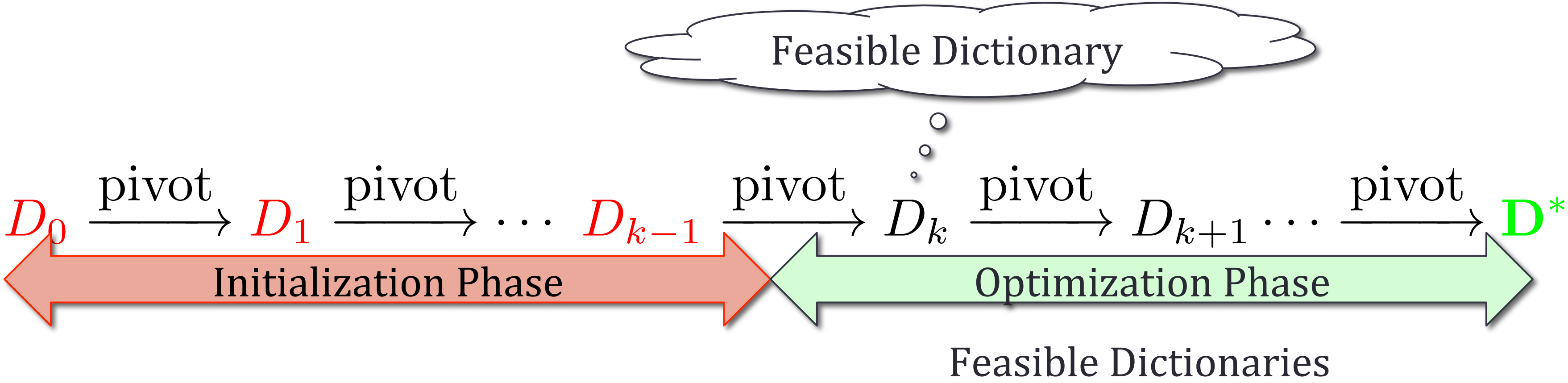
SIMPLEX METHOD: INITIALIZATION PHASE

How to find a feasible point using Simplex.

Simplex Algorithm



Initial Dictionary

$$\begin{array}{lccc} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}$$

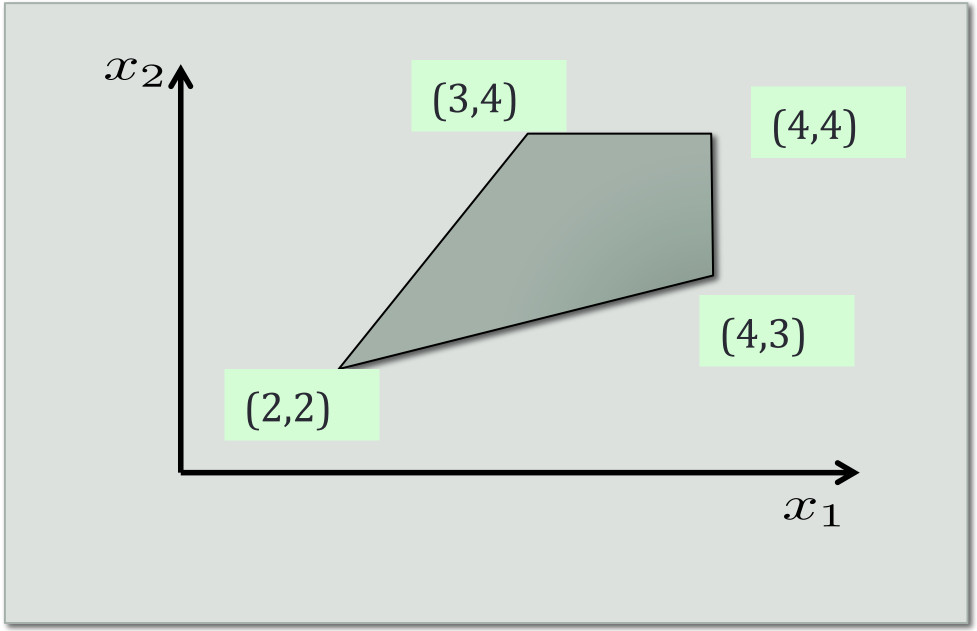
$$\begin{array}{lccc} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} + \color{red}{\mathbf{x_s}} & = & \mathbf{b} \\ & \mathbf{x}, \color{red}{\mathbf{x_s}} & \geq & 0 \end{array}$$

Feasible?

$$\begin{array}{ccc} \mathbf{x_s} & = & \mathbf{b} - A\,\mathbf{x} \\ \hline z & = & 0 + \mathbf{c}^\intercal\,\mathbf{x} \end{array}$$

Example

$$
\begin{array}{rrcl}
\text{max.} & x_1 + 2x_2 & & \\
\text{s.t.} & -2x_1 + x_2 & \leq & -2 \\
 & x_2 & \leq & 4 \\
 & x_1 - 2x_2 & \leq & -2 \\
 & x_1 & \leq & 4 \\
 & x_1, x_2 & \geq & 0
\end{array}
$$



Step 1: Adding Slack

$$\begin{array}{llcl}
\text{max.} & x_1 + 2x_2 & & \\
\text{s.t.} & -2x_1 + x_2 + \color{red}{x_3} & = & -2 \\
& x_2 + \color{red}{x_4} & = & 4 \\
& x_1 - 2x_2 + \color{red}{x_5} & = & -2 \\
& x_1 + \color{red}{x_6} & = & 4 \\
& x_1, x_2, \color{red}{x_3, \ldots, x_6} & \geq & 0
\end{array}$$

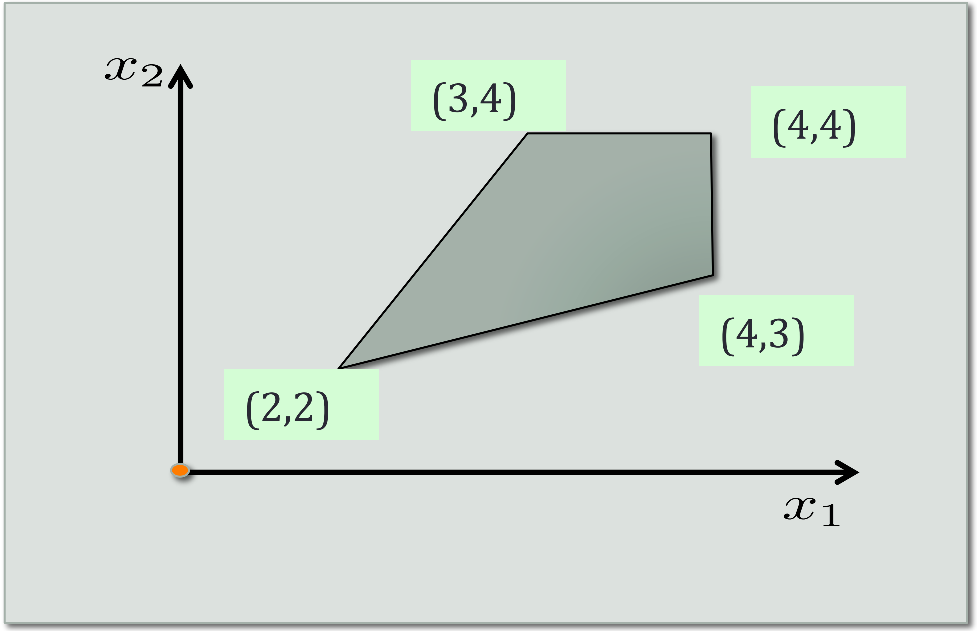
Step 2: Initial Dictionary

$$
\begin{array}{llcr}
\text{max.} & x_1 + 2x_2 & & \\
\text{s.t.} & -2x_1 + x_2 + x_3 & = & -2 \\
 & x_2 + x_4 & = & 4 \\
 & x_1 - 2x_2 + x_5 & = & -2 \\
 & x_1 + x_6 & = & 4 \\
 & x_1, x_2, x_3, \ldots, x_6 & \geq & 0
\end{array}
$$

$$
\begin{array}{rcrrr}
x_3 & = & -2 & +2x_1 & -x_2 \\
x_4 & = & 4 & +0x_1 & -x_2 \\
x_5 & = & -2 & -x_1 & +2x_2 \\
x_6 & = & 4 & -x_1 & +0x_2 \\
\hline
z & = & 0 & +x_1 & +2x_2
\end{array}
$$

