LINEAR AND INTEGER PROGRAMMING

The Simplex Method: A tutorial in three acts.

Three to Simplex

1. The Standard Form Linear Program.
2. “Dictionaries”.
3. Pivoting.

Putting it all together in an example.

ACT I: THE STANDARD FORM.

Revision of material covered under LP formulations.

Some words of wisdom

If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem?

- George Pólya (How to Solve It?)

try some easy examples

George Pólya

(source: mactutor)

Linear Program

$$
\begin{array}{rl}
\text{maximize} & c_1x_1 + \ldots + c_nx_n \\
\text{subj.to.} & a_{11}x_1 + \ldots + a_{1n}x_n \leq b_1 \\
& a_{21}x_1 + \ldots + a_{2n}x_n \leq b_2 \\
& \ddots \qquad \vdots \\
& a_{m1}x_1 + \ldots + a_{mn}x_n \leq b_m
\end{array}
$$

Linear Program in Matrix Form

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{array}{llll} \text{maximize} & c_1x_1 + \ldots + c_nx_n & & \\ \text{subj.to.} & a_{11}x_1 + \ldots + a_{1n}x_n & \leq & b_1 \\ & a_{21}x_1 + \ldots + a_{2n}x_n & \leq & b_2 \\ & \ddots & \vdots & \\ & a_{m1}x_1 + \ldots + a_{mn}x_n & \leq & b_m \end{array}$$

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^\mathsf{T}\,\mathbf{x} \\ \text{subj.to} & A\,\mathbf{x} \leq \mathbf{b} \end{array}$$

$$A = \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{mn} \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Standard Form (Definition)

$$
\begin{array}{rrcl}
\text{maximize} & c_1x_1 + \ldots + c_nx_n & & \\
\text{subj.to.} & a_{11}x_1 + \ldots + a_{1n}x_n & \leq & b_1 \\
& a_{21}x_1 + \ldots + a_{2n}x_n & \leq & b_2 \\
& \ddots & \vdots & \\
& a_{m1}x_1 + \ldots + a_{mn}x_n & \leq & b_m \\
\text{non-negative} & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

Standard Form (Matrix Notation)

$$
\begin{array}{llcl}
\text{maximize} & c_1x_1 + \ldots + c_nx_n & & \\
\text{subj.to.} & a_{11}x_1 + \ldots + a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 + \ldots + a_{2n}x_n & \leq & b_2 \\
 & \ddots & \vdots & \\
 & a_{m1}x_1 + \ldots + a_{mn}x_n & \leq & b_m \\
 & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

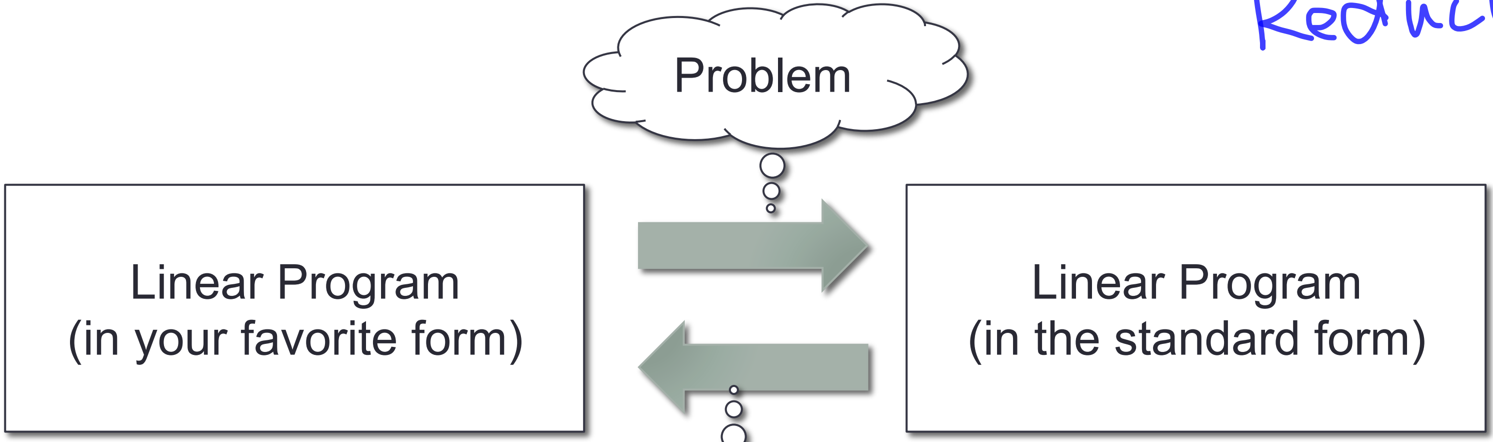
$$
\begin{array}{rr}
\text{maximize} & \mathbf{c}^\intercal \mathbf{x} \\
\text{subj.to.} & \mathbf{A}\ \mathbf{x} \leq \mathbf{b} \\
 & \mathbf{x} \geq 0
\end{array}
$$

Standard Form LP (Example)

$$
\begin{array}{rrcl}
\text{maximize} & -5x_1 + 4x_2 - 3x_3 & & \\
\text{s.t.} & 2x_1 - 3x_2 + x_3 & \leq & 5 \\
 & 4x_1 + x_2 + 2x_3 & \leq & 11 \\
 & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\
 & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Converting LPs to Standard Form

Reduction



$$\begin{array}{ll} \text{maximize} & \mathbf{c}^\intercal \mathbf{x} \\ \text{subj.to} & A\,\mathbf{x} \le \mathbf{b} \end{array}$$

$$\begin{array}{ll} \text{maximize} & \mathbf{e}^\intercal \mathbf{z} \\ \text{subj.to.} & \mathbf{C}\,\mathbf{z} \le \mathbf{d} \\ & \mathbf{z} \ge 0 \end{array}$$

Converting to Standard Form

$$
\begin{array}{lrcl}
\text{minimize} & -5x_1 + 4x_2 - 3x_3 & & \\
\text{s.t.} & 2x_1 - 3x_2 + x_3 & = & 5 \\
 & 4x_1 + x_2 + 2x_3 & \geq & 11 \\
 & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\
 & x_1 & \geq & 0
\end{array}
$$

x_2, x_3 ?

Objective Direction

$$\text{minimize } \mathbf{c}^\intercal \mathbf{x} \xrightarrow{\text{change to}} \text{maximize } (-\mathbf{c}^\intercal \mathbf{x})$$

$$\text{min } -5x_1 + 4x_2 - 3x_3 \quad \rightarrow \quad \text{max } 5x_1 - 4x_2 + 3x_3$$

Equality Constraints

$$\mathbf{a_i}^\intercal \mathbf{x} = b_i \xrightarrow{\text{change to}} \begin{cases} \mathbf{a_i}^\intercal \mathbf{x} \le b_i \\ \mathbf{a_i}^\intercal \mathbf{x} \ge b_i \end{cases}$$

$x \geq y \longrightarrow -x \leq -y$

$$2x_1 - 3x_2 + x_3 = 5 \xrightarrow{\text{change to}} \begin{cases} 2x_1 - 3x_2 + x_3 \le 5 \\ 2x_1 - 3x_2 + x_3 \ge 5 \end{cases}$$

Missing Non-Negative Constraint

Problem: missing constraint $x_i \geq 0$

- Introduce two fresh variables: x_i^+, x_i^-
- Replace every occurrence of x_i with $x_i^+ - x_i^-$
- Add the constraints $x_i^+ \geq 0, x_i^- \geq 0$

$7 \Rightarrow 7-0$

$-7 \Rightarrow 0-7$ or $2-9$

Converting to Standard Form

$$
\begin{array}{rrcl}
\text{minimize} & -5x_1 + 4x_2 - 3x_3 & & \\
\text{s.t.} & 2x_1 - 3x_2 + x_3 & = & 5 \\
 & 4x_1 + x_2 + 2x_3 & \geq & 11 \\
 & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\
 & x_1 & \geq & 0
\end{array}
$$

Conversion to Standard Form (Result)

$$\begin{array}{lrcl} \text{minimize} & -5x_1 + 4x_2 - 3x_3 & & \\ \text{s.t.} & 2x_1 - 3x_2 + x_3 & = & 5 \\ & 4x_1 + x_2 + 2x_3 & \geq & 11 \\ & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\ & x_1 & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \max & 5x_1 - 4x_2^+ + 4x_2^- + 3x_3^+ - 3x_3^- & & \\ \text{s.t.} & 2x_1 - 3x_2^+ + 3x_2^- + x_3^+ - x_3^- & \leq & 5 \\ & -2x_1 + 3x_2^+ - 3x_2^- - x_3^+ + x_3^- & \leq & -5 \\ & -4x_1 - x_2^+ + x_2^- - 2x_3^+ + 2x_3^- & \leq & -11 \\ & 3x_1 + 4x_2 - 4x_2^- + 2x_3^+ - 2x_3^- & \leq & 8 \\ & x_1, x_2^+, x_2^-, x_3^+, x_3^- & \geq & 0 \end{array}$$

Transform Solutions Back

$$
\begin{array}{llcl}
\text{minimize} & -5x_1 + 4x_2 - 3x_3 & & \\
\text{s.t.} & 2x_1 - 3x_2 + x_3 & = & 5 \\
& 4x_1 + x_2 + 2x_3 & \geq & 11 \\
& 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\
& x_1 & \geq & 0
\end{array}
$$

$$
\begin{array}{llcl}
\text{max} & 5x_1 - 4x_2^+ + 4x_2^- + 3x_3^+ - 3x_3^- & & \\
\text{s.t.} & 2x_1 - 3x_2^+ + 3x_2^- + x_3^+ - x_3^- & \leq & 5 \\
& -2x_1 + 3x_2^+ - 3x_2^- - x_3^+ + x_3^- & \leq & -5 \\
& -4x_1 - x_2^+ + x_2^- - 2x_3^+ + 2x_3^- & \leq & -11 \\
& 3x_1 + 4x_2 - 4x_2^- + 2x_3^+ - 2x_3^- & \leq & 8 \\
& x_1, x_2^+, x_2^-, x_3^+, x_3^- & \geq & 0
\end{array}
$$

$$(x_1 = 3.43, x_2^+ = 0.143, x_2^- = 0, x_3^+ = 0, x_3^- = 1.43)$$

$$(x_1 = 3.43, x_2 = 0.143, x_3 = -1.43)$$

ACT II: THE DICTIONARY

A convenient data structure for Linear Programs and representation of solutions.

