PS5: Combinatorial Optimization Problems Using ILP

Help

The **due date** for this homework is **Mon 8 Dec 2014 3:00 PM CST**.

In this assignment, we will solve various combinatorial optimization problems by reducing to ILP. We assume that you are able to use an ILP solver: you are welcome to try GLPK (with its interfaces to MATLAB or Octave) or the recent intlinprog function in MATLAB.

- [ ] **In accordance with the Coursera Honor Code, I (Kevin Zhu) certify that the answers here are my own work.**

Question 1

Consider the following 3-SAT instance with 3 Boolean variables x_1, x_2, x_3 and 5 clauses $C_1, \ldots, C_5$:

$$\begin{array}{ll} (x_1 \text{ OR } x_2 \text{ OR } \neg x_3) & \leftarrow C_1 \\ (x_1 \text{ OR } x_2 \text{ OR } x_3) & \leftarrow C_2 \\ (x_1 \text{ OR } \neg x_2 \text{ OR } \neg x_3) & \leftarrow C_3 \\ (x_1 \text{ OR } \neg x_2 \text{ OR } x_3) & \leftarrow C_4 \\ (\neg x_1) & \leftarrow C_5 \end{array}$$

Use an ILP solver to check if the constraints are satisfiable. Type "SAT" if you find that they are satisfiable and type "UNSAT" otherwise.

Question 2

Rather than ask whether a given formula has a SATisfiable assignment or not, we ask the question what is the maximum number of clauses that can be satisfied at the same time. This is called the MAX-SAT problem. Consider again the following 3-SAT instance with variables x_1, x_2, x_3 and 5

clauses:

$$\begin{array}{ll} (x_1 \text{ OR } x_2 \text{ OR } \neg x_3) & \leftarrow C_1 \\ (x_1 \text{ OR } x_2 \text{ OR } x_3) & \leftarrow C_2 \\ (x_1 \text{ OR } \neg x_2 \text{ OR } \neg x_3) & \leftarrow C_3 \\ (x_1 \text{ OR } \neg x_2 \text{ OR } x_3) & \leftarrow C_4 \\ (\neg x_1) & \leftarrow C_5 \end{array}$$

Let y_1, y_2, y_3 be the 0-1 variables corresponding to x_1, x_2, x_3, respectively. Let z_1 be another $0-1$ (binary) variable. Which of the following statements best describe the meaning of the constraint

$$y_1 + y_2 + (1 - y_3) \geq z_1$$

- If we set z_1 to 1 then the assignment to y_1, y_2, y_3 must satisfy clause C_3.
- If we set z_1 to 0, the assignment to y_1, y_2, y_3 must satisfy clause C_1
- If we set z_1 to 1, the assignment to y_1, y_2, y_3 must satisfy clause C_1.
- If we set z_1 to 1 then the assignment to y_1, y_2, y_3 must satisfy clause C_2

Question 3

Consider again the following 3-SAT instance with variables x_1, x_2, x_3 and 5 clauses:

$$\begin{array}{ll} (x_1 \text{ OR } x_2 \text{ OR } \neg x_3) & \leftarrow C_1 \\ (x_1 \text{ OR } x_2 \text{ OR } x_3) & \leftarrow C_2 \\ (x_1 \text{ OR } \neg x_2 \text{ OR } \neg x_3) & \leftarrow C_3 \\ (x_1 \text{ OR } \neg x_2 \text{ OR } x_3) & \leftarrow C_4 \\ (\neg x_1) & \leftarrow C_5 \end{array}$$

We wish to find out the maximum number of clauses that can be simultaneously satisfied by a truth assignment. To do so, we use binary variables y_1, y_2, y_3 corresponding to $x_1, \ldots, x_3$ and the binary variables $z_1, \ldots, z_5$ corresponding to clauses $C_1, \ldots, C_5$.

We set up the following 0-1 ILP but we omit the objective function:

$$
\begin{array}{lrcl}
? & ? & & \\
\text{s.t.} & y_1 + y_2 + (1 - y_3) & \geq & z_1 \\
& y_1 + y_2 + y_3 & \geq & z_2 \\
& y_1 + (1 - y_2) + (1 - y_3) & \geq & z_3 \\
& y_1 + (1 - y_2) + y_3 & \geq & z_4 \\
& 1 - y_1 & \geq & z_5 \\
& y_1, \ldots, y_3, z_1, \ldots, z_5 & \in & \{0,1\}
\end{array}
$$

Which of the following should be the objective function to find the maximum number of simultaneously satisfiable clauses?

