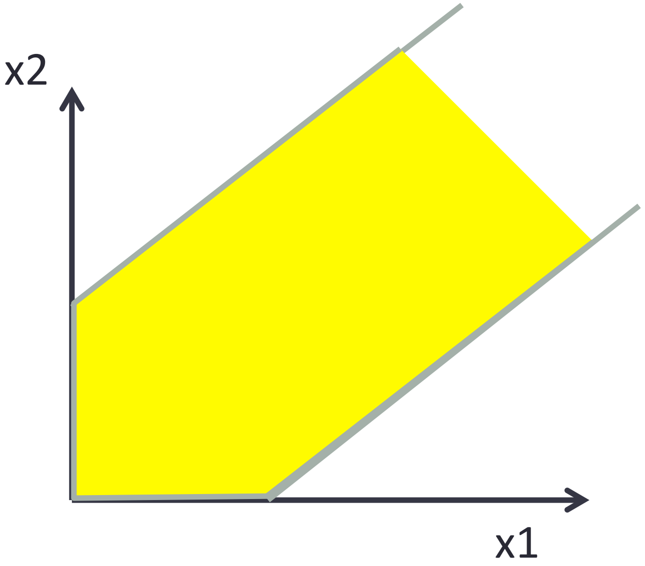
UNBOUNDED PROBLEMS

How to detect if the problem is unbounded while pivoting.

An example.

Unbounded Linear Programs

$$
\begin{array}{llll}
\max & x_1 & & \\
\text{s.t.} & x_1 & -x_2 & \leq 1 \\
 & -x_1 & +x_2 & \leq 1 \\
 & x_1 & , x_2 & \geq 0
\end{array}
$$



Example

$$
\begin{array}{rrcl}
\text{maximize} & 2x_1 + 3x_2 - 5x_3 & & \\
\text{s.t.} & x_1 - x_2 & \leq & 5 \\
& -x_1 + \quad\quad x_3 & \leq & 6 \\
& -2x_1 + \quad\quad x_3 & \leq & 2 \\
& -x_1 + x_2 & \leq & 4 \\
& x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Initial Dictionary

$$
\begin{array}{lrcl}
\text{maximize} & 2x_1 + 3x_2 - 5x_3 & & \\
\text{s.t.} & x_1 - x_2 & \leq & 5 \\
& -x_1 + x_3 & \leq & 6 \\
& -2x_1 + x_3 & \leq & 2 \\
& -x_1 + x_2 & \leq & 4 \\
& x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$
\begin{array}{rcl}
x_4 & = & 5 - x_1 + x_2 \\
x_5 & = & 6 + x_1 - x_3 \\
x_6 & = & 2 + 2x_1 - x_3 \\
x_7 & = & 4 + x_1 - x_2 \\
\hline
z & = & 0 + 2x_1 + 3x_2 - 5x_3
\end{array}
$$

Entering/Leaving Variable Analysis

$$
\begin{array}{rcl}
x_4 & = & 5 - x_1 + x_2 \\
x_5 & = & 6 + x_1 \qquad\quad - x_3 \\
x_6 & = & 2 + 2x_1 \qquad - x_3 \\
x_7 & = & 4 + x_1 - x_2 \\
\hline
z & = & 0 + 2x_1 + 3x_2 - 5x_3
\end{array}
$$

Second Dictionary

$$
\begin{array}{rcrcrcrcr}
x_2 & = & 4 & + & x_1 & - & x_7 & & \\
x_4 & = & 9 & & & - & x_7 & & \\
x_5 & = & 6 & + & x_1 & & & - & x_3 \\
x_6 & = & 2 & + & 2x_1 & & & - & x_3 \\
\hline
z & = & 12 & + & 5x_1 & - & 3x_7 & - & 5x_3
\end{array}
$$

Unbounded Dictionary

- No leaving variables.
- Alternatively: all entries in the column corr. to entering variables are non-negative.

$$
\begin{array}{cccccccc}
x_{B1} & = & b_1 & +a_{11}x_{I1} & \cdots & +{\color{red}a_{1j}}x_{Ij} & \cdots \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & \cdots & +{\color{red}a_{2j}}x_{Ij} & \cdots \\
 & \vdots & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & \cdots & +{\color{red}a_{mj}}x_{Ij} & \cdots \\
\hline
z & = & c_0 & +c_1x_{I1} & \cdots & +{\color{red}c_j}x_{Ij} & \cdots
\end{array}
$$

Unbounded Dictionary

- If we encounter an unbounded dictionary during the optimization phase,
  - Declare that the problem is unbounded and EXIT.

PIVOTING ALGORITHM

For the optimization phase.

Overview

Input: Dictionary D (Feasible)

Output: Dictionary D’ or STOP with answer.

1. Select Entering Variable.
    1. If no entering variable: dictionary is final. STOP with optimal soln.
2. For entering variable, select leaving variable.
    1. If no leaving variable: dictionary is unbounded. STOP (unbounded)
3. Perform Pivoting (Row Operations)

Dictionary Structure

$$\begin{array}{cccc} \mathbf{x_B} & = & \mathbf{b} & +A\mathbf{x_I} \\ \hline z & = & c_0 & +\mathbf{c}^\intercal\ \mathbf{x_I} \end{array}$$

$$\begin{array}{cccccc} x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1n}x_{In} \\ x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2n}x_{In} \\ & \vdots & & & & \\ x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mn}x_{In} \\ \hline z & = & c_0 & +c_1x_{I1} & +\cdots & +c_nx_{In} \end{array}$$

Entering Variable Choice

$\boxed{\text{Choose } x_{Ij} \text{ such that } c_{Ij} > 0.}$

$$
\begin{array}{rclccccc}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} \\
 & \vdots & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In}
\end{array}
$$

Leaving Variable Analysis

$$
\begin{array}{cccccccc|l}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +{\color{red}a_{1j}}x_{Ij} & +\cdots & +a_{1n}x_{In} & \rightarrow\ x_{Ij} \leq ? \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +{\color{red}a_{2j}}x_{Ij} & +\cdots & +a_{2n}x_{In} & \rightarrow\ x_{Ij} \leq ? \\
 & \vdots & & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +{\color{red}a_{mj}}x_{Ij} & +\cdots & +a_{mn}x_{In} & \rightarrow x_{Ij} \leq ? \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +{\color{red}c_j}x_{Ij} & +\cdots & +c_nx_{In} &
\end{array}
$$

$$a_{1j} < 0 \ \Rightarrow x_{Ij} \leq -\frac{b_1}{a_{1j}}$$

$$a_{1j} \geq 0 \ \Rightarrow x_{Ij} \leq \infty$$

Leaving Variable Analysis

$$
\begin{array}{ccccccc|l}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} & \rightarrow\ x_{Ij} \leq \frac{b_1}{-a_{1j}} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} & \rightarrow\ x_{Ij} \leq \infty \\
 & \vdots & & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} & \rightarrow x_{Ij} \leq \frac{b_m}{-a_{mj}} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In} &
\end{array}
$$

$$a_{ij} < 0 \ \Rightarrow x_{Ij} \leq -\frac{b_i}{a_{ij}} \qquad a_{ij} \geq 0 \ \Rightarrow x_{Ij} \leq \infty$$

Selecting the Leaving Variable

$$
\begin{array}{cccccccc|l}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} & \rightarrow\ x_{Ij} \leq \frac{b_1}{-a_{1j}} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} & \rightarrow\ x_{Ij} \leq \infty \\
 & \vdots & & & & & & & \\
x_{Bi} & = & b_i & +a_{i1}x_{I1} & +\cdots & +\mathbf{a_{ij}}x_{Ij} & +\cdots & +a_{in}x_{In} & \boxed{\rightarrow x_{Ij} \leq \frac{b_i}{-a_{ij}}} \\
 & \vdots & & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} & \rightarrow x_{Ij} \leq \frac{b_m}{-a_{mj}} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In} &
\end{array}
$$

Minimum bound

Pivoting

$$
\begin{array}{cccccccc|l}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} & \rightarrow\ x_{Ij} \leq \frac{b_1}{-a_{1j}} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} & \rightarrow\ x_{Ij} \leq \infty \\
 & \vdots & & & & & & & \\
x_{Bi} & = & b_i & +a_{i1}x_{I1} & +\cdots & +\mathbf{a_{ij}}x_{Ij} & +\cdots & +a_{in}x_{In} & \boxed{\rightarrow\ x_{Ij} \leq \frac{b_i}{-a_{ij}}} \\
 & \vdots & & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} & \rightarrow\ x_{Ij} \leq \frac{b_m}{-a_{mj}} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In} &
\end{array}
$$

$$
\boxed{x_{Ij} = -\frac{b_i}{a_{ij}} + \frac{a_{i1}}{-a_{ij}}x_{I1} + \cdots + \frac{a_{in}}{-a_{ij}}x_{In} + \frac{-1}{-a_{ij}}x_{Bi}}
$$

Pivoting Row Operations

$$x_{Ij} = -\frac{b_i}{a_{ij}} + \frac{a_{i1}}{-a_{ij}} x_{I1} + \cdots + \frac{a_{in}}{-a_{ij}} x_{In} + \frac{-1}{-a_{ij}} x_{Bi}$$

$$x_{B1} \quad = \quad b_1 \quad + a_{11} x_{I1} \quad + \cdots \quad + a_{1j} x_{Ij} \quad + \cdots \quad + a_{1n} x_{In}$$

Overview

Input: Dictionary D (Feasible)

Output: Dictionary D’ or STOP with answer.

