CSCI5654 (Linear Programming, Fall 2013)
Lecture-7

$\boxed{\text{Duality}}$

Linear Program (standard form)

$$
\begin{array}{rccccccl}
\text{maximize} & c_1 x_1 & + & \cdots & + & c_n x_n & & \\
\text{s.t.} & a_{j1} x_1 & + & \cdots & + & a_{jn} x_n & \leq & b_j \quad j \in \{1, 2, \ldots, m\} \\
 & & & & & x_1, \ldots, x_n & \geq & 0
\end{array}
$$

Problem has n variables and m constraints.

Example Problem

$$
\begin{array}{rrcrcr}
\text{maximize} & x_1 & + & 2x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 3 \\
 & & & x_2 & \leq & 3 \\
 & -x_1 & + & x_2 & \leq & 1 \\
 & x_1 & + & x_2 & \leq & 5 \\
 & & & x_1, x_2 & \geq & 0
\end{array}
$$

Bounds on Optimal Value of LP

$$
\begin{array}{rrcrcl}
\text{maximize} & x_1 & + & 2x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 3 \\
 & & & x_2 & \leq & 3 \\
 & -x_1 & + & x_2 & \leq & 1 \\
 & x_1 & + & x_2 & \leq & 5 \\
 & & & x_1, x_2 & \geq & 0
\end{array}
$$

Goal: Find lower and upper bounds on the solution to the LP.

Bounds

Lower Bound: Find a <u>feasible solution</u> $\vec{x}$.
Then $\vec{x}$ is a lower bound on optimum. (why?).

Eg. $(x_1 : 3, x_2 : 2)$ is feasible.

$$\text{Therefore,} \quad z = x_1 + 2x_2 \geq 7\,.$$

Upper Bounds: Can we find upper bounds to optimum?

Example (continued)

$$
\begin{array}{rrcrclcl}
\text{maximize} & x_1 & + & 2x_2 & & & \leftarrow & \mathbf{z} \\
\text{s.t.} & x_1 & & & \leq & 3 & \leftarrow & \mathbf{C1} \\
& & & x_2 & \leq & 3 & \leftarrow & \mathbf{C2} \\
& -x_1 & + & x_2 & \leq & 1 & \leftarrow & \mathbf{C3} \\
& x_1 & + & x_2 & \leq & 5 & \leftarrow & \mathbf{C4} \\
& & & x_1, x_2 & \geq & 0 & &
\end{array}
$$

Note:

$$\underbrace{x_1 \leq 3}_{C1} \wedge \underbrace{x_2 \leq 3}_{C2} \Rightarrow \underbrace{x_1 + 2x_2 \leq ???}_{z}.$$

Q: Using the inequality above what can you say about the solution to the LP?

Example (continued)

$$
\begin{array}{rrcrcrl}
\text{maximize} & x_1 & + & 2x_2 & & & \leftarrow \mathbf{z} \\
\text{s.t.} & x_1 & & & \leq & 3 & \leftarrow \mathbf{C1} \\
& & & x_2 & \leq & 3 & \leftarrow \mathbf{C2} \\
& -x_1 & + & x_2 & \leq & 1 & \leftarrow \mathbf{C3} \\
& x_1 & + & x_2 & \leq & 5 & \leftarrow \mathbf{C4} \\
& & & x_1, x_2 & \geq & 0 &
\end{array}
$$

Note:

$$\underbrace{x_1 \leq 3}_{C1} \wedge \underbrace{x_2 \leq 3}_{C2} \Rightarrow \underbrace{x_1 + 2x_2 \leq 3 + 2 \cdot 3 \leq 9}_{z}.$$

Q: Using the inequality above what can you say about the solution to the LP?

A: 9 is an upper bound to the solution (i.e., $z \leq 9$).

