CSCI5654 (Linear Programming, Fall 2013)
Lecture-8

Today's Lecture

1. Recap of dual variables and strong duality.
2. Complementary Slackness Theorem.
3. Interpretation of dual variables.
4. Primal and Dual Simplex.

Primal-Dual correspondences

Variable to constraint correspondence:

Dual variable (y_j)	$\leftrightarrow$	Primal constraint $A_j \vec{x} \leq b_j$
Primal variable (x_i)	$\leftrightarrow$	Dual constraint $A_i^{\mathrm{T}} \vec{y} \geq c_i$

Correspondence in dictionary:

Dual decision variable y_j	$\leftrightarrow$	Primal slack variable x_{n+j}.
Primal decision variable (x_i)	$\leftrightarrow$	Dual dictionary slack variable y_{m+i}.

Illustrate in class with an example.

Strong Duality Theorem: If $x_1^*, \ldots, x_m^*$ is primal optimal then there exists a dual optimal solution $y_1^*, \ldots, y_m^*$ that satisifies

$$\sum_i c_i x_i^* = \sum_j b_j y_j^* \; .$$

($\therefore$ if primal has optimal solution then the dual has optimum with the same value as primal).

Proof: This is theorem 5.1 in Chvátal (pages 58-59).

Example

$$
\begin{array}{rrcrcl}
\text{maximize} & x_1 & + & 2x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 3 \\
 & & & x_2 & \leq & 3 \\
 & -x_1 & + & x_2 & \leq & 1 \\
 & x_1 & + & x_2 & \leq & 5 \\
 & & & x_1, x_2 & \geq & 0
\end{array}
$$

Example (final dictionary)

$$
\begin{array}{rcccccc}
x_1 & = & 2 & + & \frac{x_5}{2} & - & \frac{x_6}{2} \\
x_2 & = & 3 & - & \frac{x_5}{2} & + & \frac{x_6}{2} \\
x_3 & = & \cdots & & & & \\
x_4 & = & \cdots & & & & \\
\hline
z & = & 8 & - & \frac{1}{2}x_5 & - & \frac{3}{2}x_6
\end{array}
$$

Insight: Dual variables correspond to primal slack variables.

$$x_3 \leftrightarrow y_1, x_4 \leftrightarrow y_2, x_5 \leftrightarrow y_3, x_6 \leftrightarrow y_4 \, .$$

Read off dual solution by from objective row of final dictionary:

$$y_1 : 0, y_2 : 0, y_3 : \frac{1}{2}, y_4 : \frac{3}{2} \, .$$

Proof of Strong Duality

Final Dictionary:

$$
\begin{array}{rcl}
x_i & = & b_i + \sum_{j \in \text{Independent}} a_{ij} x_j \\
 & \vdots & \\
\hline
z & = & z^* + c_1^* x_1 + \cdots + c_{n+m}^* x_{n+m}
\end{array}
$$

Note: $c_i^* = 0$ if $i \in$ Basis
$c_i^* \leq 0$ if $i \in$ Independent.

Claim: Set variable $y_i = -(c_{n+i}^*)$. This is a dual feasible solution with optimal value z^*.

Relationship Between Primal/Dual

		Primal		
		Optimal	Infeasible	Unbounded
Dual	Optimal	Possible	Impossible	Impossible
	Infeasible	Impossible	Possible	Possible
	Unbounded	Impossible	Possible	Impossible

Primal and Dual Infeasible

Primal:

$$\begin{array}{rrcr} \text{max.} & x_2 & & \\ \text{s.t.} & x_1 & \leq & -1 \\ & -x_2 & \leq & -1 \\ & x_1, x_2 & \geq & 0 \end{array}$$

Dual:

$$\begin{array}{rrcr} \text{min.} & -y_1 - y_2 & & \\ \text{s.t.} & y_1 & \geq & 0 \\ & -y_2 & \geq & 1 \\ & y_1, y_2 & \geq & 0 \end{array}$$

Complementary Slackness Theorem I

Let $x_1, \ldots, x_n, \underbrace{x_{n+1}, \ldots, x_{n+m}}_{slack}$ be a primal feasible.

Let $y_1, \ldots, y_m, \underbrace{y_{m+1}, \ldots, y_{m+n}}_{slack}$ be dual feasible.