1. Select Entering Variable.
    1. If no entering variable: dictionary is final. STOP with optimal soln.

2. For entering variable, select leaving variable.
    1. If no leaving variable: dictionary is unbounded. STOP (unbounded)

3. Perform Pivoting (Row Operations)

PIVOTING PRESERVES FEASIBILITY: PROOF SKETCH

To Prove

- Suppose D is a feasible but non-final dictionary and we perform a valid pivoting step in the Simplex algorithm to get to dictionary D’ then D’ is also feasible.

Before proof, a simple exercise.

$$
\begin{array}{rcrrr}
x_2 & = & 2 & +3x_1 & -x_3 \\
x_4 & = & 9 & -3x_1 & +x_3 \\
x_5 & = & 5 & +2x_1 & -x_3 \\
x_6 & = & 6 & -x_1 & +0x_3 \\
\hline
z & = & 4 & +7x_1 & -2x_3
\end{array}
$$

x_1 enters and x_4 leaves.

What is the solution associated with the next dictionary?

Dictionary

$$
\begin{array}{ccccccc|l}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} & \rightarrow\ x_{Ij} \leq \frac{b_1}{-a_{1j}} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} & \rightarrow\ x_{Ij} \leq \infty \\
 & \vdots & & & & & & & \\
x_{Bi} & = & b_i & +a_{i1}x_{I1} & +\cdots & +\mathbf{a_{ij}}x_{Ij} & +\cdots & +a_{in}x_{In} & \boxed{\rightarrow x_{Ij} \leq \frac{b_i}{-a_{ij}}} \\
 & \vdots & & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} & \rightarrow x_{Ij} \leq \frac{b_m}{-a_{mj}} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In} &
\end{array}
$$

Dictionary

$$
\begin{array}{ccccccccc|l}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} & \rightarrow & x_{Ij} \leq \frac{b_1}{-a_{1j}} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} & \rightarrow & x_{Ij} \leq \infty \\
 & \vdots & & & & & & & & \\
x_{Bi} & = & b_i & +a_{i1}x_{I1} & +\cdots & +\mathbf{a_{ij}}x_{Ij} & +\cdots & +a_{in}x_{In} & \rightarrow & \boxed{x_{Ij} \leq \frac{b_i}{-a_{ij}}} \\
 & \vdots & & & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} & \rightarrow & x_{Ij} \leq \frac{b_m}{-a_{mj}} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In} & &
\end{array}
$$

Another Fact About Simplex

- During the optimization phase of Simplex,
  - The value of the objective cannot decrease due to a pivoting step.

What is the value of the Objective after pivot?

$$
\begin{array}{ccccccccc|l}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} & & \rightarrow\ x_{Ij} \leq \frac{b_1}{-a_{1j}} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} & & \rightarrow\ x_{Ij} \leq \infty \\
 & \vdots & & & & & & & & \\
x_{Bi} & = & b_i & +a_{i1}x_{I1} & +\cdots & +\mathbf{a_{ij}}x_{Ij} & +\cdots & +a_{in}x_{In} & & \boxed{\rightarrow x_{Ij} \leq \frac{b_i}{-a_{ij}}} \\
 & \vdots & & & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} & & \rightarrow x_{Ij} \leq \frac{b_m}{-a_{mj}} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In} & &
\end{array}
$$

Degenerate Dictionary

$$
\begin{array}{rcrrrr}
x_3 & = & .5 & & & -.5x_4 \\
x_5 & = & 0 & -2x_1 & +4x_2 & +3x_4 \\
x_6 & = & 0 & +x_1 & -3x_2 & +2x_4 \\
\hline
z & = & 4 & +2x_1 & -x_2 & -4x_4
\end{array}
$$

Degeneracy Definition

$$
\begin{array}{ccccccc|l}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} & \rightarrow\ x_{Ij} \leq \frac{b_1}{-a_{1j}} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} & \rightarrow\ x_{Ij} \leq \infty \\
 & \vdots & & & & & & & \\
x_{Bi} & = & b_i & +a_{i1}x_{I1} & +\cdots & +\mathbf{a_{ij}}x_{Ij} & +\cdots & +a_{in}x_{In} & \boxed{\rightarrow x_{Ij} \leq \frac{b_i}{-a_{ij}}} \\
 & \vdots & & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} & \rightarrow x_{Ij} \leq \frac{b_m}{-a_{mj}} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In} &
\end{array}
$$

Interesting Fact

If value of objective remains same in next Dictionary after pivoting then the current dictionary is degenerate.

$$
\begin{array}{ccccccc|l}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} & \to\ x_{Ij} \leq \frac{b_1}{-a_{1j}} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} & \to\ x_{Ij} \leq \infty \\
 & \vdots & & & & & & & \\
x_{Bi} & = & b_i & +a_{i1}x_{I1} & +\cdots & +\mathbf{a_{ij}}x_{Ij} & +\cdots & +a_{in}x_{In} & \boxed{\to x_{Ij} \leq \frac{b_i}{-a_{ij}}} \\
 & \vdots & & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} & \to x_{Ij} \leq \frac{b_m}{-a_{mj}} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In} &
\end{array}
$$

Another Fact About Simplex

- During the optimization phase of Simplex,
  - The value of the objective cannot decrease due to a pivoting step.

What is the value of the Objective after pivot?

$$
\begin{array}{cccccccc|l}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} & \rightarrow\ x_{Ij} \leq \frac{b_1}{-a_{1j}} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} & \rightarrow\ x_{Ij} \leq \infty \\
 & \vdots & & & & & & & \\
x_{Bi} & = & b_i & +a_{i1}x_{I1} & +\cdots & +\mathbf{a_{ij}}x_{Ij} & +\cdots & +a_{in}x_{In} & \boxed{\rightarrow x_{Ij} \leq \frac{b_i}{-a_{ij}}} \\
 & \vdots & & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} & \rightarrow x_{Ij} \leq \frac{b_m}{-a_{mj}} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In} &
\end{array}
$$

Degenerate Dictionary

$$
\begin{array}{rcrrrr}
x_3 & = & .5 & & & -.5x_4 \\
x_5 & = & 0 & -2x_1 & +4x_2 & +3x_4 \\
x_6 & = & 0 & +x_1 & -3x_2 & +2x_4 \\
\hline
z & = & 4 & +2x_1 & -x_2 & -4x_4
\end{array}
$$

Degeneracy Definition

$$
\begin{array}{ccccccc|l}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} & \rightarrow\ x_{Ij} \leq \frac{b_1}{-a_{1j}} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} & \rightarrow\ x_{Ij} \leq \infty \\
 & \vdots & & & & & & & \\
x_{Bi} & = & b_i & +a_{i1}x_{I1} & +\cdots & +\mathbf{a_{ij}}x_{Ij} & +\cdots & +a_{in}x_{In} & \boxed{\rightarrow x_{Ij} \leq \frac{b_i}{-a_{ij}}} \\
 & \vdots & & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} & \rightarrow x_{Ij} \leq \frac{b_m}{-a_{mj}} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In} &
\end{array}
$$

Interesting Fact

If value of objective remains same in next Dictionary after pivoting then the current dictionary is degenerate.

$$
\begin{array}{rcllllll|l}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} & \to\ x_{Ij} \leq \frac{b_1}{-a_{1j}} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} & \to\ x_{Ij} \leq \infty \\
 & \vdots & & & & & & & \\
x_{Bi} & = & b_i & +a_{i1}x_{I1} & +\cdots & +\mathbf{a_{ij}}x_{Ij} & +\cdots & +a_{in}x_{In} & \boxed{\to x_{Ij} \leq \frac{b_i}{-a_{ij}}} \\
 & \vdots & & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} & \to x_{Ij} \leq \frac{b_m}{-a_{mj}} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In} &
\end{array}
$$

