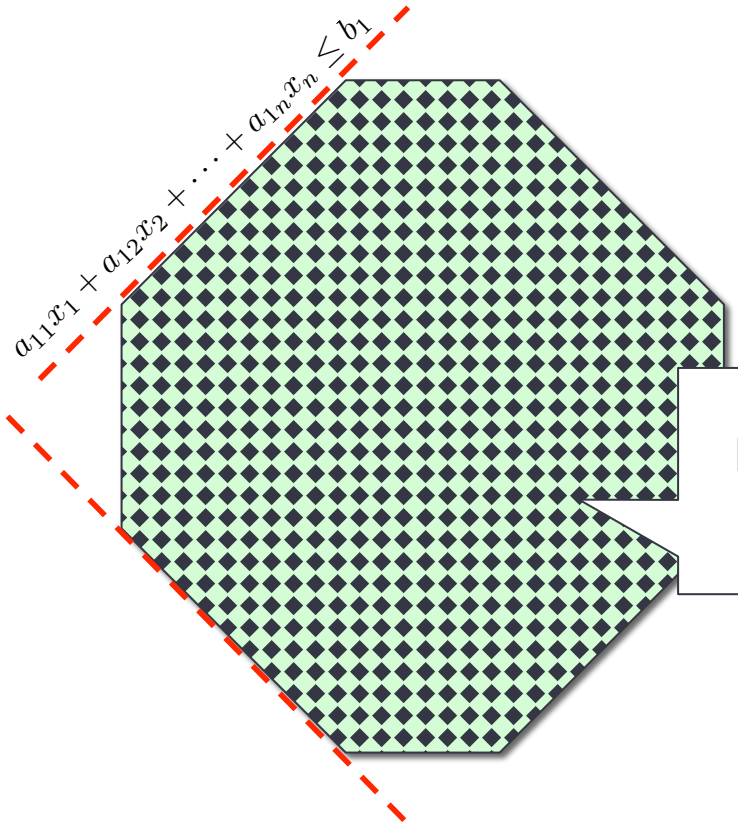
INTEGER LINEAR PROGRAMMING: LP RELAXATION

1. Relax an ILP to an LP
2. Examples with same answers and different answers.
3. Integrality gap.

Integer Linear Programming



$$
\begin{array}{lccccccc}
\max & c_1x_1 & +c_2x_2 & +\cdots+ & c_nx_n & & \\
\text{s.t.} & a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 \\
 & & & \ddots & & \vdots & \\
 & a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m \\
 & x_1, \ldots, x_n & \in & \mathbb{Z} & & &
\end{array}
$$

Feasible Region: Z-Polyhedron
(n dimensional)

Integer Linear Program

$$
\begin{array}{lccccccc}
\max & c_1x_1 & +c_2x_2 & +\cdots+ & c_nx_n & & \\
\text{s.t.} & a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 \\
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 & a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m \\
 & & & & x_1,\ldots,x_n & \in & \mathbb{Z}
\end{array}
$$

- Feasibility of ILP:
  - *Integer feasible solution.*

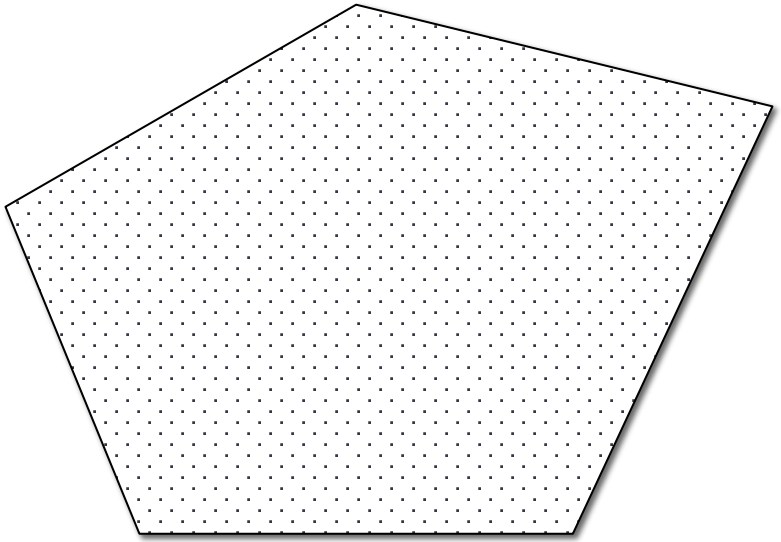
- Unbounded ILP:
  - Integer feasible solutions can achieve arbitrarily large values for the objective.