Standard Form (Matrix Notation)

$$
\begin{array}{lrcl}
\text{maximize} & c_1x_1 + \ldots + c_nx_n & & \\
\text{subj.to.} & a_{11}x_1 + \ldots + a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 + \ldots + a_{2n}x_n & \leq & b_2 \\
 & \ddots & \vdots & \\
 & a_{m1}x_1 + \ldots + a_{mn}x_n & \leq & b_m \\
 & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

$$
\begin{array}{lr}
\text{maximize} & \mathbf{c}^\intercal \mathbf{x} \\
\text{subj.to.} & \mathbf{A}\ \mathbf{x} \leq \mathbf{b} \\
 & \mathbf{x} \geq 0
\end{array}
$$

Slack Form

Trick: Change inequality constraint to an equality.

$$\mathbf{a}^\intercal \mathbf{x} \leq b \xrightarrow{\text{add slack}} \mathbf{a}^\intercal \mathbf{x} + x_s = b$$

standard Form

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^\intercal \mathbf{x} \\ \text{subj.to.} & \mathbf{A}\ \mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}$$

$\Leftrightarrow$

$$\begin{array}{llll} \text{maximize} & \mathbf{c}^\intercal \mathbf{x} & & \\ \text{s.t.} & \mathbf{A}\mathbf{x} + \mathbf{x}_{\text{slack}} & = & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \\ & \mathbf{x}_{\text{slack}} & \geq & \mathbf{0} \end{array}$$

Example (Slack Variable Addition)

$$
\begin{array}{rrcl}
\text{max.} & 5x_1 + 4x_2 + 3x_3 & & \\
\text{s.t.} & 2x_1 + 3x_2 + x_3 & \leq & 5 \\
 & 4x_1 + x_2 + 2x_3 & \leq & 11 \\
 & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\
 & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Example (Slack Variable Addition)

fresh variable

$$
\begin{array}{lrcl}
\text{max.} & 5x_1 + 4x_2 + 3x_3 & & \\
\text{s.t.} & 2x_1 + 3x_2 + x_3 + {\color{red}x_4} & = & 5 \\
 & 4x_1 + x_2 + 2x_3 + {\color{red}x_5} & = & 11 \\
 & 3x_1 + 4x_2 + 2x_3 + {\color{red}x_6} & = & 8 \\
 & x_1, x_2, x_3, {\color{red}x_4, x_5, x_6} & \geq & 0
\end{array}
$$

Dictionary

max.	$5x_1 + 4x_2 + 3x_3$	
s.t.	$2x_1 + 3x_2 + x_3 + x_4 = 5$	
	$4x_1 + x_2 + 2x_3 + x_5 = 11$	
	$3x_1 + 4x_2 + 2x_3 + x_6 = 8$	
	$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$	

basic variable

Basic set

$$
\begin{array}{rcllll}
x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\
x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\
x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\
\hline
z & = & 0 & +5x_1 & +4x_2 & +3x_3
\end{array}
$$

object function

$$\underbrace{x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0}_{\text{non-basic / co-basic / independent}}$$

Dictionary

$$\begin{array}{rcll} \text{basis} \rightarrow \mathbf{x}_b & = & \mathbf{b} & - \mathbf{A}\mathbf{x}_i \\ \hline z & = & c_0 & + \mathbf{c}^\top \underbrace{\mathbf{x}_i}_{\uparrow \text{non-basic}} \end{array}$$

Summary So Far

Transform Problem with Slack Variables

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^\intercal \mathbf{x} \\ \text{subj.to.} & \mathbf{A}\ \mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}$$

$\Leftrightarrow$

$$\begin{array}{lccc} \text{maximize} & \mathbf{c}^\intercal \mathbf{x} & & \\ \text{s.t.} & \mathbf{A}\mathbf{x} + \mathbf{x}_{\text{slack}} & = & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \\ & \mathbf{x}_{\text{slack}} & \geq & \mathbf{0} \end{array}$$

Dictionaries

Dictionary: Solution associated + Feasibility

$$
\begin{array}{rcrrrr}
x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\
x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\
x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\
\hline
z & = & 0 & +5x_1 & +4x_2 & +3x_3
\end{array}
$$

$$
\begin{array}{rccc}
\mathbf{x}_B & = & \mathbf{b} & - \; \mathbf{A}\mathbf{x}_I \\
\hline
z & = & c_0 & +\mathbf{c}^\intercal \; \mathbf{x}_I
\end{array}
$$

Solution Associated with Dictionary

- Non-basic variables have value 0. ← #1 rule
- Basic variables: read off from dictionary.

$$
\begin{array}{rcrrrr}
 & & & \overset{0}{\downarrow} & \overset{0}{\downarrow} & \overset{0}{\downarrow} \\
x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\
x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\
x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\
\hline
z & = & 0 & +5x_1 & +4x_2 & +3x_3
\end{array}
$$

Another Dictionary

$$
\begin{array}{rclllll}
x_4 & = & -28 & +10x_1 & +3x_5 & +5x_3 \\
x_2 & = & 11 & -4x_1 & -x_5 & -2x_3 \\
x_6 & = & -32 & +9x_1 & +4x_5 & +62x_3 \\
\hline
z & = & 44 & -11x_1 & -4x_5 & -5x_3
\end{array}
$$

$D: \begin{cases} x_1 = x_3 = x_5 = 0 \\ x_2 = 11,\ \underline{x_4 = -28},\ \underline{x_6 = -32} \leftarrow \times \quad x_i \geq 0 \\ z = 44 \end{cases}$

Feasible vs. Infeasible Dictionary

$$
\begin{array}{rcrrrr}
x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\
x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\
x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\
\hline
z & = & 0 & +5x_1 & +4x_2 & +3x_3
\end{array}
$$

Feasible

Infeasible

$$
\begin{array}{rcrrrr}
x_4 & = & \overset{<0}{-28} & +10x_1 & +3x_5 & +5x_3 \\
x_2 & = & 11 & -4x_1 & -x_5 & -2x_3 \\
x_6 & = & -32 & +9x_1 & +4x_5 & +62x_3 \\
\hline
z & = & 44 & -11x_1 & -4x_5 & -5x_3
\end{array}
$$

Why Dictionaries?

- Data structure for Linear Programs.
  - Organize the data in the problem

- Represents candidate solutions to the problem.

Simplex Algorithm

Problem Given

feasible

$$D_0 \xrightarrow{\text{pivot}} D_1 \xrightarrow{\text{pivot}} \cdots \; D_{k-1} \xrightarrow{\text{pivot}} D_k \xrightarrow{\text{pivot}} D_{k+1} \cdots \xrightarrow{\text{pivot}} \mathbf{D}^*$$

Initialization Phase

Optimization Phase

Infeasible Dictionaries

Feasible Dictionaries

Summary (1)

Transform Problem with Slack Variables

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^\intercal \mathbf{x} \\ \text{subj.to.} & \mathbf{A}\ \mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{array} \Leftrightarrow \begin{array}{llll} \text{maximize} & \mathbf{c}^\intercal \mathbf{x} & & \\ \text{s.t.} & \mathbf{A}\mathbf{x} + {\color{red}\mathbf{x}_{\mathrm{slack}}} & = & \mathbf{b} \\ & \mathbf{x} & \ge & \mathbf{0} \\ & {\color{red}\mathbf{x}_{\mathrm{slack}}} & \ge & \mathbf{0} \end{array}$$

Summary (2)

Dictionary: Solution associated + Feasibility

basic

$$\begin{array}{rcrrrr}
x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\
x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\
x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\
\hline
z & = & 0 & +5x_1 & +4x_2 & +3x_3
\end{array}$$

non-basic

$= 0$

$$\begin{array}{rcccc}
\mathbf{x}_B & = & \mathbf{b} & - & \mathbf{A}\mathbf{x}_I \\
\hline
z & = & c_0 & + & \mathbf{c}^{\mathsf{T}}\, \mathbf{x}_I
\end{array}$$

$x_i \geq 0$ implicate

ACT III: PIVOTING

Going from one dictionary to the next.

ACT III: PIVOTING

Going from one dictionary to the next.

Pivoting (from 2K feet)

$$\begin{array}{rcll} \mathbf{x}_B & = & \mathbf{b} & - \mathbf{A}\mathbf{x}_I \\ \hline z & = & c_0 & + \mathbf{c}^\intercal \, \mathbf{x}_I \end{array}$$

2. Choose Leaving Variable

m basic variables

$$\begin{array}{rcllllll}
x_{B1} & = & b_1 & + a_{11}x_{I1} & + \cdots & + a_{1j}x_{Ij} & + \cdots & + a_{1n}x_{In} \\
 & \vdots & & & & & & \\
\boxed{x_{Bk}} & = & b_k & + a_{k1}x_{I1} & + \cdots & + a_{kj}x_{Ij} & + \cdots & + a_{kn}x_{In} \\
 & \vdots & & & & & & \\
x_{Bm} & = & b_m & + a_{m1}x_{I1} & + \cdots & + a_{mj}x_{Ij} & + \cdots & + a_{mn}x_{In} \\
\hline
z & = & c_0 & + c_1x_{I1} & + \cdots & \boxed{+ c_jx_{Ij}} & + \cdots & + c_nx_{In}
\end{array}$$

1. Choose Entering Variable

n non-basic variables

Pivoting Steps

1. Choose an entering variable.

2. For the choice of entering variable, find a leaving variable.

3. Perform substitutions and obtain next dictionary.

Example

Chvátal, Chapter 2

$$
\begin{array}{rrcl}
\text{max.} & 5x_1 + 4x_2 + 3x_3 & & \\
\text{s.t.} & 2x_1 + 3x_2 + x_3 & \leq & 5 \\
& 4x_1 + x_2 + 2x_3 & \leq & 11 \\
& 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\
& x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Already in standard form.

Example (add slack)

$$\begin{array}{lrcl} \text{max.} & 5x_1 + 4x_2 + 3x_3 & & \\ \text{s.t.} & 2x_1 + 3x_2 + x_3 & \leq & 5 \\ & 4x_1 + x_2 + 2x_3 & \leq & 11 \\ & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\ & x_1, x_2, x_3 & \geq & 0 \end{array}$$

$$\begin{array}{rcrrrr} x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\ x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\ x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\ \hline z & = & 0 & +5x_1 & +4x_2 & +3x_3 \end{array}$$

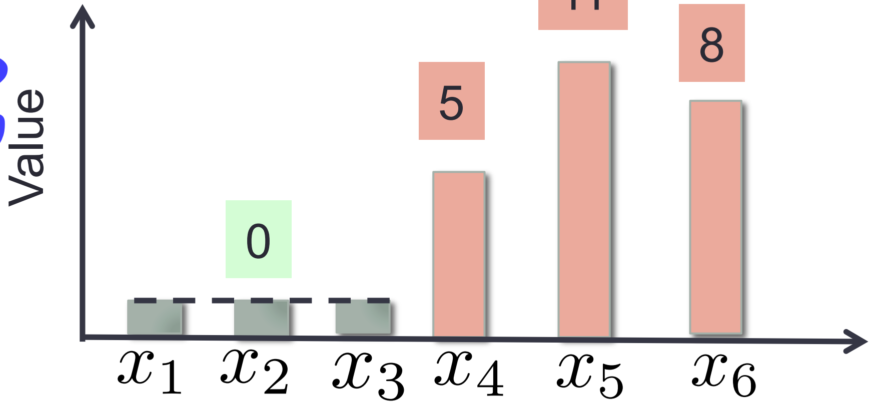
feasible

Choosing an entering variable

$$\begin{array}{rclllll}
x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\
x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\
x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\
\hline
z & = & 0 & +5x_1 & +4x_2 & +3x_3
\end{array}$$

$x_1 \uparrow$
$x_2 \uparrow \Rightarrow z \uparrow$
$x_3 \uparrow$

$$\begin{cases} \max z \\ x_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 0 \end{cases}$$



Leaving Variable

$$
\begin{array}{rcrrrr}
x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\
x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\
x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\
\hline
z & = & 0 & +5x_1 & +4x_2 & +3x_3
\end{array}
$$

Leaving Variable Analysis

$$\begin{array}{cccccc} x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\ x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\ x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\ \hline z & = & 0 & +5x_1 & +4x_2 & +3x_3 \end{array}$$

$x_2 = x_3 = 0$

$$\begin{array}{rcll} x_4 & = & \overset{\geq 0}{5 - 2x_1} - 3x_2 - x_3 & \Rightarrow \boxed{x_1 \leq \frac{5}{2}} \\ x_5 & = & 11 - 4x_1 - x_2 - 2x_3 & \Rightarrow x_1 \leq \frac{11}{4} \\ x_6 & = & 8 - 3x_1 - 4x_2 - 2x_3 & \Rightarrow x_1 \leq \frac{8}{3} \\ \hline z & = & 0 + 5x_1 + 4x_2 + 3x_3 & \end{array}$$

min

Exercise: Find Leaving Variables

Modified Problem

$$
\begin{array}{rclllll}
x_4 & = & 7 & +2x_1 & -3x_2 & +x_3 \\
x_5 & = & 12 & -4x_1 & +x_2 & +2x_3 \\
x_6 & = & 9 & -3x_1 & -4x_2 & +2x_3 \\
\hline
z & = & 10 & +2x_1 & +2x_2 & -3x_3
\end{array}
$$

Note: This dictionary is **not for** the same problem as in slide 2.