ENTERING/LEAVING VARIABLE CHOICE HEURISTICS

Entering Variable Choice

RECAP

Choose x_{Ij} such that $c_{Ij} > 0$.

$$
\begin{array}{rclccccc}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} \\
 & \vdots & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In}
\end{array}
$$

Selecting the Leaving Variable

RECAP

$$
\begin{array}{cccccccc|l}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +{\color{red}a_{1j}}x_{Ij} & +\cdots & +a_{1n}x_{In} & \rightarrow\ x_{Ij} \le \frac{b_1}{-a_{1j}} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +{\color{red}a_{2j}}x_{Ij} & +\cdots & +a_{2n}x_{In} & \rightarrow\ x_{Ij} \le \infty \\
 & \vdots & & & & & & & \\
{\color{blue}x_{Bi}} & = & {\color{blue}b_i} & {\color{blue}+a_{i1}x_{I1}} & {\color{blue}+\cdots} & {\color{blue}+\mathbf{a_{ij}}}x_{Ij} & {\color{blue}+\cdots} & {\color{blue}+a_{in}}x_{In} & \boxed{\rightarrow x_{Ij} \le {\color{blue}\frac{b_i}{-a_{ij}}}} \\
 & \vdots & & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +{\color{red}a_{mj}}x_{Ij} & +\cdots & +a_{mn}x_{In} & \rightarrow x_{Ij} \le \frac{b_m}{-a_{mj}} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +{\color{red}c_j}x_{Ij} & +\cdots & +c_nx_{In} &
\end{array}
$$

Minimum bound

Entering Variable Choice

- Often, we can have more than one choice for entering variable.
  - For each entering variable, more than one choice of leaving.

- Question: If multiple choices, then how to choose?

Heuristic #1: Largest Objective Coefficient

$$\begin{array}{ccccccccc}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +{\color{red}a_{1j}}x_{Ij} & +\cdots & +a_{1n}x_{In} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +{\color{red}a_{2j}}x_{Ij} & +\cdots & +a_{2n}x_{In} \\
 & \vdots & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +{\color{red}a_{mj}}x_{Ij} & +\cdots & +a_{mn}x_{In} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +{\color{red}c_j}x_{Ij} & +\cdots & +c_nx_{In}
\end{array}$$

Idea: Larger the value of c_j, the more increase in z per unit increase in x_{Ij}.

Heuristic #2: Greedy Choice

- Explore all possible choices.
- Choose the entering/leaving combination that yield the most increase in z.

Bland's Rule

- Associate unique index with each problem/slack variable.

$$x_j \text{ has index } j.$$

- Amongst all possible entering variables, choose least possible index.

- Amongst all possible leaving choices, choose least index choice.

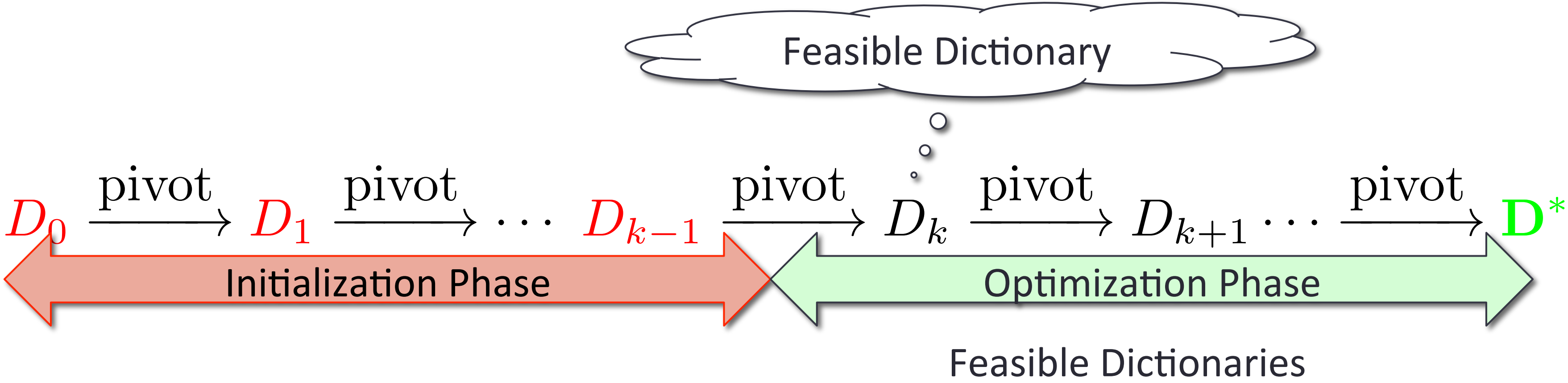
Example: Bland's Rule

$$
\begin{array}{rcrrr}
x_3 & = & 1 & +3x_1 & -25x_2 \\
x_4 & = & 11 & +0x_1 & -x_2 \\
x_5 & = & 3 & -x_1 & +x_2 \\
x_6 & = & 3 & -x_1 & +0x_2 \\
\hline
z & = & 0 & +x_1 & +25x_2
\end{array}
$$

CYCLING IN SIMPLEX

Does simplex always terminate?

Simplex Overview



Termination of Simplex

Case -1

Infinitely many dictionaries?

$$D_1 \to D_2 \to D_3 \to D_4 \to \cdots \to D_j \to \cdots$$

Case -2

$$D_1 \to D_2 \to D_3 \to D_4 \to \cdots \to D_j$$

Cycling in Simplex

$$\begin{tikzcd} D_1 \arrow[r] & D_2 \arrow[r] & D_3 \arrow[r] & D_4 \arrow[r] & \cdots \arrow[r] & D_j \arrow[llll, bend right] \end{tikzcd}$$

Only possible case for non-termination of Simplex.
Is this possible?

Cycling and Degeneracy

$$D_1 \to D_2 \to D_3 \to D_4 \to \cdots D_i \to \cdots \to D_j$$

All repeating dictionaries $D_i, \ldots, D_j$ are *degenerate*

Cycling in Simplex

- Depends on heuristic for choosing entering/leaving variables.
  - Lots of examples
  - See on-line or consult Chvatal/Vanderbei book.

Anti-Cycling Rule

- Can the choice of entering or leaving variable avoid cycling and guarantee termination of Simplex?

- Good news ☺
  - Bland’s Rule is anti-cycling.

- Recall Bland’s Rule:
  - If multiple choices for entering, choose least index.
  - If multiple choices for leaving, once again choose least index.

Practical Considerations

- Stalling

Analysis of Bland’s Rule vs. Other Heuristics

- Classic Paper by Avis + Chvatal (1974)
- On small randomly generated problems:
  - Bland’s rule performs worse than other heuristics such as largest objective coefficient and greedy.
  - Stalling seems to be made worse by Bland’s rule in many situations?
- Suggestion:
  1. Cycling never seems to happen in practice.
  2. Use Bland’s rule selectively.

Example of Mixed Heuristic

- Use largest objective coefficient rule for selecting entering variable.

- If last K dictionaries are all degenerate
  - apply Bland’s rule continuously until final or non-degenerate dictionary.

COMPLEXITY OF SIMPLEX

Klee-Minty Cubes.

Simplex Complexity

- Focus on maximum number of pivots required.
- Fortunately, Bland’s rule guarantees eventual termination.
- Upper bound is given by

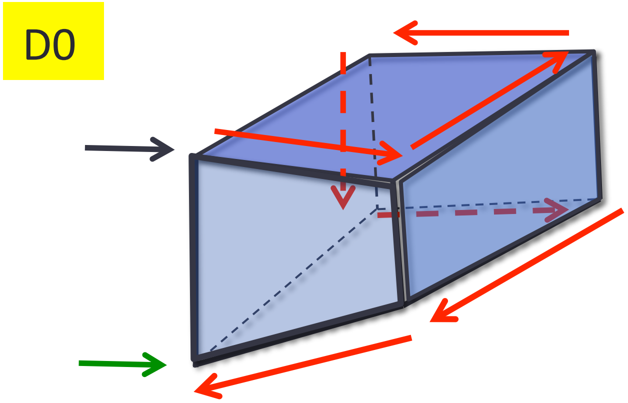
$$\binom{n+m}{m} = \frac{(n+m)!}{m!n!}$$

Klee-Minty Example

$$
\begin{array}{lrcl}
\text{max.} & 10^{n-1}x_1 + 10^{n-2}x_2 + \ldots + 10x_{n-1} + 1x_n & & \\
\text{s.t.} & x_1 & \leq & 1 \\
 & 20x_1 + x_2 & \leq & 100 \\
 & 200x_1 + 20x_2 + x_3 & \leq & 100^2 \\
 & \ldots & & \\
 & 2(\sum_{j=1}^{i-1} 10^{i-j} x_j) + x_i & \leq & 100^{i-1} \\
 & \ldots & & \\
 & x_1, \ldots, x_n & \geq & 0
\end{array}
$$

Klee-Minty Cube

- Feasible region is a “distorted” cube.



