INITIALIZATION PHASE SIMPLEX

AUXILLIARY PROBLEM

$$\begin{array}{lccc} \max & \mathbf{c}^{\mathsf{T}}\ \mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} + \mathbf{x_s} & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x_s} & \geq & 0 \end{array} \quad \Longrightarrow \quad \begin{array}{lccc} \max & -x_0 & & \\ \text{s.t.} & A\mathbf{x} + \mathbf{x_s} - x_0\mathbf{1} & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x_s}, x_0 & \geq & 0 \end{array}$$

1. Aux. problem cannot be unbounded.

2. Aux. problem is always feasible.

Initialization Phase Simplex

$$
\begin{array}{lrcl}
\max & -x_0 & & \\
\text{s.t.} & A\mathbf{x} + \mathbf{x_s} - x_0\mathbf{1} & = & \mathbf{b} \\
& \mathbf{x}, \mathbf{x_s}, x_0 & \geq & 0
\end{array}
$$

If opt. value = 0 then form initial feasible dictionary for original problem.

Initial Aux. Dictionary $\xrightarrow{\text{Pivoting Steps*}}$ Final Aux. Dictionary

If opt. value < 0 then problem infeasible.

Initialization Phase Simplex

1. How to form initial dictionary for Aux. Problem?

2. How to perform pivoting?
   - Minor modifications.

3. How to find initial dictionary for original problem?

Initial Dictionary For Aux Problem.

$$\begin{array}{llcl} \max & -x_0 & & \\ \text{s.t.} & A\mathbf{x} + \mathbf{x_s} - x_0\mathbf{1} & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x_s}, x_0 & \geq & 0 \end{array}$$

$$\begin{array}{rcrrr} \mathbf{x_s} & = & \mathbf{b} & +\mathbf{1}x_0 & -A\mathbf{x} \\ \hline z & = & 0 & -x_0 & \end{array}$$

Special Move:

Entering Variable is x_0

Leaving variable is the variable with least value of b

Initial Dictionary For Aux. Problem

$$
\begin{array}{rclllll}
x_{B1} & = & b_1 & \color{red}{+1x_0} & +a_{11}x_{I1} & + \cdots & +a_{1n}x_{In} \\
x_{B2} & = & b_2 & \color{red}{+1x_0} & +a_{21}x_{I1} & + \cdots & +a_{2n}x_{In} \\
 & \vdots & & & & & \\
\mathbf{x_{Bj}} & = & \mathbf{b_j} & \color{red}{+1x_0} & +\mathbf{a_{j1}x_{I1}} & + \cdots & +\mathbf{a_{jn}x_{In}} \\
 & \vdots & & & & & \\
x_{Bm} & = & b_m & \color{red}{+1x_0} & +a_{m1}x_{I1} & + \cdots & +a_{mn}x_{In} \\
\hline
z & = & 0 & \color{red}{-x_0} & & &
\end{array}
$$

$$b_j = \min(b_1, \ldots, b_m)$$

Example

$$
\begin{array}{lllllll}
\max & -x_0 & & & & & \\
\text{s.t.} & -2x_1 & +x_2 & +x_3 & = & -2 + x_0 \\
& & x_2 & +x_4 & = & 4 + x_0 \\
& x_1 & -2x_2 & +x_5 & = & -2 + x_0 \\
& x_1 & & +x_6 & = & 4 + x_0 \\
& & & x_1, \ldots, x_6, x_0 & \geq & 0
\end{array}
$$

x0 is entering
x3 or x5 is leaving.

$$
\begin{array}{cccccc}
x_3 & = & -2 & +x_0 & +2x_1 & -x_2 \\
x_4 & = & 4 & +x_0 & +0x_1 & -x_2 \\
x_5 & = & -2 & +x_0 & -x_1 & +2x_2 \\
x_6 & = & 4 & +x_0 & -x_1 & +0x_2 \\
\hline
w & = & 0 & -x_0 & &
\end{array}
$$

Example (Cont.)

x0 is entering
x3 is leaving.

$$
\begin{array}{rcrrrr}
x_3 & = & -2 & +x_0 & +2x_1 & -x_2 \\
x_4 & = & 4 & +x_0 & +0x_1 & -x_2 \\
x_5 & = & -2 & +x_0 & -x_1 & +2x_2 \\
x_6 & = & 4 & +x_0 & -x_1 & +0x_2 \\
\hline
w & = & 0 & -x_0 & &
\end{array}
$$

$$
\begin{array}{rcrrrr}
x_0 & = & 2 & +x_3 & -2x_1 & +x_2 \\
x_4 & = & 6 & +x_3 & -2x_1 & +0x_2 \\
x_5 & = & 0 & +x_3 & -3x_1 & +3x_2 \\
x_6 & = & 6 & +x_3 & -3x_1 & +x_2 \\
\hline
w & = & -2 & -x_3 & +2x_1 & -x_2
\end{array}
$$

Claim

For initial aux. dictionary

1. Choose x_0 as entering variable.
2. Choose basic variable x_j with least value of b_j as leaving.

The resulting dictionary is always feasible.

$$
\begin{array}{rcllll}
x_{B1} & = & b_1 & +1x_0 & +a_{11}x_{I1} & +\cdots & +a_{1n}x_{In} \\
x_{B2} & = & b_2 & +1x_0 & +a_{21}x_{I1} & +\cdots & +a_{2n}x_{In} \\
 & \vdots & & & & & \\
x_{Bm} & = & b_m & +1x_0 & +a_{m1}x_{I1} & +\cdots & +a_{mn}x_{In} \\
\hline
z & = & 0 & -x_0 & & &
\end{array}
$$

Initial Dictionary For Aux. Problem

$$
\begin{array}{rclllll}
x_{B1} & = & b_1 & +1x_0 & +a_{11}x_{I1} & +\cdots & +a_{1n}x_{In} \\
x_{B2} & = & b_2 & +1x_0 & +a_{21}x_{I1} & +\cdots & +a_{2n}x_{In} \\
 & \vdots & & & & & \\
\mathbf{x_{Bj}} & = & \mathbf{b_j} & +1x_0 & +\mathbf{a_{j1}x_{I1}} & +\cdots & +\mathbf{a_{jn}x_{In}} \\
 & \vdots & & & & & \\
x_{Bm} & = & b_m & +1x_0 & +a_{m1}x_{I1} & +\cdots & +a_{mn}x_{In} \\
\hline
z & = & 0 & -x_0 & & &
\end{array}
$$

$$b_j = \min(b_1, \ldots, b_m)$$

Pivoting

Initial Aux. Dictionary $\xrightarrow{\text{Pivoting Steps*}}$ Final Aux. Dictionary

Special Rule:

Whenever x_0 is one possible leaving variable, preferentially choose x_0 as the leaving variable.