Pivoting: Finding new dictionary.

$$\begin{array}{lcrrrr} x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\ x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\ x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\ \hline z & = & 0 & +5x_1 & +4x_2 & +3x_3 \end{array}$$

x_1 enters and x_4 leaves

$$x_4 = 5 - 2x_1 - 3x_2 - x_3$$

$$\Downarrow$$

$$x_1 = 5/2 - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4$$

Note: This dictionary is for a different problem than the dictionary in slide 1. The rest of this PPT will keep pivoting the dictionary on this slide.

Pivoting

orig. dic.

$$\begin{array}{rcrrrr} x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\ x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\ x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\ \hline z & = & 0 & +5x_1 & +4x_2 & +3x_3 \end{array}$$

x_1 enters and x_4 leaves

$$\begin{array}{rcl} x_1 & = & \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_5 & = & 1 + 5x_2 \qquad\qquad + 2x_4 \\ x_6 & = & \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \\ \hline z & = & \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4 \end{array}$$

$x_1 = 5/2$
$x_2 = 0$
$x_3 = 0$
$x_4 = 0$
$x_5 = 1$
$x_6 = 1/2$
$z = 25/2$

Entering Variable Analysis

$$
\begin{aligned}
x_1 &= \tfrac{5}{2} - \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3 - \tfrac{1}{2}x_4 \\
x_5 &= 1 + 5x_2 \qquad\qquad + 2x_4 \\
x_6 &= \tfrac{1}{2} + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 + \tfrac{3}{2}x_4
\end{aligned}
$$

$$
z = \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4
$$

Pivot

Leaving Variable Analysis

$$
\begin{array}{rcl}
x_1 & = & \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\
x_5 & = & 1 + 5x_2 \qquad\qquad + 2x_4 \\
x_6 & = & \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \\
\hline
z & = & \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4
\end{array}
$$

$$
\begin{array}{rcll}
x_1 & = & \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 & \Rightarrow \underline{x_3 \leq 5} \\
x_5 & = & 1 + 5x_2 \qquad\qquad + 2x_4 & \Rightarrow \text{no constraint} \\
x_6 & = & \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 & \Rightarrow \underline{x_3 \leq 1}
\end{array}
$$

Pivoting

D_2

D_3

$$
\begin{aligned}
x_1 &= \tfrac{5}{2} - \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3 - \tfrac{1}{2}x_4 \\
x_5 &= 1 + 5x_2 + 2x_4 \\
x_6 &= \tfrac{1}{2} + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 + \tfrac{3}{2}x_4 \\
\hline
z &= \tfrac{25}{2} - \tfrac{7}{2}x_2 + \tfrac{1}{2}x_3 - \tfrac{5}{2}x_4
\end{aligned}
$$

x_3 enters and x_6 leaves

$$
\begin{aligned}
x_3 &= 1 + x_2 + 3x_4 - 2x_6 \\
x_1 &= 2 - 2x_2 - 2x_4 + x_6 \\
x_5 &= 1 + 5x_2 + 2x_4 \\
\hline
z &= 13 - 3x_2 - x_4 - x_6
\end{aligned}
$$

Final Dictionary

$$\begin{array}{rcl} x_3 & = & 1 + x_2 + 3x_4 - 2x_6 \\ x_1 & = & 2 - 2x_2 - 2x_4 + x_6 \\ x_5 & = & 1 + 5x_2 + 2x_4 \\ \hline z & = & 13 - 3x_2 - x_4 - x_6 \end{array}$$

$$\begin{array}{rcl} \mathbf{x}_B & = & \mathbf{b} \quad - \mathbf{A}\mathbf{x}_I \\ \hline z & = & c_0 \quad + \mathbf{c}^\intercal \mathbf{x}_I \end{array}$$

$$\mathbf{c} \leq 0$$

No choice for entering variables.

Simplex Algorithm

$$D_0 \xrightarrow{\text{pivot}} D_1 \xrightarrow{\text{pivot}} \cdots \; D_{k-1} \xrightarrow{\text{pivot}} D_k \xrightarrow{\text{pivot}} D_{k+1} \cdots \xrightarrow{\text{pivot}} \mathbf{D}^*$$

Initialization Phase | Optimization Phase

Infeasible Dictionaries | Feasible Dictionaries

Summary (1/2)

Choice of leaving variable

$$
\begin{array}{rcllll}
x_{B1} & = & b_1 & -a_{11}x_{I1} & \cdots & -a_{1n}x_{Im} \\
x_{B2} & = & b_2 & -a_{21}x_{I1} & \cdots & -a_{2n}x_{Im} \\
 & \vdots & & & \ddots & \\
x_{Bm} & = & b_n & -a_{m1}x_{I1} & \cdots & -a_{mn}x_{Im} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_nx_{In}
\end{array}
$$

Choice of entering variable $(c_i > 0)$

Summary (2/2)

Pivoting: computing next dictionary

x_3 enters and x_6 leaves

$$
\begin{array}{rcl}
x_1 & = & \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\
x_5 & = & 1 + 5x_2 \qquad\qquad + 2x_4 \\
x_6 & = & \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \\
\hline
z & = & \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4
\end{array}
$$

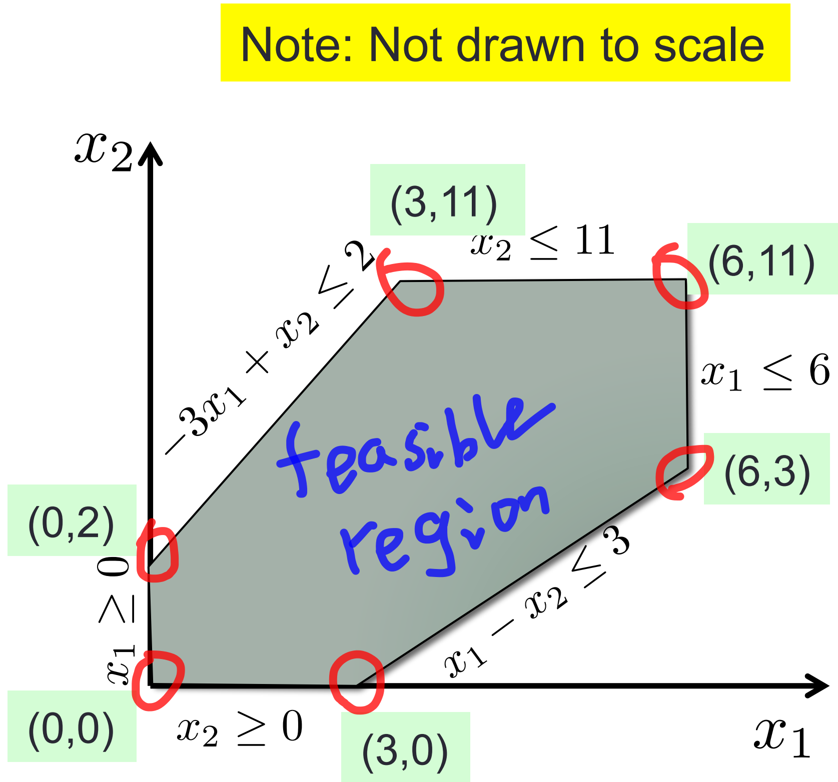
$$
\begin{array}{rcl}
x_3 & = & 1 + x_2 + 3x_4 - 2x_6 \\
x_1 & = & 2 - 2x_2 - 2x_4 + x_6 \\
x_5 & = & 1 + 5x_2 + 2x_4 \\
\hline
z & = & 13 - 3x_2 - x_4 - x_6
\end{array}
$$

SIMPLEX: EXAMPLE

A Complete Example with Visualization.

Linear Programming Problem

$$\begin{array}{rrrcl}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
& & +x_2 & \leq & 11 \\
& x_1 & -x_2 & \leq & 3 \\
& x_1 & & \leq & 6 \\
& x_1, & x_2 & \geq & 0
\end{array}$$



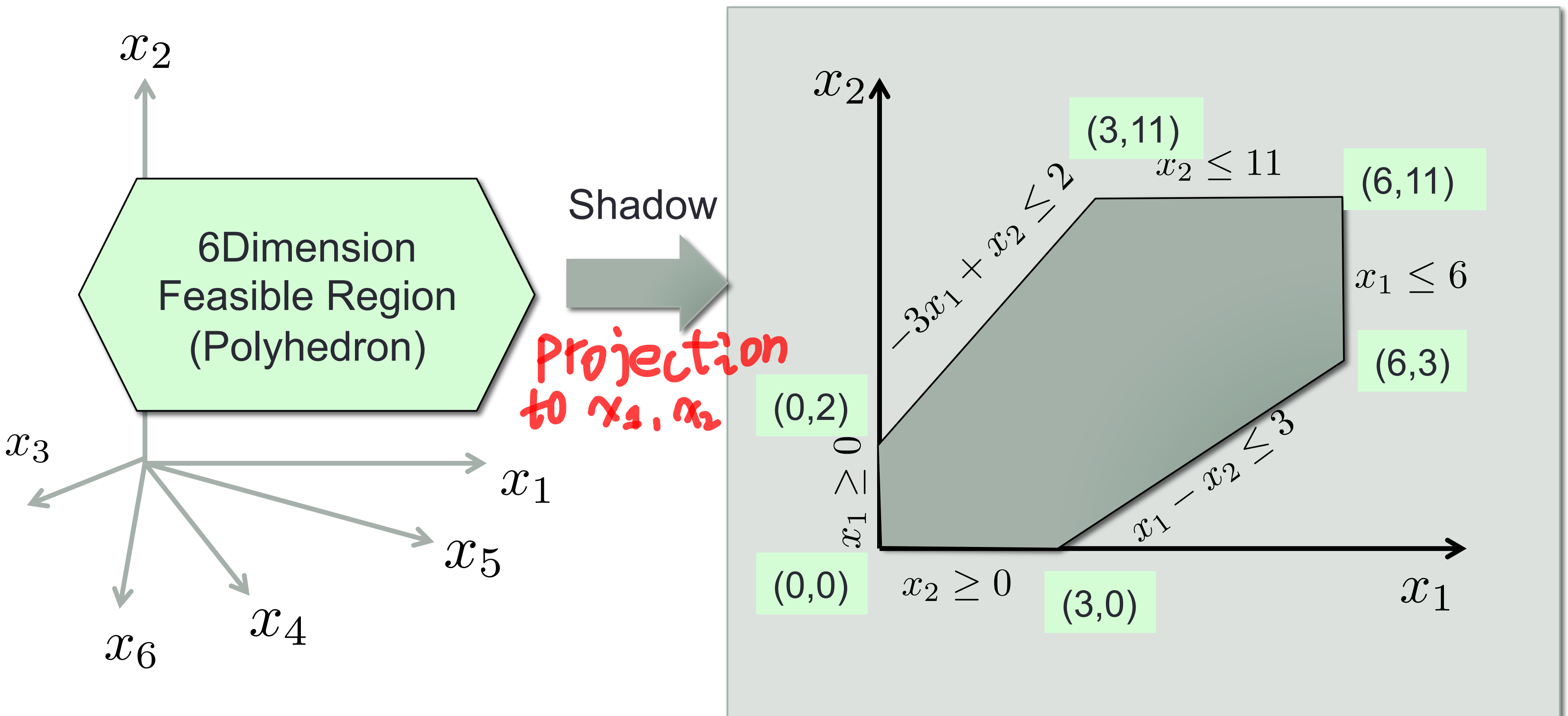
Goal: Solve LP using Simplex and visualize!

