CSCI5654 (Linear Programming, Fall 2013)
Lecture-2

Today's Lecture

- Overview of Linear Programming.
- Some initial observations.
- Standard Form for LP (Chvátal, ch. 1).
- Simplex Basics (Chvátal, ch. 2).

Linear Program

Optimization Problem: Linear (affine) objectives and constraints.

$$
\begin{array}{rrcl}
\text{maximize} & c_1x_1 + \ldots + c_nx_n + c_0 & & \\
\text{subj.to.} & a_{11}x_1 + \ldots + a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 + \ldots + a_{2n}x_n & \leq & b_2 \\
 & \ddots & \vdots & \\
 & a_{m1}x_1 + \ldots + a_{mn}x_n & \leq & b_n
\end{array}
$$

Decision Variables: $x_1, \ldots, x_n$.

decision variables take real number values.

Example #1

Consider the following linear program:

$$\begin{array}{rlcr} \text{maximize} & 5x - y & & \\ \text{s.t.} & x & \geq & 0 \\ & x & \leq & 2 \\ & y & \geq & -1 \\ & y & \leq & 5 \end{array}$$

Solution: $x = 2, y = -1$, yields optimal value of objective $z = 11$.

Q: Can there be multiple optima?

Problems with Multiple Optima

$$\begin{array}{rccc} \text{maximize} & 5x & & \\ \text{s.t.} & x & \geq & 0 \\ & x & \leq & 2 \\ & y & \geq & -1 \\ & y & \leq & 5 \end{array}$$

Optimal Solutions: There are infinitely many optima:

x	y	objective(z)
2	-1	10
2	$-.5$	10
2	0	10
$\vdots$		$\vdots$
2	5	10

This is quite common as the number of decision variables increases.

Example #2

$$\begin{array}{rlcr} \text{maximize} & 5x - y & & \\ \text{s.t.} & x & \geq & 3 \\ & x & \leq & 2 \\ & y & \geq & -1 \\ & y & \leq & 5 \end{array}$$

Outcome: The problem is "infeasible". Constraints on decision variables contradict each other.

Example #3

$$
\begin{array}{rlcr}
\text{maximize} & 5x - y & & \\
\text{s.t.} & x & \geq & 3 \\
 & y & \geq & -1 \\
 & y & \leq & 5
\end{array}
$$

Outcome: The problem is "unbounded".

For every number N, there is a feasible solution

with objective value $z \geq N$.

Outcomes for Linear Program

Every LP can result in one of three outcomes:

- **Optimal** Feasible solution $\vec{x}$ maximizes the objective function. Such a solution is an <u>optimal solution</u>. (multiple optimal solutions possible).
- **Unbounded** For every N, there is <u>feasible</u> solution $\vec{x}$, such that objective function value is $\geq N$.
  Note: Some texts denote unbounded maximum as ∞.
  unbounded minimum as $-\infty$.
- **Infeasible** No feasible solution exists.
  Note: Some texts and solvers assign a value of $-\infty$ for infeasible maximum ($+\infty$ for infeasible minimum).

Standard Form

Every LP we will consider will be of the following standard form:

$$\begin{array}{rrcl} \text{maximize} & c_1x_1 + \ldots + c_nx_n + c_0 & & \\ \text{subj.to.} & a_{11}x_1 + \ldots + a_{1n}x_n & \leq & b_1 \\ & a_{21}x_1 + \ldots + a_{2n}x_n & \leq & b_2 \\ & \ddots & \vdots & \\ & a_{m1}x_1 + \ldots + a_{mn}x_n & \leq & b_m \\ & x_1, x_2, \ldots, x_n & \geq & 0 \end{array}$$

Matrix Notation:

$$\begin{array}{rrcl} \text{maximize} & \vec{c} \cdot \vec{x} & & \\ \text{s.t.} & A\vec{x} & \leq & \vec{b} \\ & \vec{x} & \geq & \vec{0} \end{array}$$

Standard Form

Result: Any LP can be rewritten as a standard form LP.