Infeasible Initial Dictionary

$$
\begin{array}{rcrrr}
x_3 & = & -2 & +2x_1 & -x_2 \\
x_4 & = & 4 & +0x_1 & -x_2 \\
x_5 & = & -2 & -x_1 & +2x_2 \\
x_6 & = & 4 & -x_1 & +0x_2 \\
\hline
z & = & 0 & +x_1 & +2x_2
\end{array}
$$



Initialization Phase Simplex : Overview

- Goal: Get to a dictionary with feasible solution.
  - Alternatively, conclude problem infeasibility.

- Strategy:
  - Modify the problem
  - Perform Simplex on Modified Problem.

Initialization Phase Simplex : Overview

- Goal: Get to a dictionary with feasible solution.
  - Alternatively, conclude problem infeasibility.

- Strategy:
  - Modify the problem
  - Perform Simplex on Modified Problem.

CONSTRUCTING THE AUXILIARY PROBLEM

AUXILLIARY PROBLEM

$$\begin{array}{lrcl}
\text{max.} & x_1 + 2x_2 & & \\
\text{s.t.} & -2x_1 + x_2 + x_3 & = & -2 \\
& x_2 + x_4 & = & 4 \\
& x_1 - 2x_2 + x_5 & = & -2 \\
& x_1 + x_6 & = & 4 \\
& x_1, x_2, x_3, \dots, x_6 & \geq & 0
\end{array}$$

$$\begin{array}{lrrrcl}
\text{s.t.} & -2x_1 & +x_2 & +x_3 & = & -2 + x_0 \\
& & x_2 & +x_4 & = & 4 + x_0 \\
& x_1 & -2x_2 & +x_5 & = & -2 + x_0 \\
& x_1 & & +x_6 & = & 4 + x_0 \\
& & & x_1, \dots, x_6, x_0 & \geq & 0
\end{array}$$

Feasibility of Aux. Problem

$$
\begin{array}{lrrrcl}
\text{s.t.} & -2x_1 & +x_2 & +x_3 & = & -2 + x_0 \\
 & & x_2 & +x_4 & = & 4 + x_0 \\
 & x_1 & -2x_2 & +x_5 & = & -2 + x_0 \\
 & x_1 & & +x_6 & = & 4 + x_0 \\
 & & \multicolumn{2}{r}{x_1, \ldots, x_6, x_0} & \geq & 0
\end{array}
$$

Claim: Aux. Problem is Feasible.

If Original Problem is Feasible?

$$
\begin{array}{lrcr}
\text{max.} & x_1 + 2x_2 & & \\
\text{s.t.} & -2x_1 + x_2 + x_3 & = & -2 \\
& x_2 + x_4 & = & 4 \\
& x_1 - 2x_2 + x_5 & = & -2 \\
& x_1 + x_6 & = & 4 \\
& x_1, x_2, x_3, \ldots, x_6 & \geq & 0
\end{array}
$$

$$
\begin{array}{lccccc}
\text{s.t.} & -2x_1 & +x_2 & +x_3 & = & -2 + x_0 \\
& & x_2 & +x_4 & = & 4 + x_0 \\
& x_1 & -2x_2 & +x_5 & = & -2 + x_0 \\
& x_1 & & +x_6 & = & 4 + x_0 \\
& & x_1, \ldots, x_6, x_0 & & \geq & 0
\end{array}
$$

$$
\left(\begin{array}{c} x_1 : 2, x_2 : 2, \\ x_3 : 0, x_4 : 2, x_5 : 0, x_6 : 2 \end{array} \right)
$$

$$x_0 = 0$$

If Original Problem is Infeasible?

$$
\begin{array}{llcl}
\text{max.} & x_1 + 2x_2 & & \\
\text{s.t.} & -2x_1 + x_2 + x_3 & = & -2 \\
& x_2 + x_4 & = & 4 \\
& x_1 - 2x_2 + x_5 & = & -2 \\
& x_1 + x_6 & = & 4 \\
& x_1, x_2, x_3, \ldots, x_6 & \geq & 0
\end{array}
$$

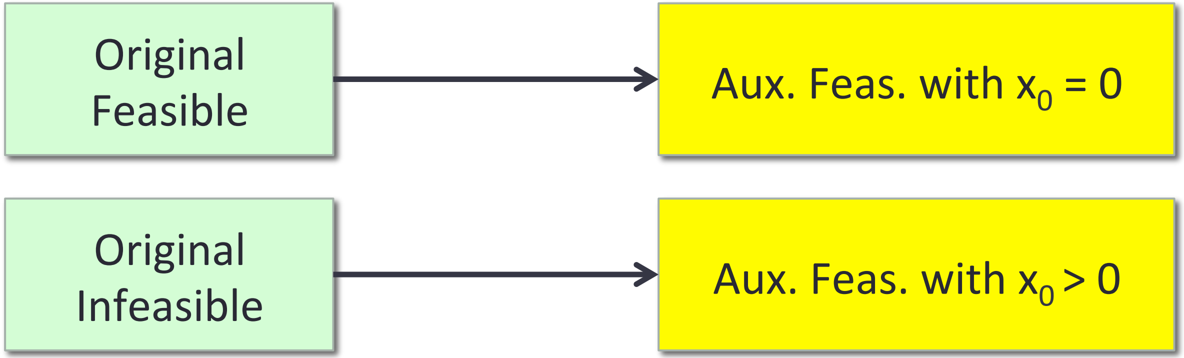
$$
\begin{array}{llllcl}
\text{s.t.} & -2x_1 & +x_2 & +x_3 & = & -2 + x_0 \\
& & x_2 & +x_4 & = & 4 + x_0 \\
& x_1 & -2x_2 & +x_5 & = & -2 + x_0 \\
& x_1 & & +x_6 & = & 4 + x_0 \\
& & & x_1, \ldots, x_6, x_0 & \geq & 0
\end{array}
$$

No Solution.

Auxiliary Problem

- Add fresh variable x_0

- Aux. Problem is always feasible.



