

# LINEAR AND INTEGER PROGRAMMING

---

The Simplex Method: A tutorial in three acts.

# Three to Simplex

1. The Standard Form Linear Program.
  2. “Dictionaries”.
  3. Pivoting.
-  Putting it all together in an example.

# ACT I: THE STANDARD FORM.

---

*Revision of material covered under LP formulations.*

# Some words of wisdom

*If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem?*

- George Pólya (*How to Solve It?*)



George Pólya  
(source: *mactutor*)

# Linear Program

$$\begin{array}{lll} \text{maximize} & c_1x_1 + \dots + c_nx_n \\ \text{subj.to.} & a_{11}x_1 + \dots + a_{1n}x_n & \leq b_1 \\ & a_{21}x_1 + \dots + a_{2n}x_n & \leq b_2 \\ & \ddots & \vdots \\ & a_{m1}x_1 + \dots + a_{mn}x_n & \leq b_m \end{array}$$

# Linear Program in Matrix Form

$$\begin{array}{llll} \text{maximize} & c_1x_1 + \dots + c_nx_n \\ \text{subj.to.} & a_{11}x_1 + \dots + a_{1n}x_n & \leq & b_1 \\ & a_{21}x_1 + \dots + a_{2n}x_n & \leq & b_2 \\ & \ddots & \vdots & \\ & a_{m1}x_1 + \dots + a_{mn}x_n & \leq & b_m \end{array}$$

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T \mathbf{x} \\ \text{subj.to} & A \mathbf{x} \leq \mathbf{b} \end{array}$$

# Standard Form (Definition)

$$\begin{array}{lll} \text{maximize} & c_1x_1 + \dots + c_nx_n & \\ \text{subj.to.} & a_{11}x_1 + \dots + a_{1n}x_n & \leq b_1 \\ & a_{21}x_1 + \dots + a_{2n}x_n & \leq b_2 \\ & \ddots & \vdots \\ & a_{m1}x_1 + \dots + a_{mn}x_n & \leq b_m \\ & x_1, x_2, \dots, x_n & \geq 0 \end{array}$$

# Standard Form (Matrix Notation)

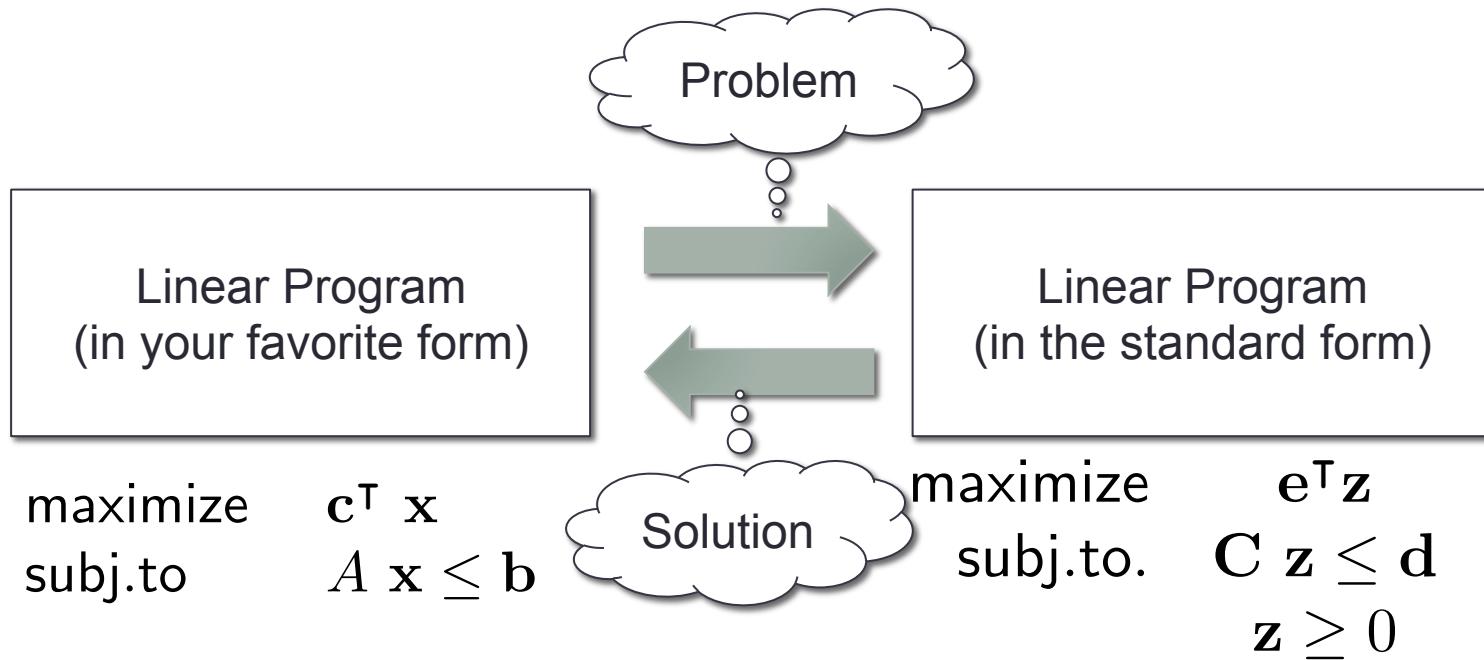
$$\begin{array}{lll} \text{maximize} & c_1x_1 + \dots + c_nx_n \\ \text{subj.to.} & a_{11}x_1 + \dots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + \dots + a_{2n}x_n \leq b_2 \\ & \ddots \quad \vdots \\ & a_{m1}x_1 + \dots + a_{mn}x_n \leq b_m \\ & \underline{x_1, x_2, \dots, x_n} \geq 0 \end{array}$$

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T \mathbf{x} \\ \text{subj.to.} & \mathbf{A} \mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}$$

# Standard Form LP (Example)

$$\begin{array}{lll} \text{maximize} & -5x_1 + 4x_2 - 3x_3 \\ \text{s.t.} & 2x_1 - 3x_2 + x_3 & \leq 5 \\ & 4x_1 + x_2 + 2x_3 & \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 & \leq 8 \\ & x_1, x_2, x_3 & \geq 0 \end{array}$$

# Converting LPs to Standard Form



# Converting to Standard Form

$$\begin{aligned} \text{minimize} \quad & -5x_1 + 4x_2 - 3x_3 \\ \text{s.t.} \quad & 2x_1 - 3x_2 + x_3 = 5 \\ & 4x_1 + x_2 + 2x_3 \geq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & x_1 \geq 0 \end{aligned}$$

# Objective Direction

$$\text{minimize } \mathbf{c}^T \mathbf{x} \quad \xrightarrow{\text{change to}} \quad \text{maximize } (-\mathbf{c}^T \mathbf{x})$$

$$\min -5x_1 + 4x_2 - 3x_3 \quad \rightarrow \quad \max 5x_1 - 4x_2 + 3x_3$$

# Equality Constraints

$$\mathbf{a_i}^\top \mathbf{x} = b_i \quad \xrightarrow{\text{change to}} \quad \begin{cases} \mathbf{a_i}^\top \mathbf{x} \leq b_i \\ \mathbf{a_i}^\top \mathbf{x} \geq b_i \end{cases}$$

$$2x_1 - 3x_2 + x_3 = 5 \quad \xrightarrow{\text{change to}} \quad \begin{cases} 2x_1 - 3x_2 + x_3 \leq 5 \\ 2x_1 - 3x_2 + x_3 \geq 5 \end{cases}$$

# Missing Non-Negative Constraint

**Problem:** missing constraint  $x_i \geq 0$

- Introduce two fresh variables:  $x_i^+, x_i^-$
- Replace every occurrence of  $x_i$  with  $x_i^+ - x_i^-$
- Add the constraints  $x_i^+ \geq 0, x_i^- \geq 0$

# Converting to Standard Form

$$\begin{aligned} \text{minimize} \quad & -5x_1 + 4x_2 - 3x_3 \\ \text{s.t.} \quad & 2x_1 - 3x_2 + x_3 = 5 \\ & 4x_1 + x_2 + 2x_3 \geq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & x_1 \geq 0 \end{aligned}$$

# Conversion to Standard Form (Result)

$$\begin{array}{lll} \text{minimize} & -5x_1 + 4x_2 - 3x_3 \\ \text{s.t.} & 2x_1 - 3x_2 + x_3 = 5 \\ & 4x_1 + x_2 + 2x_3 \geq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & x_1 \geq 0 \end{array}$$

$$\begin{array}{llll} \max & 5x_1 - 4x_2^+ + 4x_2^- + 3x_3^+ - 3x_3^- & & \\ \text{s.t.} & 2x_1 - 3x_2^+ + 3x_2^- + x_3^+ - x_3^- & \leq & 5 \\ & -2x_1 + 3x_2^+ - 3x_2^- - x_3^+ + x_3^- & \leq & -5 \\ & -4x_1 - x_2^+ + x_2^- - 2x_3^+ + 2x_3^- & \leq & -11 \\ & 3x_1 + 4x_2 - 4x_2^- + 2x_3^+ - 2x_3^- & \leq & 8 \\ & x_1, x_2^+, x_2^-, x_3^+, x_3^- & \geq & 0 \end{array}$$

# Transform Solutions Back

$$\begin{array}{ll}
 \text{minimize} & -5x_1 + 4x_2 - 3x_3 \\
 \text{s.t.} & \begin{array}{lcl} 2x_1 - 3x_2 + x_3 & = & 5 \\ 4x_1 + x_2 + 2x_3 & \geq & 11 \\ 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\ x_1 & \geq & 0 \end{array}
 \end{array}$$



$$\begin{array}{ll}
 \max & 5x_1 - 4x_2^+ + 4x_2^- + 3x_3^+ - 3x_3^- \\
 \text{s.t.} & \begin{array}{lcl} 2x_1 - 3x_2^+ + 3x_2^- + x_3^+ - x_3^- & \leq & 5 \\ -2x_1 + 3x_2^+ - 3x_2^- - x_3^+ + x_3^- & \leq & -5 \\ -4x_1 - x_2^+ + x_2^- - 2x_3^+ + 2x_3^- & \leq & -11 \\ 3x_1 + 4x_2^- + 2x_3^+ - 2x_3^- & \leq & 8 \\ x_1, x_2^+, x_2^-, x_3^+, x_3^- & \geq & 0 \end{array}
 \end{array}$$

$$(x_1 = 3.43, x_2^+ = 0.143, x_2^- = 0, x_3^+ = 0, x_3^- = 1.43)$$



$$(x_1 = 3.43, x_2 = 0.143, x_3 = -1.43)$$

# ACT II: THE DICTIONARY

---

*A convenient data structure for Linear  
Programs and representation of solutions.*

# Standard Form (Matrix Notation)

$$\begin{array}{lll} \text{maximize} & c_1x_1 + \dots + c_nx_n \\ \text{subj.to.} & a_{11}x_1 + \dots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + \dots + a_{2n}x_n \leq b_2 \\ & \ddots \quad \vdots \\ & a_{m1}x_1 + \dots + a_{mn}x_n \leq b_m \\ & x_1, x_2, \dots, x_n \geq 0 \end{array}$$

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^\top \mathbf{x} \\ \text{subj.to.} & \mathbf{A} \mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

# Slack Form

Trick: Change inequality constraint to an equality.

$$\mathbf{a}^\top \mathbf{x} \leq b \xrightarrow{\text{add slack}} \mathbf{a}^\top \mathbf{x} + x_s = b$$

$$\begin{array}{ll}\text{maximize} & \mathbf{c}^\top \mathbf{x} \\ \text{subj.to.} & \mathbf{A} \mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}\end{array}$$



$$\begin{array}{lll}\text{maximize} & \mathbf{c}^\top \mathbf{x} & \\ \text{s.t.} & \mathbf{A} \mathbf{x} + \mathbf{x}_{\text{slack}} & = \mathbf{b} \\ & \mathbf{x} & \geq \mathbf{0} \\ & \mathbf{x}_{\text{slack}} & \geq \mathbf{0}\end{array}$$

