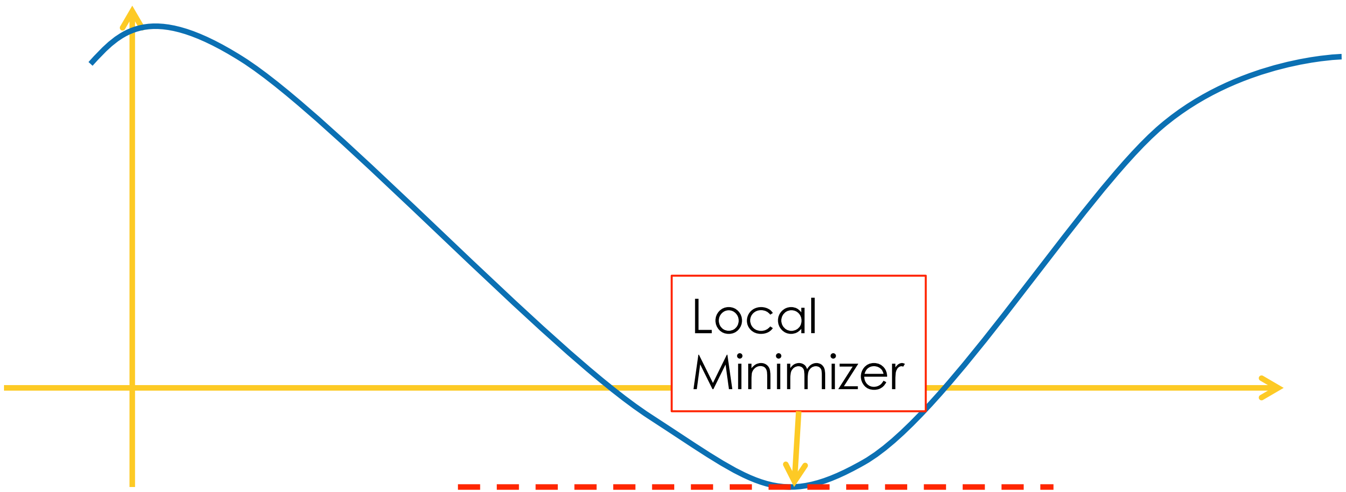
EQUALITY CONSTRAINED OPTIMIZATION

Lagrange Multiplier Method

Unconstrained Optimization

- Goal: minimize function F(x) for all x.
- Unconstrained minimization problem.



Equality Constrained Optimization

$$
\begin{aligned}
\min \quad & f(\mathbf{x}) \\
\text{s.t.} \quad & g_1(\mathbf{x}) = 0 \\
& g_2(\mathbf{x}) = 0 \\
& \vdots \\
& g_m(\mathbf{x}) = 0
\end{aligned}
$$

$$L(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) + \sum_{i=1}^{m} y_i g_i(\mathbf{x})$$

$$
\boxed{
\begin{aligned}
\nabla_{\mathbf{x}} L(\mathbf{x}, \mathbf{y}) &= 0 \\
\nabla_{\mathbf{y}} L(\mathbf{x}, \mathbf{y}) &= 0
\end{aligned}
}
$$

First order Necessary Conditions

Lagrange Multiplier Method

$$\begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & g_1(\mathbf{x}) = 0 \\ & g_2(\mathbf{x}) = 0 \\ & \vdots \\ & g_m(\mathbf{x}) = 0 \end{array}$$

$$L(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) + \sum_{i=1}^{m} y_i g_i(\mathbf{x})$$

$$\boxed{\begin{aligned} \nabla_{\mathbf{x}} L(\mathbf{x}, \mathbf{y}) &= 0 \\ \nabla_{\mathbf{y}} L(\mathbf{x}, \mathbf{y}) &= 0 \end{aligned}}$$

$$\frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} y_i \frac{\partial g_i}{\partial x_j} = 0$$

$$g_i(\mathbf{x}) = 0$$

Solve using Newton's method

Example

$$\min \ \sin(x) + \cos(y) + z^2$$
$$\text{s.t. } x^2 + y^2 + z^2 = 1$$

$$L(x, y, z, \lambda) = \sin(x) + \cos(y) + z^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

$$\begin{aligned} \cos(x) + 2\lambda x &= 0 \\ -\sin(y) + 2\lambda y &= 0 \\ 2z + 2\lambda z &= 0 \\ x^2 + y^2 + z^2 &= 1 \end{aligned}$$

Solve using Newton's method