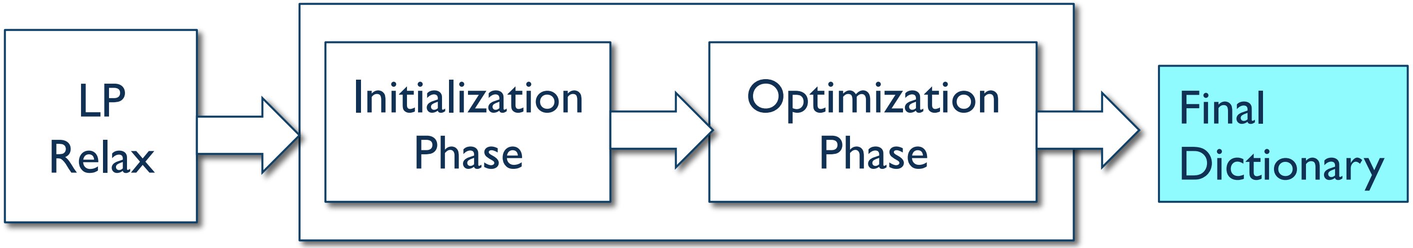
GOMORY-CHVATAL CUTS

Overall Idea

$$\begin{array}{llcl} \max & \mathbf{c}^\intercal\ \mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} + \mathbf{x_s} & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x_s} & \geq & \mathbf{0} \\ & \mathbf{x}, \mathbf{x_s} & \in & \mathbb{Z} \end{array}$$

1. Solve the LP relaxation using Simplex algorithm.



Final Dictionary

$$\begin{array}{c|c c}
\mathbf{x_B} & \tilde{\mathbf{b}} & +\tilde{A}\mathbf{x_N} \\
\hline
z & z_0 &
\end{array}$$

all integer entries?

Optimal solution to ILP found!!

fractional entries

Derive a cutting plane

Example: Final Dictionary

$$
\begin{array}{c|cccc}
x_1 & \boxed{1.2} & -3.1x_2 & +4.3x_3 & -0.5x_5 \\
x_4 & 1 & -x_2 & +x_3 & -x_5 \\
x_6 & \boxed{2.5} & +1.3x_2 & -2.1x_3 & +x_5 \\
\hline
z & 1.7 & -1.2x_2 & -2.3x_3 & -2.1x_5
\end{array}
$$

$$\boxed{x_1 = 1.2,\ x_2 = 0,\ x_3 = 0,\ x_4 = 1,\ x_5 = 0,\ x_6 = 2.5}$$

Cutting Plane Derivation: Step 1

Step 1 Identify row with fractional constant.

$$\begin{array}{c|cccc} x_1 & 1.2 & -3.1x_2 & +4.3x_3 & -0.5x_5 \\ x_4 & 1 & -x_2 & +x_3 & -x_5 \\ x_6 & 2.5 & +1.3x_2 & -2.1x_3 & +x_5 \\ \hline z & 1.7 & -1.2x_2 & -2.3x_3 & -2.1x_5 \end{array}$$

$$x_1 + 3.1x_2 - 4.3x_3 + 0.5x_5 = 1.2$$

Cutting Plane Derivation: Step 2

$$x_1 + 3.1x_2 - 4.3x_3 + 0.5x_5 = 1.2$$

$$\Downarrow$$

$$\underbrace{(x_1 + 3x_2 - 5x_3 + 0x_5)}_{A} + \underbrace{(0.1x_2 + 0.7x_3 + 0.5x_5)}_{B} = 1 + 0.2$$

A: integer

B: integer + fraction

B ≥ 0

A + B = 1.2

Claim

$$(\ldots {}_2 + \underbrace{0.7x_3 + 0.5x_5)}_{B} = 1 + 0.2$$

A

B I

B ≥

A

Conclusion:

1. Fractional part of B is 0.2
2. B ≥ 0.2

In other words,

1. B – 0.2 is an integer.
2. B - 0.2 ≥ 0

B	A+B	Possible
0.2	1.2	YES
-0.8	1.2	No
0.1	1.2	No
-2.8	1.2	No
4.2	1.2	Yes
101.2	1.2	Yes

Cutting Plane

$$\begin{array}{c|cccc}
x_1 & 1.2 & -3.1x_2 & +4.3x_3 & -0.5x_5 \\
x_4 & 1 & -x_2 & +x_3 & -x_5 \\
x_6 & 2.5 & +1.3x_2 & -2.1x_3 & +x_5 \\
\hline
z & 1.7 & -1.2x_2 & -2.3x_3 & -2.1x_5
\end{array}$$

Cutting Plane:

$$\boxed{0.1x_2 + 0.7x_3 + 0.5x_5 \geq 0.2}$$

Cutting Plane: Definition.