# Example (Slack Variable Addition)

$$\begin{array}{lllll}\text{max.} & 5x_1 + 4x_2 + 3x_3 & & & \\ \text{s.t.} & 2x_1 + 3x_2 + x_3 & \leq & 5 & \\ & 4x_1 + x_2 + 2x_3 & \leq & 11 & \\ & 3x_1 + 4x_2 + 2x_3 & \leq & 8 & \\ & x_1, x_2, x_3 & \geq & 0 & \end{array}$$

# Example (Slack Variable Addition)

$$\begin{array}{llllll} \text{max.} & 5x_1 + 4x_2 + 3x_3 & & & & \\ \text{s.t.} & 2x_1 + 3x_2 + x_3 + x_4 & = & 5 & & \\ & 4x_1 + x_2 + 2x_3 + x_5 & = & 11 & & \\ & 3x_1 + 4x_2 + 2x_3 + x_6 & = & 8 & & \\ & x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0 & & \end{array}$$

# Dictionary

$$\begin{array}{lllll} \max. & 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} & 2x_1 + 3x_2 + x_3 + x_4 & = & 5 \\ & 4x_1 + x_2 + 2x_3 + x_5 & = & 11 \\ & 3x_1 + 4x_2 + 2x_3 + x_6 & = & 8 \\ & x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0 \end{array}$$

$$\begin{array}{rcl} x_4 & = & 5 - 2x_1 - 3x_2 - x_3 \\ x_5 & = & 11 - 4x_1 - x_2 - 2x_3 \\ x_6 & = & 8 - 3x_1 - 4x_2 - 2x_3 \\ \hline z & = & 0 + 5x_1 + 4x_2 + 3x_3 \end{array}$$

# Dictionary

$$\frac{\mathbf{x}_b = \mathbf{b} - \mathbf{A}\mathbf{x}_i}{z = c_0 + \mathbf{c}^\top \mathbf{x}_i}$$

# Summary So Far

## Transform Problem with Slack Variables

$$\begin{array}{ll}\text{maximize} & \mathbf{c}^T \mathbf{x} \\ \text{subj.to.} & \mathbf{A} \mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}\end{array}$$



$$\begin{array}{lll}\text{maximize} & \mathbf{c}^T \mathbf{x} & \\ \text{s.t.} & \mathbf{A} \mathbf{x} + \mathbf{x}_{\text{slack}} & = \mathbf{b} \\ & \mathbf{x} & \geq \mathbf{0} \\ & \mathbf{x}_{\text{slack}} & \geq \mathbf{0}\end{array}$$

# Dictionaries

Dictionary: Solution associated + Feasibility

$$\begin{array}{rcl} x_4 & = & 5 - 2x_1 - 3x_2 - x_3 \\ x_5 & = & 11 - 4x_1 - x_2 - 2x_3 \\ x_6 & = & 8 - 3x_1 - 4x_2 - 2x_3 \\ \hline z & = & 0 + 5x_1 + 4x_2 + 3x_3 \end{array}$$

$$\frac{\mathbf{x}_B = \mathbf{b} - \mathbf{A}\mathbf{x}_I}{z = c_0 + \mathbf{c}^\top \mathbf{x}_I}$$

# Solution Associated with Dictionary

- Non-basic variables have value 0.
- Basic variables: read off from dictionary.

$$\begin{array}{rcl} x_4 & = & 5 \quad -2x_1 \quad -3x_2 \quad -x_3 \\ x_5 & = & 11 \quad -4x_1 \quad -x_2 \quad -2x_3 \\ x_6 & = & 8 \quad -3x_1 \quad -4x_2 \quad -2x_3 \\ \hline z & = & 0 \quad +5x_1 \quad +4x_2 \quad +3x_3 \end{array}$$

# Another Dictionary

$$\begin{array}{rcl} x_4 & = & -28 + 10x_1 + 3x_5 + 5x_3 \\ x_2 & = & 11 - 4x_1 - x_5 - 2x_3 \\ x_6 & = & -32 + 9x_1 + 4x_5 + 62x_3 \\ \hline z & = & 44 - 11x_1 - 4x_5 - 5x_3 \end{array}$$

# Feasible vs. Infeasible Dictionary

$$\begin{array}{rcccc} x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\ x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\ x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\ \hline z & = & 0 & +5x_1 & +4x_2 & +3x_3 \end{array}$$



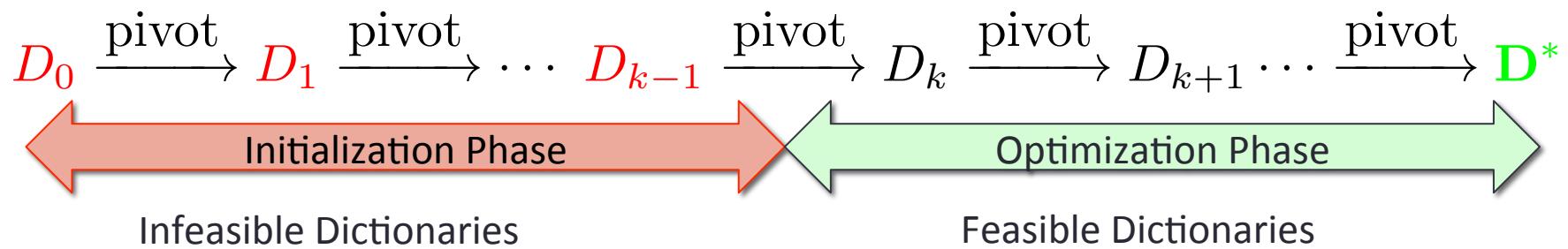
Infeasible

$$\begin{array}{rcccc} x_4 & = & -28 & +10x_1 & +3x_5 & +5x_3 \\ x_2 & = & 11 & -4x_1 & -x_5 & -2x_3 \\ x_6 & = & -32 & +9x_1 & +4x_5 & +62x_3 \\ \hline z & = & 44 & -11x_1 & -4x_5 & -5x_3 \end{array}$$

# Why Dictionaries?

- Data structure for Linear Programs.
  - Organize the data in the problem
- Represents candidate solutions to the problem.

# Simplex Algorithm



# Summary (1)

## Transform Problem with Slack Variables

$$\begin{array}{ll}\text{maximize} & \mathbf{c}^T \mathbf{x} \\ \text{subj.to.} & \mathbf{A} \mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}\end{array}$$



$$\begin{array}{lll}\text{maximize} & \mathbf{c}^T \mathbf{x} & \\ \text{s.t.} & \mathbf{A} \mathbf{x} + \mathbf{x}_{\text{slack}} & = \mathbf{b} \\ & \mathbf{x} & \geq \mathbf{0} \\ & \mathbf{x}_{\text{slack}} & \geq \mathbf{0}\end{array}$$

# Summary (2)

Dictionary: Solution associated + Feasibility

$$\begin{array}{rcl} x_4 & = & 5 - 2x_1 - 3x_2 - x_3 \\ x_5 & = & 11 - 4x_1 - x_2 - 2x_3 \\ x_6 & = & 8 - 3x_1 - 4x_2 - 2x_3 \\ \hline z & = & 0 + 5x_1 + 4x_2 + 3x_3 \end{array}$$

$$\frac{\mathbf{x}_B = \mathbf{b} - \mathbf{A}\mathbf{x}_I}{z = c_0 + \mathbf{c}^\top \mathbf{x}_I}$$

# ACT III: PIVOTING

---

*Going from one dictionary to the next.*

# ACT III: PIVOTING

---

*Going from one dictionary to the next.*

# Pivoting (from 2K feet)

2. Choose  
Leaving Variable

$$\begin{aligned}\mathbf{x}_B &= \mathbf{b} - \mathbf{A}\mathbf{x}_I \\ z &= c_0 + \mathbf{c}^\top \mathbf{x}_I\end{aligned}$$

m basic variables

$$\begin{array}{lcl}x_{B1} & = & b_1 + a_{11}x_{I1} + \cdots + \color{red}{a_{1j}}x_{Ij} + \cdots + a_{1n}x_{In} \\ \vdots & & \downarrow \\ x_{Bk} & = & b_k + a_{k1}x_{I1} + \cdots + \color{red}{a_{kj}}x_{Ij} + \cdots + a_{kn}x_{In} \\ \vdots & & \downarrow \\ x_{Bm} & = & b_m + a_{m1}x_{I1} + \cdots + \color{red}{a_{mj}}x_{Ij} + \cdots + a_{mn}x_{In} \\ z & = & c_0 + c_1x_{I1} + \cdots + \color{red}{+ c_jx_{Ij}} + \cdots + c_nx_{In}\end{array}$$

1. Choose  
Entering Variable  
in non-basic variables

# Pivoting Steps

1. Choose an entering variable.
2. For the choice of entering variable, find a leaving variable.
3. Perform substitutions and obtain next dictionary.

# Example

Chvátal, Chapter 2

$$\begin{array}{lllll}\text{max.} & 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} & 2x_1 + 3x_2 + x_3 & \leq & 5 \\ & 4x_1 + x_2 + 2x_3 & \leq & 11 \\ & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\ & x_1, x_2, x_3 & \geq & 0\end{array}$$

Already in standard form.

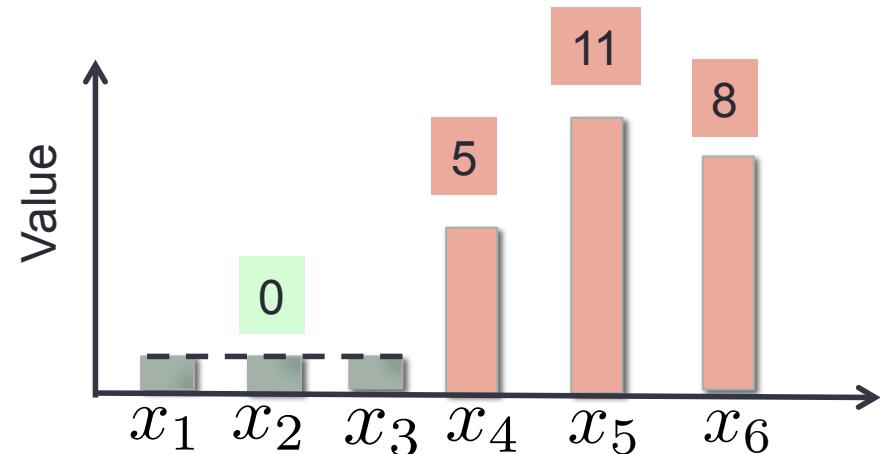
# Example (add slack)

$$\begin{array}{lllll} \text{max.} & 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} & 2x_1 + 3x_2 + x_3 & \leq & 5 \\ & 4x_1 + x_2 + 2x_3 & \leq & 11 \\ & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\ & x_1, x_2, x_3 & \geq & 0 \end{array}$$

$$\begin{array}{rcl} x_4 & = & 5 - 2x_1 - 3x_2 - x_3 \\ x_5 & = & 11 - 4x_1 - x_2 - 2x_3 \\ x_6 & = & 8 - 3x_1 - 4x_2 - 2x_3 \\ \hline z & = & 0 + 5x_1 + 4x_2 + 3x_3 \end{array}$$

# Choosing an entering variable

$$\begin{array}{rcccc} x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\ x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\ x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\ \hline z & = & 0 & +5x_1 & +4x_2 & +3x_3 \end{array}$$



# Leaving Variable

$$\begin{array}{rcl} x_4 & = & 5 \quad -2x_1 \quad -3x_2 \quad -x_3 \\ x_5 & = & 11 \quad -4x_1 \quad -x_2 \quad -2x_3 \\ x_6 & = & 8 \quad -3x_1 \quad -4x_2 \quad -2x_3 \\ \hline z & = & 0 \quad +5x_1 \quad +4x_2 \quad +3x_3 \end{array}$$

# Leaving Variable Analysis

$$\begin{array}{rcccc} x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\ x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\ x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\ \hline z & = & 0 & +5x_1 & +4x_2 & +3x_3 \end{array}$$

$$x_4 = 5 - 2x_1 - 3x_2 - x_3 \Rightarrow x_1 \leq \frac{5}{2}$$

$$x_5 = 11 - 4x_1 - x_2 - 2x_3 \Rightarrow x_1 \leq \frac{11}{4}$$

$$x_6 = 8 - 3x_1 - 4x_2 - 2x_3 \Rightarrow x_1 \leq \frac{8}{3}$$

---

$$z = 0 + 5x_1 + 4x_2 + 3x_3$$

# Exercise: Find Leaving Variables

Modified Problem

$$\begin{array}{rcl} x_4 & = & 7 + 2x_1 - 3x_2 + x_3 \\ x_5 & = & 12 - 4x_1 + x_2 + 2x_3 \\ \hline x_6 & = & 9 - 3x_1 - 4x_2 + 2x_3 \\ \hline z & = & 10 + 2x_1 + 2x_2 - 3x_3 \end{array}$$

Note: This dictionary  
is **not for** the same problem as  
in slide 2.