Linear Programming Relaxation

$$
\begin{array}{lccccccc}
\max & c_1x_1 & +c_2x_2 & +\cdots+ & c_nx_n & & \\
\text{s.t.} & a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 \\
 & & & \ddots & & \vdots & \\
 & a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m \\
 & x_1,\ldots,x_n & \in & \mathbb{Z} & & &
\end{array}
$$

Q: What happens to the answer if we take away the integrality constraints?

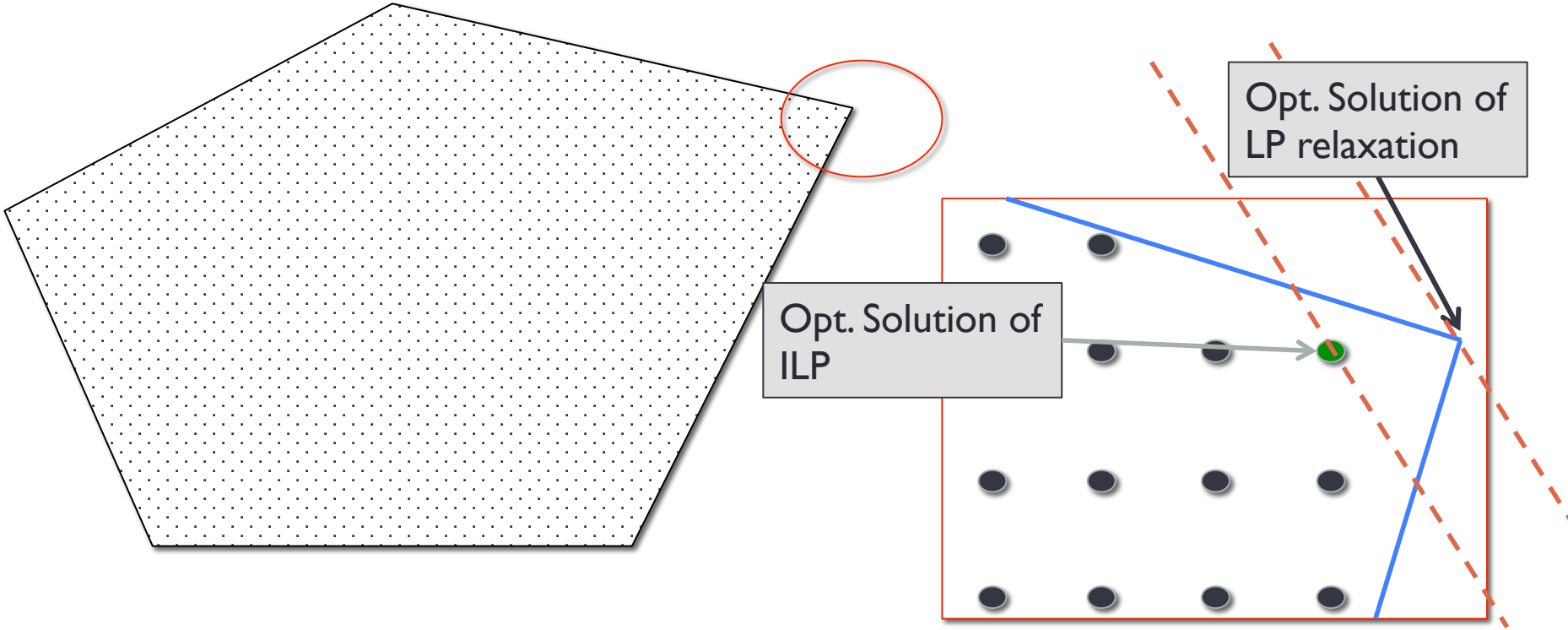
Feasible Regions



$$
\begin{array}{lccccc}
\text{max} & c_1x_1 & +c_2x_2 & +\cdots+ & c_nx_n & \\
\text{s.t.} & a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq b_1 \\
 & & & \ddots & & \vdots \\
 & a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq b_m \\
 & & & x_1,\ldots,x_n & \in \mathbb{Z} &
\end{array}
$$