Klee-Minty Example

- If we follow the largest coefficient rule, then simplex requires $2^{n}-1$ iterations to converge.

- Worst-Case Complexity of Simplex (with largest objective coefficient heuristic) is exponential.

- Other pivoting rules have similar worst-case results.
  - Avis and Chvatal's exponential lower bound for Bland's rule on Klee-Minty Cubes.

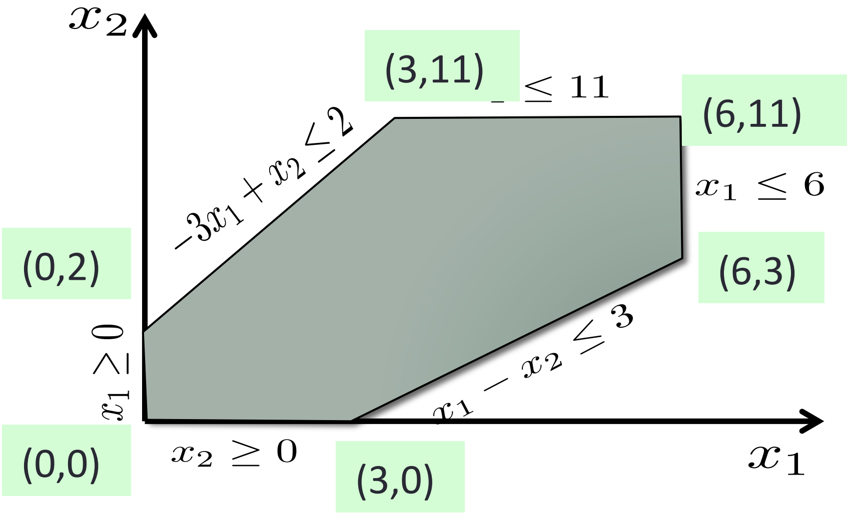
POLYHEDRA: VERTICES

Linear Programming Problem

From Two Weeks Ago.

$$\begin{array}{lrrcr}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}$$

Note: Not drawn to scale

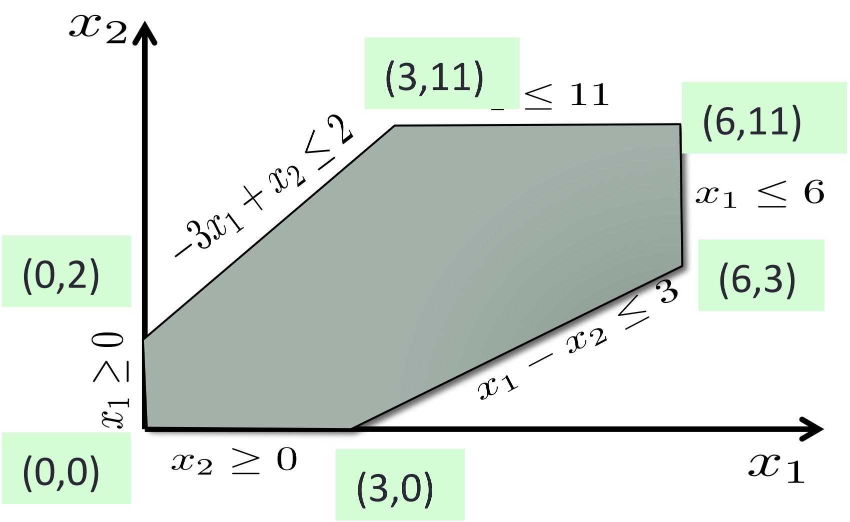


Goal: Solve LP using Simplex and visualize!

Active Constraints

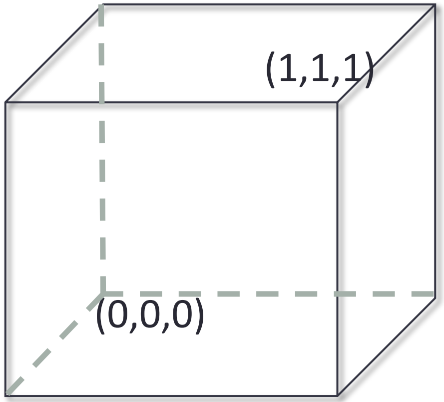
$$\begin{array}{llllll}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}$$

Note: Not drawn to scale



Active Constraints

$$
\begin{array}{rrrcl}
x_1 & & & \leq & 1 \\
 & x_2 & & \leq & 1 \\
 & & x_3 & \leq & 1 \\
x_1 & & & \geq & 0 \\
 & x_2 & & \geq & 0 \\
 & & x_3 & \geq & 0
\end{array}
$$



Basic Geometric Facts

- Intersection of 2 lines in 2D yields a point.
  - Lines must be non-parallel.
- Intersection of 3 planes in 3D yields a point.
  - Exclude parallel planes, or other corner cases.
- Intersection of 4 hyper-planes in 4D yields a point.
  - Again, some corner cases.

Intersection of n hyper-planes

$$\begin{array}{cccccccc} a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & = & b_1 & \leftarrow & \color{red}{\mathcal{H}_1} \\ & & \ddots & & \vdots & & & \\ a_{n1}x_1 & +a_{n2}x_2 & +\cdots+ & a_{nn}x_n & = & b_n & \leftarrow & \color{red}{\mathcal{H}_n} \end{array}$$

$$\textbf{rank}\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = n$$

Vertex (Definition)

A feasible solution **x** to the constraints is a vertex iff

$$\boxed{\begin{array}{cccccc} a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 \\ & & \ddots & & \vdots & \\ a_{j1}x_1 & +a_{j2}x_2 & +\cdots+ & a_{jn}x_n & \leq & b_j \\ & & \ddots & & \vdots & \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m \end{array}}$$

at least n ineqs.
are active for x.

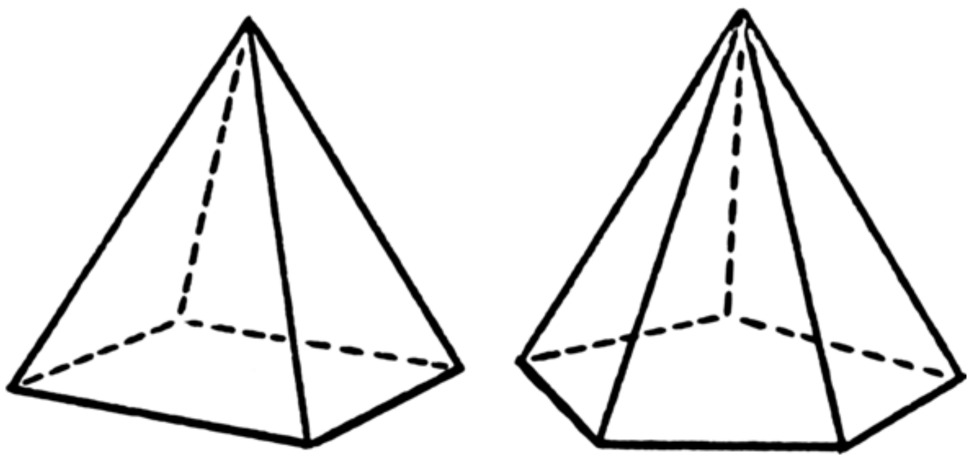
rank of the active constraints for **x** is n

Vertex Issue #1

- Does every point **x** that activates n constraints form a vertex?

Vertex Issue #2

- Can a vertex activate more than n constraints?



Vertex Issue #3

- What if there are more variables than constraints?

Number of Vertices

- n-dimensional hyper cube has 2^n vertices.