# Pivoting: Finding new dictionary.

$$\begin{array}{rcccccc} x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\ x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\ \hline x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\ z & = & 0 & +5x_1 & +4x_2 & +3x_3 \end{array}$$

$x_1$  enters and  $x_4$  leaves

Note: This dictionary is for a different problem than the dictionary in slide 1.  
The rest of this PPT will keep pivoting the dictionary on this slide.

# Pivoting

$$\begin{array}{rcccc} \textcolor{red}{x}_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\ \textcolor{red}{x}_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\ \textcolor{red}{x}_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\ \hline \textcolor{blue}{z} & = & 0 & +5x_1 & +4x_2 & +3x_3 \end{array}$$

$$\textcolor{red}{x}_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \quad x_1 \text{ enters and } x_4 \text{ leaves}$$

$$\textcolor{red}{x}_5 = 1 + 5x_2 + 2x_4$$

$$\textcolor{red}{x}_6 = \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4$$

---

$$\textcolor{blue}{z} = \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4$$

# Entering Variable Analysis

$$x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4$$

$$x_5 = 1 + 5x_2 + 2x_4$$

$$x_6 = \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4$$

---

$$z = \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4$$

# Leaving Variable Analysis

$$\begin{array}{rcl} \boxed{\begin{array}{l} x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_5 = 1 + 5x_2 + 2x_4 \\ x_6 = \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \end{array}} \\[10pt] \boxed{z = \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4} \end{array}$$

$$x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \Rightarrow \underline{x_3 \leq 5}$$

$$x_5 = 1 + 5x_2 + 2x_4 \Rightarrow \text{no constraint}$$

$$x_6 = \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \Rightarrow \underline{x_3 \leq 1}$$

# Pivoting

$$\begin{array}{rcl} \textcolor{red}{x_1} & = & \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ \textcolor{red}{x_5} & = & 1 + 5x_2 \quad + 2x_4 \\ \textcolor{red}{x_6} & = & \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \end{array}$$

$$\begin{array}{rcl} \textcolor{blue}{z} & = & \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4 \\ x_3 \text{ enters and } x_6 \text{ leaves} \end{array}$$

$$\textcolor{red}{x_3} = 1 + x_2 + 3x_4 - 2x_6$$

$$\textcolor{red}{x_1} = 2 - 2x_2 - 2x_4 + x_6$$

$$\textcolor{red}{x_5} = 1 + 5x_2 + 2x_4$$

---

$$\textcolor{blue}{z} = 13 - 3x_2 - x_4 - x_6$$

# Final Dictionary

$$x_3 = 1 + x_2 + 3x_4 - 2x_6$$

$$x_1 = 2 - 2x_2 - 2x_4 + x_6$$

$$x_5 = 1 + 5x_2 + 2x_4$$

---

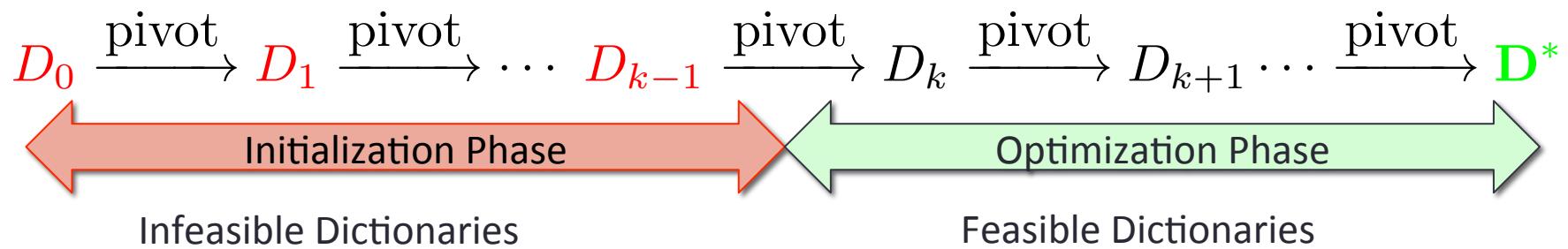
$$z = 13 - 3x_2 - x_4 - x_6$$

No choice for entering variables.

$$\begin{array}{rcl} \mathbf{x}_B & = & \mathbf{b} - \mathbf{A}\mathbf{x}_I \\ z & = & c_0 + \mathbf{c}^\top \mathbf{x}_I \end{array}$$

$$\mathbf{c} \leq 0$$

# Simplex Algorithm



# Summary (1/2)

Choice of  
leaving  
variable

$$\begin{array}{rcl} x_{B1} & = & b_1 - a_{11}x_{I1} \cdots - a_{1n}x_{Im} \\ x_{B2} & = & b_2 - a_{21}x_{I1} \cdots - a_{2n}x_{Im} \\ & \vdots & \ddots \\ x_{Bm} & = & b_n - a_{m1}x_{I1} \cdots - a_{mn}x_{Im} \\ \hline z & = & c_0 + c_1x_{I1} + \cdots + c_nx_{In} \end{array}$$

Choice of entering  
variable

# Summary (2/2)

Pivoting: computing next dictionary

$x_3$  enters and  $x_6$  leaves

$$x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4$$

$$x_5 = 1 + 5x_2 + 2x_4$$

$$x_6 = \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4$$

---

$$z = \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4$$

$$x_3 = 1 + x_2 + 3x_4 - 2x_6$$

$$x_1 = 2 - 2x_2 - 2x_4 + x_6$$

$$x_5 = 1 + 5x_2 + 2x_4$$

---

$$z = 13 - 3x_2 - x_4 - x_6$$

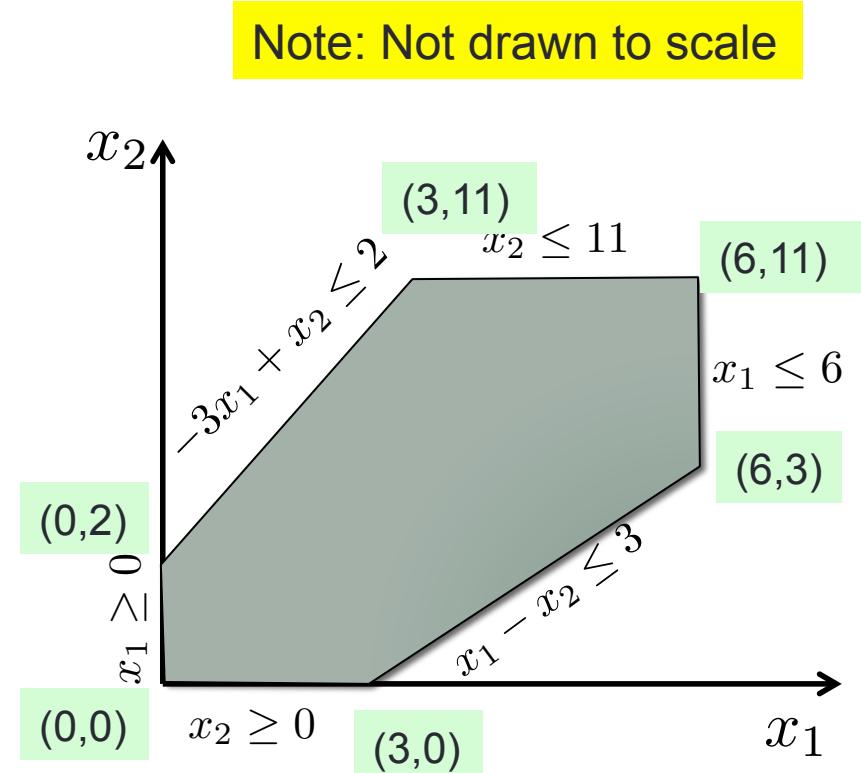
# SIMPLEX: EXAMPLE

---

A Complete Example with Visualization.

# Linear Programming Problem

$$\begin{array}{llllll} \text{max.} & x_1 & +2x_2 & & & \\ \text{s.t.} & -3x_1 & +x_2 & \leq & 2 & \\ & & +x_2 & \leq & 11 & \\ & x_1 & -x_2 & \leq & 3 & \\ & x_1 & & & 6 & \\ & x_1, & x_2 & \geq & 0 & \end{array}$$



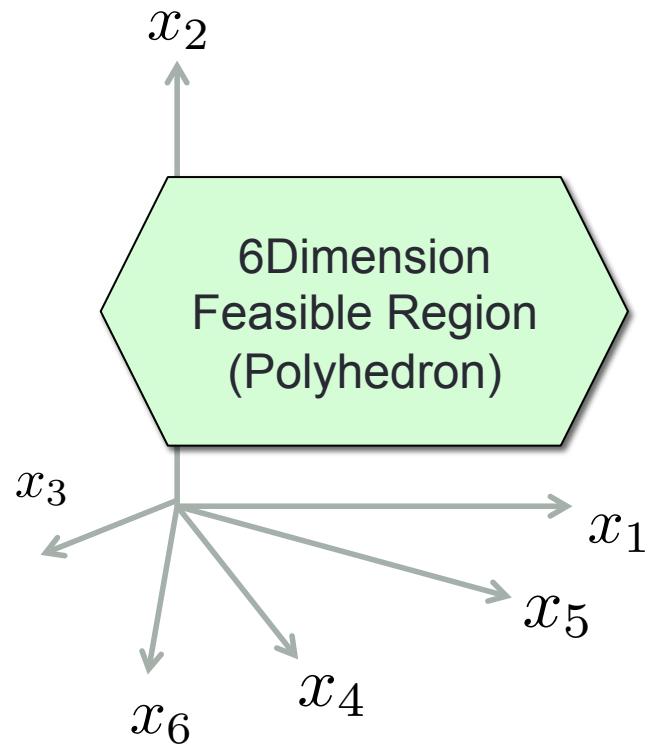
Goal: Solve LP using Simplex and visualize!

