REVISED SIMPLEX: BASIS FACTORIZATION

Reducing the complexity of revised Simplex method.

$$\pi A_B = \mathbf{c}_B^\mathsf{T}$$

$$\hat{\mathbf{c}} = \mathbf{c}_I^\mathsf{T} - \pi A_I$$

Entering Variable Analysis

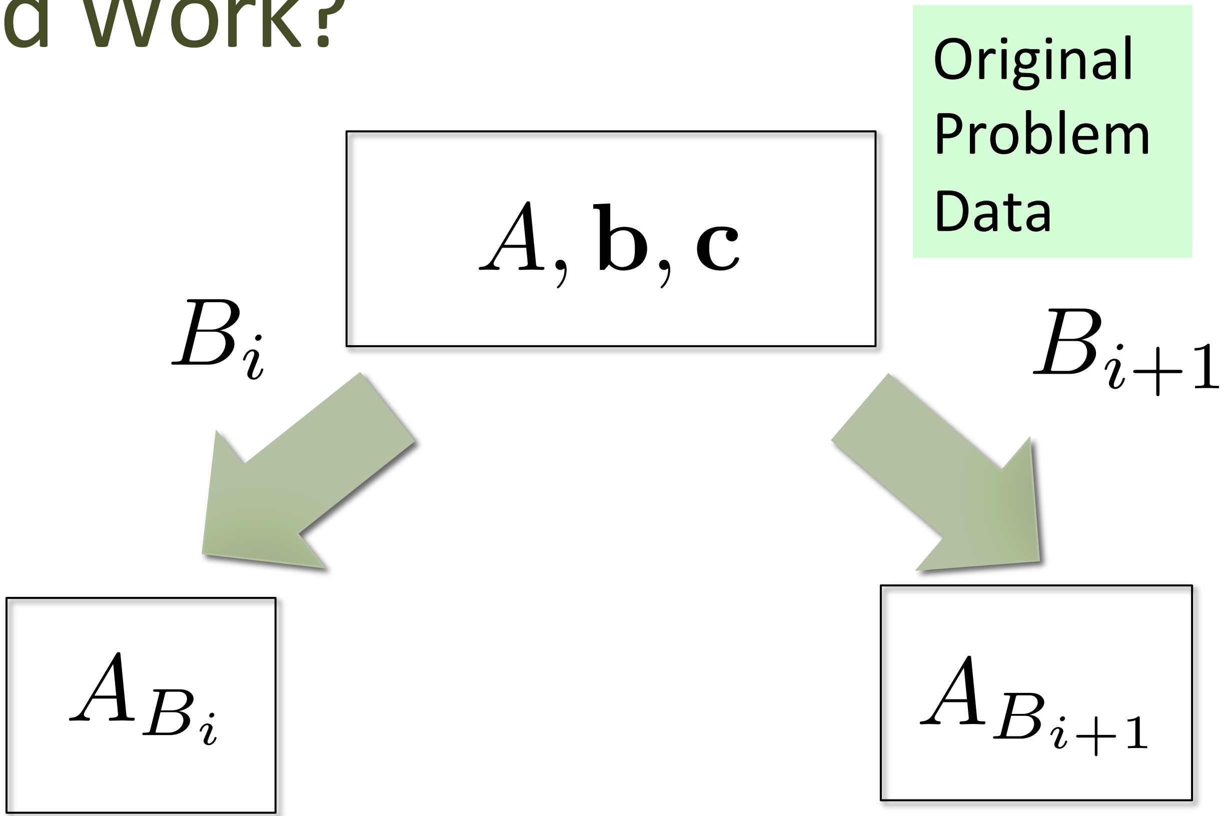
$$A_B \hat{\mathbf{b}} = \mathbf{b}$$

$$A_B \hat{\mathbf{a}}_j = -A_j$$

Leaving Variable Analysis

Update New Basis

Wasted Work?



Ideas

1. How does basis matrix A_B ...?

2. Can we reuse A ...

- Using ... (SMW) formula.
- Eta Matr...
- Updating t... (Forrest-Tomlin Method).

Practical Simplex implementations use these ideas + a lot more!

