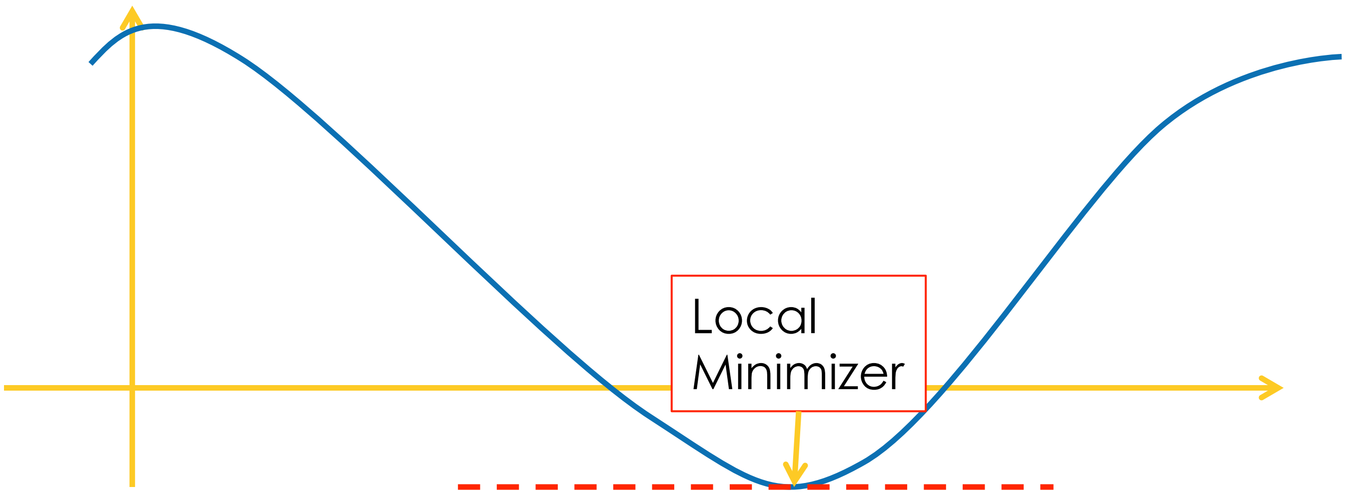
Newton Method for Optimization

- Goal: minimize function F(x) for all x.
- Unconstrained minimization problem.



Minimization of smooth function.

$$F : \mathbb{R}^n \rightarrow \mathbb{R}$$

- F is a C^2 function.
- Continuous, first and second derivatives.

If $\mathbf{x} \in \mathbb{R}^n$ is a local minimizer of F then $\nabla F = 0$

First-Order Necessary Conditions

If $\nabla F(\mathbf{x}) = 0$ and $\nabla^2 F$ is positive definite at $\mathbf{x}$, then $\mathbf{x}$ is an isolated local minimum of F.

Second-Order Sufficient Condition

Newton method for finding minima

$$F : \mathbb{R}^n \to \mathbb{R}$$

$$\boxed{\nabla F(\mathbf{x}) = \begin{pmatrix} \frac{\partial F}{\partial x_1} \\ \vdots \\ \frac{\partial F}{\partial x_n} \end{pmatrix} = 0}$$

Solve

$$\text{Newton Step: } \Delta = \boxed{-(\nabla^2 F)^{-1}}(\nabla F)$$

Hessian Matrix

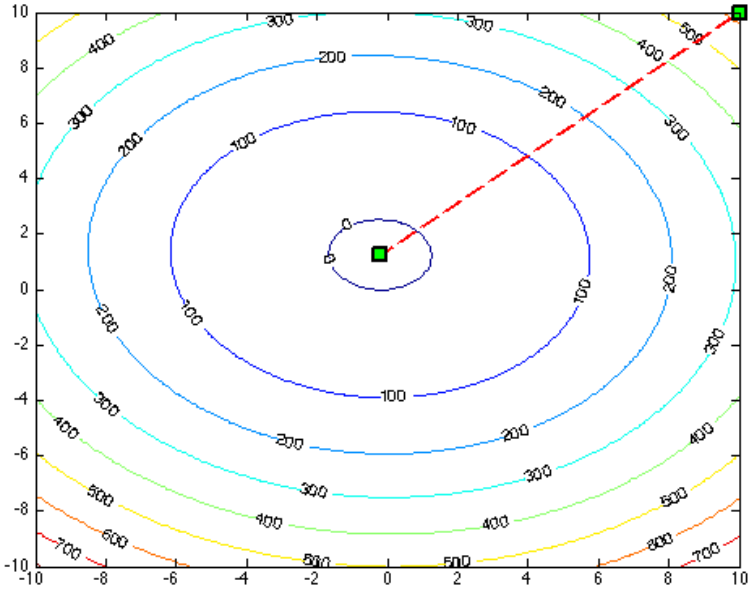
Hessian Matrix

$$\nabla^2 F = \begin{bmatrix} \frac{\partial^2 F}{\partial \ x_1^2} & \frac{\partial^2 F}{\partial \ x_1 x_2} & \cdots & \frac{\partial^2 F}{\partial \ x_1 x_n} \\ \frac{\partial^2 F}{\partial \ x_2 x_1} & \frac{\partial^2 F}{\partial \ x_2^2} & \cdots & \frac{\partial^2 F}{\partial \ x_2 x_n} \\ \vdots & & & \vdots \\ \frac{\partial^2 F}{\partial \ x_n x_1} & \frac{\partial^2 F}{\partial \ x_n x_2} & \cdots & \frac{\partial^2 F}{\partial \ x_n^2} \end{bmatrix}$$

Does inverse always exist?

$$\text{Newton Step: } \Delta = -(\nabla^2 F)^{-1}(\nabla F)$$

Newton's Method Example



$$\min_{(x,y)}(3x^2 + 4y^2 + 0.2xy + x - 10y)$$