Step 1: Add Slack Variables

$$\begin{array}{lrrcr}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}$$

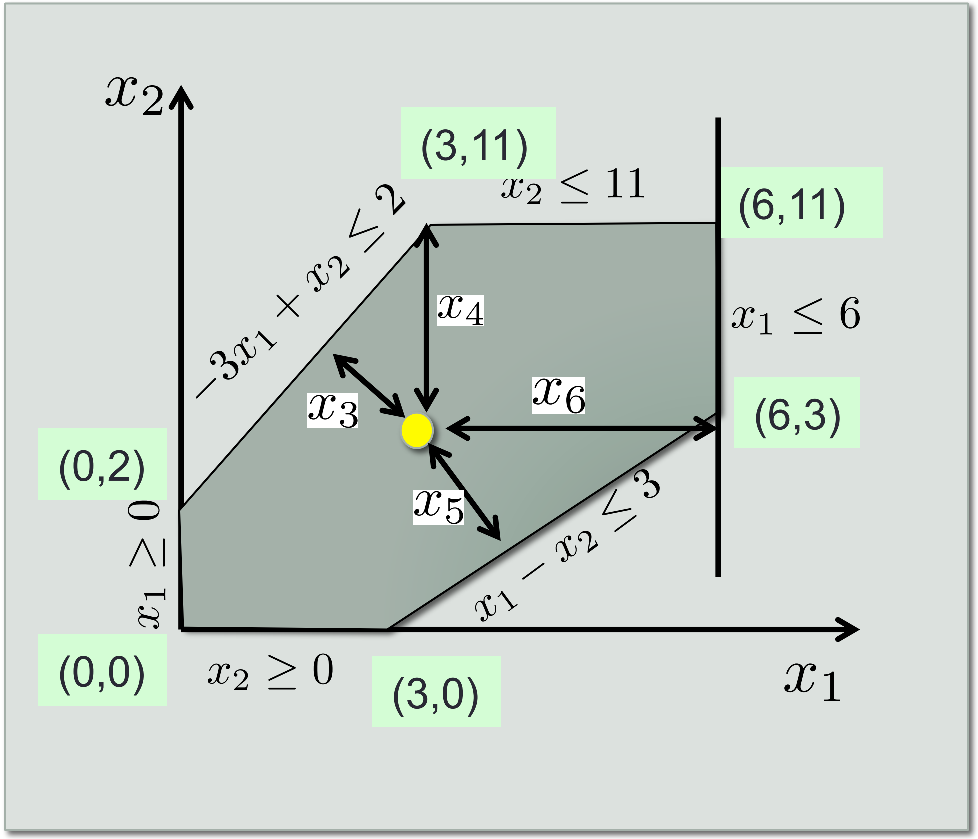
$$\begin{array}{lrrrcr}
\text{max.} & x_1 & +2x_2 & & & \\
\text{s.t.} & -3x_1 & +x_2 & \color{red}{+x_3} & = & 2 \\
 & & +x_2 & \color{red}{+x_4} & = & 11 \\
 & x_1 & -x_2 & \color{red}{+x_5} & = & 3 \\
 & x_1 & & \color{red}{+x_6} & = & 6 \\
 & \multicolumn{3}{r}{x_1, x_2, \color{red}{x_3, \ldots, x_6}} & \geq & 0
\end{array}$$

Visualizing with Slack



Schematic Drawing

Alternative Visualization of Slack



Initial Dictionary

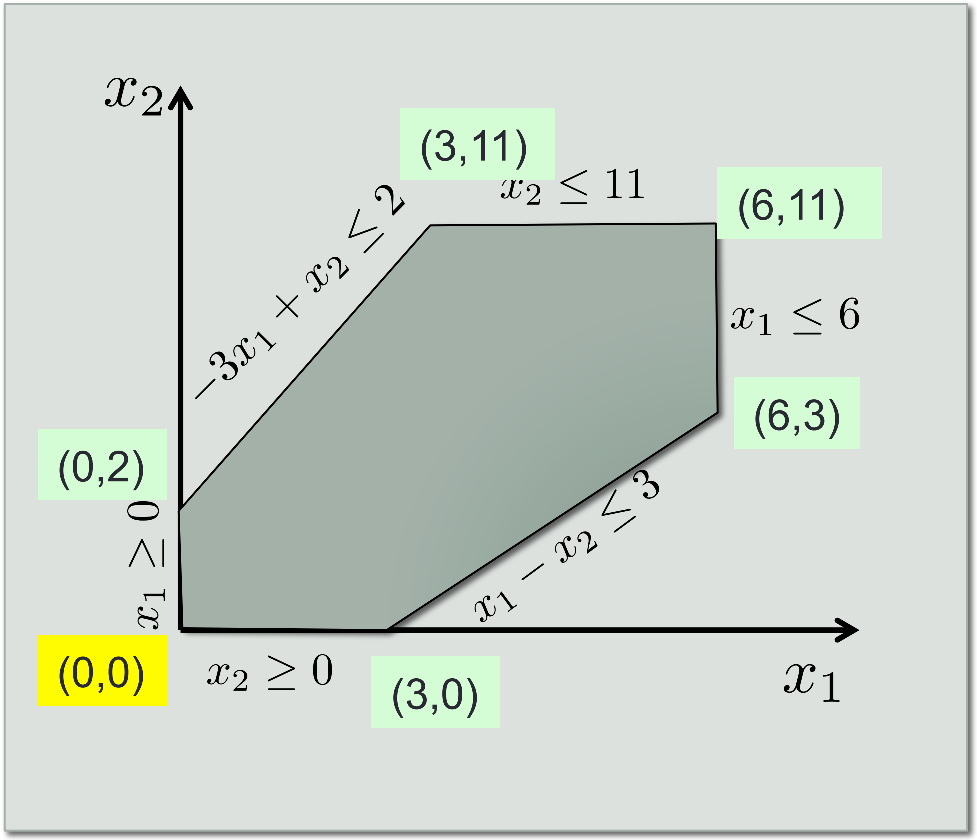
max.	x_1	$+2x_2$			
s.t.	$-3x_1$	$+x_2$	$+x_3$	$=$	2
		$+x_2$	$+x_4$	$=$	11
	x_1	$-x_2$	$+x_5$	$=$	3
	x_1		$+x_6$	$=$	6
		$x_1, x_2, x_3, \ldots, x_6$		$\geq$	0

Dict. D1

$$
\begin{array}{rcrrr}
x_3 & = & 2 & +3x_1 & -x_2 \\
x_4 & = & 11 & +0x_1 & -x_2 \\
x_5 & = & 3 & -x_1 & +x_2 \\
x_6 & = & 6 & -x_1 & +0x_2 \\
\hline
z & = & 0 & +x_1 & +2x_2
\end{array}
$$

Solution Associated (Dict. D1)

$$\begin{array}{rcrrr} x_3 & = & 2 & +3x_1 & -x_2 \\ x_4 & = & 11 & +0x_1 & -x_2 \\ x_5 & = & 3 & -x_1 & +x_2 \\ x_6 & = & 6 & -x_1 & +0x_2 \\ \hline z & = & 0 & +x_1 & +2x_2 \end{array}$$



Entering/Leaving Variable Analysis

$$
\begin{array}{rcrrr}
x_3 & = & 2 & +3x_1 & -x_2 \\
x_4 & = & 11 & +0x_1 & -x_2 \\
x_5 & = & 3 & -x_1 & +x_2 \\
x_6 & = & 6 & -x_1 & +0x_2 \\
\hline
z & = & 0 & +x_1 & +2x_2
\end{array}
$$

x_1 enters and x_5 leaves. $\left\{ \begin{array}{l} x_1 \le 3 \\ x_1 \le 6 \end{array} \right.$

OR

x_2 enters and x_3 leaves.

Pivoting

$$\begin{array}{rcrrr} x_3 & = & 2 & +3x_1 & -x_2 \\ x_4 & = & 11 & +0x_1 & -x_2 \\ x_5 & = & 3 & -x_1 & +x_2 \\ x_6 & = & 6 & -x_1 & +0x_2 \\ \hline z & = & 0 & +x_1 & +2x_2 \end{array}$$

x_2 enters and x_3 leaves.

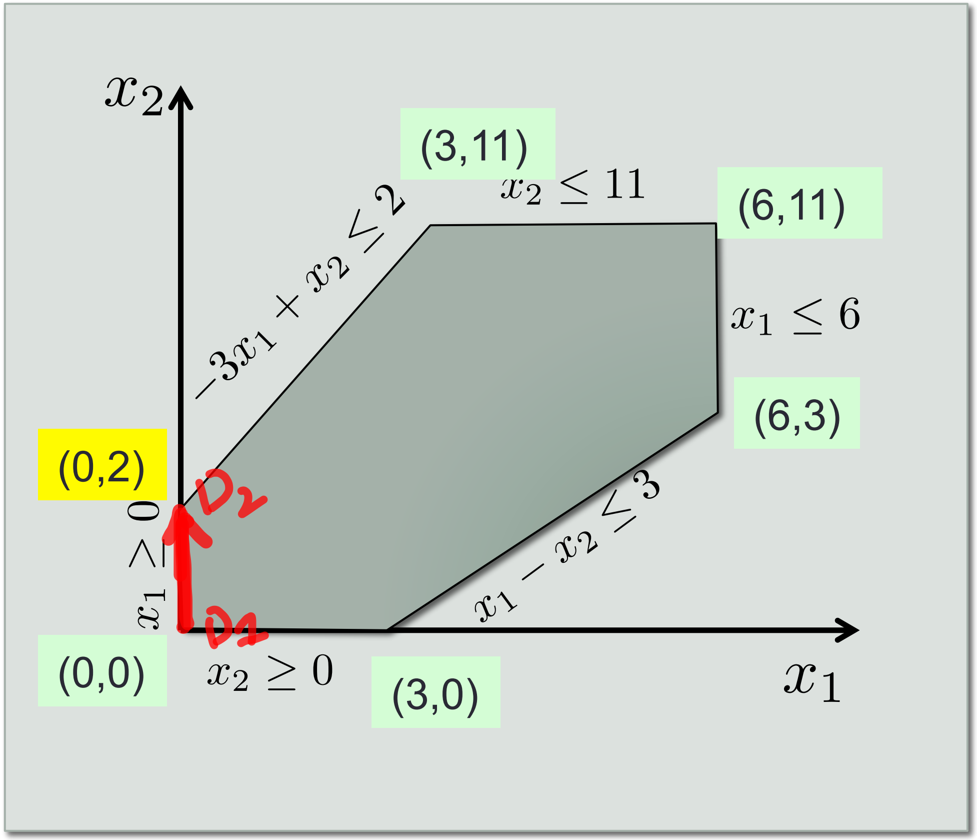
$\to x_2 = 2 + 3x_1 - x_3$

Dict. D2

$$\begin{array}{rcrrr} x_2 & = & 2 & +3x_1 & -x_3 \\ x_4 & = & 9 & -3x_1 & +x_3 \\ x_5 & = & 5 & +2x_1 & -x_3 \\ x_6 & = & 6 & -x_1 & +0x_3 \\ \hline z & = & 4 & +7x_1 & -2x_3 \end{array}$$

