdots & +{\color{red}a_{mj}}x_{Ij} & \cdots \\
\hline
z & = & c_0 & +c_1x_{I1} & \cdots & +{\color{red}c_j}x_{Ij} & \cdots
\end{array}
$$

Unbounded Dictionary and Ray

$$
\begin{array}{cccccccc}
x_{B1} & = & b_1 & +a_{11}x_{I1} & \cdots & +a_{1j}x_{Ij} & \cdots \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & \cdots & +a_{2j}x_{Ij} & \cdots \\
 & \vdots & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & \cdots & +a_{mj}x_{Ij} & \cdots \\
\hline
z & = & c_0 & +c_1x_{I1} & \cdots & +c_jx_{Ij} & \cdots
\end{array}
$$