AUXILLIARY PROBLEM

$$
\begin{array}{llll}
\text{max.} & x_1 + 2x_2 & & \\
\text{s.t.} & -2x_1 + x_2 + x_3 & = & -2 \\
& x_2 + x_4 & = & 4 \\
& x_1 - 2x_2 + x_5 & = & -2 \\
& x_1 + x_6 & = & 4 \\
& x_1, x_2, x_3, \ldots, x_6 & \geq & 0
\end{array}
$$

Same as min. x0

$$
\begin{array}{lllllll}
\max & -x_0 & & & & & \\
\text{s.t.} & -2x_1 & +x_2 & +x_3 & = & -2 + x_0 \\
& & x_2 & +x_4 & = & 4 + x_0 \\
& x_1 & -2x_2 & +x_5 & = & -2 + x_0 \\
& x_1 & & +x_6 & = & 4 + x_0 \\
& & & x_1, \ldots, x_6, x_0 & \geq & 0
\end{array}
$$

AUXILLIARY PROBLEM

$$\begin{array}{lrcl} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} + \mathbf{x_s} & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x_s} & \geq & 0 \end{array} \quad \Longrightarrow \quad \begin{array}{lrcl} \max & -x_0 & & \\ \text{s.t.} & A\mathbf{x} + \mathbf{x_s} - x_0 \mathbf{1} & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x_s}, x_0 & \geq & 0 \end{array}$$

1. Aux. problem cannot be unbounded.

2. Aux. problem is always feasible.

Aux. Problem Always Feasible (Proof)

$$\begin{array}{lrcl} \max & \mathbf{c}^\intercal\, \mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} + \mathbf{x_s} & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x_s} & \geq & 0 \end{array}
\quad \Rightarrow \quad
\begin{array}{lrcl} \max & -x_0 & & \\ \text{s.t.} & A\mathbf{x} + \mathbf{x_s} - x_0\mathbf{1} & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x_s}, x_0 & \geq & 0 \end{array}$$

$$\mathbf{x} = 0,\ x_0 = -\min(\mathbf{b}, 0), x_s = \mathbf{b} + x_0\mathbf{1}$$

Initialization Phase Simplex

$$
\begin{array}{lrcl}
\max & -x_0 & & \\
\text{s.t.} & A\mathbf{x} + \mathbf{x_s} - x_0\mathbf{1} & = & \mathbf{b} \\
& \mathbf{x}, \mathbf{x_s}, x_0 & \geq & 0
\end{array}
$$

If opt. value = 0 then form initial feasible dictionary for original problem.

Initial Aux. Dictionary → Pivoting Steps* → Final Aux. Dictionary

If opt. value < 0 then problem infeasible.

INITIALIZATION PHASE SIMPLEX

AUXILLIARY PROBLEM

$$\begin{array}{llcl} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} + \mathbf{x_s} & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x_s} & \geq & 0 \end{array} \quad \Rightarrow \quad \begin{array}{llcl} \max & -x_0 & & \\ \text{s.t.} & A\mathbf{x} + \mathbf{x_s} - x_0 \mathbf{1} & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x_s}, x_0 & \geq & 0 \end{array}$$

1. Aux. problem cannot be unbounded.

2. Aux. problem is always feasible.

Initialization Phase Simplex

$$\begin{array}{lrcl} \max & -x_0 & & \\ \text{s.t.} & A\mathbf{x} + \mathbf{x_s} - x_0\mathbf{1} & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x_s}, x_0 & \geq & 0 \end{array}$$

If opt. value = 0 then form initial feasible dictionary for original problem.

Initial Aux. Dictionary

Pivoting Steps*

Final Aux. Dictionary

If opt. value < 0 then problem infeasible.

Initialization Phase Simplex

1. How to form initial dictionary for Aux. Problem?

2. How to perform pivoting?
   - Minor modifications.

3. How to find initial dictionary for original problem?

Initial Dictionary For Aux Problem.

$$\begin{array}{lrcl} \max & -x_0 & & \\ \text{s.t.} & A\mathbf{x} + \mathbf{x_s} - x_0\mathbf{1} & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x_s}, x_0 & \geq & 0 \end{array}$$

$$\begin{array}{ccccc} \mathbf{x_s} & = & \mathbf{b} & +\mathbf{1}x_0 & -A\mathbf{x} \\ \hline z & = & 0 & -x_0 & \end{array}$$

Special Move:

Entering Variable is x_0

Leaving variable is the variable with least value of b

Initial Dictionary For Aux. Problem

$$
\begin{array}{rclllll}
x_{B1} & = & b_1 & {\color{red}+1x_0} & +a_{11}x_{I1} & +\cdots & +a_{1n}x_{In} \\
x_{B2} & = & b_2 & {\color{red}+1x_0} & +a_{21}x_{I1} & +\cdots & +a_{2n}x_{In} \\
 & \vdots & & & & & \\
\mathbf{x_{Bj}} & = & \mathbf{b_j} & {\color{red}+1x_0} & +\mathbf{a_{j1}x_{I1}} & +\cdots & +\mathbf{a_{jn}x_{In}} \\
 & \vdots & & & & & \\
x_{Bm} & = & b_m & {\color{red}+1x_0} & +a_{m1}x_{I1} & +\cdots & +a_{mn}x_{In} \\
\hline
z & = & 0 & {\color{red}-x_0} & & &
\end{array}
$$

$$b_j = \min(b_1, \ldots, b_m)$$

Example

$$
\begin{array}{llllll}
\max & -x_0 & & & & \\
\text{s.t.} & -2x_1 & +x_2 & +x_3 & = & -2 + x_0 \\
 & & x_2 & +x_4 & = & 4 + x_0 \\
 & x_1 & -2x_2 & +x_5 & = & -2 + x_0 \\
 & x_1 & & +x_6 & = & 4 + x_0 \\
 & & x_1, \ldots, x_6, x_0 & & \geq & 0
\end{array}
$$

x0 is entering
x3 or x5 is leaving.

$$
\begin{array}{cccccc}
x_3 & = & -2 & +x_0 & +2x_1 & -x_2 \\
x_4 & = & 4 & +x_0 & +0x_1 & -x_2 \\
x_5 & = & -2 & +x_0 & -x_1 & +2x_2 \\
x_6 & = & 4 & +x_0 & -x_1 & +0x_2 \\
\hline
w & = & 0 & -x_0 & &
\end{array}
$$

Example (Cont.)

$$
\begin{array}{rcrrrr}
x_3 & = & -2 & +x_0 & +2x_1 & -x_2 \\
x_4 & = & 4 & +x_0 & +0x_1 & -x_2 \\
x_5 & = & -2 & +x_0 & -x_1 & +2x_2 \\
x_6 & = & 4 & +x_0 & -x_1 & +0x_2 \\
\hline
w & = & 0 & -x_0 & &
\end{array}
$$

x0 is entering
x3 is leaving.

$$
\begin{array}{rcrrrr}
x_0 & = & 2 & +x_3 & -2x_1 & +x_2 \\
x_4 & = & 6 & +x_3 & -2x_1 & +0x_2 \\
x_5 & = & 0 & +x_3 & -3x_1 & +3x_2 \\
x_6 & = & 6 & +x_3 & -3x_1 & +x_2 \\
\hline
w & = & -2 & -x_3 & +2x_1 & -x_2
\end{array}
$$

Claim

For initial aux. dictionary

1. Choose x_0 as entering variable.
2. Choose basic variable x_j with least value of b_j as leaving.

The resulting dictionary is always feasible.

