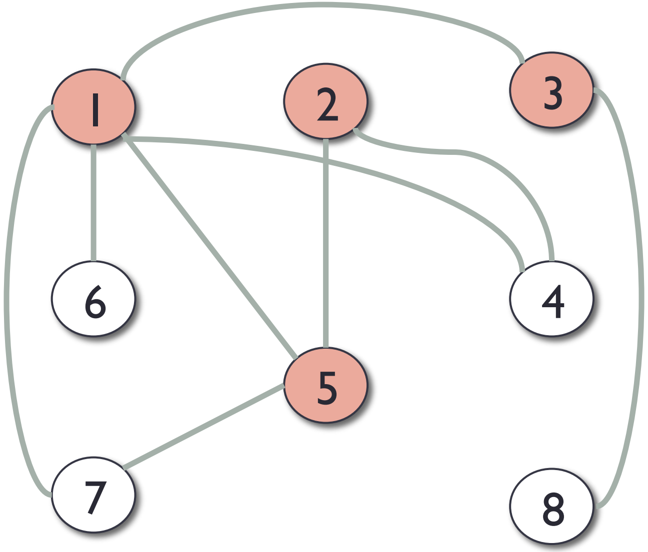
ILP AND VERTEX COVER

A flavor of approximation algorithms

Rounding Schemes

- LP relaxation yields solutions with fractional parts.
- However, ILP asks for integer solution.
- In some cases, we can approximate ILP optimum by "rounding"
  - Take optimal solution of LP relaxation
  - Round the answer to an integer answer using rounding scheme.
  - Deduce something about the ILP optimal solution.

Vertex Cover Problem



Choose smallest subset of vertices

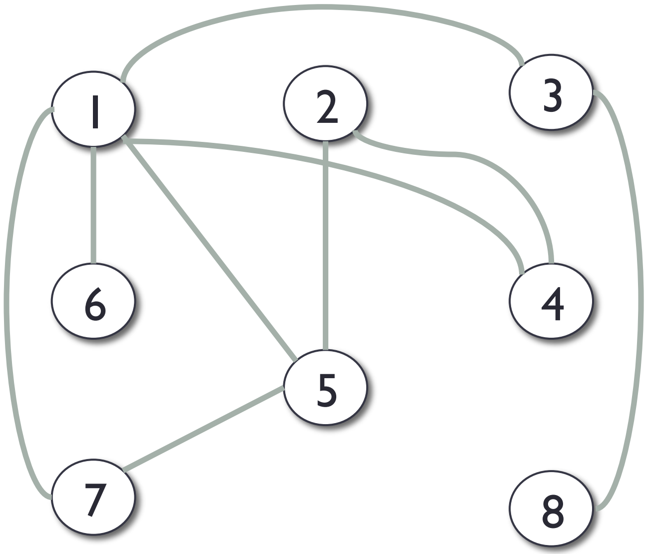
Every edge must be "covered"

Eg, { 1, 2, 3, 5 }

or

{1, 2, 3, 7 }

ILP for the vertex cover problem (Example)

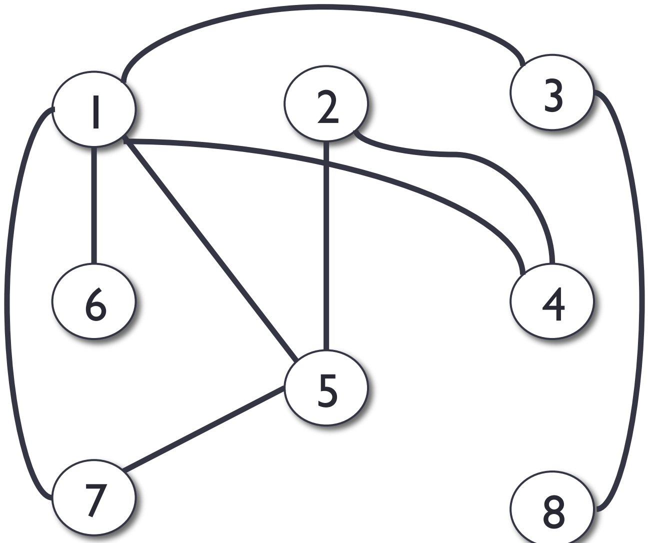


ILP decision variables

$$x_1, \ldots, x_8$$

$$x_i = \begin{cases} 0 & \text{Vertex } \# \, i \text{ not chosen in subset} \\ 1 & \text{Vertex } \# \, i \text{ is chosen in subset} \end{cases}$$