Solution Associated (D2)

x_2	$=$	2	$+3x_1$	$-x_3$
x_4	$=$	9	$-3x_1$	$+x_3$
x_5	$=$	5	$+2x_1$	$-x_3$
x_6	$=$	6	$-x_1$	$+0x_3$
z	$=$	4	$+7x_1$	$-2x_3$



Entering/Leaving Variable Analysis

$$
\begin{array}{rcllr}
x_2 & = & 2 & +3x_1 & -x_3 \\
x_4 & = & 9 & -3x_1 & +x_3 \\
x_5 & = & 5 & +2x_1 & -x_3 \\
x_6 & = & 6 & -x_1 & +0x_3 \\
\hline
z & = & 4 & +7x_1 & -2x_3
\end{array}
$$

x_1 enters and x_4 leaves.

Pivoting

$$\begin{array}{rclll}
x_2 & = & 2 & +3x_1 & -x_3 \\
x_4 & = & 9 & -3x_1 & +x_3 \\
x_5 & = & 5 & +2x_1 & -x_3 \\
x_6 & = & 6 & -x_1 & +0x_3 \\
\hline
z & = & 4 & +7x_1 & -2x_3
\end{array}$$

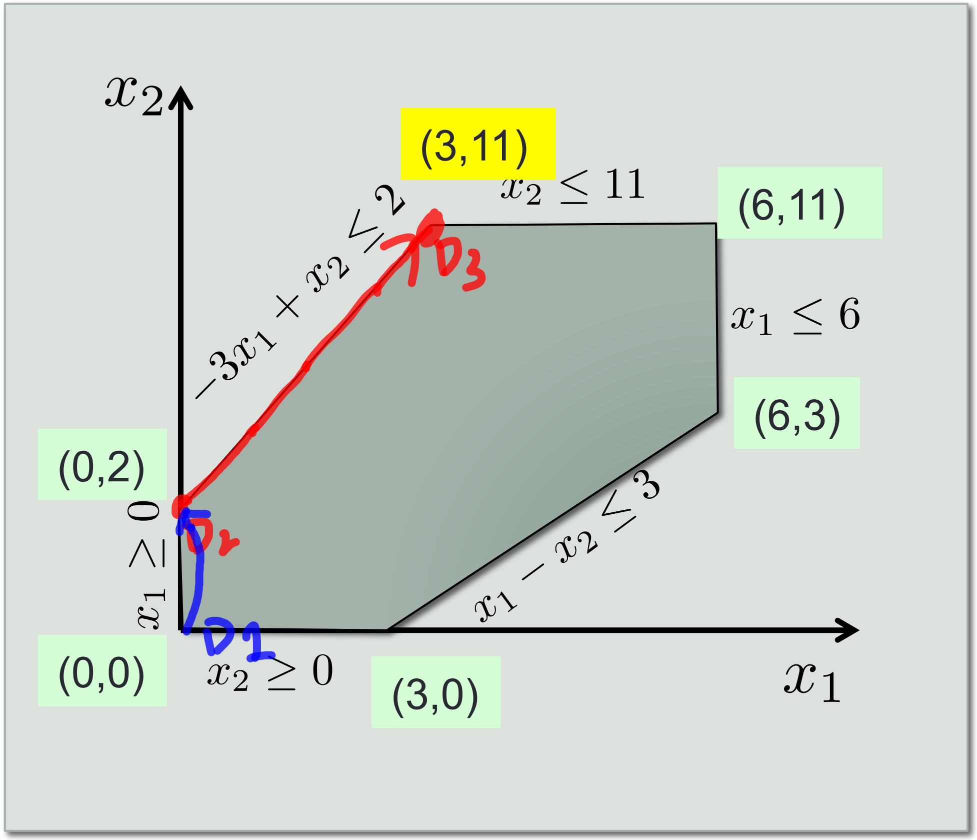
x_1 enters and x_4 leaves.

Dict. D3

$$\begin{array}{rclll}
x_1 & = & 3 & -\frac{1}{3}x_4 & +\frac{1}{3}x_3 \\
x_2 & = & 11 & -x_4 & +0x_3 \\
x_5 & = & 11 & -\frac{2}{3}x_4 & -\frac{1}{3}x_3 \\
x_6 & = & 3 & +\frac{1}{3}x_4 & -\frac{1}{3}x_3 \\
\hline
z & = & 25 & -\frac{7}{3}x_4 & +\frac{1}{3}x_3
\end{array}$$

Solution Associated with D3

$$
\begin{array}{rcrrr}
x_1 & = & 3 & -\frac{1}{3}x_4 & +\frac{1}{3}x_3 \\
x_2 & = & 11 & -x_4 & +0x_3 \\
x_5 & = & 11 & -\frac{2}{3}x_4 & -\frac{1}{3}x_3 \\
x_6 & = & 3 & +\frac{1}{3}x_4 & -\frac{1}{3}x_3 \\
\hline
z & = & 25 & -\frac{7}{3}x_4 & +\frac{1}{3}x_3
\end{array}
$$



Entering/Leaving Variable Analysis

$$
\begin{array}{rcrrr}
x_1 & = & 3 & -\frac{1}{3}x_4 & +\frac{1}{3}x_3 \\
x_2 & = & 11 & -x_4 & +0x_3 \\
x_5 & = & 11 & -\frac{2}{3}x_4 & -\frac{1}{3}x_3 \\
x_6 & = & 3 & +\frac{1}{3}x_4 & -\frac{1}{3}x_3 \\
\hline
z & = & 25 & -\frac{7}{3}x_4 & +\frac{1}{3}x_3 \\
 & & & \times & \checkmark
\end{array}
$$

x_3 enters and x_6 leaves.

Pivoting

$$\begin{array}{rcrrr} x_1 & = & 3 & -\frac{1}{3}x_4 & +\frac{1}{3}x_3 \\ x_2 & = & 11 & -x_4 & +0x_3 \\ x_5 & = & 11 & -\frac{2}{3}x_4 & -\frac{1}{3}x_3 \\ x_6 & = & 3 & +\frac{1}{3}x_4 & -\frac{1}{3}x_3 \\ \hline z & = & 25 & -\frac{7}{3}x_4 & +\frac{1}{3}x_3 \end{array}$$

x_3 enters and x_6 leaves.

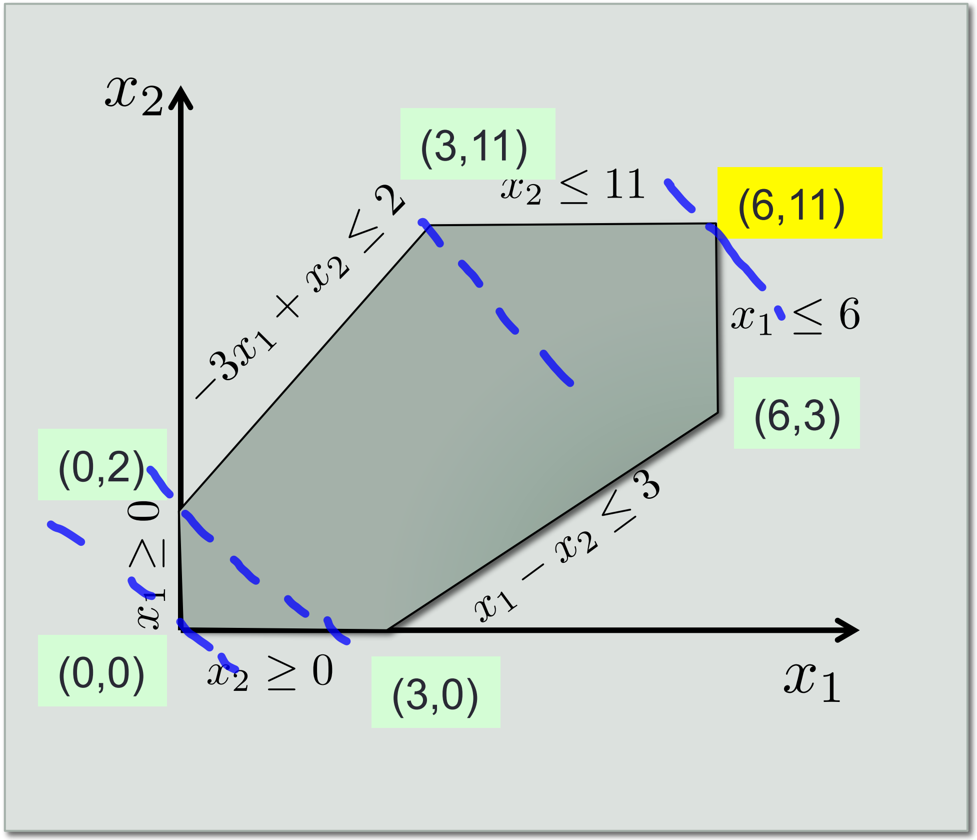
Dict. D4

$$\begin{array}{rcrrr} x_3 & = & 9 & +x_4 & -3x_6 \\ x_1 & = & 6 & & -x_6 \\ x_2 & = & 11 & -x_4 & +0x_6 \\ x_5 & = & 8 & -x_4 & +x_6 \\ \hline z & = & 28 & -2x_4 & -x_6 \end{array}$$