$$
\begin{array}{rcllll}
x_{B1} & = & b_1 & +1x_0 & +a_{11}x_{I1} & +\cdots & +a_{1n}x_{In} \\
x_{B2} & = & b_2 & +1x_0 & +a_{21}x_{I1} & +\cdots & +a_{2n}x_{In} \\
 & \vdots & & & & & \\
x_{Bm} & = & b_m & +1x_0 & +a_{m1}x_{I1} & +\cdots & +a_{mn}x_{In} \\
\hline
z & = & 0 & -x_0 & & &
\end{array}
$$

Initial Dictionary For Aux. Problem

$$
\begin{array}{rclllll}
x_{B1} & = & b_1 & \color{red}{+1x_0} & +a_{11}x_{I1} & +\cdots & +a_{1n}x_{In} \\
x_{B2} & = & b_2 & \color{red}{+1x_0} & +a_{21}x_{I1} & +\cdots & +a_{2n}x_{In} \\
 & \vdots & & & & & \\
\mathbf{x_{Bj}} & = & \mathbf{b_j} & \color{red}{+1x_0} & +\mathbf{a_{j1}x_{I1}} & +\cdots & +\mathbf{a_{jn}x_{In}} \\
 & \vdots & & & & & \\
x_{Bm} & = & b_m & \color{red}{+1x_0} & +a_{m1}x_{I1} & +\cdots & +a_{mn}x_{In} \\
\hline
z & = & 0 & \color{red}{-x_0} & & &
\end{array}
$$

$$b_j = \min(b_1, \ldots, b_m)$$

Pivoting

Initial Aux. Dictionary $\xrightarrow{\text{Pivoting Steps*}}$ Final Aux. Dictionary

Special Rule:

Whenever x_0 is one possible leaving variable, preferentially choose x_0 as the leaving variable.

Pivoting

Initial Aux. Dictionary $\xrightarrow{\text{Pivoting Steps*}}$ Final Aux. Dictionary

Special Rule:

Whenever x_0 is one possible leaving variable, preferentially choose x_0 as the leaving variable.

Example

$$
\begin{array}{rcrrrr}
x_0 & = & 2 & +x_3 & -2x_1 & +x_2 \\
x_4 & = & 6 & +x_3 & -2x_1 & +0x_2 \\
x_5 & = & 0 & +x_3 & -3x_1 & +3x_2 \\
x_6 & = & 6 & +x_3 & -3x_1 & +x_2 \\
\hline
w & = & -2 & -x_3 & +2x_1 & -x_2
\end{array}
$$

x1 enters + x5 leaves

$$
\begin{array}{rcrrrr}
x_1 & = & 0 & +\frac{1}{3}x_3 & -\frac{1}{3}x_5 & +x_2 \\
x_0 & = & 2 & +\frac{1}{3}x_3 & +\frac{2}{3}x_5 & -x_2 \\
x_4 & = & 6 & +\frac{1}{3}x_3 & +\frac{2}{3}x_5 & -2x_2 \\
x_6 & = & 6 & +0x_3 & -x_5 & -2x_2 \\
\hline
w & = & -2 & -\frac{1}{3}x_3 & -\frac{2}{3}x_5 & +x_2
\end{array}
$$

Example (Cont)

$$\begin{array}{rclllll} x_1 & = & 0 & +\frac{1}{3}x_3 & -\frac{1}{3}x_5 & +x_2 \\ x_0 & = & 2 & +\frac{1}{3}x_3 & +\frac{2}{3}x_5 & -x_2 \\ x_4 & = & 6 & +\frac{1}{3}x_3 & +\frac{2}{3}x_5 & -2x_2 \\ x_6 & = & 6 & +0x_3 & -x_5 & -2x_2 \\ \hline w & = & -2 & -\frac{1}{3}x_3 & -\frac{2}{3}x_5 & +x_2 \end{array}$$

x2 enters and
x0 leaves

$$\begin{array}{rclllll} x_1 & = & 2 & +\frac{2}{3}x_3 & +\frac{1}{3}x_5 & -x_0 \\ x_2 & = & 2 & +\frac{1}{3}x_3 & +\frac{2}{3}x_5 & -x_0 \\ x_4 & = & 2 & -\frac{1}{3}x_3 & -\frac{2}{3}x_5 & +2x_0 \\ x_6 & = & 2 & -\frac{2}{3}x_3 & +\frac{4}{3}x_5 & +2x_0 \\ \hline w & = & 0 & +0x_3 & +0x_5 & -x_0 \end{array}$$

Finding Initial Dictionary (Orig. Problem)

$$
\begin{array}{rcllll}
x_1 & = & 2 & +\frac{2}{3}x_3 & +\frac{1}{3}x_5 & -x_0 \\
x_2 & = & 2 & +\frac{1}{3}x_3 & +\frac{2}{3}x_5 & -x_0 \\
x_4 & = & 2 & -\frac{1}{3}x_3 & -\frac{2}{3}x_5 & +2x_0 \\
x_6 & = & 2 & -\frac{2}{3}x_3 & +\frac{4}{3}x_5 & +2x_0 \\
\hline
w & = & 0 & +0x_3 & +0x_5 & -x_0
\end{array}
$$

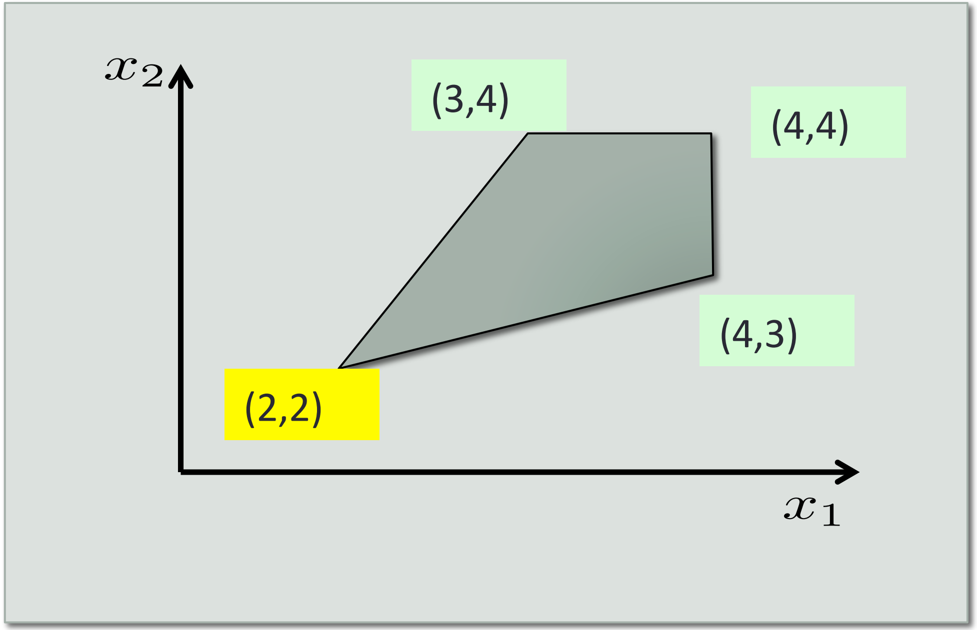
$$
\begin{array}{lrcl}
\text{max.} & x_1 + 2x_2 & & \\
\text{s.t.} & -2x_1 + x_2 & \leq & -2 \\
 & x_2 & \leq & 4 \\
 & x_1 - 2x_2 & \leq & -2 \\
 & x_1 & \leq & 4 \\
 & x_1, x_2 & \geq & 0
\end{array}
$$

$$
\begin{array}{rclll}
x_1 & = & 2 & +\frac{2}{3}x_3 & +\frac{1}{3}x_5 \\
x_2 & = & 2 & +\frac{1}{3}x_3 & +\frac{2}{3}x_5 \\
x_4 & = & 2 & -\frac{1}{3}x_3 & -\frac{2}{3}x_5 \\
x_6 & = & 2 & -\frac{2}{3}x_3 & +\frac{4}{3}x_5 \\
\hline
z & = & & &
\end{array}
$$

$$
\begin{array}{rcl}
z & = & x_1 + 2x_2 \\
 & = & (2 + \frac{2}{3}x_3 + \frac{1}{3}x_5) + 2(2 + \frac{1}{3}x_3 + \frac{2}{3}x_5) \\
 & = & 6 + \frac{4}{3}x_3 + \frac{5}{3}x_5
\end{array}
$$

