Piecewise - Linear Optimization

- Linear vs. Affine

- Piecewise - Linear (or Affine) Functions

- LP Equivalent (General)

- LP Equivalent ℓ_∞ - Norm Minimization

- LP Equivalent ℓ_1 - Norm Minimization

Linear vs. Affine

- Linear:

$$f(\boldsymbol{x}) = \boldsymbol{a}^T\boldsymbol{x} + 0$$

- Affine:

$$f(\boldsymbol{x}) = \boldsymbol{a}^T\boldsymbol{x} + b$$

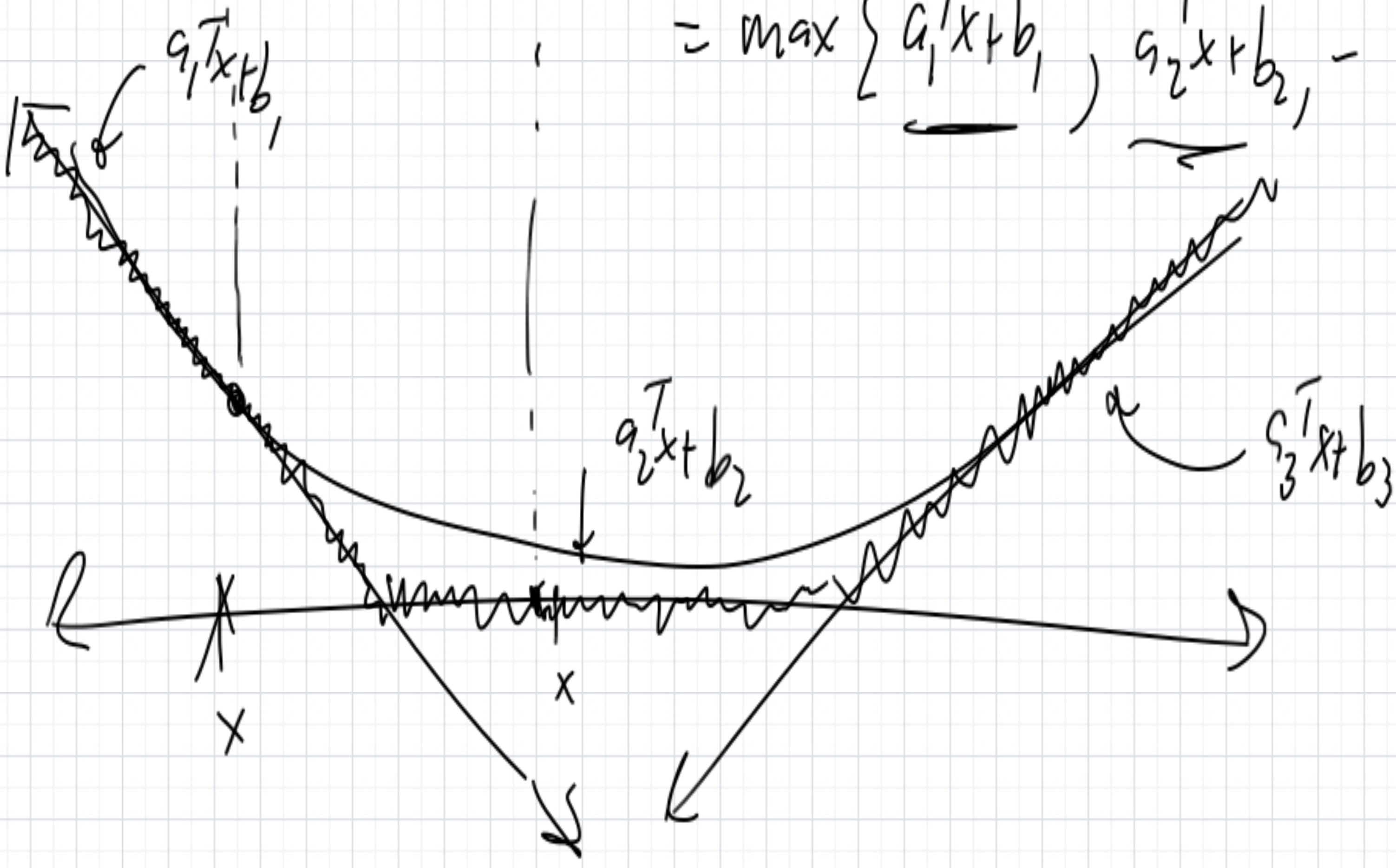
Piecewise - Linear (or Affine) Function

- Convex Function!

- Non - Differentiable Function.

$$f(\boldsymbol{x}) = \max_{i=1,2,\cdots,n} (\boldsymbol{a_i}^T \boldsymbol{x} + b_i)$$

$$= \max\{a_1^T x + b_1,\ a_2^T x + b_2,\ \cdots\}$$



LP Equivalent

$$\min \; \max \{ a_1^T x + b_1, \; a_2^T x + b_2, \; \cdots \}$$

- Need to add an auxiliary variable (scalar) t

$=$

$$\begin{aligned} &\text{minimize} && t \\ &\text{subject to} && a_i^T x + b_i \le t \end{aligned}$$

$a_1^T x + b_1 \le t$

$a_2^T x + b_2 \le t$

$\vdots$

$\hat{A}\hat{x} \le \hat{b}$

- Matrix Form: $\hat{x} = \begin{bmatrix} x \\ t \end{bmatrix}$ - new state-vector

$$\min \; \underbrace{\begin{bmatrix} \vec{0} & 1 \end{bmatrix}}_{\hat{c}^T} \begin{bmatrix} x \\ t \end{bmatrix} = t$$

s.t.

$$\begin{bmatrix} a_1^T & -1 \\ a_2^T & -1 \\ \vdots & -1 \end{bmatrix} \le \begin{bmatrix} -b_1 \\ -b_2 \\ \vdots \end{bmatrix}$$

$$\min \underbrace{\begin{bmatrix} \vec{0} & 1 \end{bmatrix}}_{\hat{c}^T} \underbrace{\begin{bmatrix} x \\ t \end{bmatrix}}_{\hat{x}}$$

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix}$$

$$\text{s.t.} \quad \underbrace{\begin{bmatrix} A & -\mathbb{1} \end{bmatrix}}_{\hat{A}} \underbrace{\begin{bmatrix} x \\ t \end{bmatrix}}_{\hat{x}} \leq \underbrace{\begin{bmatrix} -b \end{bmatrix}}_{\hat{b}}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

>> linprog$(\hat{c}, \hat{A}, \hat{b})$