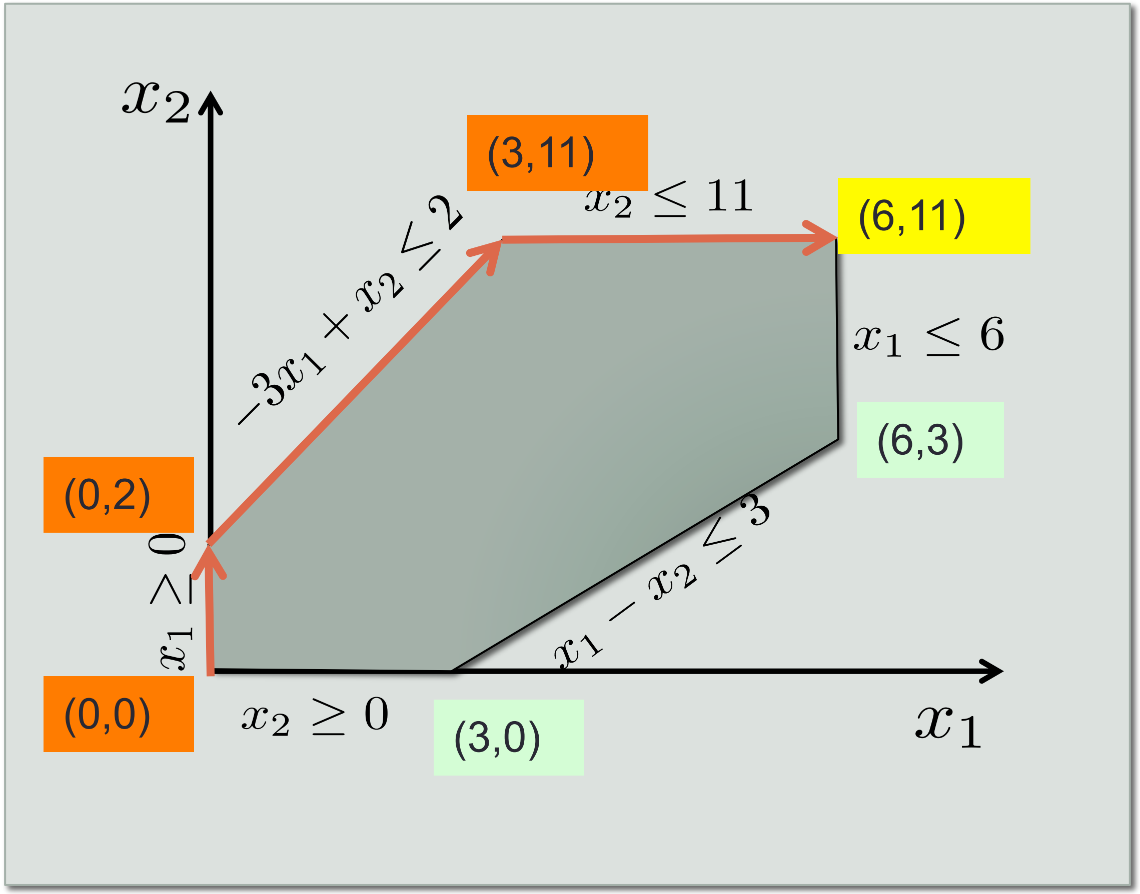
Done !!!

Final Dictionary

$$\begin{array}{ccccc} x_3 & = & 9 & +x_4 & -3x_6 \\ x_1 & = & 6 & & -x_6 \\ x_2 & = & 11 & -x_4 & +0x_6 \\ x_5 & = & 8 & -x_4 & +x_6 \\ \hline z & = & 28 & -2x_4 & -x_6 \end{array}$$

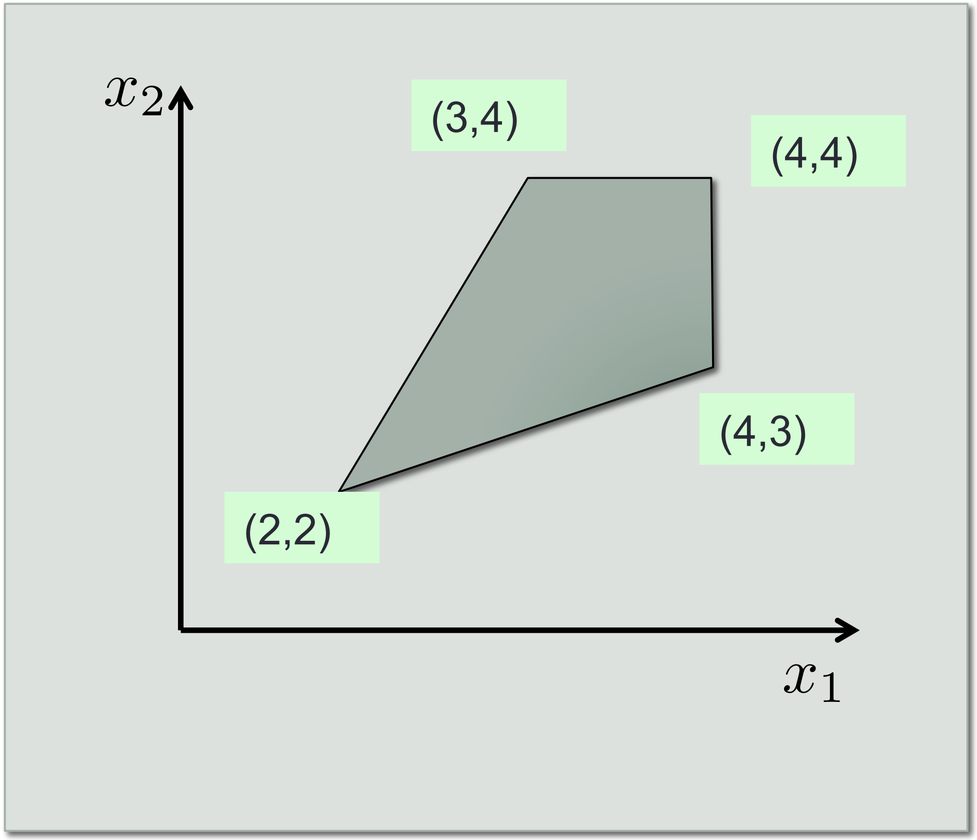


Simplex Dictionaries



Example # 2

$$
\begin{array}{rrcl}
\text{max.} & x_1 + 2x_2 & & \\
\text{s.t.} & -2x_1 + x_2 & \leq & -2 \\
 & x_2 & \leq & 4 \\
 & x_1 - 2x_2 & \leq & -2 \\
 & x_1 & \leq & 4 \\
 & x_1, x_2 & \geq & 0
\end{array}
$$



Step 1: Adding Slack

$$
\begin{array}{lrcl}
\text{max.} & x_1 + 2x_2 & & \\
\text{s.t.} & -2x_1 + x_2 + \color{red}{x_3} & = & -2 \\
& x_2 + \color{red}{x_4} & = & 4 \\
& x_1 - 2x_2 + \color{red}{x_5} & = & -2 \\
& x_1 + \color{red}{x_6} & = & 4 \\
& x_1, x_2, \color{red}{x_3, \ldots, x_6} & \geq & 0
\end{array}
$$

Step 2: Initial Dictionary

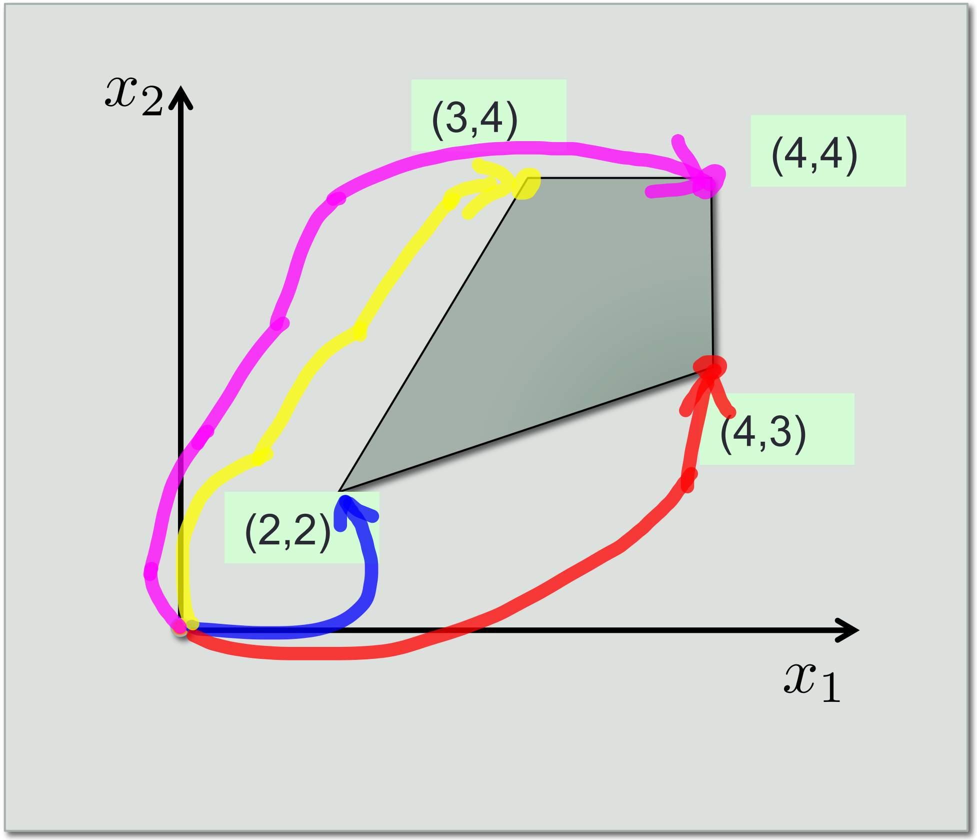
$$
\begin{array}{lrcl}
\text{max.} & x_1 + 2x_2 & & \\
\text{s.t.} & -2x_1 + x_2 + \color{red}{x_3} & = & -2 \\
 & x_2 + \color{red}{x_4} & = & 4 \\
 & x_1 - 2x_2 + \color{red}{x_5} & = & -2 \\
 & x_1 + \color{red}{x_6} & = & 4 \\
 & x_1, x_2, \color{red}{x_3, \ldots, x_6} & \geq & 0
\end{array}
$$

Infeasible

$$
\begin{array}{rcrrr}
x_3 & = & \color{red}{-2} & +2x_1 & -x_2 \\
x_4 & = & 4 & +0x_1 & -x_2 \\
x_5 & = & \color{red}{-2} & -x_1 & +2x_2 \\
x_6 & = & 4 & -x_1 & +0x_2 \\
\hline
z & = & 0 & +x_1 & +2x_2
\end{array}
$$

Infeasible Initial Dictionary

$$
\begin{array}{rclll}
x_3 & = & {\color{red}-2} & +2x_1 & -x_2 \\
x_4 & = & 4 & +0x_1 & -x_2 \\
x_5 & = & {\color{red}-2} & -x_1 & +2x_2 \\
x_6 & = & 4 & -x_1 & +0x_2 \\
\hline
z & = & 0 & +x_1 & +2x_2
\end{array}
$$



Diet Problem

- Variables

- Constraints

- Objective

- Formulate an LP (Inequality Form)

Problem Description

X_{18} - serving of carrots

- Variables

variable x_j (out of n variables) is the # of servings (units) of food j

- Healthy Diet Requirement (Constraints)

$i = 8$ - Vit A

- For nutrient i (of m nutrients) you should have at least l_i amount and at most u_i amount.

$l_8 = 5000$

$u_8 = 50000$

- In 1 serving of food j there is a_{ij} amount of nutrient i

$a_{8,18}$ - amount of VitA in a serving of carrots

- Find the Cheapest Diet (Objective)

1 serving of food j, costs c_j

C_{18} - cost per servings of carrots

Inequality Form

$$\begin{array}{ll} \text{minimize} & \boxed{c^T x} \\ \text{subject to} & \cancel{Ax \leq b} \end{array}$$

- Objective

$$\text{Total cost} = c_1 x_1 + \cdots c_n x_n$$

$$\min \underbrace{\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}^T}_{c^T} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{x}$$

Inequality Form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b \end{array}$$

- Non-Negative Constraint

$x_1 \ge 0$

$x_2 \ge 0$

$\vdots$

$\cancel{x_{18} < 0}$

$$\boxed{-\overset{n\times n}{I}\, x \le \overset{n\times 1}{\vec{0}}}$$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Inequality Form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \boxed{A}x \leq \boxed{b} \end{array}$$

- Nutrient Bounds

$$l_1 \leq a_{11} x_1 + a_{12} x_2 + \cdots a_{1n} x_n \leq u_1$$

$$\vdots$$

$$l_m \leq a_{m1} x_1 + a_{m2} x_2 + \cdots a_{mn} x_n \leq u_m$$

$$\begin{bmatrix} l_1 \\ \vdots \\ l_m \end{bmatrix} = l \leq \hat{A} x \leq u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

Inequality Form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b \end{array}$$

$-Ix \le \vec{0}$

$\hat{A}x \ge l \iff -\hat{A}x \le -l$

$\hat{A}x \le u$

$$\underbrace{\begin{bmatrix} \hat{A} \\ -\hat{A} \\ -I \end{bmatrix}}_{A} x \le \underbrace{\begin{bmatrix} u \\ -l \\ \vec{0} \end{bmatrix}}_{b}$$

food →

$$\hat{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

SOLVING LINEAR PROGRAMS WITH GLPK

Tutorial with Examples

GNU
Linear
Programming
Kits

GLPK

- Why GLPK?
  - Open source, available free of cost to everyone.
  - API for programmers + Input languages that are easy to use.