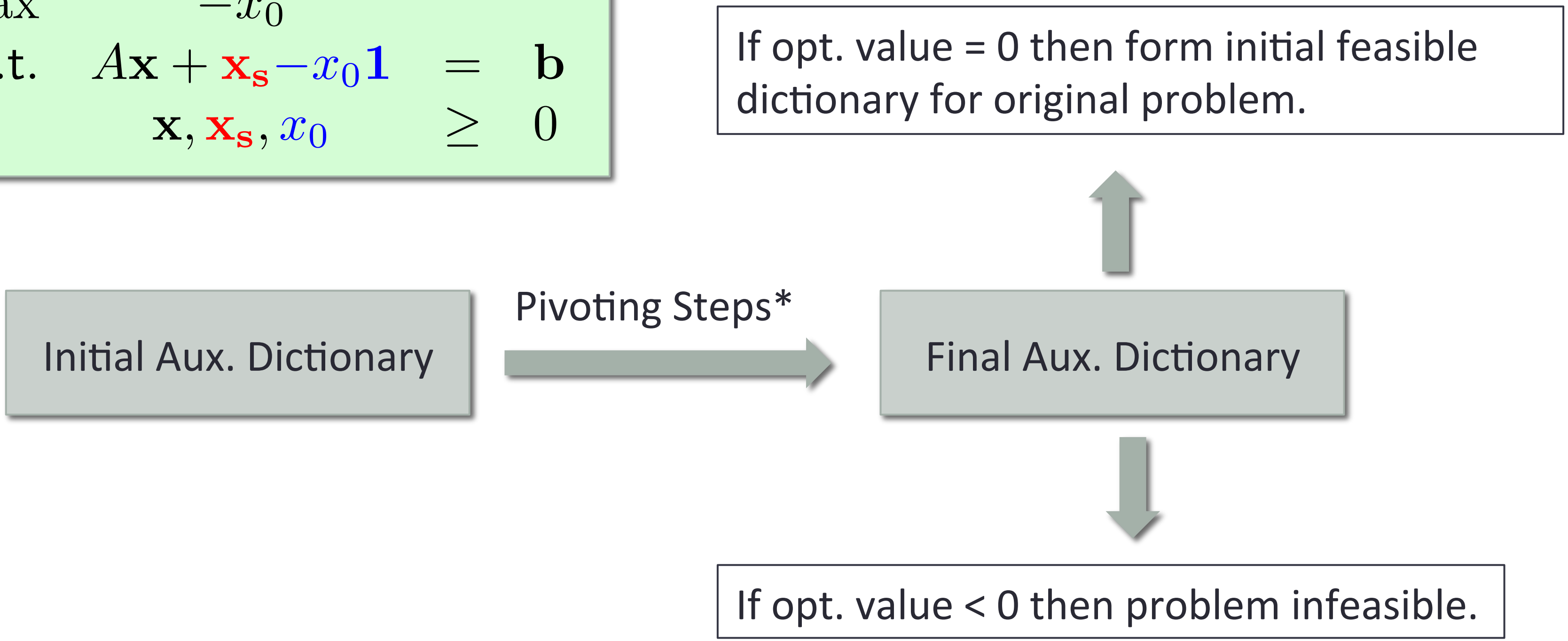
Initial Dictionary

$$
\begin{array}{rclll}
x_1 & = & 2 & +\frac{2}{3}x_3 & +\frac{1}{3}x_5 \\
x_2 & = & 2 & +\frac{1}{3}x_3 & +\frac{2}{3}x_5 \\
x_4 & = & 2 & -\frac{1}{3}x_3 & -\frac{2}{3}x_5 \\
x_6 & = & 2 & -\frac{2}{3}x_3 & +\frac{4}{3}x_5 \\
\hline
z & = & 6 & +\frac{4}{3}x_3 & +\frac{5}{3}x_5
\end{array}
$$



Initialization Phase Simplex

$$
\begin{array}{llcl}
\max & -x_0 & & \\
\text{s.t.} & A\mathbf{x} + \color{red}{\mathbf{x_s}} \color{blue}{- x_0 \mathbf{1}} & = & \mathbf{b} \\
& \mathbf{x}, \color{red}{\mathbf{x_s}}, \color{blue}{x_0} & \geq & 0
\end{array}
$$



Non-Standard Pivots

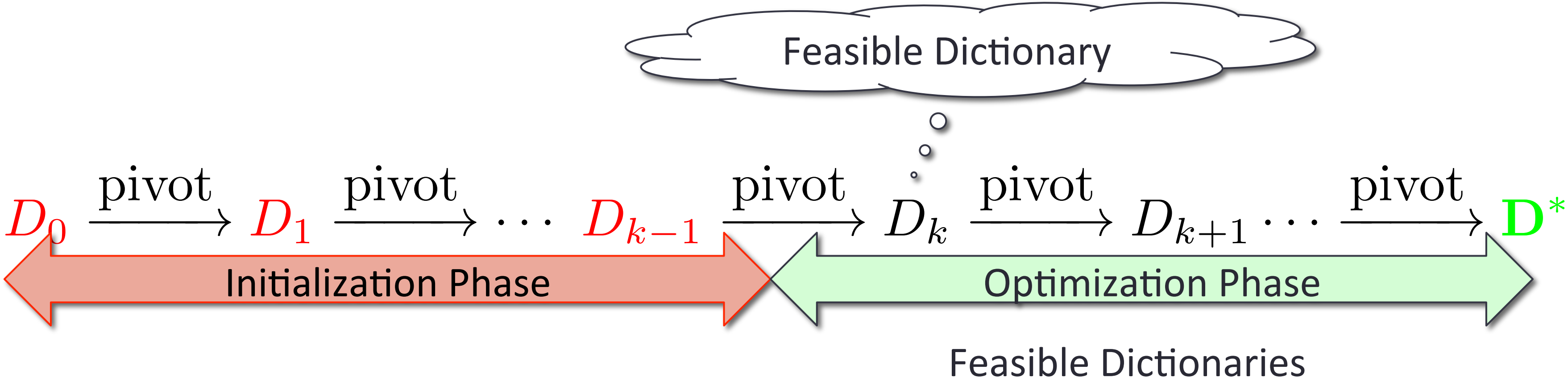
- At the first step:
  - x_0 is entering.
  - Variable with least b_j is leaving.

- During initialization phase:
  - Whenever x_0 can be a leaving variable: we preferentially choose it.

