KARUSH-KUHN-TUCKER (KKT) CONDITIONS FOR LP

Karush-Kuhn-Tucker Conditions

- Very important for many optimization problems.

$$\begin{array}{lrcl} \max & \mathbf{c}^\intercal\ \mathbf{x} & & \\ & A\ \mathbf{x} + \mathbf{x_s} & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x_s} & \geq & \mathbf{0} \end{array} \qquad \begin{array}{lrcl} \min & \mathbf{b}^\intercal\ \mathbf{y} & & \\ & A^\intercal\ \mathbf{y} - \mathbf{y_s} & = & \mathbf{c} \\ & \mathbf{y}, \mathbf{y_s} & \geq & \mathbf{0} \end{array}$$

Primal   Dual

Necessary and Sufficient Conditions for optimal solution

$$(\mathbf{x}, \mathbf{x_s}, \mathbf{y}, \mathbf{y_s})$$

KKT conditions for Linear Programs

The primal-dual solution $(\mathbf{x}, \mathbf{x_s}, \mathbf{y}, \mathbf{y_s})$ is optimal iff it satisfies the following conditions:

$$\begin{array}{rcl} A\,\mathbf{x} + \mathbf{x_s} & = & \mathbf{b} \\ \mathbf{x}, \mathbf{x_s} & \geq & \mathbf{0} \end{array}$$

(x,x_s) is primal feasible

$$\begin{array}{rcl} A^{\mathsf{T}}\,\mathbf{y} - \mathbf{y_s} & = & \mathbf{c} \\ \mathbf{y}, \mathbf{y_s} & \geq & \mathbf{0} \end{array}$$

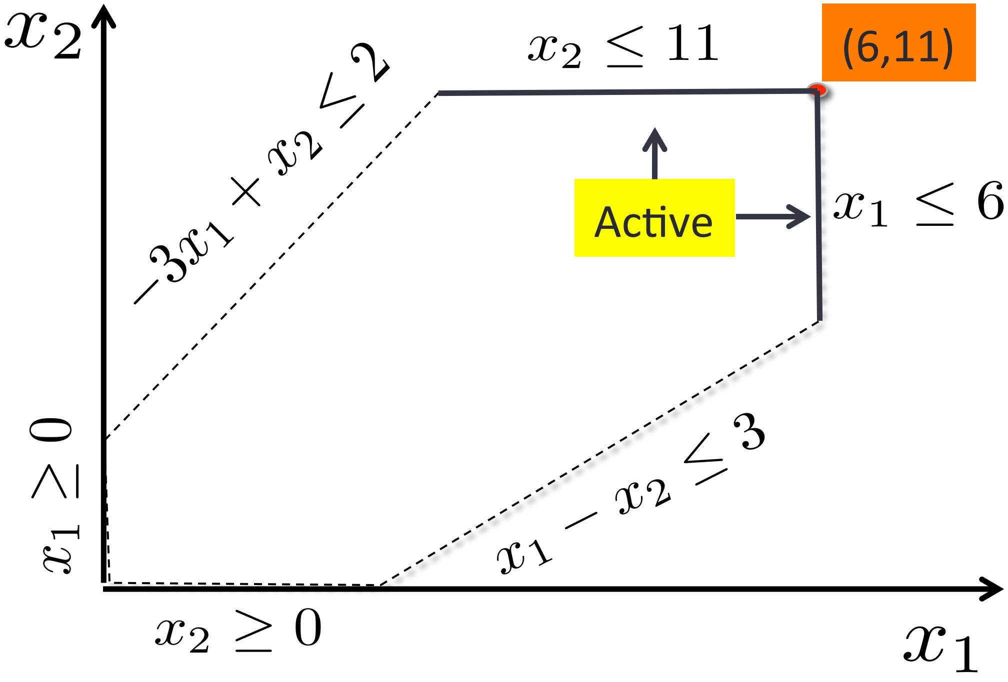
(y,y_s) is dual feasible

$$x_j y_{s,j} = 0$$
$$y_j x_{s,j} = 0$$

Product of all complementary pairs is zero.

An Example

$$
\begin{array}{lcccc}
\text{max.} & x_1 & +2x_2 & & \\
x_3 \rightarrow & -3x_1 & +x_2 & \leq & 2 \leftarrow y_1 \\
x_4 \rightarrow & & +x_2 & \leq & 11 \leftarrow y_2 \\
x_5 \rightarrow & x_1 & -x_2 & \leq & 3 \leftarrow y_3 \\
x_6 \rightarrow & x_1 & & \leq & 6 \leftarrow y_4 \\
& x_1, & x_2 & \geq & 0 \leftarrow y_5, y_6
\end{array}
$$



Primal	Dual
$x_1 : 6$	$y_5 : 0$
$x_2 : 11$	$y_6 : 0$
$x_3 : 9$	$y_1 : 0$
$x_4 : 0$	$y_2 : 2$
$x_5 : 8$	$y_3 : 0$
$x_6 : 0$	$y_4 : 1$

Final Dictionary and KKT conditions

$$\begin{array}{c|cc} \mathbf{x_B} & \mathbf{b} & +A\mathbf{x_I} \\ \hline z & z_0 & +\mathbf{c}^\intercal \mathbf{x_I} \end{array} \qquad \begin{array}{c|cc} \mathbf{x_I}^c & -\mathbf{c} & -A^\intercal \mathbf{x_B}^c \\ \hline d & -z_0 & -\mathbf{b}^\intercal \mathbf{x_B}^c \end{array}$$

$$x_I = \mathbf{0}, x_B = \mathbf{b} \qquad x_B^c = \mathbf{0}, x_I^c = -\mathbf{c}$$

Claim: The solutions represented by final primal and dual dictionaries satisfy the KKT condition!