DICTIONARY RECONSTRUCTION

Surgery ☺

Challenge

$$\begin{array}{lrcl} \max & \mathbf{c}^\intercal\ \mathbf{x} & & \\ & A\mathbf{x} + \mathbf{x_s} & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x_s} & \geq & 0 \end{array}$$

$$B = \{x_{b_1}, \ldots, x_{b_m}\}$$

$$\begin{array}{rcl} \mathbf{x_B} & = & \mathbf{p} + R\ \mathbf{x_I} \\ \hline z & = & e_0 + \mathbf{e}^\intercal \mathbf{x_I} \end{array}$$

Reconstruct Dictionary
Given B

Example Problem

$$
\begin{array}{lrrcr}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$

Original Problem

$$
\begin{array}{lrrrcr}
\text{max.} & x_1 & +2x_2 & & & \\
\text{s.t.} & -3x_1 & +x_2 & +x_3 & = & 2 \\
 & & +x_2 & +x_4 & = & 11 \\
 & x_1 & -x_2 & +x_5 & = & 3 \\
 & x_1 & & +x_6 & = & 6 \\
 & \multicolumn{3}{r}{x_1, x_2, x_3, \ldots, x_6} & \geq & 0
\end{array}
$$

Problem with Slack

Example

$$\boxed{\begin{array}{lrrrrr} \text{max.} & x_1 & +2x_2 & & & \\ \text{s.t.} & -3x_1 & +x_2 & \color{red}{+x_3} & = & 2 \\ & & +x_2 & \color{red}{+x_4} & = & 11 \\ & x_1 & -x_2 & \color{red}{+x_5} & = & 3 \\ & x_1 & & \color{red}{+x_6} & = & 6 \\ & & \multicolumn{2}{r}{x_1, x_2, \color{red}{x_3, \ldots, x_6}} & \geq & 0 \end{array}}$$

$$B = \{1, 2, 5, 6\}.$$

Example (Desired Goal)

$$
\begin{array}{rcrrr}
x_1 & = & ? & +?x_3 & +?x_4 \\
x_2 & = & ? & +?x_3 & +?x_4 \\
x_5 & = & ? & +?x_3 & +?x_4 \\
x_6 & = & ? & +?x_3 & +?x_4 \\
\hline
z & = & ? & +?x_3 & +?x_4
\end{array}
$$

$$
\boxed{\begin{array}{rrrrcr}
\text{max.} & x_1 & +2x_2 & & & \\
\text{s.t.} & -3x_1 & +x_2 & \color{red}{+x_3} & = & 2 \\
 & & +x_2 & \color{red}{+x_4} & = & 11 \\
 & x_1 & -x_2 & \color{red}{+x_5} & = & 3 \\
 & x_1 & & \color{red}{+x_6} & = & 6 \\
 & \multicolumn{3}{r}{x_1, x_2, \color{red}{x_3, \ldots, x_6}} & \geq & 0
\end{array}}
$$

$$B = \{1, 2, 5, 6\}.$$

Key Principle

Dictionary is just a restatement of the problem.

We express basic variables in terms of non-basic.

Dict. Reconstruction (Step 1)

$$B = \{1, 2, 5, 6\}.$$

$$\begin{array}{lrrrrrl}
\text{max.} & x_1 & +2x_2 & & & & \\
\text{s.t.} & -3x_1 & +x_2 & +x_3 & = & 2 \\
 & & +x_2 & +x_4 & = & 11 \\
 & x_1 & -x_2 & +x_5 & = & 3 \\
 & x_1 & & +x_6 & = & 6 \\
 & & x_1, x_2, & x_3, \ldots, x_6 & \geq & 0
\end{array}$$

$$\begin{array}{rrrrcrrr}
-3x_1 & +x_2 & & & = & 2 & -x_3 & \\
 & x_2 & & & = & 11 & & -x_4 \\
x_1 & -x_2 & +x_5 & & = & 3 & & \\
x_1 & & & +x_6 & = & 6 & &
\end{array}$$

Dict. Reconstruction (Step 2)

$$\boxed{\begin{array}{cccccccc} -3x_1 & +x_2 & & & = & 2 & -x_3 & \\ & x_2 & & & = & 11 & & -x_4 \\ x_1 & -x_2 & +x_5 & & = & 3 & & \\ x_1 & & & +x_6 & = & 6 & & \end{array}}$$

$$\begin{pmatrix} -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 2 \\ 11 \\ 3 \\ 6 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