- **Fact:** If x_0 is a leaving variable

  then next dictionary has to be final.

Simplex Algorithm



EXAMPLE OF INFEASIBLE PROBLEM.

Example

$$
\begin{array}{rrrrrr}
\max & x_1 & +2x_2 & -x_3 & & \\
\text{s.t.} & x_1 & -x_2 & & \leq & 5 \\
& & x_2 & +x_3 & \leq & 14 \\
& -x_1 & & +x_3 & \leq & -6 \\
& & & -x_3 & \leq & -7
\end{array}
$$

Initial Dictionary

$$\begin{array}{lllll} \max & x_1 & +2x_2 & -x_3 & & \\ \text{s.t.} & x_1 & -x_2 & & \leq & 5 \\ & & x_2 & +x_3 & \leq & 14 \\ & -x_1 & & +x_3 & \leq & -6 \\ & & & -x_3 & \leq & -7 \end{array}$$

$$\begin{array}{rclllll} x_4 & = & 5 & -x_1 & +x_2 & \\ x_5 & = & 14 & & -x_2 & -x_3 \\ x_6 & = & \color{red}{-6} & +x_1 & & -x_3 \\ x_7 & = & \color{red}{-7} & & & +x_3 \\ \hline z & = & 0 & +x_1 & +2x_2 & -x_3 \end{array}$$

Auxiliary Problem

$$\begin{array}{llllll} \max & -x_0 & & & & \\ \text{s.t.} & x_1 & -x_2 & & \leq & 5 & +x_0 \\ & & x_2 & +x_3 & \leq & 14 & +x_0 \\ & -x_1 & & +x_3 & \leq & -6 & +x_0 \\ & & & -x_3 & \leq & -7 & +x_0 \end{array}$$

Aux. Problem (Initial Dictionary)

$$
\begin{array}{rclccccc}
x_4 & = & 5 & +x_0 & -x_1 & +x_2 & \\
x_5 & = & 14 & +x_0 & & -x_2 & -x_3 \\
x_6 & = & \color{red}{-6} & +x_0 & +x_1 & & -x_3 \\
x_7 & = & \color{red}{-7} & +x_0 & & & +x_3 \\
\hline
w & = & 0 & -x_0 & & &
\end{array}
$$

x_0 enters and x_7 leaves!

Dictionary D1

$$
\begin{array}{rcrrrrr}
x_4 & = & 5 & +x_0 & -x_1 & +x_2 & \\
x_5 & = & 14 & +x_0 & & -x_2 & -x_3 \\
x_6 & = & \color{red}{-6} & +x_0 & +x_1 & & -x_3 \\
x_7 & = & \color{red}{-7} & +x_0 & & & +x_3 \\
\hline
w & = & 0 & -x_0 & & &
\end{array}
$$

$$
\begin{array}{rcrrrrr}
x_0 & = & 7 & +x_7 & & & -x_3 \\
x_4 & = & 12 & +x_7 & -x_1 & +x_2 & -x_3 \\
x_5 & = & 21 & +x_7 & & -x_2 & -2x_3 \\
x_6 & = & 1 & +x_7 & +x_1 & & -2x_3 \\
\hline
w & = & -7 & -x_7 & & & +x_3
\end{array}
$$

x_3 enters and x_6 leaves

Dictionary D2

$$
\begin{array}{rcrrrrr}
x_3 & = & \frac{1}{2} & +\frac{1}{2}x_7 & +\frac{1}{2}x_1 & & -\frac{1}{2}x_6 \\
x_0 & = & \frac{13}{2} & +\frac{1}{2}x_7 & -\frac{1}{2}x_1 & & +\frac{1}{2}x_6 \\
x_4 & = & \frac{23}{2} & +\frac{1}{2}x_7 & -\frac{3}{2}x_1 & +x_2 & +\frac{1}{2}x_6 \\
x_5 & = & 20 & & -x_1 & -x_2 & +x_6 \\
\hline
w & = & -\frac{13}{2} & -\frac{1}{2}x_7 & +\frac{1}{2}x_1 & & -\frac{1}{2}x_6
\end{array}
$$

x_1 enters and x_4 leaves

Dictionary D3

$$
\begin{array}{rclrrrr}
x_1 & = & \frac{23}{3} & +\frac{1}{3}x_7 & -\frac{2}{3}x_4 & +\frac{2}{3}x_2 & +\frac{1}{3}x_6 \\
x_0 & = & \frac{8}{3} & +\frac{1}{3}x_7 & +\frac{1}{3}x_4 & -\frac{1}{3}x_2 & +\frac{1}{3}x_6 \\
x_3 & = & \frac{13}{3} & +\frac{2}{3}x_7 & -\frac{1}{3}x_4 & +\frac{1}{3}x_2 & -\frac{1}{3}x_6 \\
x_5 & = & \frac{37}{3} & -\frac{1}{3}x_7 & +\frac{2}{3}x_4 & -\frac{5}{3}x_2 & +\frac{2}{3}x_6 \\
\hline
w & = & -\frac{8}{3} & -\frac{1}{3}x_7 & -\frac{1}{3}x_4 & +\frac{1}{3}x_2 & -\frac{1}{3}x_6
\end{array}
$$

x_2 enters and x_5 leaves

Dictionary D4

$$
\begin{array}{lclllll}
x_1 & = & \frac{63}{5} & +\frac{1}{5}x_7 & -\frac{2}{5}x_4 & -\frac{2}{5}x_5 & +\frac{3}{5}x_6 \\
x_0 & = & \frac{1}{5} & +\frac{2}{5}x_7 & +\frac{2}{5}x_4 & +\frac{2}{5}x_5 & +\frac{2}{5}x_6 \\
x_3 & = & \frac{34}{5} & +\frac{3}{5}x_7 & -\frac{2}{5}x_4 & -\frac{2}{5}x_5 & -\frac{2}{5}x_6 \\
x_2 & = & \frac{37}{5} & -\frac{1}{5}x_7 & +\frac{2}{5}x_4 & -\frac{3}{5}x_5 & +\frac{2}{5}x_6 \\
\hline
w & = & -\frac{1}{5} & -\frac{2}{5}x_7 & -\frac{2}{5}x_4 & -\frac{2}{5}x_5 & -\frac{2}{5}x_6
\end{array}
$$

Aux. Problem

$$
\begin{array}{llllllll}
\max & -x_0 & & & & & \\
\text{s.t.} & x_1 & -x_2 & & \leq & 5 & +x_0 \\
 & & x_2 & +x_3 & \leq & 14 & +x_0 \\
 & -x_1 & & +x_3 & \leq & -6 & +x_0 \\
 & & & -x_3 & \leq & -7 & +x_0
\end{array}
$$

Least possible value of x_0 is $\frac{1}{5}$.