# Step 1: Add Slack Variables

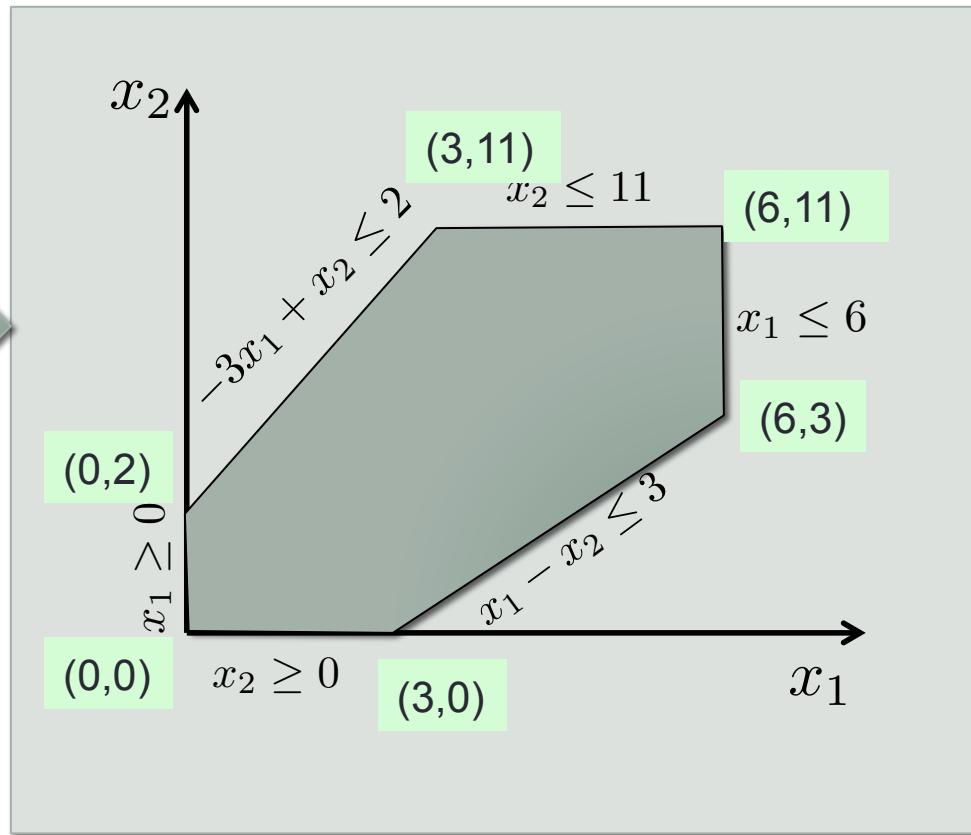
$$\begin{array}{lllll} \text{max.} & x_1 & +2x_2 & & \\ \text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\ & & +x_2 & \leq & 11 \\ & x_1 & -x_2 & \leq & 3 \\ & x_1 & & \leq & 6 \\ & x_1, & x_2 & \geq & 0 \end{array}$$

$$\begin{array}{llllll} \text{max.} & x_1 & +2x_2 & & & \\ \text{s.t.} & -3x_1 & +x_2 & +x_3 & = & 2 \\ & & +x_2 & +x_4 & = & 11 \\ & x_1 & -x_2 & +x_5 & = & 3 \\ & x_1 & & +x_6 & = & 6 \\ & x_1, x_2, & x_3, \dots, x_6 & \geq & 0 \end{array}$$

# Visualizing with Slack

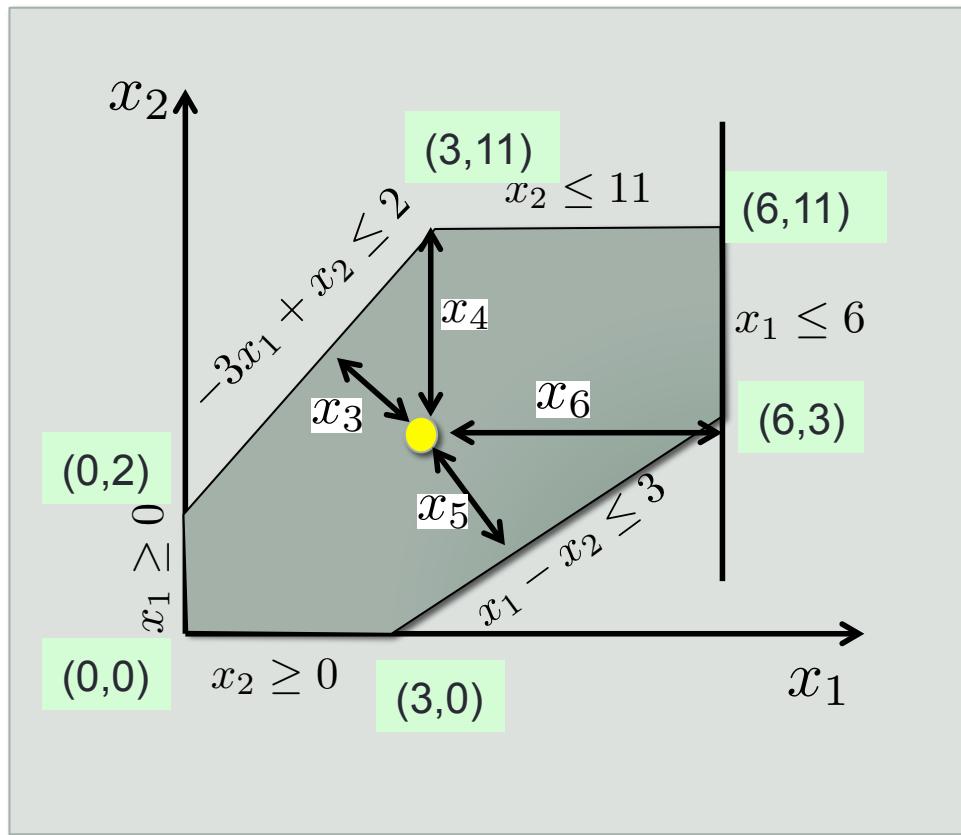


Shadow



Schematic Drawing

# Alternative Visualization of Slack



# Initial Dictionary

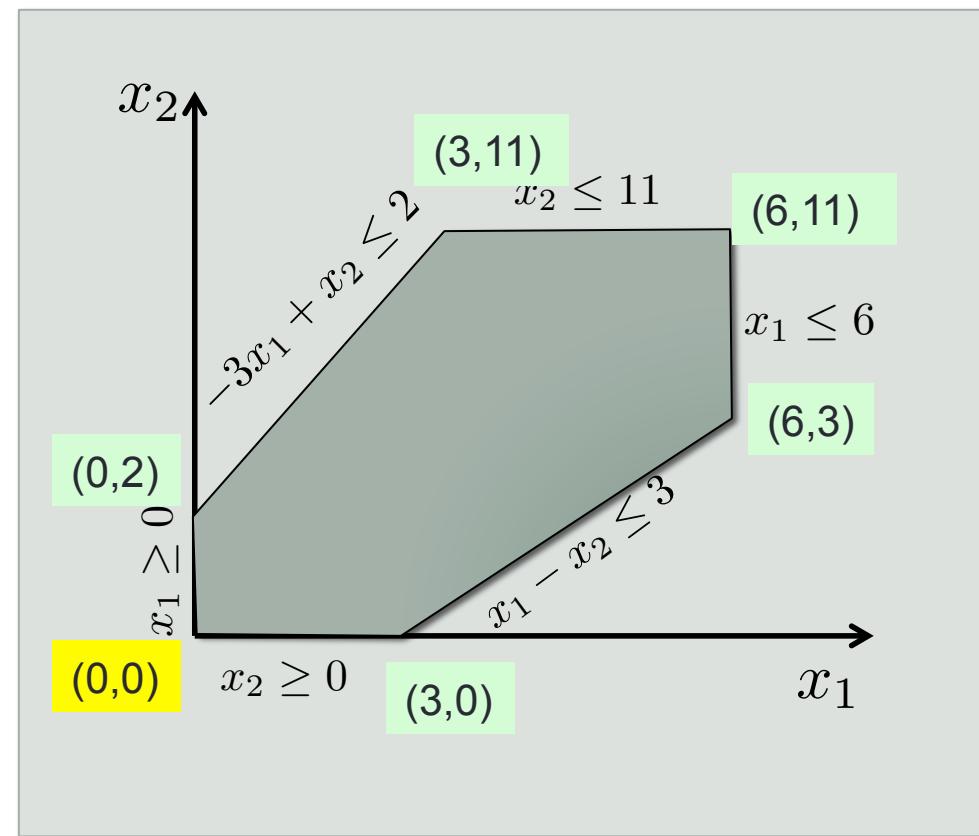
$$\begin{array}{llllll}
 \text{max.} & x_1 & +2x_2 & & & \\
 \text{s.t.} & -3x_1 & +x_2 & +x_3 & = & 2 \\
 & & +x_2 & +x_4 & = & 11 \\
 & x_1 & -x_2 & +x_5 & = & 3 \\
 & x_1 & & +x_6 & = & 6 \\
 & x_1, x_2, x_3, \dots, x_6 & \geq & 0 & &
 \end{array}$$

Dict. D1

$$\begin{array}{rcl}
 x_3 & = & 2 & +3x_1 & -x_2 \\
 x_4 & = & 11 & +0x_1 & -x_2 \\
 x_5 & = & 3 & -x_1 & +x_2 \\
 x_6 & = & 6 & -x_1 & +0x_2 \\
 \hline
 z & = & 0 & +x_1 & +2x_2
 \end{array}$$

# Solution Associated (Dict. D1)

$x_3$	=	2	$+3x_1$	$-x_2$
$x_4$	=	11	$+0x_1$	$-x_2$
$x_5$	=	3	$-x_1$	$+x_2$
$x_6$	=	6	$-x_1$	$+0x_2$
$z$	=	0	$+x_1$	$+2x_2$



# Entering/Leaving Variable Analysis

$$\begin{array}{rcl} x_3 & = & 2 + 3x_1 - x_2 \\ x_4 & = & 11 + 0x_1 - x_2 \\ x_5 & = & 3 - x_1 + x_2 \\ x_6 & = & 6 - x_1 + 0x_2 \\ \hline z & = & 0 + x_1 + 2x_2 \end{array}$$

$x_1$  enters and  $x_5$  leaves.

$x_2$  enters and  $x_3$  leaves.

# Pivoting

$$\begin{array}{rcccccc} x_3 & = & 2 & +3x_1 & -x_2 \\ x_4 & = & 11 & +0x_1 & -x_2 \\ x_5 & = & 3 & -x_1 & +x_2 \\ x_6 & = & 6 & -x_1 & +0x_2 \\ \hline z & = & 0 & +x_1 & +2x_2 \end{array}$$

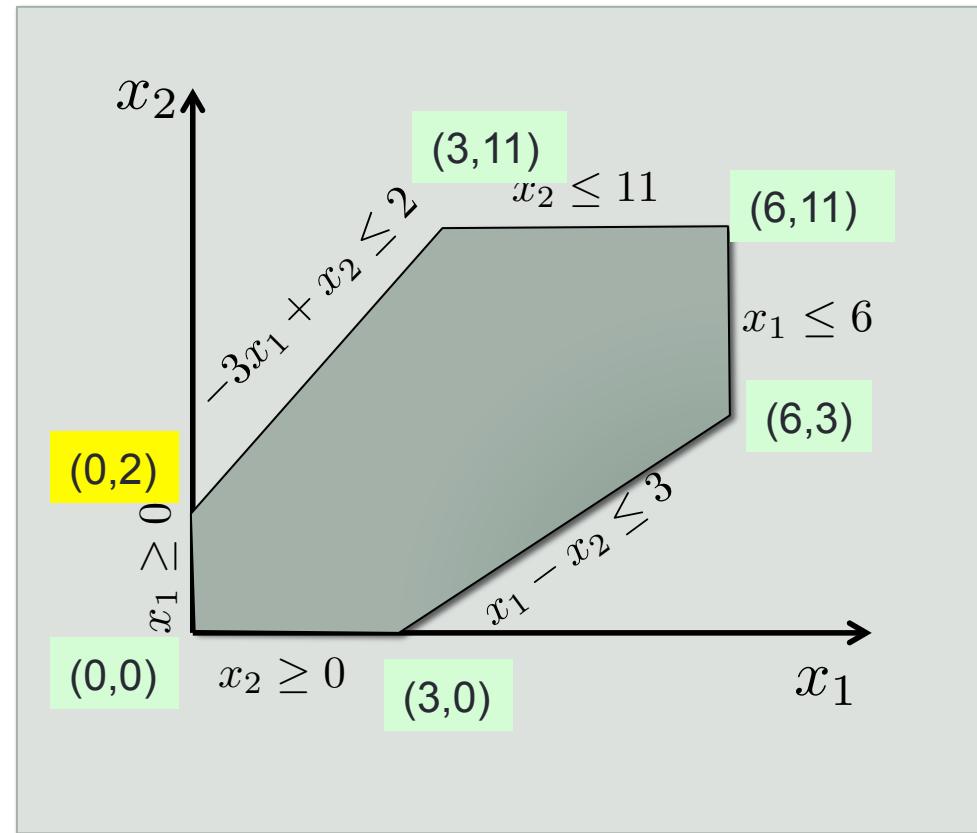
$x_2$  enters and  $x_3$  leaves.

