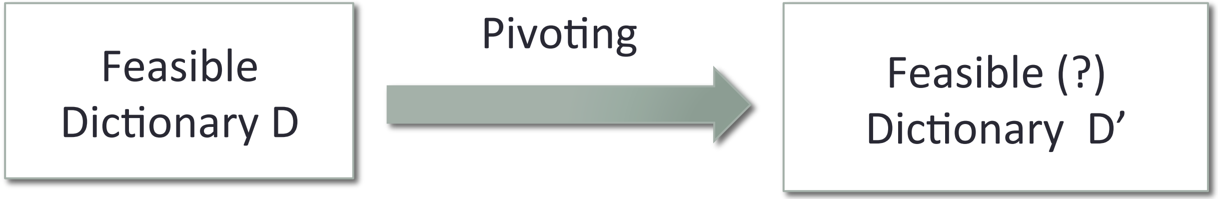
PIVOTING PRESERVES FEASIBILITY: PROOF SKETCH

To Prove

- Suppose D is a feasible but non-final dictionary and we perform a valid pivoting step in the Simplex algorithm to get to dictionary D’ then D’ is also feasible.



Before proof, a simple exercise.

$$\begin{array}{rcrrr} x_2 & = & 2 & +3x_1 & -x_3 \\ x_4 & = & 9 & -3x_1 & +x_3 \\ x_5 & = & 5 & +2x_1 & -x_3 \\ x_6 & = & 6 & -x_1 & +0x_3 \\ \hline z & = & 4 & +7x_1 & -2x_3 \end{array}$$

x_1 enters and x_4 leaves.

What is the solution associated with the next dictionary?

Dictionary

$$
\begin{array}{ccccccccc|l}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} & & \rightarrow\ x_{Ij} \leq \frac{b_1}{-a_{1j}} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} & & \rightarrow\ x_{Ij} \leq \infty \\
 & \vdots & & & & & & & & \\
x_{Bi} & = & b_i & +a_{i1}x_{I1} & +\cdots & +\mathbf{a_{ij}}x_{Ij} & +\cdots & +a_{in}x_{In} & & \boxed{\rightarrow x_{Ij} \leq \frac{b_i}{-a_{ij}}} \\
 & \vdots & & & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} & & \rightarrow x_{Ij} \leq \frac{b_m}{-a_{mj}} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In} & &
\end{array}
$$

Dictionary

$$
\begin{array}{ccccccc|l}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} & \rightarrow x_{Ij} \le \frac{b_1}{-a_{1j}} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} & \rightarrow x_{Ij} \le \infty \\
 & \vdots & & & & & & & \\
x_{Bi} & = & b_i & +a_{i1}x_{I1} & +\cdots & +\mathbf{a_{ij}}x_{Ij} & +\cdots & +a_{in}x_{In} & \boxed{\rightarrow x_{Ij} \le \frac{b_i}{-a_{ij}}} \\
 & \vdots & & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} & \rightarrow x_{Ij} \le \frac{b_m}{-a_{mj}} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In} &
\end{array}
$$

Another Fact About Simplex

- During the optimization phase of Simplex,
  - The value of the objective cannot decrease due to a pivoting step.

What is the value of the Objective after pivot?

$$
\begin{array}{cclllllll|l}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} & \rightarrow & x_{Ij} \leq \frac{b_1}{-a_{1j}} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} & \rightarrow & x_{Ij} \leq \infty \\
 & \vdots & & & & & & & & \\
x_{Bi} & = & b_i & +a_{i1}x_{I1} & +\cdots & +\mathbf{a_{ij}}x_{Ij} & +\cdots & +a_{in}x_{In} & & \boxed{\rightarrow x_{Ij} \leq \frac{b_i}{-a_{ij}}} \\
 & \vdots & & & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} & \rightarrow & x_{Ij} \leq \frac{b_m}{-a_{mj}} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In} & &
\end{array}
$$

Degenerate Dictionary

$$
\begin{array}{rcrrrr}
x_3 & = & .5 & & & -.5x_4 \\
x_5 & = & 0 & -2x_1 & +4x_2 & +3x_4 \\
x_6 & = & 0 & +x_1 & -3x_2 & +2x_4 \\
\hline
z & = & 4 & +2x_1 & -x_2 & -4x_4
\end{array}
$$

Degeneracy Definition

$$
\begin{array}{ccccccc|l}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} & \rightarrow\ x_{Ij} \leq \frac{b_1}{-a_{1j}} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} & \rightarrow\ x_{Ij} \leq \infty \\
 & \vdots & & & & & & & \\
x_{Bi} & = & b_i & +a_{i1}x_{I1} & +\cdots & +\mathbf{a_{ij}}x_{Ij} & +\cdots & +a_{in}x_{In} & \boxed{\rightarrow x_{Ij} \leq \frac{b_i}{-a_{ij}}} \\
 & \vdots & & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} & \rightarrow x_{Ij} \leq \frac{b_m}{-a_{mj}} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In} &
\end{array}
$$

Interesting Fact

If value of objective remains same in next Dictionary after pivoting then the current dictionary is degenerate.

$$
\begin{array}{ccccccc|l}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} & \rightarrow\ x_{Ij} \leq \frac{b_1}{-a_{1j}} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} & \rightarrow\ x_{Ij} \leq \infty \\
 & \vdots & & & & & & & \\
x_{Bi} & = & b_i & +a_{i1}x_{I1} & +\cdots & +\mathbf{a_{ij}}x_{Ij} & +\cdots & +a_{in}x_{In} & \boxed{\rightarrow x_{Ij} \leq \frac{b_i}{-a_{ij}}} \\
 & \vdots & & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} & \rightarrow x_{Ij} \leq \frac{b_m}{-a_{mj}} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In} &
\end{array}
$$