Infeasibility: Conclusion

$$\begin{array}{rrrrrr} \max & x_1 & +2x_2 & -x_3 & & \\ \text{s.t.} & x_1 & -x_2 & & \leq & 5 \\ & & x_2 & +x_3 & \leq & 14 \\ & -x_1 & & +x_3 & \leq & -6 \\ & & & -x_3 & \leq & -7 \end{array}$$

SIMPLEX: NUMBER OF POSSIBLE DICTIONARIES

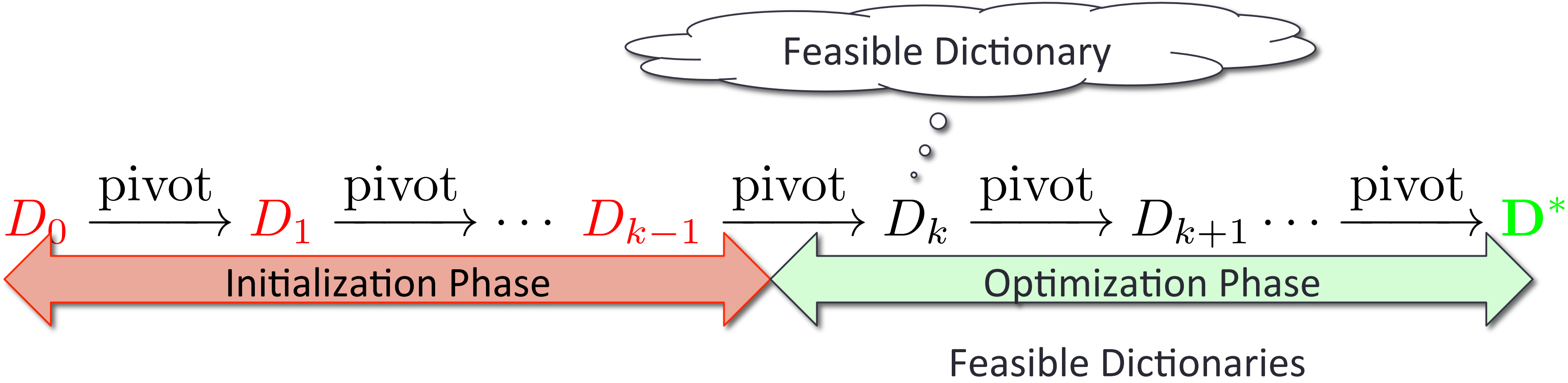
Original LP

n decision variables
m rows in A.

$$\begin{array}{lrcl} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ & A\mathbf{x} + \color{red}{\mathbf{x_s}} & = & \mathbf{b} \\ & \mathbf{x}, \color{red}{\mathbf{x_s}} & \geq & 0 \end{array}$$

Simplex Overview



Dictionary (Opt. Phase)

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix}$$

Basis $B = \{x_{b_1}, \ldots, x_{b_m}\}$

$$\begin{array}{rcl} \mathbf{x_B} & = & \mathbf{p} + R\,\mathbf{x_I} \\ \hline z & = & e_0 + \mathbf{e}^\intercal \mathbf{x_I} \end{array}$$

Overview

For any LP instance, the number of possible dictionaries is finite.

1. Every dictionary uniquely reconstructed from basis set.

2. # of basis set is upper bounded by $\binom{n+m}{m}$

DICTIONARY RECONSTRUCTION

Surgery ☺

DICTIONARY RECONSTRUCTION

Surgery ☺

Challenge

$$
\begin{array}{lrcl}
\max & \mathbf{c}^\intercal \ \mathbf{x} & & \\
 & A\mathbf{x} + \mathbf{x_s} & = & \mathbf{b} \\
 & \mathbf{x}, \mathbf{x_s} & \geq & 0
\end{array}
$$

$$B = \{x_{b_1}, \ldots, x_{b_m}\}$$

$$
\begin{array}{rcl}
\mathbf{x_B} & = & \mathbf{p} + R\ \mathbf{x_I} \\
\hline
z & = & e_0 + \mathbf{e}^\intercal \mathbf{x_I}
\end{array}
$$

Reconstruct Dictionary Given B

Example Problem

$$
\begin{array}{lrrrr}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$

Original Problem

$$
\begin{array}{lrrrrr}
\text{max.} & x_1 & +2x_2 & & & \\
\text{s.t.} & -3x_1 & +x_2 & \color{red}{+x_3} & = & 2 \\
 & & +x_2 & \color{red}{+x_4} & = & 11 \\
 & x_1 & -x_2 & \color{red}{+x_5} & = & 3 \\
 & x_1 & & \color{red}{+x_6} & = & 6 \\
 & & \multicolumn{2}{r}{x_1, x_2, \color{red}{x_3, \ldots, x_6}} & \geq & 0
\end{array}
$$

Problem with Slack

Example

$$
\begin{array}{lrrrcr}
\text{max.} & x_1 & +2x_2 & & & \\
\text{s.t.} & -3x_1 & +x_2 & \color{red}{+x_3} & = & 2 \\
 & & +x_2 & \color{red}{+x_4} & = & 11 \\
 & x_1 & -x_2 & \color{red}{+x_5} & = & 3 \\
 & x_1 & & \color{red}{+x_6} & = & 6 \\
 & \multicolumn{3}{r}{x_1, x_2, \color{red}{x_3, \ldots, x_6}} & \geq & 0
\end{array}
$$

$$B = \{1, 2, 5, 6\}.$$

Example (Desired Goal)

$$
\begin{array}{rcccc}
x_1 & = & ? & +?x_3 & +?x_4 \\
x_2 & = & ? & +?x_3 & +?x_4 \\
x_5 & = & ? & +?x_3 & +?x_4 \\
x_6 & = & ? & +?x_3 & +?x_4 \\
\hline
z & = & ? & +?x_3 & +?x_4
\end{array}
$$

$$
\boxed{
\begin{array}{lccccc}
\text{max.} & x_1 & +2x_2 & & & \\
\text{s.t.} & -3x_1 & +x_2 & +x_3 & = & 2 \\
& & +x_2 & +x_4 & = & 11 \\
& x_1 & -x_2 & +x_5 & = & 3 \\
& x_1 & & +x_6 & = & 6 \\
& & & x_1, x_2, x_3, \ldots, x_6 & \geq & 0
\end{array}
}
$$

$$B = \{1, 2, 5, 6\}.$$

Key Principle

Dictionary is just a restatement of the problem.

We express basic variables in terms of non-basic.

Dict. Reconstruction (Step 1)

$$B = \{1, 2, 5, 6\}.$$

$$\begin{array}{lrrrrrl}
\text{max.} & x_1 & +2x_2 & & & \\
\text{s.t.} & -3x_1 & +x_2 & +x_3 & = & 2 \\
 & & +x_2 & +x_4 & = & 11 \\
 & x_1 & -x_2 & +x_5 & = & 3 \\
 & x_1 & & +x_6 & = & 6 \\
 & & x_1, x_2, & x_3, \ldots, x_6 & \geq & 0
\end{array}$$

$$\begin{array}{rrrrcrrr}
-3x_1 & +x_2 & & & = & 2 & -x_3 & \\
 & x_2 & & & = & 11 & & -x_4 \\
x_1 & -x_2 & +x_5 & & = & 3 & & \\
x_1 & & & +x_6 & = & 6 & &
\end{array}$$

Dict. Reconstruction (Step 2)

$$\boxed{\begin{array}{ccccccc} -3x_1 & +x_2 & & & = & 2 & -x_3 & \\ & x_2 & & & = & 11 & & -x_4 \\ x_1 & -x_2 & +x_5 & & = & 3 & & \\ x_1 & & & +x_6 & = & 6 & & \end{array}}$$

$$\begin{pmatrix} -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 2 \\ 11 \\ 3 \\ 6 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