Finding Upper Bounds

Strategy: Multiply constraint rows by positive quantities.

$$
\begin{array}{rrcrcrl}
\text{maximize} & x_1 & + & 2x_2 & & & \leftarrow \mathbf{z} \\
\text{s.t.} & x_1 & & & \leq & 3 & \leftarrow \mathbf{C1} \\
 & & & x_2 & \leq & 3 & \leftarrow \mathbf{C2} \\
 & -x_1 & + & x_2 & \leq & 1 & \leftarrow \mathbf{C3} \\
 & x_1 & + & x_2 & \leq & 5 & \leftarrow \mathbf{C4} \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}
$$

- $C1 + 2 \cdot C2 \;\Rightarrow\; z \leq 9$.
- $2 \cdot C4 \;\Rightarrow\; z \leq 10$.
- **Question #1:** $-1 \cdot C3 + 3 \cdot C2 \Rightarrow\; z \leq ?$
- **Question #2:** $2 \cdot C3 + C1 \;\Rightarrow\; z \leq ?$
- **Question #3:** Can you think of other combinations?

Upper Bounds to LP Solution

$$
\begin{array}{rcccccccl}
\text{maximize} & c_1 x_1 & + & \cdots & + & c_n x_n & & & \mathbf{z} \\
\text{s.t.} & a_{11} x_1 & + & \cdots & + & a_{1n} x_n & \leq & b_1 & \leftarrow \mathbf{C1} \\
& a_{21} x_1 & + & \cdots & + & a_{2n} x_n & \leq & b_2 & \leftarrow \mathbf{C2} \\
& & & \ddots & & & \vdots & & \\
& a_{m1} x_1 & & \cdots & + & a_{mn} x_n & \leq & b_m & \leftarrow \mathbf{Cm} \\
& & & & & x_1, \ldots, x_n & \geq & 0 &
\end{array}
$$

Consider $y_1 \mathbf{C1} + y_2 \mathbf{C2} + \cdots + y_m \mathbf{Cm}$.

Q: What are the conditions on $y_1, \ldots, y_m$ so that this combination upper bounds z?

Upper Bounds to LP Solution

Conditions:

$$\begin{array}{rcl}
a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m & \geq & c_1 \\
a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m & \geq & c_2 \\
& \vdots & \\
a_{1n}y_1 + a_{1n}y_2 + \cdots + a_{mn}y_m & \geq & c_n \\
y_1, y_2, \ldots, y_m & \geq & 0
\end{array}$$

Upper Bound: $b_1y_1 + \ldots + b_my_m$

Q: What values of $y_1, \ldots, y_m$ can yield best bound on optimum??

Best upper bound on LP Solution

$$
\begin{array}{rrcl}
\text{minimize} & b_1 y_1 + b_2 y_2 + \cdots + b_m y_m & & \\
\text{s.t.} & a_{11} y_1 + a_{21} y_2 + \cdots + a_{m1} y_m & \geq & c_1 \\
 & a_{12} y_1 + a_{22} y_2 + \cdots + a_{m2} y_m & \geq & c_2 \\
 & & \vdots & \\
 & a_{1n} y_1 + a_{1n} y_2 + \cdots + a_{mn} y_m & \geq & c_n \\
 & y_1, y_2, \ldots, y_m & \geq & 0
\end{array}
$$

Note: This is called the dual problem.

Dual Problem

Primal Problem: decision variables are $x_1, \ldots, x_n$.

$$\begin{array}{rccccccl} \text{maximize} & c_1 x_1 & + & \cdots & + & c_n x_n & & \\ \text{s.t.} & a_{j1} x_1 & + & \cdots & + & a_{jn} x_n & \leq & b_j \quad j \in \{1, 2, \ldots, m\} \\ & & & & & x_1, \ldots, x_n & \geq & 0 \end{array}$$

Dual Problem: decision variables are $y_1, \ldots, y_m$.