$$\begin{tikzcd}
\boxed{\text{Original Problem}} \arrow[r, bend left] & \boxed{\text{Standard Form}} \arrow[l, bend left]
\end{tikzcd}$$

Standard Form Conversion

Transform given LP into standard form.
Objective:

$$\text{minimize } (\vec{c} \cdot \vec{x}) \quad \rightarrow \quad \text{maximize } -(\vec{c} \cdot \vec{x})\,.$$

Constraints: Transform equalities.

$$\vec{a} \cdot \vec{x} = b \;\; \rightarrow \;\; \left(\begin{array}{ccc} \vec{a} \cdot \vec{x} & \leq & b \\ -\vec{a} \cdot \vec{x} & \leq & -b \end{array} \right)\,.$$

Variables: If $x_i \geq 0$ does not appear as a constraint,

$$\text{Rewrite: } x_i = x_i^+ - x_i^-\,, \text{ wherein, } x_i^+, x_i^- \geq 0\,.$$

Example

Original LP: Transform this to standard form.

$$\begin{array}{rcrl} \min & -5x_1 + 4x_2 - 3x_3 & & \\ \text{s.t.} & 2x_1 - 3x_2 + x_3 & = & 5 \\ & 4x_1 + x_2 + 2x_3 & \geq & 11 \\ & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\ & x_1 & \geq & 0 \end{array}$$

Steps for Conversion:

1. Transform objective from min to max.

$$\min \; -5x_1 + 4x_2 - 3x_3 \quad \to \quad \max 5x_1 - 4x_2 + 3x_3$$

2. Change equalities to inequalities:

$$2x_1 - 3x_2 + x_3 = 5 \;\; \to \;\; \left(\begin{array}{l} 2x_1 - 3x_2 + x_3 \leq 5 \\ -2x_1 + 3x_2 - x_3 \leq -5 \end{array} \right)$$

3. Change $\geq$ constraints by $\leq$:

$$4x_1 + x_2 + 2x_3 \geq 11 \quad \to \quad -4x_1 - x_2 - 2x_3 \leq -11$$

4. x_2, x_3 can be negative. Need to transform:

$$x_2 \mapsto x_2^+ - x_2^- \quad x_3 \mapsto x_3^+ - x_3^- \,.$$

Resulting LP:

$$
\begin{array}{lrcr}
\max & 5x_1 - 4x_2^+ + 4x_2^- + 3x_3^+ - 3x_3^- & & \\
\text{s.t.} & 2x_1 - 3x_2^+ + 3x_2^- + x_3^+ - x_3^- & \leq & 5 \\
 & -2x_1 + 3x_2^+ - 3x_2^- - x_3^+ + x_3^- & \leq & -5 \\
 & -4x_1 - x_2^+ + x_2^- - 2x_3^+ + 2x_3^- & \leq & -11 \\
 & 3x_1 + 4x_2 - 4x_2^- + 2x_3^+ - 2x_3^- & \leq & 8 \\
 & x_1, x_2^+, x_2^-, x_3^+, x_3^- & \geq & 0
\end{array}
$$

Transform Solutions Back:

$$
\begin{array}{c}
(x_1 = 3.43, x_2^+ = 0.143, x_2^- = 0, x_3^+ = 0, x_3^- = 1.43) \\
\downarrow \\
(x_1 = 3.43, x_2 = 0.143, x_3 = -1.43)
\end{array}
$$

Slack Variables

Basic Idea: Rewrite inequality of the form

$$a_1x_1 + \cdots + a_nx_n \leq b_n$$

into equality

$$x_{n+i} = b_n - a_1x_1 - a_2x_2 - \cdots - a_nx_n ,$$

where x_{n+i} is a new <u>slack variable</u>.

Note: We add constraint $x_{n+i} \geq 0$.