- Availability: http://www.gnu.org/software/glpk/

Downloading + Installing GLPK

www.gnu.org/software/glpk/

Introduction | Downloading | Documentation | Mailing Lists/Newsgroups | Request an Enhancement | Report a Bug | Maintainer

Introduction to GLPK

The GLPK (GNU Linear Programming Kit) package is intended for solving large-scale linear programming (LP), mixed integer programming (MIP), and other related problems. It is a set of routines written in ANSI C and organized in the form of a callable library.

GLPK supports the *GNU MathProg modeling language*, which is a subset of the AMPL language.

The GLPK package includes the following main components:

- primal and dual simplex methods
- primal-dual interior-point method
- branch-and-cut method
- translator for GNU MathProg
- application program interface (API)
- stand-alone LP/MIP solver

Downloading GLPK

The GLPK distribution tarball can be found on http://ftp.gnu.org/gnu/glpk/ [via http] and ftp://ftp.gnu.org/gnu/glpk/ [via FTP]. It can also be found on one of our FTP mirrors; please use a mirror if possible.

To make sure that the GLPK distribution tarball you have downloaded is intact you need to download the corresponding .sig file and run a command like this:

```
gpg --verify glpk-4.32.tar.gz.sig
```

If that command fails because you do not have the required public key, run the following command to import it:

```
gpg --keyserver keys.gnupg.net --recv-keys 5981E818
```

and then re-run the previous command.

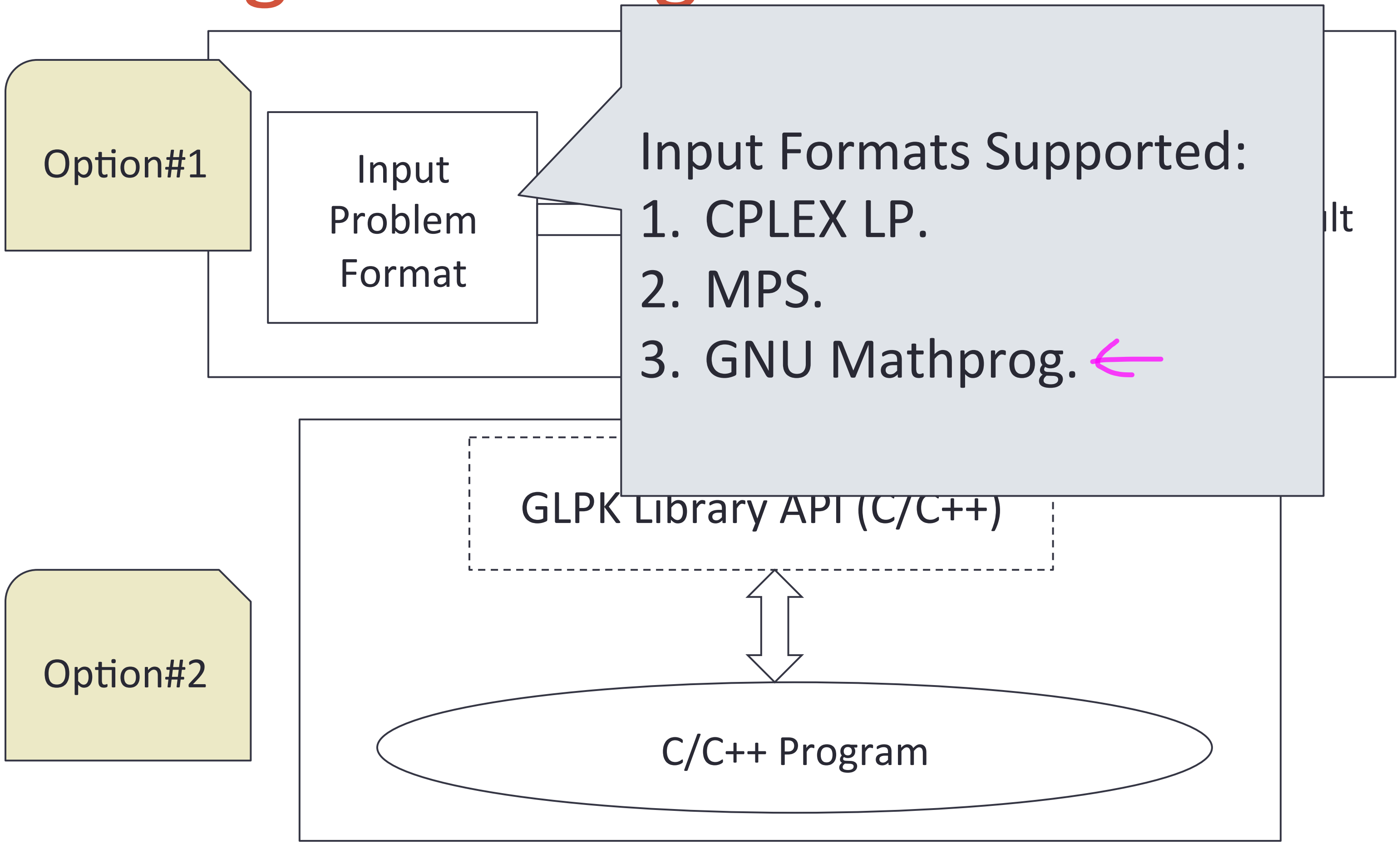
Documentation

The GLPK documentation consists of the Reference Manual and the description of the GNU MathProg modeling language. Both these documents are included in the distribution (in LaTeX, DVI, and PostScript formats).

Mailing Lists/Newsgroups

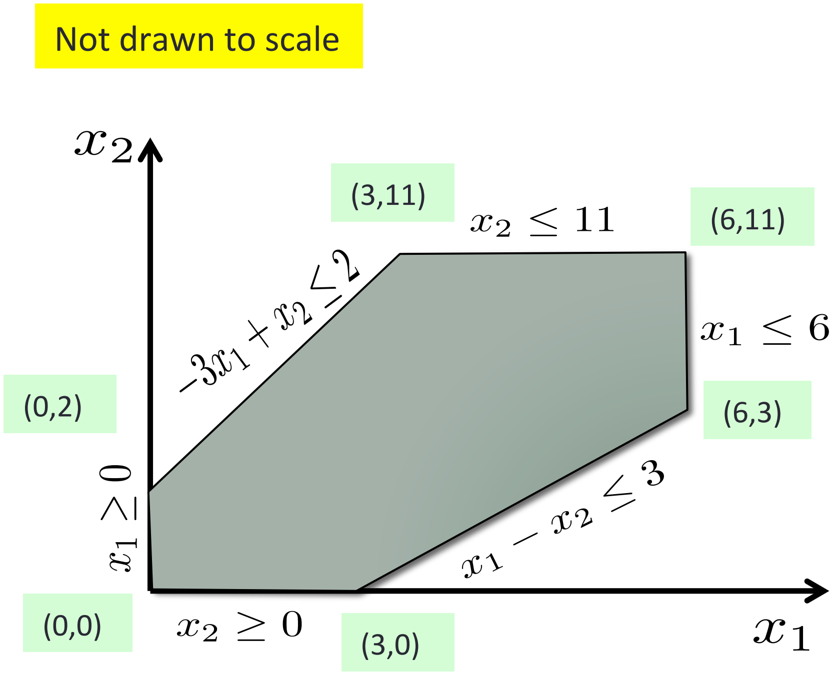
GLPK has two mailing lists: *help-glpk@gnu.org* and *bug-glpk@gnu.org*.

Solving LPs through GLPK



Example #1

$$
\begin{array}{llllll}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$



Solution: $x_1 = 6$, $x_2 = 11$Opt. Objective Value: 28

Specifying Problem: Mathprog Format

$$
\begin{array}{lrrcr}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$

ex1.ampl

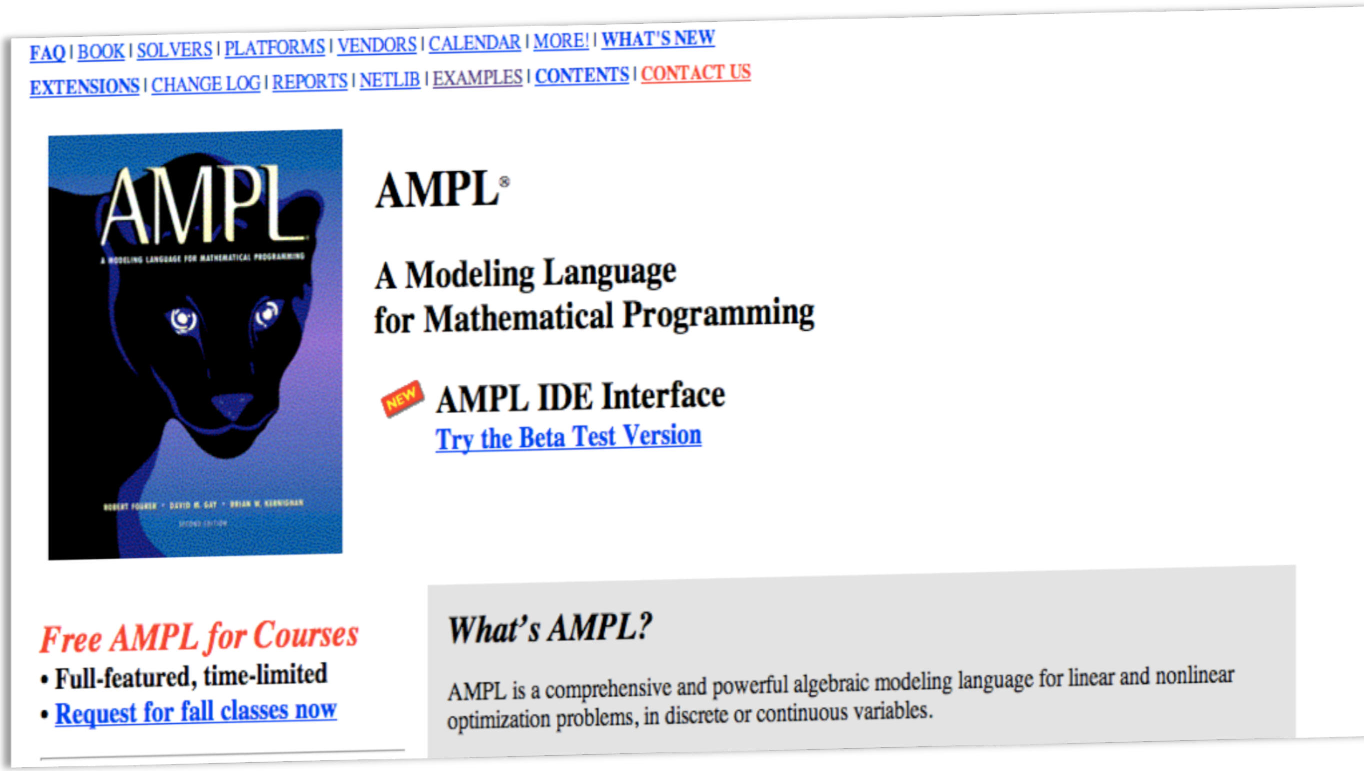
```
var x1 >= 0;
var x2 >= 0;
maximize obj: x1 + 2 * x2;
c1: -3 * x1 + x2 <= 2;
c2: x2 <= 11;
c3: x1 - x2 <= 3;
c4: x1 <= 6;
solve;
display x1;
display x2;
end;
```