$$\begin{array}{rccccccl} \text{minimize} & b_1 y_1 & + & \cdots & + & b_m y_m & & \\ \text{s.t.} & a_{1j} y_1 & + & \cdots & + & a_{mj} y_m & \geq & c_j \quad j \in \{1, 2, \ldots, n\} \\ & y_1, \ldots, y_n & \geq & 0 & & & & \end{array}$$

Dual Problem (matrix)

Primal LP:

$$
\begin{array}{rrcl}
\text{max.} & \vec{c}\cdot\vec{x} & & \\
\text{s.t.} & A\vec{x} & \leq & \vec{b} \\
 & \vec{x} & \geq 0 &
\end{array}
$$

Dual LP:

$$
\begin{array}{rrcl}
\text{min.} & \vec{b}\cdot\vec{y} & & \\
 & A^{\mathrm{T}}\vec{y} & \geq & \vec{c} \\
 & \vec{y} & \geq & 0
\end{array}
$$

(dual in <u>standard form</u>)

$$
\begin{array}{rrcl}
\text{max} & -\vec{b}\cdot\vec{y} & & \\
 & -A^{\mathrm{T}}\vec{y} & \leq & -\vec{c} \\
 & \vec{y} & \geq & 0
\end{array}
$$

Exercise: Use the information in this frame to prove:

The dual of dual problem is the same as the original primal problem.

Example

$$
\begin{array}{rrcrcll}
\text{maximize} & x_1 & + & 2x_2 & & & \\
\text{s.t.} & x_1 & & & \leq & 3 & \leftarrow y_1 \\
 & & & x_2 & \leq & 3 & \leftarrow y_2 \\
 & -x_1 & + & x_2 & \leq & 1 & \leftarrow y_3 \\
 & x_1 & + & x_2 & \leq & 5 & \leftarrow y_4 \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}
$$

Q: Write down the dual problem.

Example (dual)

$$
\begin{array}{rcccccccccc}
\textbf{min.} & 3y_1 & + & 3y_2 & + & y_3 & + & 5y_4 & & \\
\textbf{s.t.} & y_1 & & & - & y_3 & + & y_4 & \geq & 1 \\
& & & y_2 & + & y_3 & + & y_4 & \geq & 2 \\
& & & \multicolumn{3}{r}{y_1, y_2, y_3, y_4} & \geq & 0 & &
\end{array}
$$

Dual optimal: $y_1 = 0, y_2 = 0, y_3 = \frac{1}{2}, y_4 = \frac{3}{2}$ yields optimal value 8.

Example (dictionary)

$$
\begin{array}{rcccccc}
x_3 & = & 3 & - & x_1 & & \\
x_4 & = & 3 & - & & & x_2 \\
x_5 & = & 1 & + & x_1 & - & x_2 \\
x_6 & = & 5 & - & x_1 & - & x_2 \\
\hline
z & = & 0 & + & x_1 & + & 2x_2
\end{array}
$$

Example (final dictionary)

$$
\begin{array}{rcccccc}
x_1 & = & 2 & + & \frac{x_5}{2} & - & \frac{x_6}{2} \\
x_2 & = & 3 & - & \frac{x_5}{2} & + & \frac{x_6}{2} \\
x_3 & = & \cdots & & & & \\
x_4 & = & \cdots & & & & \\
\hline
z & = & 8 & - & \frac{1}{2}x_5 & - & \frac{3}{2}x_6
\end{array}
$$

Dual Certificate

Situation: I ask X to solve a large LP for me. After sometime X presents a solution $\vec{x} = (x_1^*, \ldots, x_n^*)$ that is claimed to be optimal.
Q: How do I verify that X's solution is indeed optimal?
A1: OK solve the LP yourself!!
(not acceptable)

Dual Certificate

Situation: I ask X to solve a large LP for me. After sometime X presents a solution $\vec{x} = (x_1^*, \ldots, x_n^*)$ that is claimed to be optimal.
Q: How do I verify that X's solution is indeed optimal?
A2: Provide dual variables $\vec{y} : \ (y_1^*, \ldots, y_n^*)$.