Slack Variables:

$$\begin{array}{rcl} x_{n+i} & = & b_i - \sum_{k=1}^{n} a_{ik} x_j \\ y_{m+j} & = & c_k - \sum_{k=1}^{m} a_{kj} y_j \end{array}$$

The following conditions are necessary and sufficient for $\vec{x}$ and $\vec{y}$ to be optimal solutions to the primal and dual:

$$\begin{array}{c} x_i y_{m+i} = 0,\ j = 1, \ldots, n \\ y_j x_{n+j} = 0,\ i = 1, \ldots, m \end{array}$$

Proof: $\vec{x}$ is a primal feasible solution. Therefore,

$$A\vec{x} \leq \vec{b},\ \vec{x} \geq 0\,.$$

Complementary Slackness Theorem II

Similarly, $\vec{y}$ is dual feasible. Therefore,

$$A^{\mathrm{T}}\vec{y} \geq \vec{c},\ \ \vec{y} \geq 0\,.$$

Applying these facts, we obtain

$$\begin{aligned} \vec{c}^{\,\mathrm{T}}\vec{x} &\leq (A^{\mathrm{T}}\vec{y})^{\mathrm{T}}\vec{x} (\equiv \vec{y}^{\,\mathrm{T}}A\vec{x}) \\ &\leq \vec{y}^{\,\mathrm{T}}\vec{b} \end{aligned}$$

$\vec{x}$ and $\vec{y}$ can be optimal solutions for primal/dual resp. iff the inequalities above all hold with equalities.
Therefore,

$$\vec{y}^{\,\mathrm{T}}\vec{b} = \vec{y}^{\,\mathrm{T}}A\vec{x} \tag{1}$$

$$\vec{c}^{\,\mathrm{T}}\vec{x} = \vec{y}^{\,\mathrm{T}}A\vec{x} \tag{2}$$

Complementary Slackness Theorem III

From the equation (1), we obtain

$$\vec{y}^{\mathrm{T}}(\vec{b} - Ax) = 0, \text{ or} \sum_{i=1}^{m} y_i(b_i - A_i\vec{x}) = 0\,.$$

Note that $y_i \geq 0$ and $b_i - A_i\vec{x} \geq 0$. Sum of non-negative terms is zero iff each of the individual terms are zero. Therefore

$$y_i(b_i - A_i\vec{x}) = 0$$

Note that $b_i - A_i\vec{x}$ is just the slack variable x_{n+i}.
Considering the first equation will derive the remainder of the required complementary slackness conditions.

Duality: Interpretation

Diet problem: Minimize cost, while satisfying dietary constraints.

	FX ($2)	FY ($5)	FZ ($ 15)
Carbs	20	1	1
Protein	1	30	40
Vitamins	1	10	5

1. At least 200units of carbs.
2. At least 50 units of protien.
3. At least 40 units of vitamins.
4. No more than 20 units of FX.
5. No more than 10 units of FY.
6. No more than 5 units of FZ.

Diet Problem

$$
\begin{array}{lrcrcl}
\text{max.} & -2x_1 - 5x_2 - 15x_3 & & & & \\
& -20x_1 - x_2 - x_3 & \leq & -200 & \leftarrow & \text{CARBS} \\
& -x_1 - 30x_2 - 40x_3 & \leq & -50 & \leftarrow & \text{PROT} \\
& -x_1 - 10x_2 - 5x_3 & \leq & -40 & \leftarrow & \text{VIT} \\
& x_1 & \leq & 20 & \leftarrow & \text{LIM FX} \\
& x_2 & \leq & 10 & \leftarrow & \text{LIM FY} \\
& x_3 & \leq & 5 & \leftarrow & \text{LIM FZ} \\
& x_1, x_2, x_3 & \geq & 0 & &
\end{array}
$$

Optimal: $x_1 \sim 9.84, x_2 \sim 3, x_3 : 0.$

Dual: $y_1 : 0.07, y_2 : 0, y_3 : 0.5, y_4, \ldots, y_6 : 0.$

Shadow costs

$$\begin{array}{lrcrcl}
\text{max.} & -2x_1 - 5x_2 - 15x_3 & & & & \\
& -20x_1 - x_2 - x_3 & \leq & -200 & \leftarrow & \text{CARBS(0.07)} \\
& -x_1 - 30x_2 - 40x_3 & \leq & -50 & \leftarrow & \text{PROT(0)} \\
& -x_1 - 10x_2 - 5x_3 & \leq & -40 & \leftarrow & \text{VIT(0.5)} \\
& x_1 & \leq & 20 & \leftarrow & \text{LIM FX 0} \\
& x_2 & \leq & 10 & \leftarrow & \text{LIM FY 0} \\
& x_3 & \leq & 5 & \leftarrow & \text{LIM FZ 0} \\
& x_1, x_2, x_3 & \geq & 0 & &
\end{array}$$

Experiment: Play around with RHS of inequalities and observe solution.