Dict. D2

$$\begin{array}{rcccccc} x_2 & = & 2 & +3x_1 & -x_3 \\ x_4 & = & 9 & -3x_1 & +x_3 \\ x_5 & = & 5 & +2x_1 & -x_3 \\ x_6 & = & 6 & -x_1 & +0x_3 \\ \hline z & = & 4 & +7x_1 & -2x_3 \end{array}$$

# Solution Associated (D2)

$$\begin{array}{rclcl} x_2 & = & 2 & +3x_1 & -x_3 \\ x_4 & = & 9 & -3x_1 & +x_3 \\ x_5 & = & 5 & +2x_1 & -x_3 \\ x_6 & = & 6 & -x_1 & +0x_3 \\ \hline z & = & 4 & +7x_1 & -2x_3 \end{array}$$



# Entering/Leaving Variable Analysis

$$\begin{array}{rcl} x_2 & = & 2 + 3x_1 - x_3 \\ x_4 & = & 9 - 3x_1 + x_3 \\ x_5 & = & 5 + 2x_1 - x_3 \\ x_6 & = & 6 - x_1 + 0x_3 \\ \hline z & = & 4 + 7x_1 - 2x_3 \end{array}$$

$x_1$  enters and  $x_4$  leaves.

# Pivoting

$x_2$	=	2	$+3x_1$	$-x_3$
$x_4$	=	9	$-3x_1$	$+x_3$
$x_5$	=	5	$+2x_1$	$-x_3$
$x_6$	=	6	$-x_1$	$+0x_3$
$z$	=	4	$+7x_1$	$-2x_3$

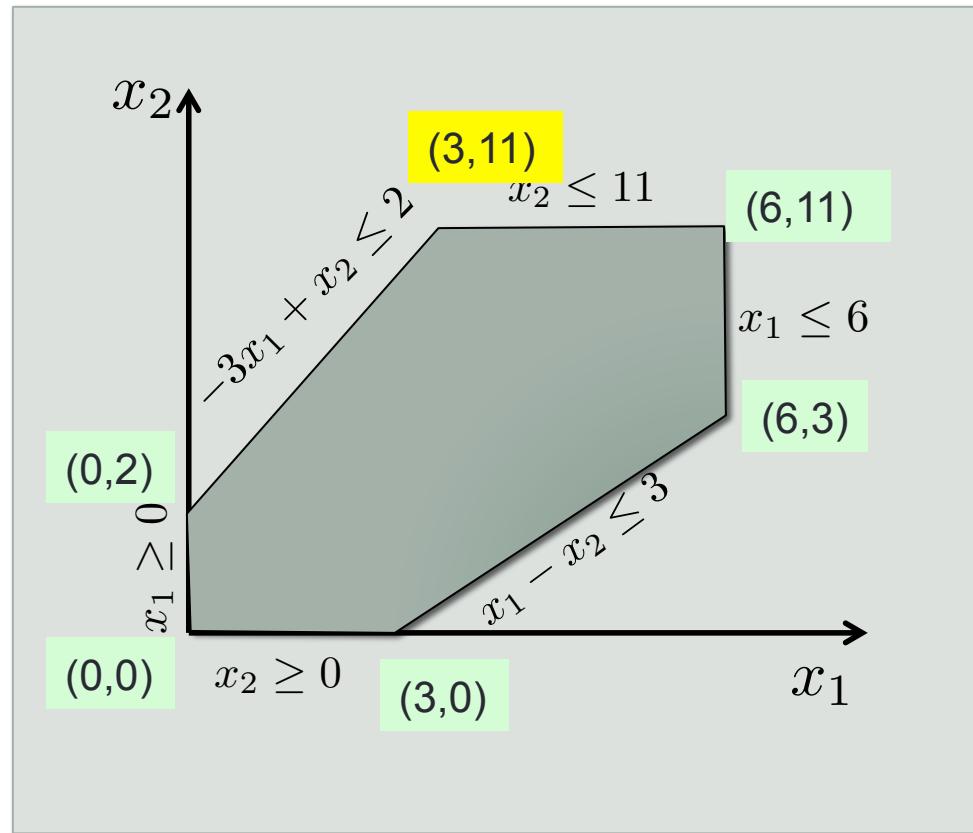
$x_1$  enters and  $x_4$  leaves.

Dict. D3

$$\begin{array}{rcl}
 x_1 & = & 3 - \frac{1}{3}x_4 + \frac{1}{3}x_3 \\
 x_2 & = & 11 - x_4 + 0x_3 \\
 x_5 & = & 11 - \frac{2}{3}x_4 - \frac{1}{3}x_3 \\
 x_6 & = & 3 + \frac{1}{3}x_4 - \frac{1}{3}x_3 \\
 \hline
 z & = & 25 - \frac{7}{3}x_4 + \frac{1}{3}x_3
 \end{array}$$

# Solution Associated with D3

$$\begin{array}{rcl}
 x_1 & = & 3 - \frac{1}{3}x_4 + \frac{1}{3}x_3 \\
 x_2 & = & 11 - x_4 + 0x_3 \\
 x_5 & = & 11 - \frac{2}{3}x_4 - \frac{1}{3}x_3 \\
 x_6 & = & 3 + \frac{1}{3}x_4 - \frac{1}{3}x_3 \\
 \hline
 z & = & 25 - \frac{7}{3}x_4 + \frac{1}{3}x_3
 \end{array}$$



# Entering/Leaving Variable Analysis

$$\begin{array}{rcl} x_1 & = & 3 - \frac{1}{3}x_4 + \frac{1}{3}x_3 \\ x_2 & = & 11 - x_4 + 0x_3 \\ x_5 & = & 11 - \frac{2}{3}x_4 - \frac{1}{3}x_3 \\ x_6 & = & 3 + \frac{1}{3}x_4 - \frac{1}{3}x_3 \\ \hline z & = & 25 - \frac{7}{3}x_4 + \frac{1}{3}x_3 \end{array}$$

$x_3$  enters and  $x_6$  leaves.

# Pivoting

$x_1$	=	3	$-\frac{1}{3}x_4$	$+\frac{1}{3}x_3$
$x_2$	=	11	$-x_4$	$+0x_3$
$x_5$	=	11	$-\frac{2}{3}x_4$	$-\frac{1}{3}x_3$
$x_6$	=	3	$+\frac{1}{3}x_4$	$-\frac{1}{3}x_3$
$z$	=	25	$-\frac{7}{3}x_4$	$+\frac{1}{3}x_3$

$x_3$  enters and  $x_6$  leaves.

Dict. D4

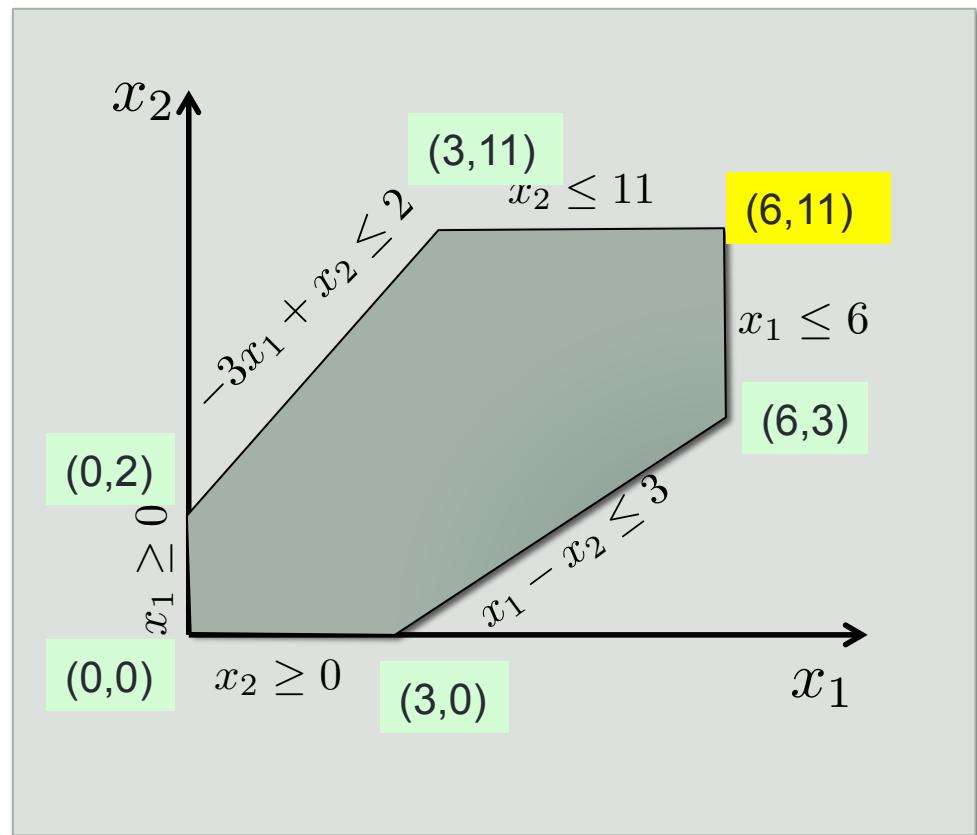
$$\begin{array}{rclcrcl} x_3 & = & 9 & +x_4 & -3x_6 \\ x_1 & = & 6 & & & -x_6 \\ x_2 & = & 11 & -x_4 & +0x_6 \\ x_5 & = & 8 & -x_4 & +x_6 \\ \hline z & = & 28 & -2x_4 & -x_6 \end{array}$$