1. Check primal feasibility of $\vec{x}$.
2. Check dual feasibility of $\vec{y}$.
3. Check that the objective values are the same.

Primal-Dual Certificate: In the presence of both, each solution can serve as a certificate to the optimality of the other.

Primal dual correspondences

Variable to constraint correspondence:

Dual variable (y_j)	Primal constraint $A_j\vec{x} \leq b_j$
Primal variable (x_i)	Dual constraint $A \cdot_i \vec{y} \geq c_i$

Correspondence in dictionary:

Dual variable y_j	Primal slack variable x_{n+j}.
Primal variable (x_i)	Dual dictionary slack variable y_{m+i}.

Strong Duality Theorem: If $x_1^*, \ldots, x_m^*$ is primal optimal then there exists a dual optimal solution $y_1^*, \ldots, y_m^*$ that satisifies

$$\sum_i c_i x_i^* = \sum_j b_j y_j^* \, .$$

($\therefore$ if primal has optimal solution then the dual has optimum with the same value as primal).

Proof: This is theorem 5.1 in Chvátal (pages 58-59).

Example

$$
\begin{array}{rcccccl}
\text{maximize} & x_1 & + & 2x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 3 \\
& & & x_2 & \leq & 3 \\
& -x_1 & + & x_2 & \leq & 1 \\
& x_1 & + & x_2 & \leq & 5 \\
& & & x_1, x_2 & \geq & 0
\end{array}
$$

Example (final dictionary)

$$\begin{array}{rcrcrcr} x_1 & = & 2 & + & \frac{x_5}{2} & - & \frac{x_6}{2} \\ x_2 & = & 3 & - & \frac{x_5}{2} & + & \frac{x_6}{2} \\ x_3 & = & \cdots & & & & \\ x_4 & = & \cdots & & & & \\ \hline z & = & 8 & - & \frac{1}{2}x_5 & - & \frac{3}{2}x_6 \end{array}$$

Insight: Dual variables correspond to primal slack variables.

$$x_3 \leftrightarrow y_1, x_4 \leftrightarrow y_2, x_5 \leftrightarrow y_3, x_6 \leftrightarrow y_4\,.$$

Read off dual solution by from objective row of final dictionary:

$$y_1 : 0, y_2 : 0, y_3 : \frac{1}{2}, y_4 : \frac{3}{2}\,.$$

Proof of Strong Duality

Final Dictionary:

$$
\begin{array}{rcl}
x_i & = & b_i + \sum_{j \in \text{Independent}} a_{ij} x_j \\
 & \vdots & \\
\hline
z & = & z^* + c_1^* x_1 + \cdots + c_{n+m}^* x_{n+m}
\end{array}
$$

Note: $c_i^* = 0$ if $i \in$ Basis
$c_i^* \leq 0$ if $i \in$ Independent.
Claim: Set variable $y_i = -(c_{n+i}^*)$. This is a dual feasible solution with optimal value z^*.

Relationship Between Primal/Dual

		Primal		
		Optimal	Infeasible	Unbounded
Dual	Optimal	Possible	Impossible	Impossible
	Infeasible	Impossible	Possible	Possible
	Unbounded	Impossible	Possible	Impossible

Primal and Dual Infeasible

Primal:

$$\begin{array}{rrcr} \text{max.} & x_2 & & \\ \text{s.t.} & x_1 & \leq & -1 \\ & -x_2 & \leq & -1 \\ & x_1, x_2 & \geq & 0 \end{array}$$

Dual:

$$\begin{array}{rrcr} \text{min.} & -y_1 - y_2 & & \\ \text{s.t.} & y_1 & \geq & 0 \\ & -y_2 & \geq & 1 \\ & y_1, y_2 & \geq & 0 \end{array}$$