- In general, combinatorial explosion of vertices.
  - m constraints, n variables: $\binom{m}{n}$ upper bound on vertices

UNBOUNDED POLYHEDRA: RAYS

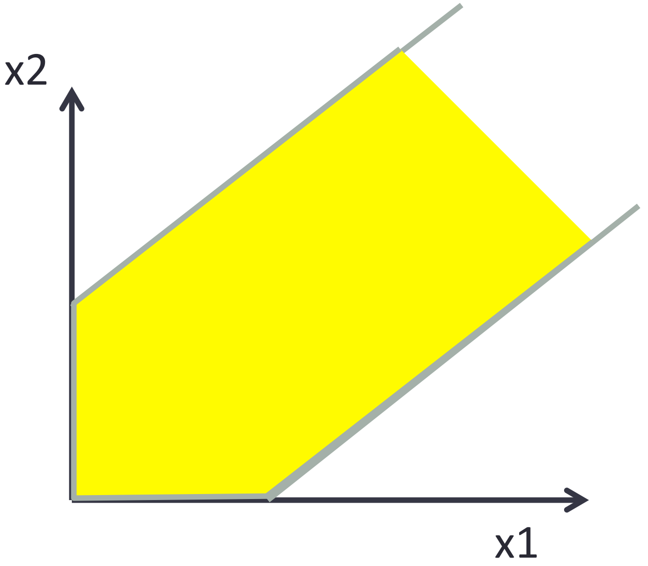
Thus far...

- Feasible Region: Polyhedra
- Vertices:
  - Activate at l
  - Acti
- Vertice
- Simplex ertex.
  - Degene generate

Unbounded Problems.

Unbounded Linear Programs and Rays

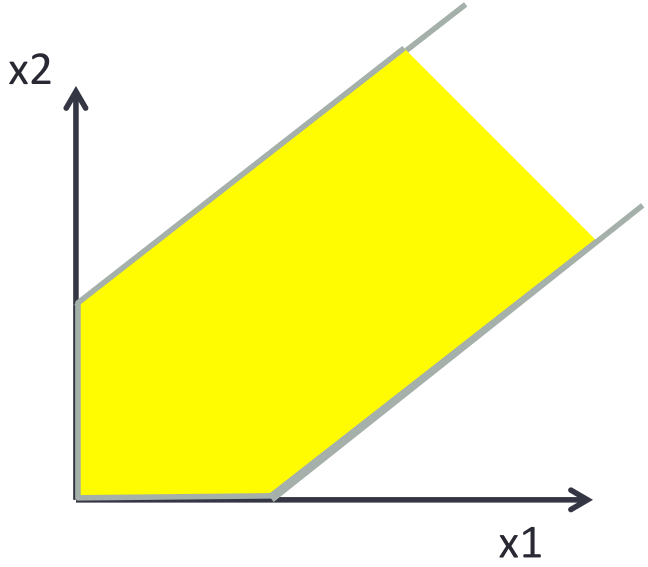
$$\begin{array}{llll} \max & x_1 & & \\ \text{s.t.} & x_1 & -x_2 & \leq 1 \\ & -x_1 & +x_2 & \leq 1 \\ & x_1 & , x_2 & \geq 0 \end{array}$$



Ray

Vector $\mathbf{r}$ is a ray of polyhedron P iff for every $\mathbf{x} \in P$ and every $\lambda \geq 0$,

$$\mathbf{x} + \lambda \mathbf{r} \in P$$



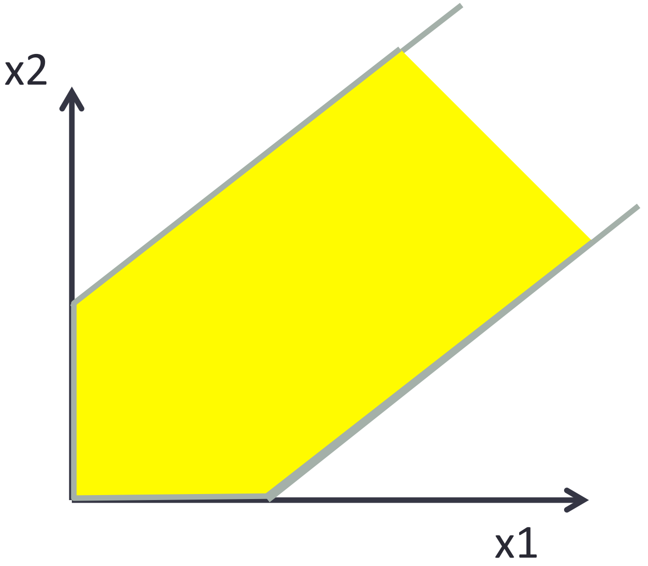
Ray (Fundamental Property)

Polyhedron: $A\mathbf{x} \leq \mathbf{b}$

$\mathbf{r}$ is a ray if and only if $A\mathbf{r} \leq \mathbf{0}$

Ray (Fundamental Property)

$$\begin{array}{llcl} \max & x_1 & & \\ \text{s.t.} & x_1 & -x_2 & \leq & 1 \\ & -x_1 & +x_2 & \leq & 1 \\ & x_1 & , x_2 & \geq 0 & \end{array}$$



Is (1,1) a ray of this polyhedron?

Example

$$
\begin{array}{rlcl}
\text{maximize} & 2x_1 + 3x_2 - 5x_3 & & \\
\text{s.t.} & x_1 - x_2 & \leq & 5 \\
& -x_1 + \quad x_3 & \leq & 6 \\
& -2x_1 + \quad x_3 & \leq & 2 \\
& -x_1 + x_2 & \leq & 4 \\
& x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Second Dictionary

$$
\begin{array}{rcrcrcrcr}
x_2 & = & 4 & + & x_1 & - & x_7 & & \\
x_4 & = & 9 & & & - & x_7 & & \\
x_5 & = & 6 & + & x_1 & & & - & x_3 \\
x_6 & = & 2 & + & 2x_1 & & & - & x_3 \\
\hline
z & = & 12 & + & 5x_1 & - & 3x_7 & - & 5x_3
\end{array}
$$

Unbounded Dictionary

- No leaving variables.
- Alternatively: all entries in the column corr. to entering variables are non-negative.

$$
\begin{array}{cccccccc}
x_{B1} & = & b_1 & +a_{11}x_{I1} & \cdots & +\color{red}{a_{1j}}x_{Ij} & \cdots \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & \cdots & +\color{red}{a_{2j}}x_{Ij} & \cdots \\
 & \vdots & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & \cdots & +\color{red}{a_{mj}}x_{Ij} & \cdots \\
\hline
z & = & c_0 & +c_1x_{I1} & \cdots & +\color{red}{c_j}x_{Ij} & \cdots
\end{array}
$$

Unbounded Dictionary and Ray

$$
\begin{array}{cccccccc}
x_{B1} & = & b_1 & +a_{11}x_{I1} & \cdots & +a_{1j}x_{Ij} & \cdots \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & \cdots & +a_{2j}x_{Ij} & \cdots \\
 & \vdots & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & \cdots & +a_{mj}x_{Ij} & \cdots \\
\hline
z & = & c_0 & +c_1x_{I1} & \cdots & +c_jx_{Ij} & \cdots
\end{array}
$$

DEGENERATE POLYHEDRA

Degenerate Dictionaries

$$\begin{array}{rcrrr} x_1 & = & 3 & -\frac{1}{3}x_4 & +\frac{1}{3}x_3 \\ x_2 & = & 11 & -x_4 & +0x_3 \\ x_5 & = & 11 & -\frac{2}{3}x_4 & -\frac{1}{3}x_3 \\ x_6 & = & 0 & +\frac{1}{3}x_4 & -\frac{1}{3}x_3 \\ \hline z & = & 25 & -\frac{7}{3}x_4 & +\frac{1}{3}x_3 \end{array}$$

1. Understand geometry of degeneracy
2. Highly degenerate polyhedra.

Vertex (Definition)

A feasible solution **x** to the constraints is a vertex iff

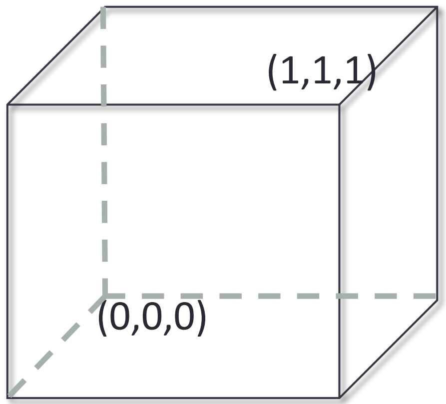
$$
\begin{array}{cccccc}
a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 \\
 & & \ddots & & \vdots & \\
a_{j1}x_1 & +a_{j2}x_2 & +\cdots+ & a_{jn}x_n & \leq & b_j \\
 & & \ddots & & \vdots & \\
a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m
\end{array}
$$

at least n ineqs.
are active for x.