Standard Form

Every LP we will consider will be of the following standard form:

$$
\begin{array}{rrcl}
\text{maximize} & c_1 x_1 + \ldots + c_n x_n + c_0 & & \\
\text{subj.to.} & a_{11} x_1 + \ldots + a_{1n} x_n & \leq & b_1 \\
& a_{21} x_1 + \ldots + a_{2n} x_n & \leq & b_2 \\
& \ddots & \vdots & \\
& a_{m1} x_1 + \ldots + a_{mn} x_n & \leq & b_n \\
& x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

Matrix Notation:

$$
\begin{array}{rrcl}
\text{maximize} & \vec{c} \cdot \vec{x} & & \\
\text{s.t.} & A\vec{x} & \leq & \vec{b} \\
& \vec{x} & \geq & \vec{0}
\end{array}
$$

Transformation using Slack Variables

$$
\begin{array}{rccc}
\text{maximize} & c_1 x_1 + \ldots + c_n x_n + c_0 & & \\
\text{subj.to.} & x_{n+1} & = & b_1 - a_{11} x_1 - \ldots - a_{1n} x_n \\
 & \cdots & \cdots & \\
 & x_{n+m} & = & b_n - a_{m1} x_1 + \ldots - a_{mn} x_n \\
 & x_1, x_2, \ldots, x_n & \geq & 0 \\
 & x_{n+1}, \ldots, x_{n+m} & \geq & 0
\end{array}
$$

Matrix Notation (constraints):

$$
\begin{array}{rcl}
\vec{x}_{\text{slack}} & = & \vec{b} - A \cdot \vec{x}_{\text{decision}} \\
\vec{x}_{\text{decision}} & \geq & 0 \\
\vec{x}_{\text{slack}} & \geq & 0
\end{array}
$$

Simplex Method

Overall Strategy.

Concepts: Basic feasible solution, dictionary, pivoting.

1. Discover the first <u>basic feasible solution</u> (initial dictionary).
2. <u>Pivot</u> from one basis (basic feasible solution) until optimum achieved.
3. Each <u>pivot</u> increases the value of the objective function.

Example (from Chvátal, ch. 2)

Solve the following LP:

$$
\begin{array}{rrcl}
\text{max.} & 5x_1 + 4x_2 + 3x_3 & & \\
\text{s.t.} & 2x_1 + 3x_2 + x_3 & \leq & 5 \\
 & 4x_1 + x_2 + 2x_3 & \leq & 11 \\
 & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\
 & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Introduce Slack Variables

$$\begin{array}{rcl} x_4 & = & 5 - 2x_1 - 3x_2 - x_3 \\ x_5 & = & 11 - 4x_1 - x_2 - 2x_3 \\ x_6 & = & 8 - 3x_1 - 4x_2 - 2x_3 \\ \hline z & = & 5x_1 + 4x_2 + 3x_3 \end{array}$$

Independent Variables: x_1, x_2, x_3 (on the RHS of equalities).
Basic Variables: x_4, x_5, x_6 on the LHS of equalities.
Basic Feasible Solution: Set independent variables to zero.

$$\underbrace{x_1 = 0,\ x_2 = 0,\ x_3 = 0}_{independent},\ \underbrace{x_4 = 5, x_5 = 11, x_6 = 8}_{dependent(basic)}.$$

Note: This does not work in <u>all cases</u> (more about <u>initialization</u>, next week).

Dictionary

A general way to organize our problem and solution.

$$
\begin{array}{rcl}
x_4 & = & 5 - 2x_1 - 3x_2 - x_3 \\
x_5 & = & 11 - 4x_1 - x_2 - 2x_3 \\
x_6 & = & 8 - 3x_1 - 4x_2 - 2x_3 \\
\hline
z & = & 5x_1 + 4x_2 + 3x_3
\end{array}
$$

Associated Solution: Every dictionary yields an associated basic feasible solution.

1. Set all independent variables to 0.
2. Set all dependent variables appropriately.

Initial Dictionary

$$
\begin{array}{rcl}
\color{red}{x_4} & = & \underline{5} & - & 2x_1 - 3x_2 - x_3 \\
\color{red}{x_5} & = & \underline{11} & - & 4x_1 - x_2 - 2x_3 \\
\color{red}{x_6} & = & \underline{8} & - & 3x_1 - 4x_2 - 2x_3 \\
\hline
\color{blue}{z} & = & 0 & + & 5x_1 + 4x_2 + 3x_3
\end{array}
$$

$$
\underbrace{x_1 = 0,\ x_2 = 0,\ x_3 = 0}_{basic(independent)},\ \underbrace{x_4 = 5,\ x_5 = 11,\ x_6 = 8}_{dependent}.
$$

Dictionary Objective Value: $z = 0$.

