LINEAR AND INTEGER PROGRAMMING

The Simplex Method: A tutorial in three acts.

Three to Simplex

1. The Standard Form Linear Program.
2. “Dictionaries”.
3. Pivoting.

Putting it all together in an example.

ACT I: THE STANDARD FORM.

Revision of material covered under LP formulations.

Some words of wisdom

If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem?

- George Pólya (How to Solve It?)

George Pólya
(source: mactutor)

Linear Program

$$
\begin{array}{rlcl}
\text{maximize} & c_1x_1 + \ldots + c_nx_n & & \\
\text{subj.to.} & a_{11}x_1 + \ldots + a_{1n}x_n & \leq & b_1 \\
& a_{21}x_1 + \ldots + a_{2n}x_n & \leq & b_2 \\
& \ddots & \vdots & \\
& a_{m1}x_1 + \ldots + a_{mn}x_n & \leq & b_m
\end{array}
$$

Linear Program in Matrix Form

$$\begin{array}{llcl} \text{maximize} & c_1x_1 + \ldots + c_nx_n & & \\ \text{subj.to.} & a_{11}x_1 + \ldots + a_{1n}x_n & \leq & b_1 \\ & a_{21}x_1 + \ldots + a_{2n}x_n & \leq & b_2 \\ & \ddots & \vdots & \\ & a_{m1}x_1 + \ldots + a_{mn}x_n & \leq & b_m \end{array}$$

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^\intercal\ \mathbf{x} \\ \text{subj.to} & A\ \mathbf{x} \leq \mathbf{b} \end{array}$$

Standard Form (Definition)

$$
\begin{array}{rrcl}
\text{maximize} & c_1x_1 + \ldots + c_nx_n & & \\
\text{subj.to.} & a_{11}x_1 + \ldots + a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 + \ldots + a_{2n}x_n & \leq & b_2 \\
 & \ddots & \vdots & \\
 & a_{m1}x_1 + \ldots + a_{mn}x_n & \leq & b_m \\
 & \color{red}{x_1, x_2, \ldots, x_n} & \color{red}{\geq} & \color{red}{0}
\end{array}
$$

Standard Form (Matrix Notation)

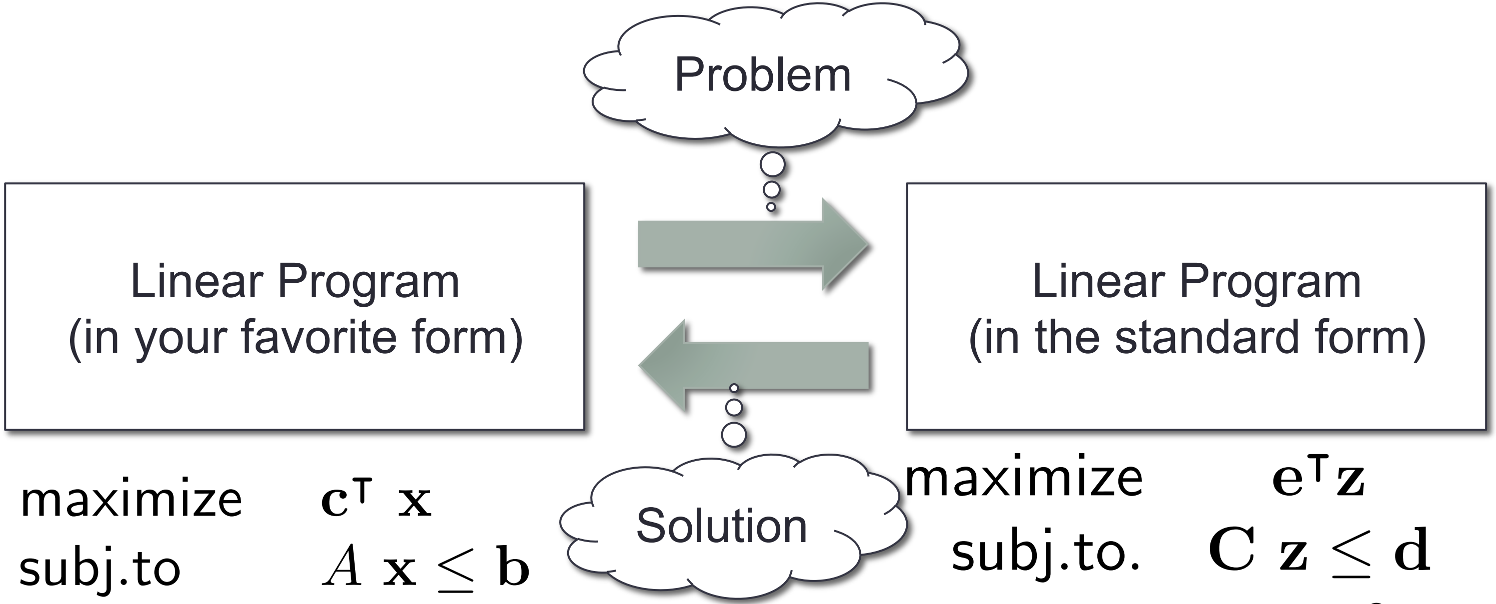
$$
\begin{array}{llcl}
\text{maximize} & c_1x_1 + \ldots + c_nx_n & & \\
\text{subj.to.} & a_{11}x_1 + \ldots + a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 + \ldots + a_{2n}x_n & \leq & b_2 \\
 & \ddots & \vdots & \\
 & a_{m1}x_1 + \ldots + a_{mn}x_n & \leq & b_m \\
 & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

$$
\begin{array}{ll}
\text{maximize} & \mathbf{c}^\intercal \mathbf{x} \\
\text{subj.to.} & \mathbf{A}\ \mathbf{x} \leq \mathbf{b} \\
 & \mathbf{x} \geq 0
\end{array}
$$