Dict Reconstruction (Step 3)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}^{-1} \left[\begin{pmatrix} 2 \\ 11 \\ 3 \\ 6 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \right]$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & -\frac{1}{3} & 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 2 \\ 11 \\ 3 \\ 6 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \right]$$

Dict. Reconstruction

$$\begin{pmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ 11 \\ 3 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ 0 & -1 \\ \frac{-1}{3} & \frac{-2}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{pmatrix} \times \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

Dictionary Reconstruction (Final Result)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ 11 \\ 3 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ 0 & -1 \\ \frac{-1}{3} & \frac{-2}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{pmatrix} \times \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

$$\begin{array}{cclll} x_1 & = & 3 & +\frac{1}{3}x_3 & -\frac{1}{3}x_4 \\ x_2 & = & 11 & +0x_3 & -x_4 \\ x_5 & = & 11 & -\frac{1}{3}x_3 & -\frac{2}{3}x_4 \\ x_6 & = & 3 & -\frac{1}{3}x_3 & +\frac{1}{3}x_4 \\ \hline z & = & & & \end{array}$$

$$//\text{ORIG: } x_1 + 2x_2$$

Dict. Reconstruction

$$\boxed{\begin{array}{lccc} \max & \mathbf{c}^\intercal\ \mathbf{x} & & \\ & A\mathbf{x} + \mathbf{x_s} & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x_s} & \geq & 0 \end{array}}$$

$$\text{Basis set: } B = \{x_{b1}, \ldots, x_{bm}\}.$$

Splitting the A matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1,b1} & \cdots & a_{1,b2} & \cdots & a_{1,bm} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,b1} & \cdots & a_{2,b2} & \cdots & a_{2,bm} & \cdots & a_{2n} \\ & \vdots & & \vdots & & \ddots & & & \\ a_{m1} & \cdots & a_{m,b1} & \cdots & a_{m,b2} & \cdots & a_{m,bm} & \cdots & a_{mn} \end{pmatrix} \times \begin{pmatrix} x_1 \\ \vdots \\ x_{b1} \\ \vdots \\ x_{b2} \\ \vdots \\ x_{bm} \\ \vdots \\ x_m \end{pmatrix}$$

$$A\mathbf{x} = A_B\mathbf{x_B} + A_I\mathbf{x_I}$$

Rewriting the Equation

$$A\mathbf{x} + \mathbf{x_s} = \mathbf{b}$$

Step 1

$$A_B\mathbf{x_B} + A_I\mathbf{x_I} = b$$

Step 2

$$A_B\ \mathbf{x_B} = \mathbf{b} - A_I\mathbf{x_I}$$

Is A_B always invertible?

Dictionary Reconstruction

$$A_B\ \mathbf{x_B} = \mathbf{b} - A_I \mathbf{x_I}$$

$$\Downarrow$$

$$\mathbf{x_B} = A_B^{-1}\mathbf{b} - A_B^{-1} A_I \mathbf{x_I}$$

Is A_B always invertible?

Result Dictionary

$$\mathbf{c}^\intercal \ \mathbf{x} = \mathbf{c_B}^\intercal \mathbf{x_B} + \mathbf{c_I}^\intercal \mathbf{x_I}$$

$$\begin{array}{ccccc} \mathbf{x_B} & = & A_B^{-1}\mathbf{b} & & -A_B^{-1}A_I\mathbf{x_I} \\ \hline \mathbf{c} & = & \mathbf{c_B}^\intercal \ A_B^{-1}\mathbf{b} & + & \left(-\mathbf{c_B}^\intercal A_B^{-1}A_I + \mathbf{c_I}^\intercal\right) \ \mathbf{x_I} \end{array}$$

Claim #1

Given Original Problem +
Desired Basis

Can reconstruct dictionary uniquely.

Key Insight # 2

For any given problem, number of possible dictionaries is finite.

$$\#\text{Dictionaries} \;\leq\; \#\text{Basis Set}$$

$$\#\text{Basis Set} \;=\; \binom{n+m}{m}$$