ILP for the vertex cover problem (Example)



$$
\begin{array}{lrcll}
\min & x_1 + x_2 + \cdots + x_8 & & & \\
\text{s.t.} & x_1 + x_7 & \geq & 1 & \leftarrow \text{ Edge: } (1,7) \\
& x_1 + x_6 & \geq & 1 & \leftarrow \text{ Edge: } (1,6) \\
& x_2 + x_4 & \geq & 1 & \\
& \cdots & & & \\
& x_i + x_j & \geq & 1 & \leftarrow (i,j) \in E \\
& \cdots & & & \\
& x_1 & \leq & 1 & \\
& \vdots & & & \\
& x_8 & \leq & 1 & \\
& x_1, \ldots, x_8 & \geq & 0 & \\
& x_1, \ldots, x_8 & \in & \mathbb{Z} &
\end{array}
$$

Vertex Cover to ILP

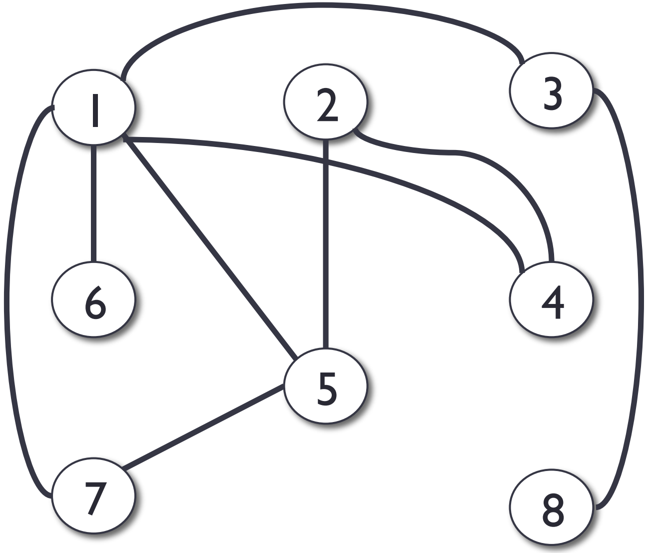
- Vertices {1,…, n}
  - Decision variables: $x_1, \ldots, x_n$

$$\boxed{x_i \in \{0,1\}}$$

$$\begin{array}{lrl}
\min & \sum_{i=1}^{n} x_i & \\
\text{s.t.} & 0 \leq x_i \leq 1 & \forall\ i \in V \\
 & x_i + x_j \geq 1 & \forall\ (i,j) \in E \\
 & x_i \in \mathbb{Z} & \forall\ i \in V
\end{array}$$

LP relaxation of a vertex cover

- Problem: we may get fractional solution.



x_1	1
x_2	1
x_3	$\frac{3}{4}$
x_4	0
x_5	$\frac{5}{6}$
x_6	0
x_7	$\frac{1}{6}$
x_8	$\frac{1}{4}$

Objective value: 4

But solution meaningless for vertex cover.

