LP Equivalent: ℓ_1 - Norm Approximation

- ℓ_1 - Norm: $||\boldsymbol{y}||_1 = \sum_i |y_i| = \sum_i \max(-y_i, y_i)$ = $\max(-y_1, y_1) + \max(-y_2, y_2)$ ...

$f_1(x)$ $f_2(x)$ ...

t_1 t_2

- Fitting/Approximation Problem:

$$\text{minimize} \quad ||A\boldsymbol{x} - \boldsymbol{b}||_1$$

$A \in \mathbb{R}^{m \times n}$

- LP Equivalent:
  - -- Need to add an auxiliary vector: $\boldsymbol{t} = [\, t_1 \;\; t_2 \;\; \cdots \;\; t_m \,]^T$

m - # of Constraint equations

$$\begin{aligned} \text{minimize} \quad & t_1 + t_2 + \cdots t_n \\ \text{subject to} \quad & A\boldsymbol{x} - \boldsymbol{b} \leq \boldsymbol{t} \\ & -(A\boldsymbol{x} - \boldsymbol{b}) \geq \boldsymbol{t} \end{aligned}$$

$-Ax + b \leq t$ $\leq$

$a_1^T x - b_1 \leq t_1$

$a_2^T x - b_1 \leq t_2$

$\vdots$

LP Equivalent: ℓ_1 - Norm Approximation

$$
\begin{array}{ll}
\text{minimize} & t_1 + t_2 + \cdots t_n = \mathbb{1}^T t \\
\text{subject to} & Ax - b \leq t \quad \longrightarrow \quad Ax - It \leq b \\
& -(Ax - b) \cancel{\geq} t \quad \leq \quad \longrightarrow \quad -Ax - It \leq -b
\end{array}
$$

$$\hat{x} = \begin{bmatrix} x \\ t \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} \vec{0} & \mathbb{1} \end{bmatrix}}_{\hat{c}^T} \begin{bmatrix} x \\ t \end{bmatrix} = \mathbb{1}^T t = t_1 + t_2 + \cdots$$

$$\overbrace{\begin{bmatrix} A & -I \\ -A & -I \end{bmatrix}}^{\hat{A}} \overbrace{\begin{bmatrix} x \\ t \end{bmatrix}}^{\hat{x}} \leq \overbrace{\begin{bmatrix} b \\ -b \end{bmatrix}}^{\hat{b}}$$

$$\boxed{\begin{array}{ll} \min & \hat{c}^T \hat{x} \\ \text{s.t.} & \hat{A}\hat{x} \leq \hat{b} \end{array}}$$

LP Equivalent: ℓ_1 - Norm Approximation

- Matrix Form:

$$
\begin{array}{ll}
\text{minimize} & \begin{bmatrix} \mathbf{0} & \mathbf{1} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{t} \end{bmatrix} \\
\text{subject to} & \begin{bmatrix} A & -I \\ -A & -I \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{t} \end{bmatrix} \leq \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix}
\end{array}
$$