rank of the active constraints for **x** is n

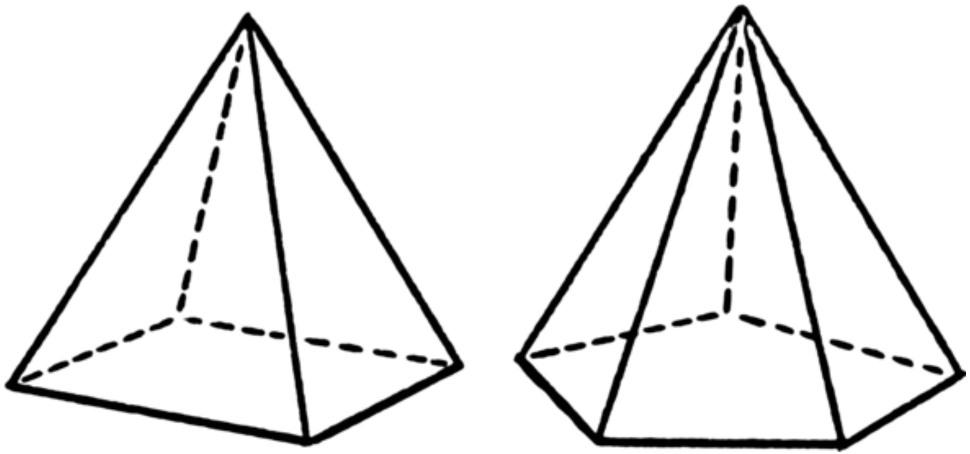
Vertices and Active Constraints

$$
\begin{array}{rcl}
x_1 & \leq & 1 \\
x_2 & \leq & 1 \\
x_3 & \leq & 1 \\
x_1 & \geq & 0 \\
x_2 & \geq & 0 \\
x_3 & \geq & 0
\end{array}
$$

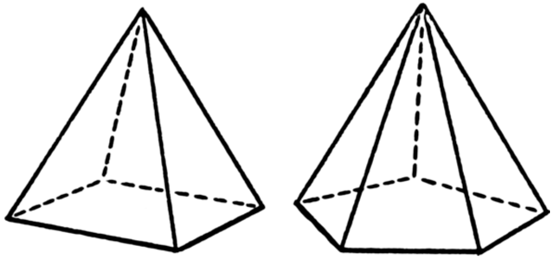


Vertex Issue #2

- Can a vertex activate more than n constraints?



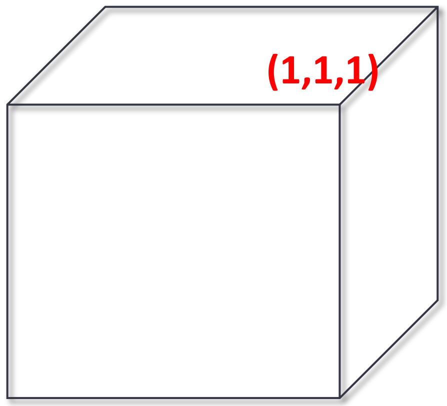
Degenerate Vertex (Definition)

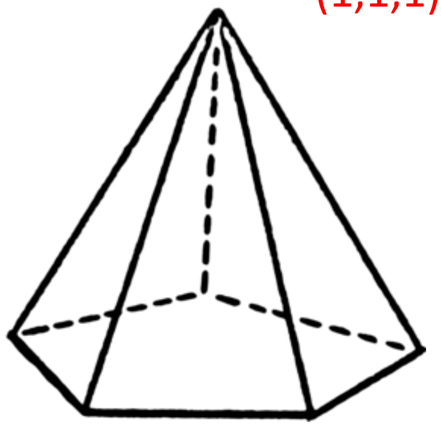


Vertex **x** is degenerate iff

it activates n+k constraints for k > 0

Degenerate vs. Non-degenerate vertex





Non-degenerate:

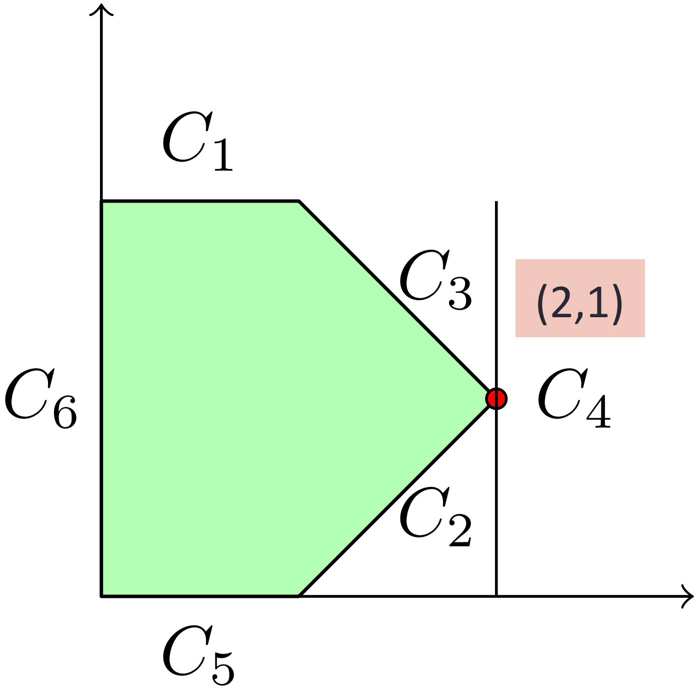
- Activates exactly n constraints.
- Exactly n faces meet at the vertex.
- Unique dictionary associated with vertex.

Degenerate:

- Activates n + k constraints (k > 0)
- More than n faces meet at the vertex.
- Multiple dictionaries associated with vertex.

Degeneracy due redundancy

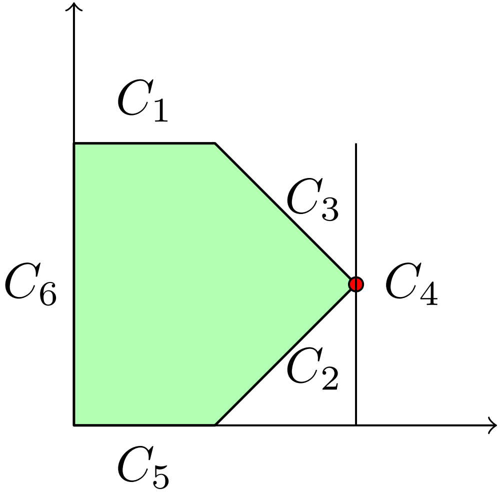
$$
\begin{array}{lrrcl}
\max & x & & & \\
\text{s.t.} & & & & \\
C_1: & & y & \leq & 2 \\
C_2: & x & -y & \leq & 1 \\
C_3: & x & +y & \leq & 3 \\
C_4: & x & & \leq & 2 \\
C_5: & -x & & \leq & 0 \\
C_6: & & -y & \leq & 0
\end{array}
$$



Degeneracy due to redundancy

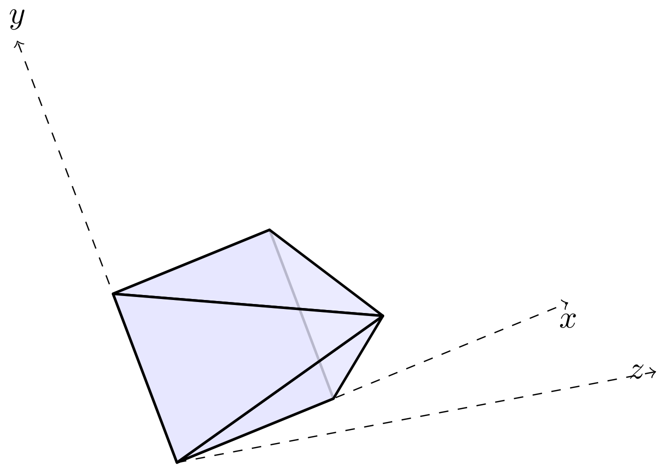
$$\begin{array}{c|cccc}
w_1 & 1 & -\frac{1}{2}w_2 & +\frac{1}{2}w_3 \\
x & 2 & -\frac{1}{2}w_2 & -\frac{1}{2}w_3 \\
y & 1 & +\frac{1}{2}w_2 & -\frac{1}{2}w_3 \\
w_4 & 0 & +\frac{1}{2}w_2 & +\frac{1}{2}w_3 \\
\hline
z & 2 & -\frac{1}{2}w_2 & -\frac{1}{2}w_3
\end{array}$$