Rounding Scheme

- Simple rounding scheme:

$$x_i^* \geq \frac{1}{2} \quad \rightarrow \quad x_i = 1$$

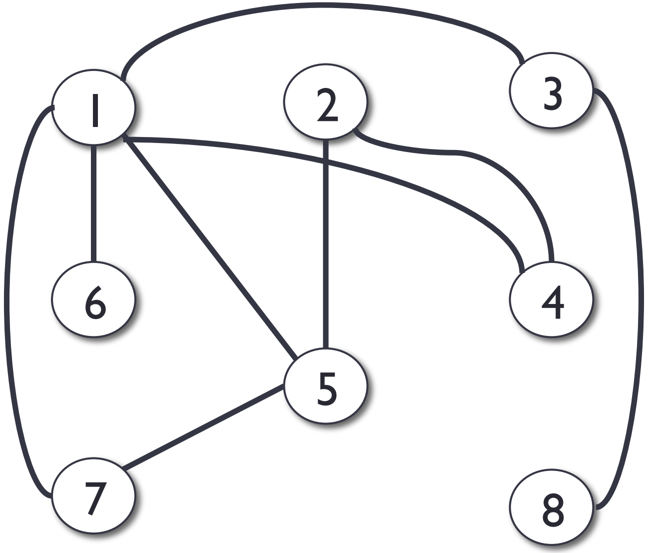
Real-Optimal Solution is at least 0.5

Include vertex in the cover.

$$x_i^* < \frac{1}{2} \quad \rightarrow \quad x_i = 0$$

LP relaxation of a vertex cover

- Problem: we may get fractional solution.



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x_7	0
x_8	0

Rounding Scheme

Rounding scheme takes optimal fractional solution from LP relaxation and produces an integral solution.

$$\mathbf{x}^* \quad \xrightarrow{\text{rounding}} \quad \hat{\mathbf{x}}$$

1. Does rounding always produces a valid vertex cover?
2. How does the rounded solution compare to the opt. solution?

Rounding Scheme Produces a Cover

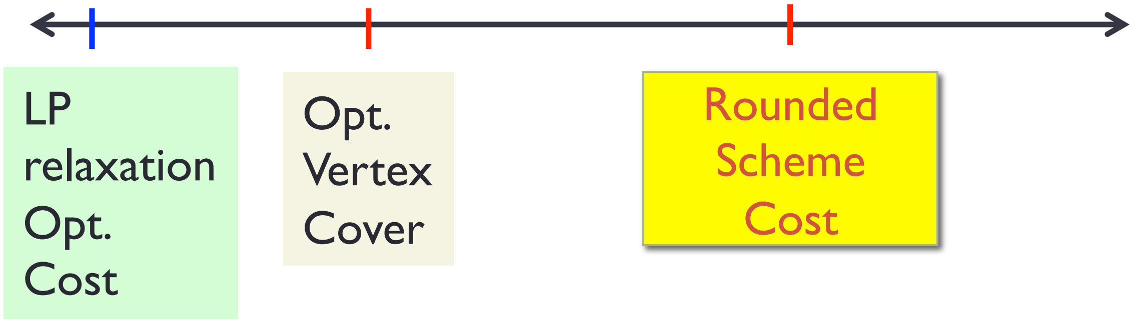
$$\mathbf{x}^* \xrightarrow{\text{rounding}} \hat{\mathbf{x}}$$

$$x_i^* + x_j^* \geq 1, \quad \text{for each } (i,j) \in E$$

$$\hat{x}_i = 1 \text{ or } \hat{x}_j = 1 \text{ for each } (i,j) \in E$$

To Prove: The solution obtained after rounding covers every edge.

Rounding Scheme Approximation Guarantee



$$\mathbf{x}^* \xrightarrow{\text{rounding}} \hat{\mathbf{x}}$$

Fact: $2x_i^* \geq \hat{x}_i$ for all vertices i.

$$2\sum_{i=1}^{n} x^* \geq \sum_{i=1}^{n} \hat{x}_i$$

2 * (Cost of LP relaxation) $\geq$ (Cost of Rounded Scheme Vertex Cover)

Approximation Guarantee

- Theorem #1: Rounding scheme yields a vertex cover.
- Cost of the solution obtained by rounding: C
- Optimal vertex cover cost: C^*

- Theorem #2: $C^* \leq C \leq 2\,C^*$

- LP relaxation + rounding scheme:
  - 2-approximation for vertex cover!!