Shadow Cost (Margin Cost)

$$
\begin{array}{lrcrcl}
\text{max.} & -2x_1 - 5x_2 - 15x_3 & & & & \\
 & -20x_1 - x_2 - x_3 & \leq & -200 & \leftarrow & \text{CARBS(0.07)} \\
 & -x_1 - 30x_2 - 40x_3 & \leq & -50 & \leftarrow & \text{PROT(0)} \\
 & -x_1 - 10x_2 - 5x_3 & \leq & -40 & \leftarrow & \text{VIT(0.5)} \\
 & x_1 & \leq & 20 & \leftarrow & \text{LIM FX 0} \\
 & x_2 & \leq & 10 & \leftarrow & \text{LIM FY 0} \\
 & x_3 & \leq & 5 & \leftarrow & \text{LIM FZ 0} \\
 & x_1, x_2, x_3 & \geq & 0 & &
\end{array}
$$

CARBS: \$0.07, PROT: free, VIT: \$0.5

Q1: If dual var. corresponding to a constraint is 0 does that mean that we can drop the constraint and still get the same answer?
A1: YES.
Q2: Can we, then, change the RHS of the inequality to arbitrary values?
A2: NO.

Sensitivity Theorem

Consider **Primal Problem:** decision variables are $x_1, \ldots, x_n$.

$$\begin{array}{rcccccccl} \text{maximize} & c_1 x_1 & + & \cdots & + & c_n x_n & & & \\ \text{s.t.} & a_{j1} x_1 & + & \cdots & + & a_{jn} x_n & \leq & b_j & j \in \{1, 2, \ldots, m\} \\ & & & & & x_1, \ldots, x_n & \geq & 0 & \end{array}$$

Optimal Value: z^*.
Primal Optimal: $x_1^*, \ldots, x_n^*$. **Dual Optimal:** $y_1^*, \ldots, y_n^*$.
Final simplex dictionary is <u>non-degenerate</u>.
Modified LP:

$$\begin{array}{rcccccccl} \text{maximize} & c_1 x_1 & + & \cdots & + & c_n x_n & & & \\ \text{s.t.} & a_{j1} x_1 & + & \cdots & + & a_{jn} x_n & \leq & b_j + t_j & j \in \{1, 2, \ldots, m\} \\ & & & & & x_1, \ldots, x_n & \geq & 0 & \end{array}$$

Optimal Value: There is $\epsilon > 0$ s.t., if $|t_i| \leq \epsilon$, $z' = z^* + \sum_i t_i y_i$.
Shadow Cost: The dual decision variables are interpreted as "shadow cost".

$$\boxed{\text{Primal-Dual Simplex}}$$

Primal and Dual

Consider the following LP:

$$\begin{array}{lrrcr}
\text{max.} & -x_1 & -x_2 & & \\
& -2x_1 & -x_2 & \leq & 4 \\
& -2x_1 & +4x_2 & \leq & -8 \\
& -x_1 & +3x_2 & \leq & -7 \\
& & x_1, x_2 & \geq & 0
\end{array}$$

Dual LP

The dual LP for our example:

$$\begin{array}{rrrrcl} \text{max.} & -4y_1 & +8y_2 & +7y_3 & & \\ & 2y_1 & +2y_2 & +y_3 & \leq & 1 \\ & y_1 & -4y_2 & -3y_3 & \leq & 1 \\ & & & y_1, y_2, y_3 & \geq & 0 \end{array}$$

Primal-Dual Dictionaries

$$
\begin{array}{r|rrr}
x_3 & 4 & +2x_1 & +x_2 \\
x_4 & -8 & +2x_1 & -4x_2 \\
x_5 & -7 & +x_1 & -3x_2 \\
\hline
z & & -x_1 & -x_2
\end{array}
$$

$$
\begin{array}{r|rrrr}
y_4 & 1 & -2y_1 & -2y_2 & -y_3 \\
y_5 & 1 & -y_1 & +4y_2 & +3y_3 \\
\hline
-w & & -4y_1 & +8y_2 & +7y_3
\end{array}
$$

Note: Dual dictionary y_2 enters and y_4 leaves.