Running GLPSOL

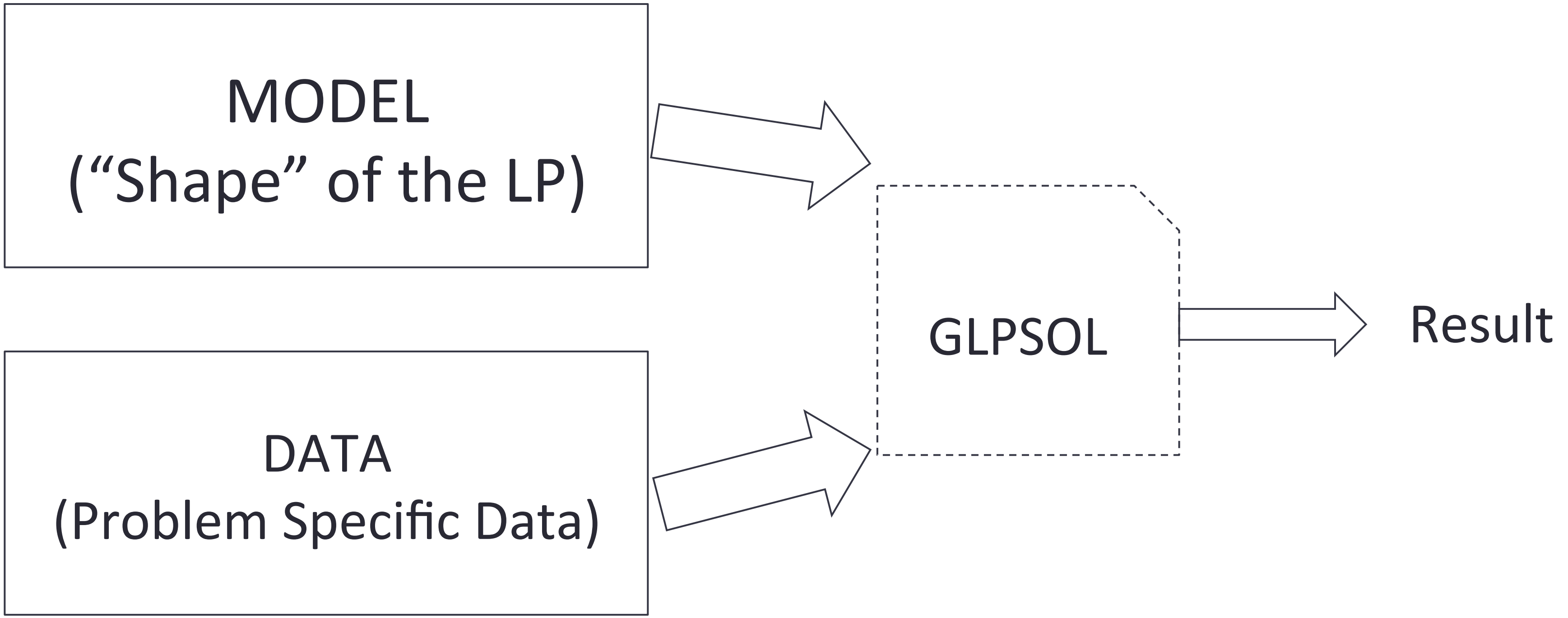
```
bash-3.2$ glpsol --math ex1.ampl
GLPSOL: GLPK LP/MIP Solver, v4.48
OPTIMAL SOLUTION FOUND
Time used:   0.0 secs
Display statement at line 12
x1.val = 6
Display statement at line 13
x2.val = 11
Model has been successfully processed
```

Using Mathprog Language

- Very close to AMPL



Mathprog input format



Example

$$\begin{array}{rrrrr} \text{max.} & x_1 & +2x_2 & & \\ \text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\ & & +x_2 & \leq & 11 \\ & x_1 & -x_2 & \leq & 3 \\ & x_1 & & \leq & 6 \\ & x_1, & x_2 & \geq & 0 \end{array}$$

"Shape of the LP"

$$\begin{array}{llll} \max & \sum_{j=1}^{n} \boxed{c_i} x_i & & \\ \text{s.t.} & \sum_{j=1}^{n} \boxed{a_{i,j}} x_j \leq \boxed{b_i} & & [i \in \{1, \ldots, \boxed{m}\}] \\ & x_j \geq 0 & & [j \in \{1, \ldots, \boxed{n}\}] \end{array}$$

Mathprog Format

```
# number of constraints
param m;
# number of decision variables
param n;

#problem parameters
param c { i in 1..n};
param A { i in 1..m, j in 1..n};
param b { i in 1..m};

#declare variables
var x { i in 1..n} >= 0;
```

c[1] .. c[n]

A[i,j]

"Shape of the LP"

$$
\begin{array}{llccl}
\max & \sum_{j=1}^{n} c_i x_i & & & \\
\text{s.t.} & \sum_{j=1}^{n} a_{i,j} x_j & \leq & b_i & [i \in \{1, \ldots, m\}] \\
 & x_j & \geq & 0 & [j \in \{1, \ldots, n\}]
\end{array}
$$

```
#objective
maximize obj: sum{ i in 1..n} c[i] * x[i];
s.t.
e{j in 1..m}: sum{i in 1..n} A[j,i] * x[i] <= b[j];

solve;
display x;
end;
```

Running GLPSOL

```
$ glpsol --model standardForm.model --data ex1.data
GLPSOL: GLPK LP/MIP Solver, v4.48
OPTIMAL SOLUTION FOUND
Time used:   0.0 secs
Memory used: 0.1 Mb (130844 bytes)
Display statement at line 20
x[1].val = 6
x[2].val = 11
Model has been successfully processed
```

SOLVING THE DIET PROBLEM IN GLPK

Modeling the Diet Problem

cost : $\begin{pmatrix} \vdots \end{pmatrix}$

- Problem Data

Name	Type	Meaning
NFoods	SCALAR (INT)	Number of Food Items
NNutrients	SCALAR (INT)	Number of Nutrients
costs	NFoods x 1	Cost per unit of each food
caloricData	Nfoods x NNutrients	Nutrients per unit of each food.
upperBnd	NNutrients x 1	Upper bound on nutrients reqd.
lowerBnd	NNutrients x 1	Lower bound on nutrients reqd.

nutri

(food |||

```
param NFoods;
param NNutrients;
param caloricDat {i in 1..NFoods, j in 1..NNutrients};
param lb {i in 1..NNutrients};
param ub {i in 1..Nnutrients};
param costs { i in 1..NFoods};

var x { i in 1..NFoods} >= 0;
```

x[1] .. x[j] . .. x[Nfood]

$$\sum_{i=1}^{Nfood} cost[i] * x_i$$

```
minimize obj: sum{i in 1..NFoods} costs[i] * x[i];

bnds {k in 1..NNutrients}:
   lb[k]<= sum{i in 1..NFoods} x[i]*caloricDat[i,k] <=
ub[k];

solve;
display x;
```

```
param NFoods := 64;
param NNutrients := 11;

param caloricMatrix :
          1      2      3      4       5      6      7      8       9      10     11
:=
     1   73.8    0     0.8    68.2    13.6    8.5    8    5867.4  160.2   159    2.3
     2   23.7    0     0.1    19.2     5.6    1.6   0.6   15471     5.1   14.9   0.3
     3    6.4    0     0.1    34.8     1.5    0.7   0.3    53.6     2.8    16    0.2
     4   72.2    0     0.6     2.5    17.1    2     2.5   106.6     5.2    3.3   0.3
     5    2.6    0     0       1.8     0.4    0.3   0.2    66       0.8    3.8   0.1
     6   20      0     0.1     1.5     4.8    1.3   0.7   467.7    66.1    6.7   0.3
     7  171.5    0     0.2    15.2    39.9    3.2   3.7     0      15.6   22.7   4.3
     8   88.2    0     5.5     8.1     2.2    1.4   9.4    98.6     0.1  121.8   6.2
     9  277.4  129.9  10.8   125.6     0      0    42.2    77.4     0     21.9   1.8
    10  358.2    0    12.3  1237.1    58.3   11.6   8.2  3055.2    27.9   80.2   2.3
```

nfood

```
$ glpsol --model diet.model --data diet.dat
GLPSOL: GLPK LP/MIP Solver, v4.48
OPTIMAL SOLUTION FOUND
Time used:   0.0 secs
Memory used: 0.4 Mb (377283 bytes)
Display statement at line 19
x[1].val = 0
x[2].val = 0.235817810889784
x[3].val = 0
x[4].val = 0
x[5].val = 0
x[6].val = 0
x[7].val = 3.54494477652071
x[8].val = 0
x[9].val = 0
x[10].val = 0
. . .
```

64

More Advanced Model

```
set Foods;
set Nutrients;

param calDat { i in Foods, j in Nutrients};
param bounds { i in Nutrients, j in 1..2};
param costs { i in Foods};

var x { i in Foods} >= 0;

minimize obj: sum{i in Foods} costs[i] * x[i];
s.t. bndConstr {k in Nutrients}:
     bounds[k,1] <= sum{i in Foods} x[i] * calDat[i,k] <= bounds[k,2];
solve;
display x;
printf '----- \n';
printf {i in Foods: x[i] >= 0.001}: 'Optimizer: %f units of food %s \n', x[i], i;
printf ' The bill for the food will be \$ %f \n', obj;
printf 'Bon appetit! \n -----\n’;
```

{ dietSet.model
dietSet.dat

Result

```
bash-3.2$ glpsol --model dietSet.model --data dietSet.data
GLPSOL: GLPK LP/MIP Solver, v4.48
-----
Optimizer says to eat 0.235818 units of food Carrots_Raw
Optimizer says to eat 3.544945 units of food Potatoes_Baked
Optimizer says to eat 2.167849 units of food Skim_Milk
Optimizer says to eat 3.600776 units of food Peanut_Butter
Optimizer says to eat 4.823229 units of food Popcorn_Air-
Popped
 The bill for the food will be $ 0.956008
Bon appetit!
 -----
Model has been successfully processed
```

SOLVING LINEAR PROGRAMS USING EXCEL

Microsoft Excel

- Spreadsheet software.
- Product marketed by Microsoft.
- Widely used in the industry.

- Supports data analysis tools:
  - Includes a “Solver” that can be used to solve LPs.

Our Favorite Example

$$
\begin{array}{rrrcr}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$

Solution: $x_1 = 6$, $x_2 = 11$
Optimal Objective Value: 28

Solving in Excel