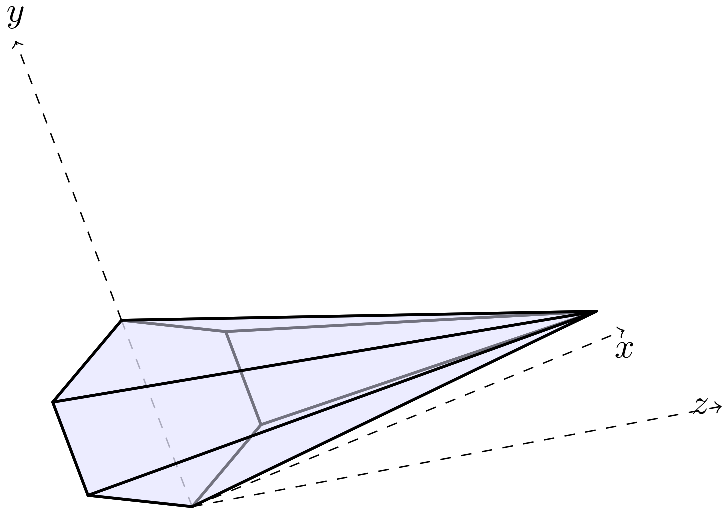
$$\begin{array}{c|cccc}
w_1 & 1 & -w_2 & +w_4 \\
x & 2 & & -w_4 \\
y & 1 & +w_2 & -w_4 \\
w_3 & 0 & -w_2 & +2w_4 \\
\hline
z & 2 & & -w_4
\end{array}$$



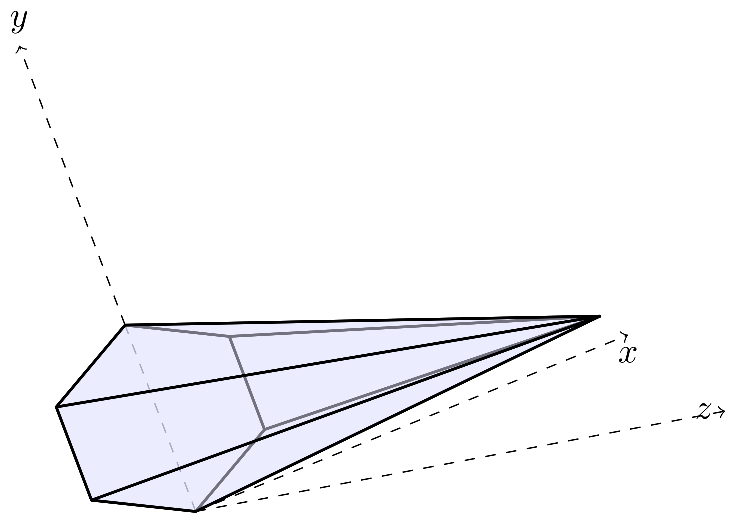
Degeneracy without redundancy

Removing any of the constraints changes feasible region.





Simplex over Degenerate Polyhedra



PIVOTING AND VERTICES

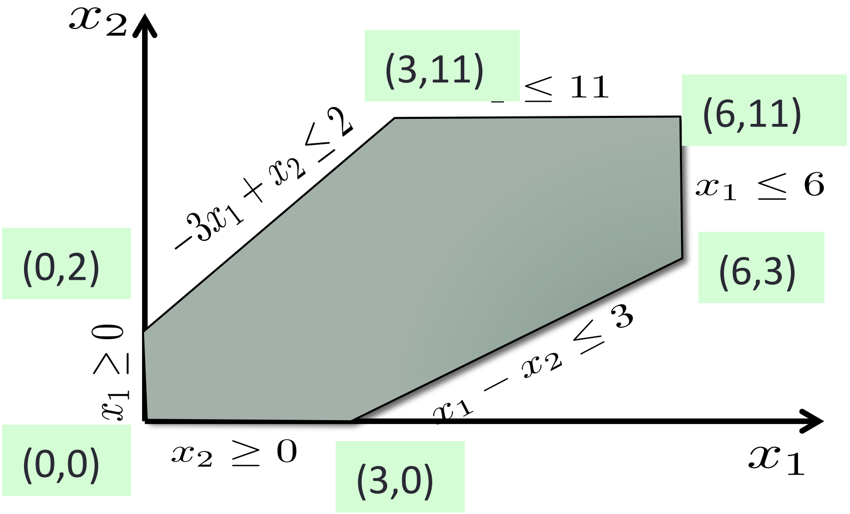
What happens when we pivot?

- Entering variable leaves non-basic set.
  - Leaving variable becomes non-basic.

Adjacent Vertices

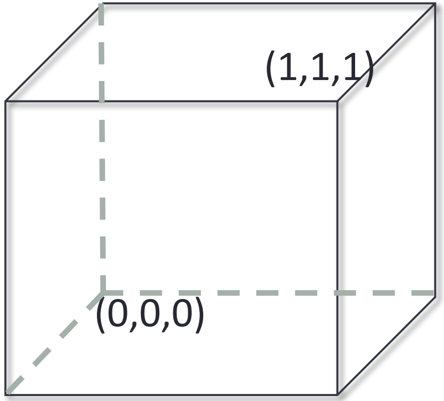
$$
\begin{array}{llllll}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$

Note: Not drawn to scale



Example #2: Adjacent Vertices

$$
\begin{array}{rrrcl}
x_1 & & & \leq & 1 \\
& x_2 & & \leq & 1 \\
& & x_3 & \leq & 1 \\
x_1 & & & \geq & 0 \\
& x_2 & & \geq & 0 \\
& & x_3 & \geq & 0
\end{array}
$$



Adjacent Vertices

Definition: Two vertices are adjacent if and only if

- At least (n-1) active constraints are common.
- Rank of common active constraints is (n-1).

$$
\begin{array}{cccccc}
a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 \\
 & & \ddots & & \vdots & \\
a_{j1}x_1 & +a_{j2}x_2 & +\cdots+ & a_{jn}x_n & \leq & b_j \\
 & & \ddots & & \vdots & \\
a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m
\end{array}
$$

Active for
both vertices

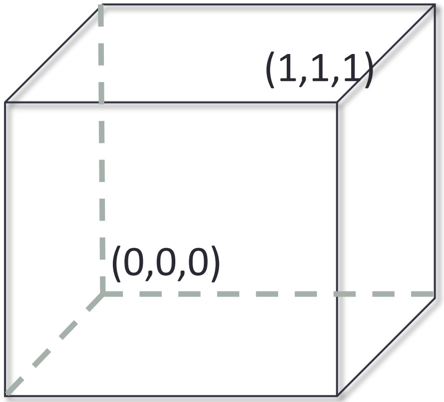
Claim

For non-degenerate/non-final dictionary D_1 if D_2 is obtained on pivot, then the vertices corr. to D_1 and D_2 are adjacent.

$$\begin{array}{c|cc} x_{B1} & \mathbf{b}_1 & \cdots \\ \hline z & c_0 & +c_{N1}x_{N1} \end{array} \qquad \begin{array}{c|cc} x_{B2} & \mathbf{b}_2 & \cdots \\ \hline z & c_2 & +c_{N2}x_{N1} \end{array}$$

Simplex Pivoting Visualization

$$
\begin{array}{lrcl}
\max & x_1 + x_2 - x_3 & & \\
& x_1 & \leq & 1 \\
& x_2 & \leq & 1 \\
& x_3 & \leq & 1 \\
& x_1 & \geq & 0 \\
& x_2 & \geq & 0 \\
& x_3 & \geq & 0
\end{array}
$$



Pivoting Issues

- Does pivoting always move to an adjacent vertex?
  - Yes, if the current dictionary is non-degenerate.

- What happens in the degenerate case?
  - Case-1: Move to an adjacent vertex.
  - Case-2: Remain in the same vertex (?)

- What happens if a dictionary is unbounded?

DICTIONARIES AND VERTICES

Main Message

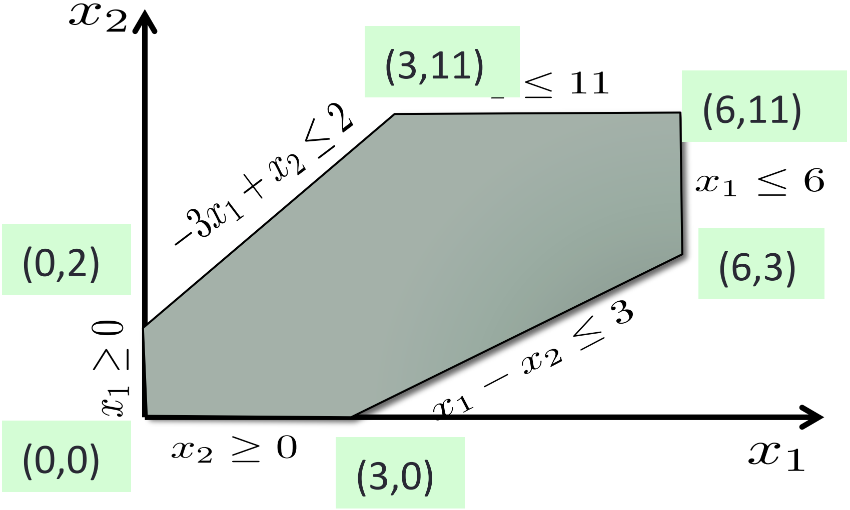
Dictionaries of Simplex = Vertices of the feasible region.