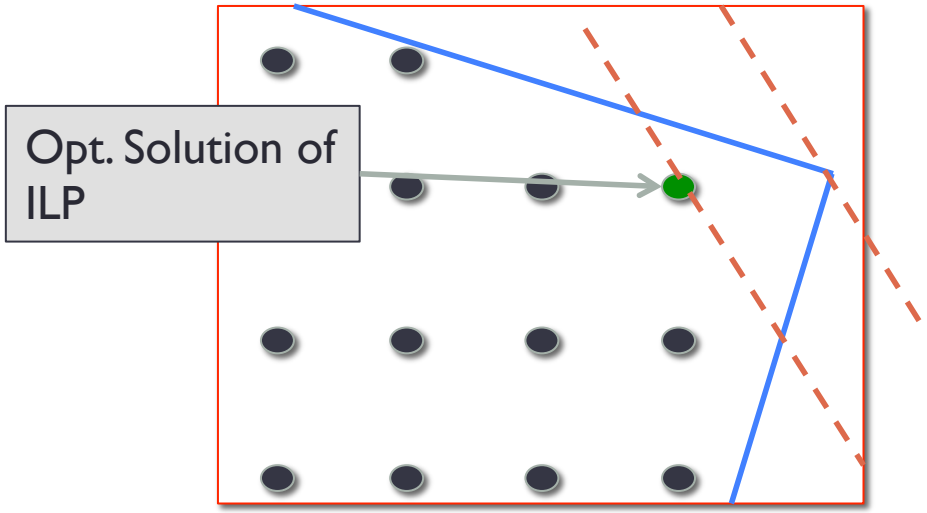
ILP feasible region $\subseteq$ LP feasible region

Case-1: Both LP and ILP are feasible.



Case-I

Optimal Objective of ILP ≤ Optimal solution of LP relaxation.



Example-1

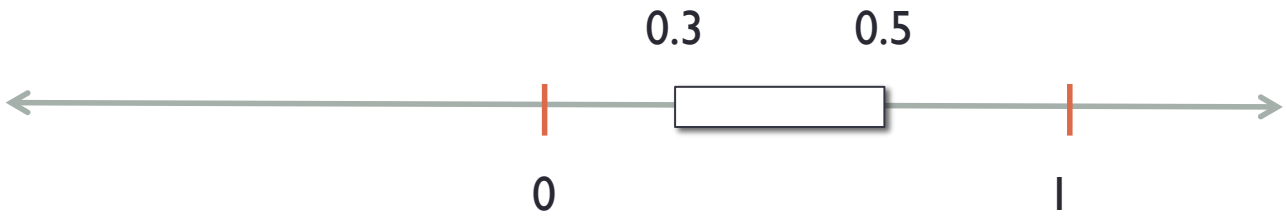
Example-2

Case-II: LP relaxation is feasible, ILP is infeasible.

$$\begin{aligned} \max \quad & x \\ \text{s.t.} \quad & \\ 3 \leq \quad & 10x \quad \leq 5 \end{aligned}$$

ILP is infeasible.

LP relaxation has optimal solution: 0.5

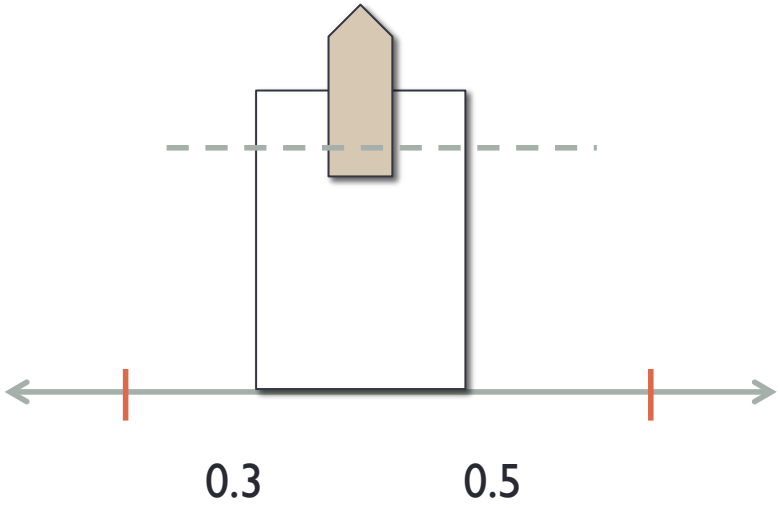


Case III: ILP is infeasible, LP is unbounded.

Example:

$$\begin{array}{lll} \max & y & \\ 3 \leq & 10x & \leq 5 \\ 0 \leq & y & \end{array}$$

ILP is infeasible.
LP relaxation is unbounded



ILP outcomes vs. LP relaxation outcomes

Integer Linear Program (ILP)

LP Relaxation

	Infeasible	**Unbounded**	**Optimal**
Infeasible	Possible	Impossible	Impossible
Unbounded	Possible	Possible	Possible (*)
Optimal	Possible	Impossible	Possible

(*) Impossible if ILP has rational coefficients

Summary (LP relaxation)

- LP relaxation: ILP minus the integrality constraints.
- LP relaxation's feasible region is a super-set of ILP feasible region.
- Analysis of various outcomes for ILP vs. outcomes for LP relaxations.