# Final Dictionary

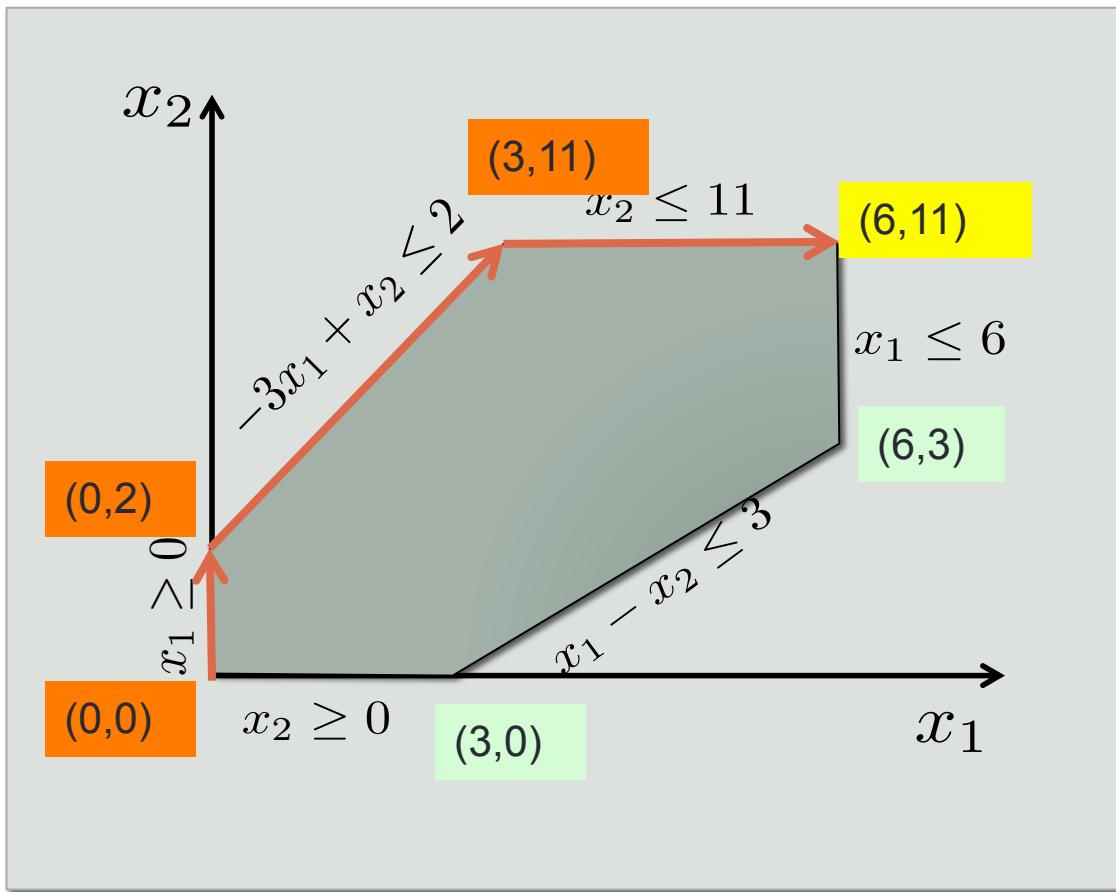
$$\begin{array}{rcl} x_3 & = & 9 \\ x_1 & = & 6 \\ x_2 & = & 11 \\ x_5 & = & 8 \end{array} \quad \begin{array}{rcl} +x_4 & & -3x_6 \\ & & -x_6 \\ -x_4 & & +0x_6 \\ -x_4 & & +x_6 \end{array}$$

---

$$z = 28 - 2x_4 - x_6$$

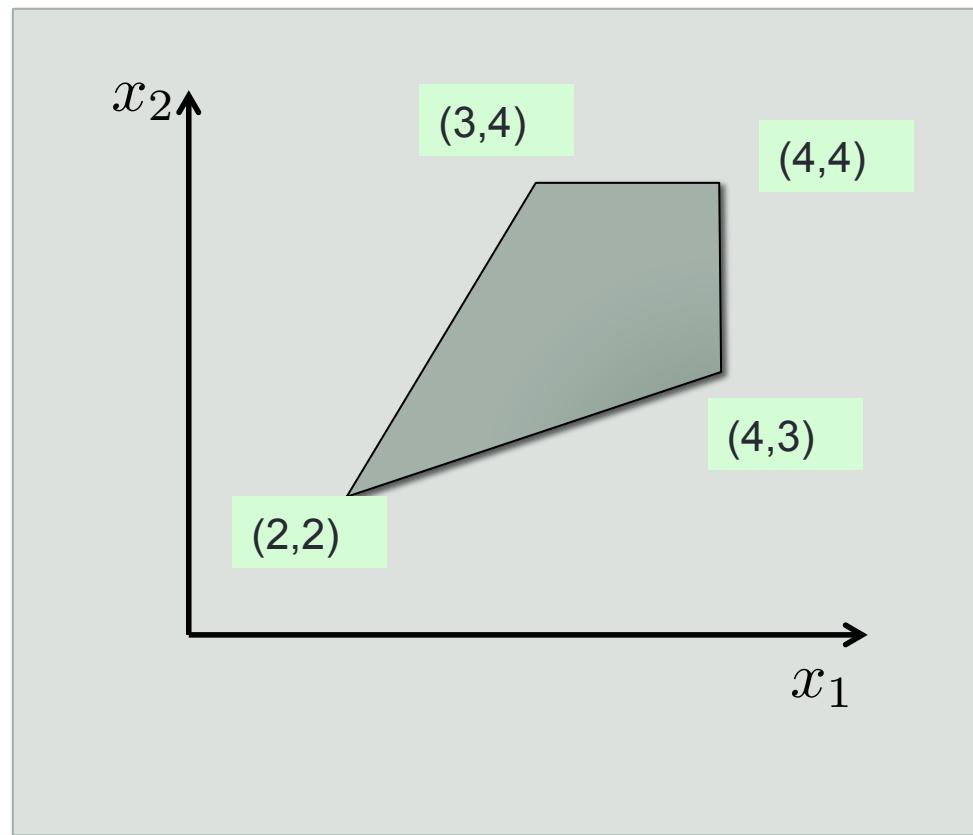


# Simplex Dictionaries



## Example # 2

$$\begin{array}{lll} \text{max.} & x_1 + 2x_2 \\ \text{s.t.} & -2x_1 + x_2 \leq -2 \\ & x_2 \leq 4 \\ & x_1 - 2x_2 \leq -2 \\ & x_1 \leq 4 \\ & x_1, x_2 \geq 0 \end{array}$$



# Step 1: Adding Slack

$$\begin{array}{lllll} \text{max.} & x_1 + 2x_2 & & & \\ \text{s.t.} & -2x_1 + x_2 + x_3 & = & -2 & \\ & x_2 + x_4 & = & 4 & \\ & x_1 - 2x_2 + x_5 & = & -2 & \\ & x_1 + x_6 & = & 4 & \\ & x_1, x_2, x_3, \dots, x_6 & \geq & 0 & \end{array}$$

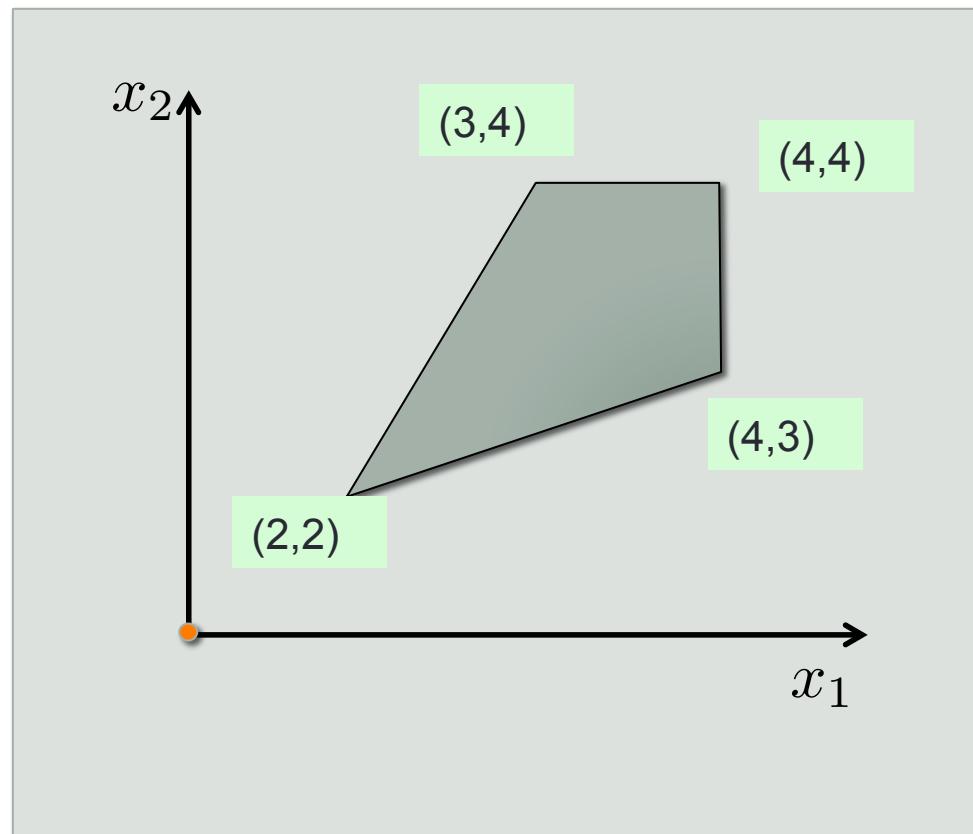
## Step 2: Initial Dictionary

$$\begin{array}{lll} \text{max.} & x_1 + 2x_2 \\ \text{s.t.} & -2x_1 + x_2 + x_3 = -2 \\ & x_2 + x_4 = 4 \\ & x_1 - 2x_2 + x_5 = -2 \\ & x_1 + x_6 = 4 \\ & x_1, x_2, x_3, \dots, x_6 \geq 0 \end{array}$$

$$\begin{array}{rcl} x_3 & = & -2 + 2x_1 - x_2 \\ x_4 & = & 4 + 0x_1 - x_2 \\ x_5 & = & -2 - x_1 + 2x_2 \\ x_6 & = & 4 - x_1 + 0x_2 \\ \hline z & = & 0 + x_1 + 2x_2 \end{array}$$

# Infeasible Initial Dictionary

$$\begin{array}{rclcl} x_3 & = & -2 & +2x_1 & -x_2 \\ x_4 & = & 4 & +0x_1 & -x_2 \\ x_5 & = & -2 & -x_1 & +2x_2 \\ x_6 & = & 4 & -x_1 & +0x_2 \\ \hline z & = & 0 & +x_1 & +2x_2 \end{array}$$



# Diet Problem

- Variables
  - Constraints
  - Objective
  - Formulate an LP (Inequality Form)
-

# Problem Description

## - Variables

variable  $x_j$  (out of  $n$  variables)  
is the # of servings (units) of food j

$x_{18}$  - serving of  
carrots

## - Healthy Diet Requirement (Constraints)

- For nutrient  $i$  (of  $m$  nutrients) you should have at least  $l_i$  amount and at most  $u_i$  amount.
- In 1 serving of food  $j$  there is  $a_{ij}$  amount of nutrient  $i$

$i = 8$  - Vit A

$l_8 = 5000$

$u_8 = 50000$

$a_{8,18}$  - amount of  
Vit A in a  
Serving of Carrots

## - Find the Cheapest Diet (Objective)

1 serving of food  $j$ , costs  $c_j$

$C_{18}$  - cost per servings  
of carrots

# Inequality Form

$$\begin{array}{l} \text{minimize} \\ \text{subject to} \end{array} \underbrace{\left[ \begin{array}{l} c^T x \\ Ax \leq b \end{array} \right]}$$

- Objective

$$\text{Total cost} = C_1 X_1 + \dots + C_n X_n$$

$$\text{Min} \quad [C_1 \ C_2 \ \dots \ C_n]^T \underbrace{\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}}_x$$

# Inequality Form

minimize  $c^T x$   
subject to  $Ax \leq b$

- Non-Negative Constraint

$$-I^{h \times n} x \leq 0^{n \times 1}$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

:

$$\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \geq \begin{bmatrix} 0 \\ \vdots \\ 0 \\ c \end{bmatrix}$$

# Inequality Form

$$\begin{array}{l} \text{minimize} \\ \text{subject to} \end{array} \begin{array}{l} \overbrace{c^T x} \\ \boxed{Ax \leq b} \end{array}$$

