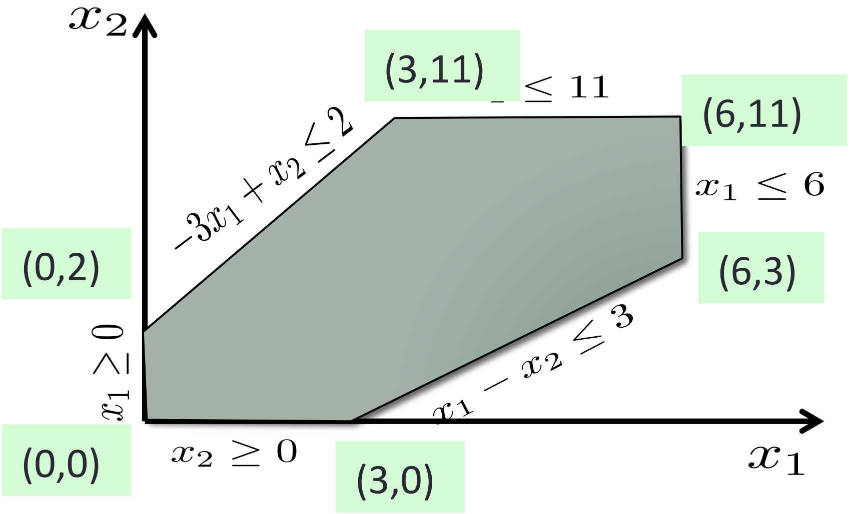
POLYHEDRA: VERTICES

Linear Programming Problem

From Two Weeks Ago.

$$
\begin{array}{llllll}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$

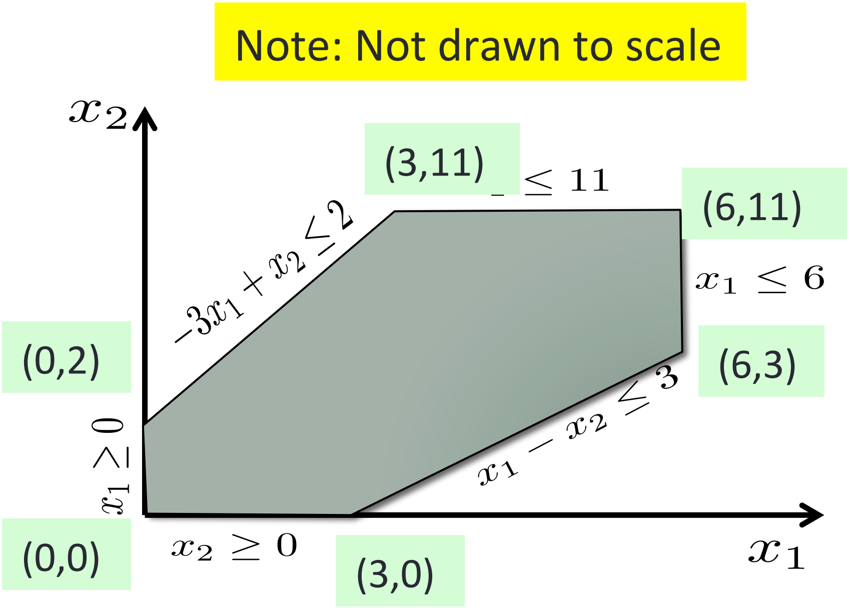
Note: Not drawn to scale



Goal: Solve LP using Simplex and visualize!

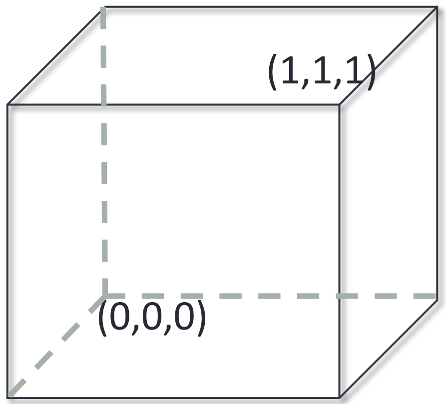
Active Constraints

$$
\begin{array}{rrrcr}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$



Active Constraints

$$
\begin{array}{rcl}
x_1 \qquad\qquad\qquad & \leq & 1 \\
x_2 \qquad\quad & \leq & 1 \\
x_3 & \leq & 1 \\
x_1 \qquad\qquad\qquad & \geq & 0 \\
x_2 \qquad\quad & \geq & 0 \\
x_3 & \geq & 0
\end{array}
$$



Basic Geometric Facts

- Intersection of 2 lines in 2D yields a point.
  - Lines must be non-parallel.
- Intersection of 3 planes in 3D yields a point.
  - Exclude parallel planes, or other corner cases.
- Intersection of 4 hyper-planes in 4D yields a point.
  - Again, some corner cases.

Intersection of n hyper-planes

$$\begin{array}{cccccccc} a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & = & b_1 & \leftarrow & \color{red}{\mathcal{H}_1} \\ & & \ddots & & \vdots & & & \\ a_{n1}x_1 & +a_{n2}x_2 & +\cdots+ & a_{nn}x_n & = & b_n & \leftarrow & \color{red}{\mathcal{H}_n} \end{array}$$

$$\textbf{rank} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = n$$

Vertex (Definition)

A feasible solution **x** to the constraints is a vertex iff

$$
\begin{array}{cccccc}
a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 \\
 & & \ddots & & \vdots & \\
a_{j1}x_1 & +a_{j2}x_2 & +\cdots+ & a_{jn}x_n & \leq & b_j \\
 & & \ddots & & \vdots & \\
a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m
\end{array}
$$

at least n ineqs.
are active for x.

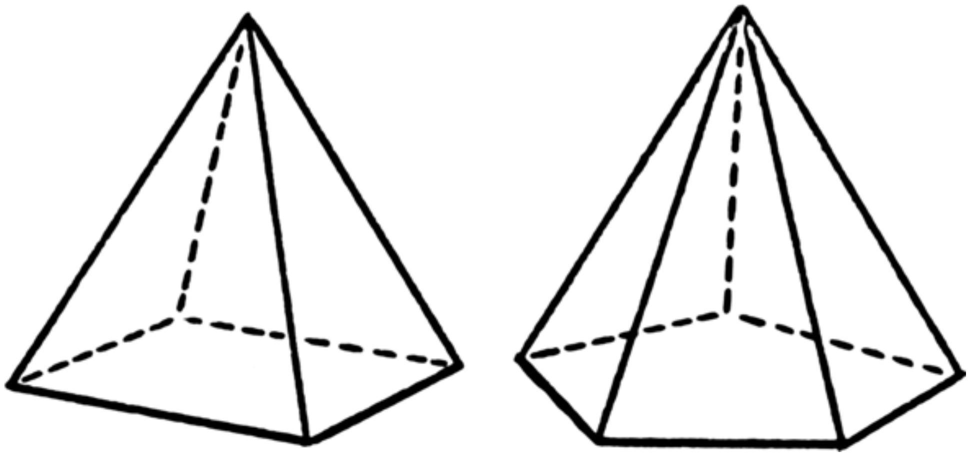
rank of the active constraints for **x** is n

Vertex Issue #1

- Does every point **x** that activates n constraints form a vertex?

Vertex Issue #2

- Can a vertex activate more than n constraints?



Vertex Issue #3

- What if there are more variables than constraints?

Number of Vertices

- n-dimensional hyper cube has 2^n vertices.

- In general, combinatorial explosion of vertices.
  - m constraints, n variables: $\binom{m}{n}$ upper bound on vertices