- $\min\ y_1 + y_2 + y_3$
- $\min z_1 + z_2 + z_3 + z_4 + z_5$.
- $\max z_1 + 2z_2 + 3z_3 + 4z_4 + 5z_5$
- $\max\ y_1 + y_2 + y_3$
- $\max\ z_1 + z_2 + z_3 + z_4 + z_5$

Question 4

Solve the ILP in the previous question using your favourite ILP solver, and enter the optimal value of the objective function.

Question 5

We solved a vertex cover problem with 500 nodes and 12400 edges using the LP relaxation and the rounding procedure in our lecture. The LP relaxation yielded an optimal solution with a cost of 123.4. What can we say about the minimal vertex cover for this problem? Select all the true options.

- The cost of the optimal vertex cover is exactly 124 vertices.
- The cost of the minimal vertex cover is at most 123
- The cost of the minimal vertex cover is at least 124.

- [ ] The minimal vertex cover cost is at most 246

- [ ] The LP relaxation optimal value tells us nothing about the optimal cover since the vertex cover can be ILP infeasible.

Question 6

A county has 7 towns $T_1, \ldots, T_7$. A corporation wishes to build supermarkets to serve the people in the county.

- For each town, the corporation can choose to or not to build a super market there. No town can have more than one super-market.
- We would like to minimize the number of super-markets built.
- Each town must have a super-market within a 20 minute driving time.
- The table below shows the driving times between the towns:

	T_1	T_2	T_3	T_4	T_5	T_6	T_7
T_1	0						
T_2	19	0					
T_3	18	17	0				
T_4	22	22	14	0			
T_5	13	23	21	29	0		
T_6	31	14	31	13	22	0	
T_7	23	32	32	17	19	26	0

Note that driving time from T_i to T_j is assumed to be the same as the reverse driving distance.

Let $z_1, \ldots, z_7$ denote the decision variables where z_i is 1 if we should build a super market in town T_i and 0 otherwise.

What is the objective function for this problem?

- ( ) $\max\ z_1 + \cdots + z_7$
- ( ) $\max\ 0$: this is just a feasibility problem.
- ( ) $\min\ z_1 + z_2 + \cdots + z_7$
- ( ) $\min\ z_1 + 2z_2 + 3z_3 + 4z_4 + 5z_5 + 6z_6 + 7z_7$

Question 7

Recall the drive times below:

	T_1	T_2	T_3	T_4	T_5	T_6	T_7
T_1	0						
T_2	19	0					
T_3	18	17	0				
T_4	22	22	14	0			
T_5	13	23	21	29	0		
T_6	31	14	31	13	22	0	
T_7	23	32	32	17	19	26	0

Each town must have a super-market within a 20 minute driving time.

Note from the table that T_1, T_2, T_3, T_5 are the only towns within 20 minutes of T_1.

Which of the following conclusions can be made?

- A super market must be built in T_1 or **all of** T_2, T_3 and T_5 must have a super market.
- At least one super market must be built in the towns $\{T_1, T_2, T_3, T_5\}$.
- At most one super market must be built in the towns $\{T_1, T_2, T_3, T_5\}$.
- Nothing can be concluded.

Question 8

Which of the constraints below expresses the requirement that

there must be a super market built within 20 minutes driving time from town T_6

The full table is recalled below for convenience:

	T_1	T_2	T_3	T_4	T_5	T_6	T_7
T_1	0	19	18	22	13	31	23
T_2	19	0	17	22	23	14	32
T_3	18	17	0	14	21	31	32
T_4	22	22	14	0	29	13	17
T_5	13	23	21	29	0	22	19
T_6	31	14	31	13	22	0	26
T_7	23	32	32	17	19	26	0

- $z_1 + z_3 + z_5 + z_7 \geq 1$
- $z_2 + z_4 + z_6 \geq 1$

- $z_2 + z_4 \geq 1$
- $z_6 \geq 1$

Question 9

Setup and solve the ILP for the supermarket construction given the data given in the previous problem. What is the smallest number of super markets that we need to construct?

Question 10

We modify the problem to take into account the cost of constructions (in millions of dollars) for each town:

T_1	T_2	T_3	T_4	T_5	T_6	T_7
12	8	12	10	9	10	7

Rather than minimize the number of super markets, we wish to minimize the cost of constructions while still requiring a supermarket within 20 minutes driving time from each town.

Setup and solve an ILP to find the minimal construction cost obtained? Report the answer in the box below.

In accordance with the Coursera Honor Code, I (Kevin Zhu) certify that the answers here are my own work.

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