- Nutrient Bounds

$$l_1 \leq a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq u_1$$

,

,

$$l_m \leq a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq u_n$$

$$\begin{bmatrix} l_1 \\ \vdots \\ l_m \\ \vdots \\ l_n \end{bmatrix} = l \leq \hat{A}x \leq u =$$

$$\begin{bmatrix} u_1 \\ \vdots \\ u_m \\ \vdots \\ u_n \end{bmatrix}$$

# Inequality Form

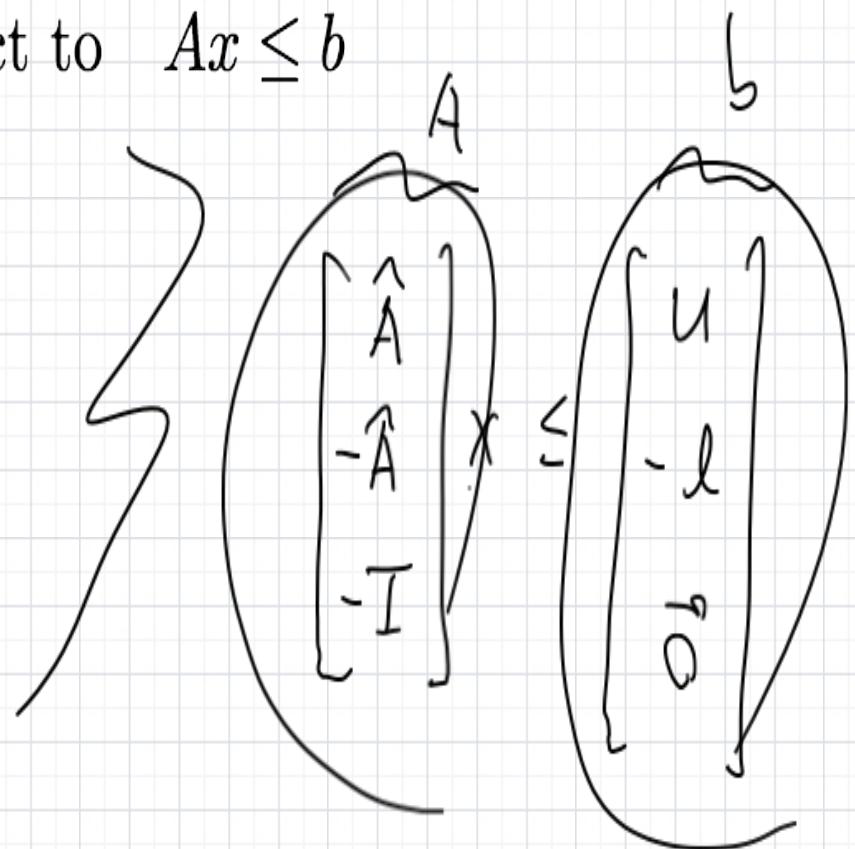
$$-I\vec{x} \leq \vec{0}$$

$$\hat{A}\vec{x} \geq l \Leftrightarrow -\hat{A}\vec{x} \leq -l$$

$$\hat{A}\vec{x} \leq u$$

minimize  $c^T x$

subject to  $Ax \leq b$



$$\hat{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

# SOLVING LINEAR PROGRAMS WITH GLPK

---

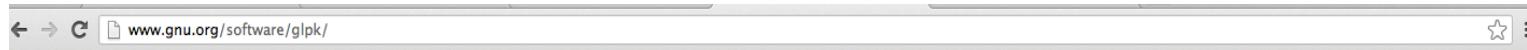
Tutorial with Examples

# GLPK

- Why GLPK?
  - Open source, available free of cost to everyone.
  - API for programmers + Input languages that are easy to use.
- Availability: <http://www.gnu.org/software/glpk/>



# Downloading + Installing GLPK



## Introduction to GLPK

The GLPK (GNU Linear Programming Kit) package is intended for solving large-scale linear programming (LP), mixed integer programming (MIP), and other related problems. It is a set of routines written in ANSI C and organized in the form of a callable library.

GLPK supports the *GNU MathProg modeling language*, which is a subset of the AMPL language.

The GLPK package includes the following main components:

- primal and dual simplex methods
- primal-dual interior-point method
- branch-and-cut method
- translator for GNU MathProg
- application program interface (API)
- stand-alone LP/MIP solver

## Downloading GLPK

The GLPK distribution tarball can be found on <http://ftp.gnu.org/gnu/glpk/> [via http] and <ftp://ftp.gnu.org/gnu/glpk/> [via FTP]. It can also be found on one of [our FTP mirrors](#); please use a mirror if possible.

To make sure that the GLPK distribution tarball you have downloaded is intact you need to download the corresponding .sig file and run a command like this:

```
gpg --verify glpk-4.32.tar.gz.sig
```

If that command fails because you do not have the required public key, run the following command to import it:

```
gpg --keyserver keys.gnupg.net --recv-keys 5981E818
```

and then re-run the previous command.

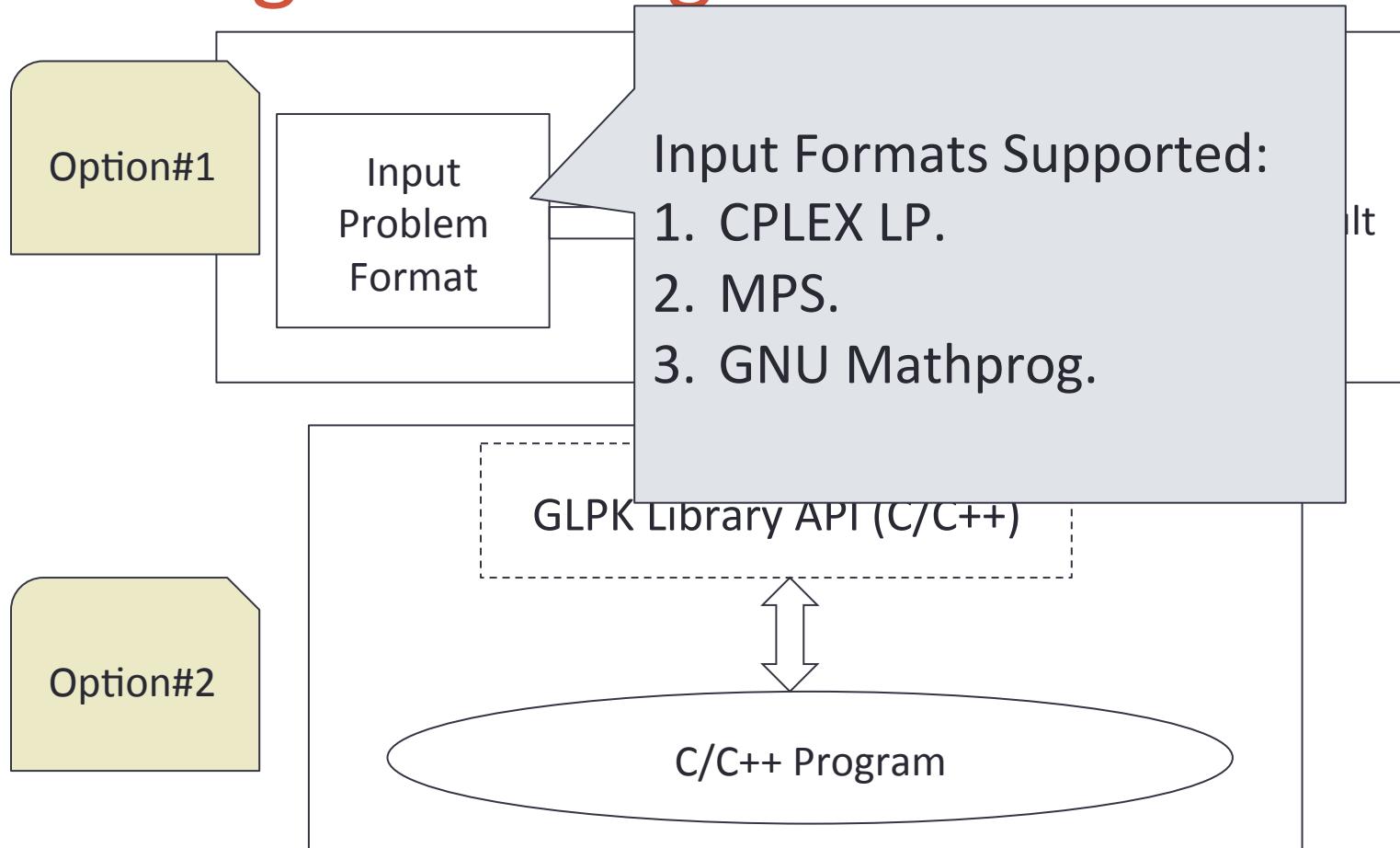
## Documentation

The GLPK documentation consists of the Reference Manual and the description of the GNU MathProg modeling language. Both these documents are included in the distribution (in LaTeX, DVI, and PostScript formats).

## Mailing Lists/Newsgroups

GLPK has two mailing lists: [help-glpk@gnu.org](mailto:help-glpk@gnu.org) and [bug-glpk@gnu.org](mailto:bug-glpk@gnu.org).

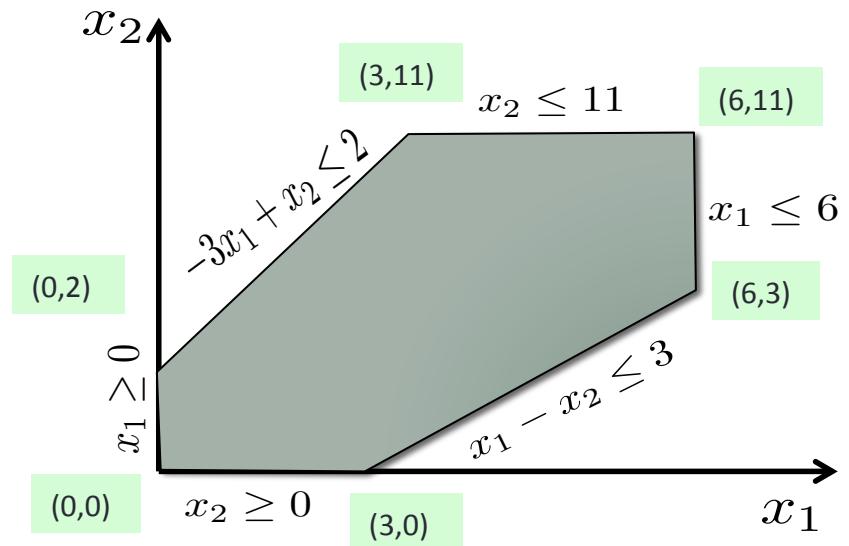
# Solving LPs through GLPK



# Example #1

$$\begin{array}{lll} \text{max.} & x_1 & + 2x_2 \\ \text{s.t.} & -3x_1 & + x_2 \leq 2 \\ & & + x_2 \leq 11 \\ & x_1 & - x_2 \leq 3 \\ & x_1 & \leq 6 \\ & x_1, x_2 & \geq 0 \end{array}$$

Not drawn to scale



Solution:  $x_1 = 6$ ,  $x_2 = 11$   
Opt. Objective Value: 28

# Specifying Problem: Mathprog Format

$$\begin{array}{llllll} \text{max.} & x_1 & +2x_2 & & & \\ \text{s.t.} & -3x_1 & +x_2 & \leq & 2 & \\ & & +x_2 & \leq & 11 & \\ & x_1 & -x_2 & \leq & 3 & \\ & x_1 & & \leq & 6 & \\ & x_1, & x_2 & \geq & 0 & \end{array}$$

```
var x1 >= 0;  
var x2 >= 0;  
maximize obj: x1 + 2 * x2;  
c1: -3 * x1 + x2 <= 2;  
c2: x2 <= 11;  
c3: x1 - x2 <= 3;  
c4: x1 <= 6;  
solve;  
display x1;  
display x2;  
end;
```