Standard Form LP (Example)

$$
\begin{array}{rrcl}
\text{maximize} & -5x_1 + 4x_2 - 3x_3 & & \\
\text{s.t.} & 2x_1 - 3x_2 + x_3 & \leq & 5 \\
 & 4x_1 + x_2 + 2x_3 & \leq & 11 \\
 & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\
 & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Converting LPs to Standard Form



$$\begin{array}{ll} \text{maximize} & \mathbf{c}^{\mathsf{T}}\,\mathbf{x} \\ \text{subj.to} & A\,\mathbf{x} \leq \mathbf{b} \end{array}$$

$$\begin{array}{ll} \text{maximize} & \mathbf{e}^{\mathsf{T}}\mathbf{z} \\ \text{subj.to.} & \mathbf{C}\,\mathbf{z} \leq \mathbf{d} \\ & \mathbf{z} \geq 0 \end{array}$$

Converting to Standard Form

$$
\begin{array}{rrcl}
\text{minimize} & -5x_1 + 4x_2 - 3x_3 & & \\
\text{s.t.} & 2x_1 - 3x_2 + x_3 & = & 5 \\
 & 4x_1 + x_2 + 2x_3 & \geq & 11 \\
 & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\
 & x_1 & \geq & 0
\end{array}
$$

Objective Direction

$$\text{minimize } \mathbf{c}^\intercal \mathbf{x} \quad \xrightarrow{\text{change to}} \quad \text{maximize } \left(-\mathbf{c}^\intercal \mathbf{x}\right)$$

$$\text{min } -5x_1 + 4x_2 - 3x_3 \quad \rightarrow \quad \text{max } 5x_1 - 4x_2 + 3x_3$$

Equality Constraints

$$\mathbf{a_i}^\intercal \mathbf{x} = b_i \xrightarrow{\text{change to}} \begin{cases} \mathbf{a_i}^\intercal \mathbf{x} \leq b_i \\ \mathbf{a_i}^\intercal \mathbf{x} \geq b_i \end{cases}$$

$$2x_1 - 3x_2 + x_3 = 5 \xrightarrow{\text{change to}} \begin{cases} 2x_1 - 3x_2 + x_3 \leq 5 \\ 2x_1 - 3x_2 + x_3 \geq 5 \end{cases}$$

Missing Non-Negative Constraint

Problem: missing constraint $x_i \geq 0$

- Introduce two fresh variables: x_i^+, x_i^-
- Replace every occurrence of x_i with $x_i^+ - x_i^-$
- Add the constraints $x_i^+ \geq 0, x_i^- \geq 0$

Converting to Standard Form

$$
\begin{array}{rrcl}
\text{minimize} & -5x_1 + 4x_2 - 3x_3 & & \\
\text{s.t.} & 2x_1 - 3x_2 + x_3 & = & 5 \\
 & 4x_1 + x_2 + 2x_3 & \geq & 11 \\
 & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\
 & x_1 & \geq & 0
\end{array}
$$

Conversion to Standard Form (Result)

$$\begin{array}{rrcr} \text{minimize} & -5x_1 + 4x_2 - 3x_3 & & \\ \text{s.t.} & 2x_1 - 3x_2 + x_3 & = & 5 \\ & 4x_1 + x_2 + 2x_3 & \geq & 11 \\ & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\ & x_1 & \geq & 0 \end{array}$$

$$\begin{array}{rrcr} \text{max} & 5x_1 - 4x_2^+ + 4x_2^- + 3x_3^+ - 3x_3^- & & \\ \text{s.t.} & 2x_1 - 3x_2^+ + 3x_2^- + x_3^+ - x_3^- & \leq & 5 \\ & -2x_1 + 3x_2^+ - 3x_2^- - x_3^+ + x_3^- & \leq & -5 \\ & -4x_1 - x_2^+ + x_2^- - 2x_3^+ + 2x_3^- & \leq & -11 \\ & 3x_1 + 4x_2 - 4x_2^- + 2x_3^+ - 2x_3^- & \leq & 8 \\ & x_1, x_2^+, x_2^-, x_3^+, x_3^- & \geq & 0 \end{array}$$

Transform Solutions Back

$$
\begin{array}{llcr}
\text{minimize} & -5x_1 + 4x_2 - 3x_3 & & \\
\text{s.t.} & 2x_1 - 3x_2 + x_3 & = & 5 \\
& 4x_1 + x_2 + 2x_3 & \geq & 11 \\
& 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\
& x_1 & \geq & 0
\end{array}
\;\Rightarrow\;
\begin{array}{llcr}
\text{max} & 5x_1 - 4x_2^+ + 4x_2^- + 3x_3^+ - 3x_3^- & & \\
\text{s.t.} & 2x_1 - 3x_2^+ + 3x_2^- + x_3^+ - x_3^- & \leq & 5 \\
& -2x_1 + 3x_2^+ - 3x_2^- - x_3^+ + x_3^- & \leq & -5 \\
& -4x_1 - x_2^+ + x_2^- - 2x_3^+ + 2x_3^- & \leq & -11 \\
& 3x_1 + 4x_2 - 4x_2^- + 2x_3^+ - 2x_3^- & \leq & 8 \\
& x_1, x_2^+, x_2^-, x_3^+, x_3^- & \geq & 0
\end{array}
$$

$$(x_1 = 3.43, x_2^+ = 0.143, x_2^- = 0, x_3^+ = 0, x_3^- = 1.43)$$

$$\downarrow$$

$$(x_1 = 3.43, x_2 = 0.143, x_3 = -1.43)$$