Final Dictionary (LP Relax)

$$
\begin{array}{ccccccc}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} \\
 & \vdots & & & & & & \\
\hline
x_{Bk} & = & b_k & +a_{k1}x_{I1} & +\cdots & +a_{kj}x_{Ij} & +\cdots & +a_{kn}x_{In} \\
\hline
 & \vdots & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In}
\end{array}
$$

$b_k \notin \mathbb{Z}$

Cutting Plane

$$\mathrm{frac}(-\mathrm{a}_{\mathrm{k1}})x_{I1} + \cdots + \mathrm{frac}(-\mathrm{a}_{\mathrm{kn}})x_{In} \geq \mathrm{frac}(\mathrm{b}_\mathrm{k})$$

$$\mathrm{frac}(\mathrm{x}) = x - \lfloor x \rfloor$$

$$
\begin{array}{ccccccccc}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} \\
 & \vdots & & & & & & \\
x_{Bk} & = & b_k & +a_{k1}x_{I1} & +\cdots & +a_{kj}x_{Ij} & +\cdots & +a_{kn}x_{In} \\
 & \vdots & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In}
\end{array}
$$

$b_k \notin \mathbb{Z}$

Cutting Plane

$$\mathrm{frac}(-\mathrm{a}_{\mathrm{k1}})x_{I1} + \cdots + \mathrm{frac}(-\mathrm{a}_{\mathrm{kn}})x_{In} \geq \mathrm{frac}(\mathrm{b}_{\mathrm{k}})$$

Claim: Every (integer) feasible point of the ILP satisfies the cutting plane constraint.

We have

$$x_{Bk} + \sum_{j=1}^{n} (-a_{kj}) x_{Ij} = b_k \qquad (1)$$

Since $-a_{kj} \geq \lfloor -a_{kj} \rfloor$ and $x_{Ij} \geq 0$, we obtain

$$\sum_{j=1}^{n} (-a_{kj}) x_{Ij} \geq \sum_{j=1}^{n} \lfloor -a_{kj} \rfloor x_{Ij}$$

Applying this to Eq.(1) we obtain:

$$x_{Bk} + \sum_{j=1}^{n} (-a_{kj}) x_{Ij} \ \geq \ x_{Bk} + \sum_{j=1}^{n} \lfloor -a_{kj} \rfloor x_{Ij} \qquad (2)$$

Therefore, we get

$$b_k \geq x_{Bk} + \sum_{j=1}^{n} \lfloor -a_{kj} \rfloor x_{Ij} \qquad (3)$$

The RHS of (3) is an integer, therefore, we conclude:

$$\lfloor b_k \rfloor \geq x_{Bk} + \sum_{j=1}^{n} \lfloor -a_{kj} \rfloor x_{Ij} \qquad (4)$$

Combining (4) with (1), we have,

$$b_k - \lfloor b_k \rfloor \leq \sum_{j=1}^{n} (-a_{kj} - \lfloor -a_{kj} \rfloor) x_{Ij}$$

In other words, $\sum_{j=1}^{n} \mathsf{frac}(-a_{kj}) x_{Ij} \geq \mathsf{frac}(b_k)$

$$\begin{array}{ccccccc}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} \\
 & \vdots & & & & & & \\
x_{Bk} & = & b_k & +a_{k1}x_{I1} & +\cdots & +a_{kj}x_{Ij} & +\cdots & +a_{kn}x_{In} \\
 & \vdots & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In}
\end{array}$$

$b_k \notin \mathbb{Z}$

Example

$$
\begin{array}{c|cccc}
x_1 & 1.2 & -3.1x_2 & +4.3x_3 & -0.5x_5 \\
x_4 & 1 & -x_2 & +x_3 & -x_5 \\
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\hline
z & 1.7 & -1.2x_2 & -2.3x_3 & -2.1x_5
\end{array}
$$

$$\boxed{0.7x_2 + 0.1x_3 + 0x_5 \geq 0.5}$$

$$\boxed{0.2x_2 + 0.3x_3 + 0.1x_5 \geq 0.7}$$