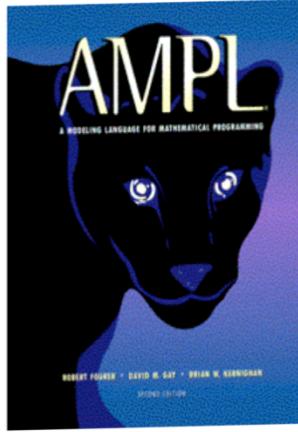
# Running GLPSOL

```
bash-3.2$ glpsol --math ex1AMPL
GLPSOL: GLPK LP/MIP Solver, v4.48
OPTIMAL SOLUTION FOUND
Time used: 0.0 secs
Display statement at line 12
x1.val = 6
Display statement at line 13
x2.val = 11
Model has been successfully processed
```

# Using Mathprog Language

- Very close to AMPL

[FAQ](#) | [BOOK](#) | [SOLVERS](#) | [PLATFORMS](#) | [VENDORS](#) | [CALENDAR](#) | [MORE!](#) | [WHAT'S NEW](#)  
[EXTENSIONS](#) | [CHANGE LOG](#) | [REPORTS](#) | [NETLIB](#) | [EXAMPLES](#) | [CONTENTS](#) | [CONTACT US](#)



**AMPL®**  
A Modeling Language  
for Mathematical Programming

**NEW!** **AMPL IDE Interface**  
[Try the Beta Test Version](#)

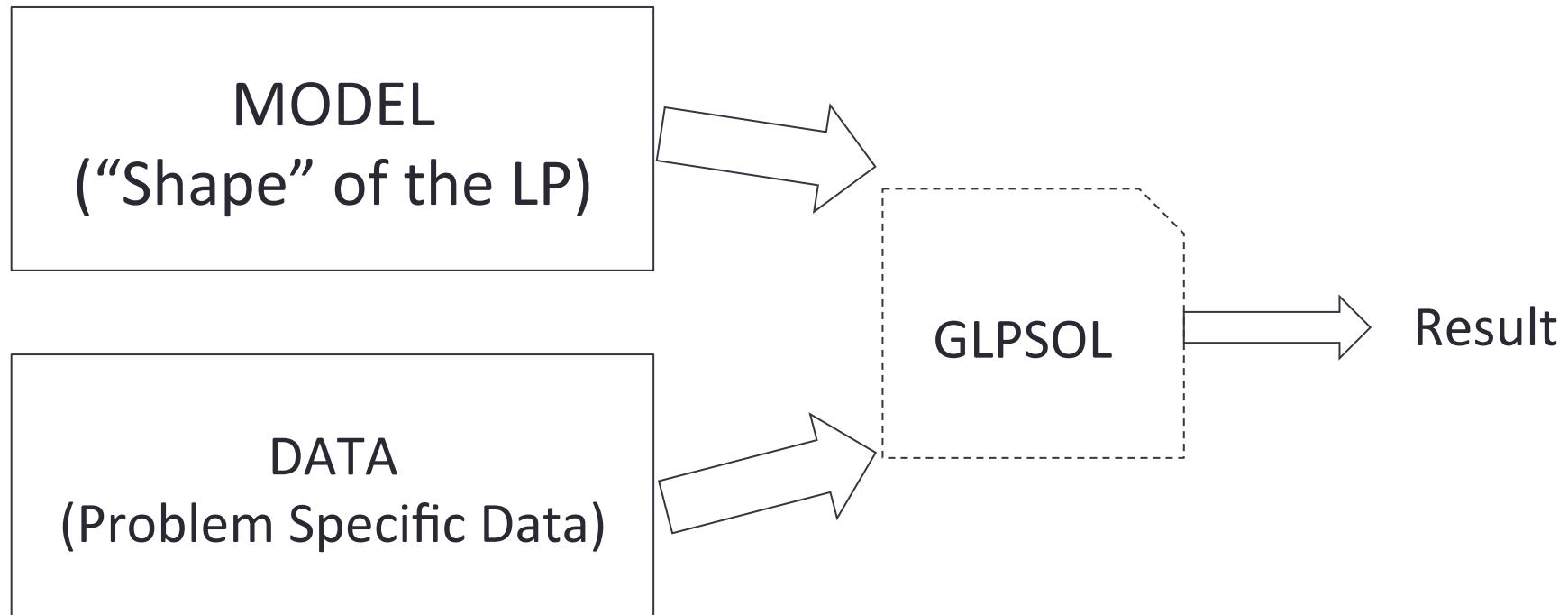
**Free AMPL for Courses**

- Full-featured, time-limited
- [Request for fall classes now](#)

**What's AMPL?**

AMPL is a comprehensive and powerful algebraic modeling language for linear and nonlinear optimization problems, in discrete or continuous variables.

# Mathprog input format



# Example

$$\begin{array}{lllll} \max. & x_1 & +2x_2 & & \\ \text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\ & & +x_2 & \leq & 11 \\ & x_1 & -x_2 & \leq & 3 \\ & x_1 & & \leq & 6 \\ & x_1, & x_2 & \geq & 0 \end{array}$$

“Shape of the LP”

$$\begin{array}{llll} \max & \sum_{j=1}^n c_i x_i & & \\ \text{s.t.} & \sum_{j=1}^n a_{i,j} x_j & \leq & b_i \quad [i \in \{1, \dots, m\}] \\ & x_j & \geq & 0 \quad [j \in \{1, \dots, n\}] \end{array}$$

# Mathprog Format

```
# number of constraints
```

```
param m;
```

```
# number of decision variables
```

```
param n;
```

```
#problem parameters
```

```
param c { i in 1..n};
```

```
param A { i in 1..m, j in 1..n};
```

```
param b { i in 1..m};
```

```
#declare variables
```

```
var x { i in 1..n} >= 0;
```

“Shape of the LP”

$$\max \quad \sum_{j=1}^n c_i x_i$$

$$\text{s.t.} \quad \begin{aligned} \sum_{j=1}^n a_{i,j} x_j &\leq b_i & [i \in \{1, \dots, m\}] \\ x_j &\geq 0 & [j \in \{1, \dots, n\}] \end{aligned}$$

#objective

```
maximize obj: sum{ i in 1..n} c[i] * x[i];
```

s.t.

```
e{j in 1..m}: sum{i in 1..n} A[j,i] * x[i] <= b[j];
```

solve;

display x;

end;

# Running GLPSOL

```
$ glpsol --model standardForm.model --data ex1.data
```

GLPSOL: GLPK LP/MIP Solver, v4.48

OPTIMAL SOLUTION FOUND

Time used: 0.0 secs

Memory used: 0.1 Mb (130844 bytes)

Display statement at line 20

x[1].val = 6

x[2].val = 11

Model has been successfully processed

# SOLVING THE DIET PROBLEM IN GLPK

---

# Modeling the Diet Problem

- Problem Data

Name	Type	Meaning
NFoods	SCALAR (INT)	Number of Food Items
NNutrients	SCALAR (INT)	Number of Nutrients
costs	NFoods x 1	Cost per unit of each food
caloricData	Nfoods x NNutrients	Nutrients per unit of each food.
upperBnd	NNutrients x 1	Upper bound on nutrients reqd.
lowerBnd	NNutrients x 1	Lower bound on nutrients reqd.

```
param NFood;
param NNutrients;
param caloricDat {i in 1..NFood, j in 1..NNutrients};
param lb {i in 1..NNutrients};
param ub {i in 1..NNutrients};
param costs { i in 1..NFood};

var x { i in 1..NFood} >= 0;

minimize obj: sum{i in 1..NFood} costs[i] * x[i];

bnds {k in 1..NNutrients}:
    lb[k]<= sum{i in 1..NFood} x[i]*caloricDat[i,k] <=
ub[k];

solve;
display x;
```

```
param NFood := 64;  
param NNutrients := 11;
```

```
param caloricMatrix :
```

	1	2	3	4	5	6	7	8	9	10	11
--	---	---	---	---	---	---	---	---	---	----	----

```
:=
```

1	73.8	0	0.8	68.2	13.6	8.5	8	5867.4	160.2	159	2.3
2	23.7	0	0.1	19.2	5.6	1.6	0.6	15471	5.1	14.9	0.3
3	6.4	0	0.1	34.8	1.5	0.7	0.3	53.6	2.8	16	0.2
4	72.2	0	0.6	2.5	17.1	2	2.5	106.6	5.2	3.3	0.3
5	2.6	0	0	1.8	0.4	0.3	0.2	66	0.8	3.8	0.1
6	20	0	0.1	1.5	4.8	1.3	0.7	467.7	66.1	6.7	0.3
7	171.5	0	0.2	15.2	39.9	3.2	3.7	0	15.6	22.7	4.3
8	88.2	0	5.5	8.1	2.2	1.4	9.4	98.6	0.1	121.8	6.2
9	277.4	129.9	10.8	125.6	0	0	42.2	77.4	0	21.9	1.8
10	358.2	0	12.3	1237.1	58.3	11.6	8.2	3055.2	27.9	80.2	2.3

```
$ glpsol --model diet.model --data diet.dat
GLPSOL: GLPK LP/MIP Solver, v4.48
OPTIMAL SOLUTION FOUND
Time used: 0.0 secs
Memory used: 0.4 Mb (377283 bytes)
Display statement at line 19
x[1].val = 0
x[2].val = 0.235817810889784
x[3].val = 0
x[4].val = 0
x[5].val = 0
x[6].val = 0
x[7].val = 3.54494477652071
x[8].val = 0
x[9].val = 0
x[10].val = 0
. . .
```

# More Advanced Model

```
set Foods;
set Nutrients;

param calDat { i in Foods, j in Nutrients};
param bounds { i in Nutrients, j in 1..2};
param costs { i in Foods};

var x { i in Foods} >= 0;

minimize obj: sum{i in Foods} costs[i] * x[i];
s.t. bndConstr {k in Nutrients}:
    bounds[k,1] <= sum{i in Foods} x[i] * calDat[i,k] <= bounds[k,2];
solve;
display x;
printf '-----\n';
printf {i in Foods: x[i] >= 0.001}: 'Optimizer: %f units of food %s \n', x[i], i;
printf ' The bill for the food will be \$ %f \n', obj;
printf 'Bon appetit! \n ----- \n';
```

# Result

```
bash-3.2$ glpsol --model dietSet.model --data dietSet.data
GLPSOL: GLPK LP/MIP Solver, v4.48
-----
Optimizer says to eat 0.235818 units of food Carrots_Raw
Optimizer says to eat 3.544945 units of food Potatoes_Baked
Optimizer says to eat 2.167849 units of food Skim_Milk
Optimizer says to eat 3.600776 units of food Peanut_Butter
Optimizer says to eat 4.823229 units of food Popcorn_Air-
Popped
The bill for the food will be $ 0.956008
Bon appetit!
-----
Model has been successfully processed
```

# SOLVING LINEAR PROGRAMS USING EXCEL

---

# Microsoft Excel

- Spreadsheet software.
  - Product marketed by Microsoft.
  - Widely used in the industry.
- 
- Supports data analysis tools:
    - Includes a “Solver” that can be used to solve LPs.

# Our Favorite Example

$$\begin{array}{lllllll}\text{max.} & x_1 & +2x_2 & & & & \\ \text{s.t.} & -3x_1 & +x_2 & \leq & 2 & & \\ & & +x_2 & \leq & 11 & & \\ & x_1 & -x_2 & \leq & 3 & & \\ & x_1 & & & & \leq & 6 \\ & x_1, & x_2 & & & \geq & 0\end{array}$$

Solution:  $x_1 = 6, x_2 = 11$

Optimal Objective Value: 28

# Solving in Excel