Q: How can we improve the optimum?
Remember: We like z to be as large as pssible

Pivoting

Entering Variable: x_1.

Q: How high can we set x_1?

$$
\begin{array}{rcl}
x_4 & = & 5 - 2x_1 - 3x_2 - x_3 & \Rightarrow & \underline{x_1 \leq \frac{5}{2}} \\
x_5 & = & 11 - 4x_1 - x_2 - 2x_3 & \Rightarrow & x_1 \leq \frac{11}{4} \\
x_6 & = & 8 - 3x_1 - 4x_2 - 2x_3 & \Rightarrow & x_1 \leq \frac{8}{3} \\
\hline
z & = & 0 + 5x_1 + 4x_2 + 3x_3 & &
\end{array}
$$

Improved Solution:

$$x_1 = \frac{5}{2}, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 1, x_6 = \frac{1}{2}$$

Improved Objective: $z = \frac{25}{2}$.

Q: What happens if we try $x_1 = \frac{8}{3}$?

New Dictionary

Replacement Rule: $x_1 \mapsto \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4$.
After pivot: x_1 enters basis, x_4 leaves basis.
New Dictionary:

$$\begin{array}{rcl} x_1 & = & \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_5 & = & 1 + 5x_2 \qquad\qquad + 2x_4 \\ x_6 & = & \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \\ \hline z & = & \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4 \end{array}$$

Property: <u>This is exactly the same as the original problem</u>.

We are just viewing it differently.

New Dictionary

$$
\begin{array}{rcl}
x_1 & = & \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\
x_5 & = & 1 + 5x_2 \qquad\qquad + 2x_4 \\
x_6 & = & \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \\
\hline
z & = & \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4
\end{array}
$$

Basic Variables: x_1, x_5, x_6.

Basic Feasible Solution: $x_1 = \frac{5}{2}, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 1, x_6 = \frac{1}{2}$

Objective Value: $z = \frac{25}{2}$.

Pivoting

Entering Variable: x_3.

$$
\begin{array}{rcll}
x_1 & = & \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 & \Rightarrow \quad \underline{x_3 \leq 5} \\
x_5 & = & 1 + 5x_2 \qquad\qquad + 2x_4 & \Rightarrow \quad \text{no constraint} \\
x_6 & = & \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 & \Rightarrow \quad \underline{x_3 \leq 1}
\end{array}
$$

New Solution: $x_1 = 2, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = 1, x_6 = 0.$

Leaving Variable: x_6.

Pivoting II

Replacement Rule: $x_3 \mapsto (1 + x_2 + 3x_4 - 2x_6)$.

$$\begin{array}{rcl} x_3 & = & 1 + x_2 + 3x_4 - 2x_6 \\ x_1 & = & 2 - 2x_2 - 2x_4 + x_6 \\ x_5 & = & 1 + 5x_2 + 2x_4 \\ \hline z & = & 13 - 3x_2 - x_4 - x_6 \end{array}$$

Solution: $x_1 = 2, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = 1, x_6 = 0$.

Objective Value: $z = 13$.

Q: Can we improve the objective function value?

Simplex

1. Convert to standard form.
2. Introduce Slack Variables.
3. Find initial feasible solution and corr. dictionary.
4. While dictionary is not final:
   4.1 Find a non-basic variable x_i to improve.
   4.2 Find a basic variable x_j that constrains x_i the most.
   4.3 Insert x_i into basis and remove x_j from basis.
5. Output final result.

Next Lecture

We will go over Simplex in detail, again.

1. Go over Simplex steps at a higher level.
2. Initialization.
3. Choosing entering variable.
4. Degeneracy.
5. Cycling.
6. Lexicographic rule.