Primal-Dual Dictionaries

Complementary pairs:

$$x_3 \leftrightarrow y_1, x_4 \leftrightarrow y_2, x_5 \leftrightarrow y_3$$
$$x_1 \leftrightarrow y_4, x_2 \leftrightarrow y_5$$

Rule:

- v_1 enters and v_2 leaves dual dictionary.
  v_1^c leaves and v_2^c enters primal dictionary.
- v_1^c enters and v_2^c leaves primal dictionary.
  v_1 leaves and v_2 enters dual dictionary.

Primal Dual Example

$$
\begin{array}{r|llll}
y_2 & .5 & -y_1 & -.5y_3 & -.5y_4 \\
y_5 & 3 & -5y_1 & +y_3 & -2y_4 \\
\hline
-w & 4 & -12y_1 & +3y_3 & -4y_4
\end{array}
$$

$$
\begin{array}{r|lll}
x_3 & 12 & +x_4 & +5x_2 \\
x_5 & -3 & +.5x_4 & -x_2 \\
x_1 & 4 & +.5x_4 & +2x_2 \\
\hline
z & -4 & -.5x_4 & -3x_2
\end{array}
$$

Primal Dual Examples

$$
\begin{array}{r|rrrr}
y_3 & 1 & -2y_1 & -2y_2 & -y_4 \\
y_5 & 4 & -7y_1 & -2y_2 & -3y_4 \\
\hline
-w & 7 & -18y_1 & -6y_2 & -7y_4
\end{array}
$$

Can you write down the primal dictionary??

Dual Simplex

- Easy transformation of primal to dual and vice versa.
- Entering variable for dual $\Rightarrow$ complementary variable leaves primal dictionary.
- Leaving variable $\Rightarrow$ complementary variable enters primal.
- Unbounded dual $\Rightarrow$ infeasible primal (and vice versa).

Why Dual Simplex?

1. When primal objective coefficients are all non-positive.
   Avoid need for an initialization phase.
2. When primal has more variables than constraints.
   Simplex usually observed to be sensitive to number of variables.
3. As an alternative to the initialization phase!

$$\boxed{\text{Initialization using Dual Simplex}}$$

Example

$$\begin{array}{lrrrr}
\text{maximize} & -x_1 & +4x_2 & & \\
\text{subjectto} & -2x_1 & -x_2 \leq & 4 & \\
 & -2x_1 & +4x_2 & \leq & -8 \\
 & -x_1 & +3x_2 & \leq & -7 \\
 & & x1, x2 & \geq & 0
\end{array}$$

Dual:

$$\begin{array}{lrrrrr}
\max & -4y_1 & +8y_2 & +7y_3 & & \\
 & 2y_1 & +2y_2 & +y_3 & \leq & 1 \\
 & y_1 & -4y_2 & -3y_3 & \leq & -4 \\
 & & & y_1, y_2, y_3 & \geq & 0
\end{array}$$

Problem: Neither primal nor dual initial dictionary is feasible.

Initialization using Dual

Idea: Change of primal objective function can make dual initial dictionary feasible!!

$$-x_1 + 4x_2 \to -x_1 - x_2$$

Dual:

$$\begin{array}{lrrrcl} \max & -4y_1 & +8y_2 & +7y_3 & & \\ & 2y_1 & +2y_2 & +y_3 & \leq & 1 \\ & y_1 & -4y_2 & -3y_3 & \leq & 1 \\ & & & y_1, y_2, y_3 & \geq & 0 \end{array}$$

The change ensures that dual starts at a feasible dictionary.

Initialization Using Dual

- Dualize given problem.
- If dual starting dictionary is infeasible, then change primal objective to make initial dual dictionary feasible.
- Perform dual simplex until a dual optimal dictionary is reached.
- Change final dual dictionary into corresponding primal dictionary.
- Replace the objective function, taking into account new basis variables.