Understanding how the basis changes.

$$A = \begin{bmatrix} 2 & -3 & 7 & -15 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 10 \\ 12 \\ 4 \\ 16 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B = \{3, 6, 7, 8\}$$

x_4 enters and x_8 leaves

How is the basis updated?

$$A_B : \begin{bmatrix} \vdots & \vdots & & {\color{red}\vdots} & & \vdots \\ A_1 & A_2 & \cdots & {\color{red}A_i} & \cdots & A_m \\ \vdots & \vdots & & {\color{red}\vdots} & & \vdots \end{bmatrix}$$

OLD

$$\widetilde{A_B} : \begin{bmatrix} \vdots & \vdots & & {\color{blue}\vdots} & & \vdots \\ A_1 & A_2 & \cdots & {\color{blue}A_k} & \cdots & A_m \\ \vdots & \vdots & & {\color{blue}\vdots} & & \vdots \end{bmatrix}$$

NEW

USING SHERMAN-MORRISON-WOODBURY FORMULA

Understanding the basis update

$$\widetilde{A_B} = A_B + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & (A_k - A_i) & \cdots & \mathbf{0} \end{bmatrix}$$

$$\tilde{A_B} : \begin{bmatrix} 7 & 0 & 0 & 15 \\ -4 & 1 & 0 & 6 \\ 1 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad A_B : \begin{bmatrix} 7 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Understanding the basis update

$$\widetilde{A_B} = A_B + \boxed{\left[\begin{array}{cccccc} \mathbf{0} & \mathbf{0} & \cdots & (A_k - A_i) & \cdots & \mathbf{0} \end{array}\right]}$$

$$(A_k - A_i) \times (0\ 0 \cdots 1\ 0 \cdots 0)$$

$$\begin{bmatrix} 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{pmatrix} 15 \\ 6 \\ -2 \\ -1 \end{pmatrix} \times (0\ 0\ 0\ 1)$$

Sherman-Morrison-Woodbury (SMW) Formula

$$\left(A + \mathbf{u}\mathbf{v}^{\intercal}\right)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^{\intercal}A^{-1}}{1+\mathbf{v}^{\intercal}A^{-1}\mathbf{u}}$$

$$\widetilde{A_B} = A_B + \underbrace{(A_k - A_i)}_{\mathbf{u}} \times \underbrace{\mathbf{e_i}^{\intercal}}_{\mathbf{v}^{\intercal}}$$

SMW formula for inverse update

$$\widetilde{A_B}^{-1} = \left(I - \frac{(\hat{a}_k - \mathbf{e_i}) \times \mathbf{e_i}^{\mathsf{T}}}{\hat{a}_k(i)}\right) A_B^{-1}$$

$$\text{where, } \hat{a_k} = A_B^{-1} A_k$$

Example

$$A_B : \begin{bmatrix} 7 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{A_B} = A_B + \begin{pmatrix} 15 \\ 6 \\ -2 \\ -1 \end{pmatrix} \times (0\ 0\ 0\ 1)$$

$$\tilde{A_B} : \begin{bmatrix} 7 & 0 & 0 & 15 \\ -4 & 1 & 0 & 6 \\ 1 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\widetilde{A_B}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 6.8 \\ 0 & 0 & 1 & -1.9333 \\ 0 & 0 & 0 & -0.4666 \end{bmatrix} \times A_B^{-1}$$

Summary

$$A_{B_i}^{-1} \xrightarrow{\text{O(m}^2\text{) steps}} A_{B_{i+1}}^{-1}$$

Matrix $A_{B_{i+1}}^{-1}$ is no longer sparse.

Extra storage cost: $O(m^2)$

ETA FACTORIZATION

SMW formula for inverse update

$$\widetilde{A_B}^{-1} = \left(I - \frac{(\hat{a}_k - \mathbf{e_i}) \times \mathbf{e_i}^\mathsf{T}}{\hat{a}_k(i)}\right) A_B^{-1}$$

$$\text{where, } \hat{a_k} = A_B^{-1} A_k$$

Eta Matrix

Basis Update

$$A_{B_i}^{-1} \xrightarrow{\text{O(m}^2\text{) steps}} A_{B_{i+1}}^{-1}$$

Matrix $A_{B_{i+1}}^{-1}$ is no longer sparse.

Extra storage cost: $O(m^2)$

Eta File: Basic Idea

$$A_{B_i}^{-1} = E_i^{-1} E_{i-1}^{-1} \cdots E_1^{-1} A_1^{-1}$$

Each E_j is sparse: requires O(m) storage.