Linear Programming Problem

From Two Weeks Ago.

$$
\begin{array}{lrrcr}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$

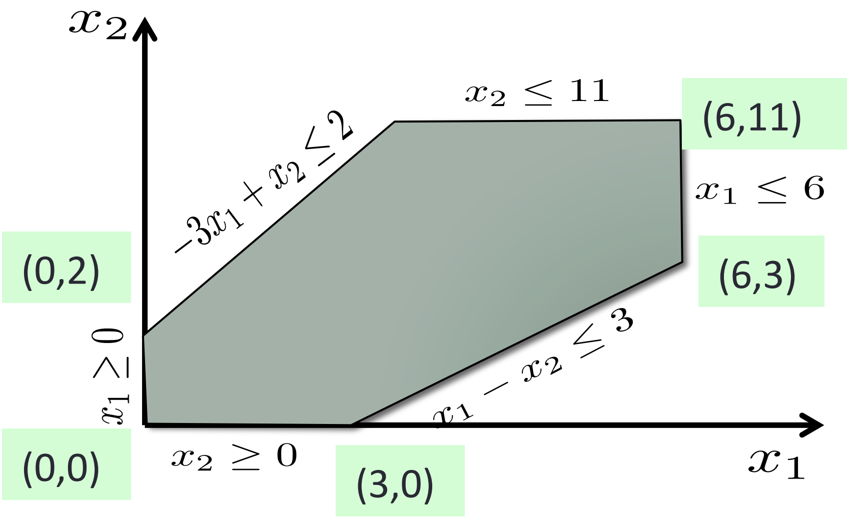
Note: Not drawn to scale



Goal: Solve LP using Simplex and visualize!

Linear Programming Problem

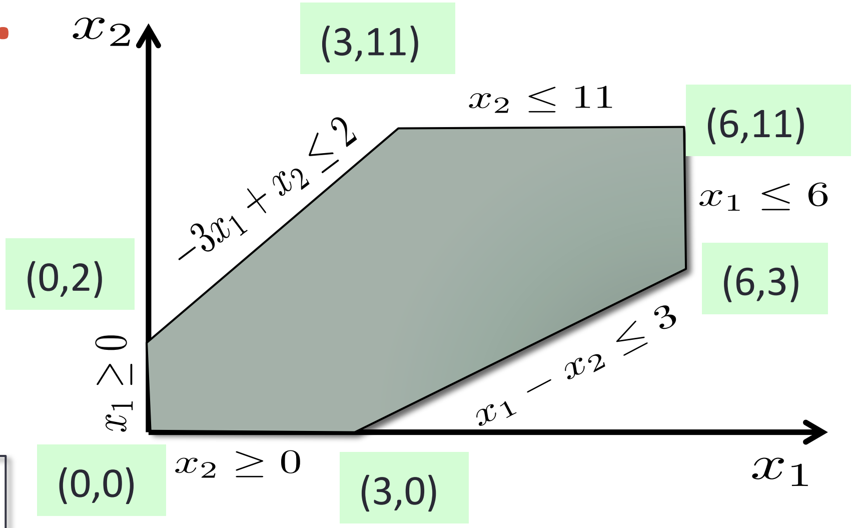
$$
\begin{array}{rrrcr}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$



Dictionary Vertex Corr.

$$
\begin{array}{llllll}
\max & x_1 & +2x_2 & & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 & \leftarrow x_3 \\
 & & x_2 & \leq & 11 & \leftarrow x_4 \\
 & x_1 & -x_2 & \leq & 3 & \leftarrow x_5 \\
 & x_1 & & \leq & 6 & \leftarrow x_6 \\
 & x_1, & x_2 & \geq & 0 &
\end{array}
$$

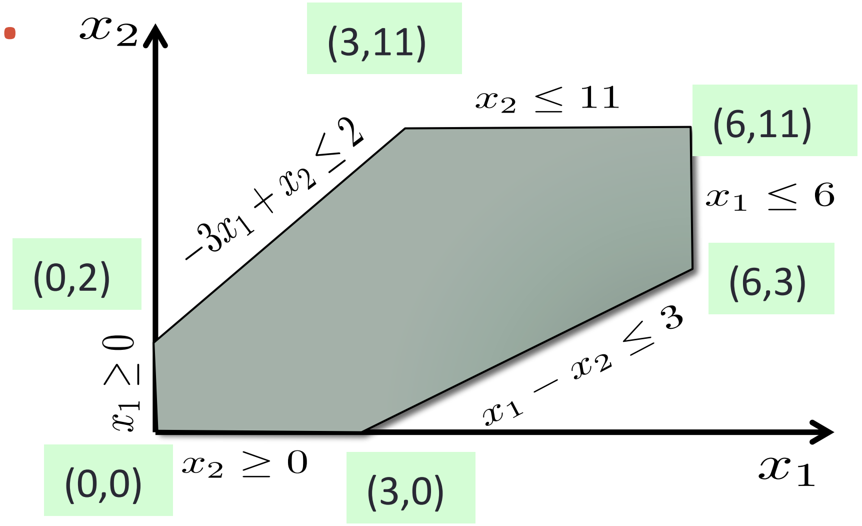
$$
\begin{array}{lcl}
x_1 & = & \\
x_2 & = & \\
x_5 & = & \\
x_6 & = & \\
\hline
z & = & \qquad\qquad x_4 \qquad x_3
\end{array}
$$



Dictionary Vertex Corr.

$$
\begin{array}{llllll}
\max & x_1 & +2x_2 & & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 & \leftarrow {\color{red}x_3} \\
 & & x_2 & \leq & 11 & \leftarrow {\color{red}x_4} \\
 & x_1 & -x_2 & \leq & 3 & \leftarrow {\color{red}x_5} \\
 & x_1 & & \leq & 6 & \leftarrow {\color{red}x_6} \\
 & x_1, & x_2 & \geq & 0 &
\end{array}
$$

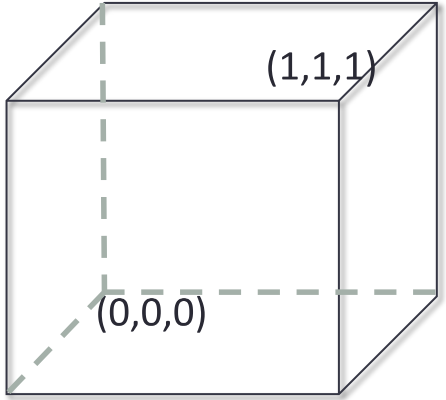
$$
\begin{array}{c|ccc}
x_1 & \cdots & & \\
x_3 & \cdots & & \\
x_4 & \cdots & & \\
x_6 & \cdots & & \\
\hline
z & ? & ?x_2 & ?x_5
\end{array}
$$



Example #3

$$\max \quad x_1 + x_2 - x_3$$

$$\begin{array}{ccccc}
x_1 & & & \leq & 1 \leftarrow \color{red}{x_4} \\
 & x_2 & & \leq & 1 \leftarrow \color{red}{x_5} \\
 & & x_3 & \leq & 1 \leftarrow \color{red}{x_6} \\
x_1 & & & \geq & 0 \\
 & x_2 & & \geq & 0 \\
 & & x_3 & \geq & 0
\end{array}$$



Linear Programming Problem (Standard Form)

$$\begin{array}{rrcl} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}$$

Feasible Dictionary

$$\begin{array}{ccccccccc} x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} \\ & \vdots & & & & & & \\ x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} \\ \hline z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In} \end{array}$$

(1) Solution associated will make at least n constraints active.

(2) Rank of active constraints is n.

Summary

- Vertex (definition).
  - A feasible point that makes at least n inequalities active.
  - The rank of active inequalities equals n.

- Feasible Dictionaries in Simplex:
  - Solution associated must be a vertex of the feasible region.

- What does pivoting do?