Dict Reconstruction (Step 3)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}^{-1} \left[\begin{pmatrix} 2 \\ 11 \\ 3 \\ 6 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \right]$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & -\frac{1}{3} & 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 2 \\ 11 \\ 3 \\ 6 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \right]$$

Dict. Reconstruction

$$\begin{pmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ 11 \\ 3 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ 0 & -1 \\ \frac{-1}{3} & \frac{-2}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{pmatrix} \times \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

Dictionary Reconstruction (Final Result)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ 11 \\ 3 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ 0 & -1 \\ \frac{-1}{3} & \frac{-2}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{pmatrix} \times \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

$$\begin{array}{lcllll} x_1 & = & 3 & +\frac{1}{3}x_3 & -\frac{1}{3}x_4 \\ x_2 & = & 11 & +0x_3 & -x_4 \\ x_5 & = & 11 & -\frac{1}{3}x_3 & -\frac{2}{3}x_4 \\ x_6 & = & 3 & -\frac{1}{3}x_3 & +\frac{1}{3}x_4 \\ \hline z & = & & & & //\mathrm{ORIG}:\ x_1 + 2x_2 \end{array}$$

Dict. Reconstruction

$$\boxed{\begin{array}{llll} \max & \mathbf{c}^\intercal \ \mathbf{x} & & \\ & A\mathbf{x} + \mathbf{x_s} & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x_s} & \geq & 0 \end{array}}$$

$$\text{Basis set: } B = \{x_{b1}, \ldots, x_{bm}\}.$$

Splitting the A matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1,b1} & \cdots & a_{1,b2} & \cdots & a_{1,bm} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,b1} & \cdots & a_{2,b2} & \cdots & a_{2,bm} & \cdots & a_{2n} \\ & \vdots & & \vdots & & \ddots & & & \\ a_{m1} & \cdots & a_{m,b1} & \cdots & a_{m,b2} & \cdots & a_{m,bm} & \cdots & a_{mn} \end{pmatrix} \times \begin{pmatrix} x_1 \\ \vdots \\ x_{b1} \\ \vdots \\ x_{b2} \\ \vdots \\ x_{bm} \\ \vdots \\ x_m \end{pmatrix}$$

$$A\mathbf{x} = A_B\mathbf{x_B} + A_I\mathbf{x_I}$$

Rewriting the Equation

$$A\mathbf{x} + \mathbf{x_s} = \mathbf{b}$$

Step 1

$$A_B\mathbf{x_B} + A_I\mathbf{x_I} = b$$

Step 2

$$A_B\ \mathbf{x_B} = \mathbf{b} - A_I\mathbf{x_I}$$

Is A_B always invertible?

Dictionary Reconstruction

$$A_B\ \mathbf{x_B} = \mathbf{b} - A_I \mathbf{x_I}$$

$$\mathbf{x_B} = A_B^{-1}\mathbf{b} - A_B^{-1} A_I \mathbf{x_I}$$

Is A_B always invertible?

Result Dictionary

$$\mathbf{c}^\intercal\ \mathbf{x} = \mathbf{c_B}^\intercal \mathbf{x_B} + \mathbf{c_I}^\intercal \mathbf{x_I}$$

$$\begin{array}{ccll}
\mathbf{x_B} & = & A_B^{-1}\mathbf{b} & -A_B^{-1}A_I\mathbf{x_I} \\
\hline
\mathbf{c} & = & \mathbf{c_B}^\intercal\ A_B^{-1}\mathbf{b} & +\left(-\mathbf{c_B}^\intercal A_B^{-1}A_I + \mathbf{c_I}^\intercal\right)\ \mathbf{x_I}
\end{array}$$

Claim #1

Given Original Problem +
Desired Basis

Can reconstruct dictionary uniquely.

Key Insight # 2

For any given problem, number of possible dictionaries is finite.

$$\#\text{Dictionaries} \;\leq\; \#\text{Basis Set}$$

$$\#\text{Basis Set} \;=\; \binom{n+m}{m}$$

REVISED SIMPLEX METHOD

Description of Pivoting

Revised Simplex: Partial Reconstruction

1. Solution for Basic Variables

3. Column for entering Variable

$$
\begin{array}{lclcccccccc}
x_{B1} & = & b_1 & & & & & + a_{1j} x_{Ij} & & & \\
x_{B2} & = & b_2 & & & & & + a_{2j} x_{Ij} & & & \\
 & \vdots & & & & & & & & & \\
x_{Bm} & = & b_m & & & & & + a_{mj} x_{Ij} & & & \\
\hline
z & = & c_0 & + c_1 x_{I1} & & + \cdots & & + c_j x_{Ij} & & + \cdots & + c_n x_{In}
\end{array}
$$

2. Objective row coefficients

Choosing Entering Variable

$$\begin{array}{ccc} \mathbf{x}_B & = & \hat{\mathbf{b}} & +\hat{A}\mathbf{x}_I \\ \hline z & = & z_0 & \boxed{+\hat{c}\ \mathbf{x}_I} \end{array}$$

Q: How do we compute the objective coefficients (aka reduced costs)?

Computing Objective Coefficients

- Original Problem Data: *A, b, c*
- Current Basis Set: $B = \{x_{B1}, x_{B2}, \ldots, x_{Bm}\}$

Choosing Entering Variable

$$
\begin{array}{ccccccccc}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} \\
 & \vdots & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In}
\end{array}
$$

$$\pi = \mathbf{c_B}^\intercal A_B^{-1}$$

$$\hat{\mathbf{c}} = \mathbf{c}_I^\intercal - \pi A_I$$

Computing the Objective Row

- Compute $\pi = \mathbf{c}_B^{\mathsf{T}} A_B^{-1}$

  by solving the system of equations

$$\boxed{\pi A_B = \mathbf{c}_B^{\mathsf{T}}}$$

- Obtain obj. row coefficients by computing

$$\boxed{\hat{\mathbf{c}} = \mathbf{c}_I^{\mathsf{T}} - \pi A_I}$$

- Objective value is $z = \pi \mathbf{b}$

Entering Variable Analysis

$$z = c_0 + c_1 x_{I1} + \cdots + c_j x_{Ij} + \cdots + c_n x_{In} \Leftarrow \hat{\mathbf{c}}$$

Choose positive coefficient c_j

If no such coefficient, then dictionary is final.

Leaving Variable Analysis

$$
\begin{array}{ccccccc}
x_{B1} & = & b_1 & & & +a_{1j}x_{Ij} & & \\
x_{B2} & = & b_2 & & & +a_{2j}x_{Ij} & & \\
 & \vdots & & & & & & \\
x_{Bm} & = & b_m & & & +a_{mj}x_{Ij} & & \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In}
\end{array}
$$

$$\hat{\mathbf{b}} \text{ and } \hat{\mathbf{a}}_j$$

Leaving Variable Analysis

- Compute $\hat{\mathbf{b}} = A_B^{-1}\mathbf{b}$ by solving the equations

$$A_B\hat{\mathbf{b}} = \mathbf{b}$$

- Compute $\hat{\mathbf{a}}_j = -A_B^{-1}A_j$ by solving the equations

$$A_B\hat{\mathbf{a}}_j = -A_j$$

We have enough data to perform leaving variable analysis.

REVISED SIMPLEX: EXAMPLE

Example Problem

$$
\begin{array}{lccccccccccc}
\max & x_1 & -x_2 & +x_3 & -x_4 & & & & & & \\
 & 2x_1 & -3x_2 & +7x_3 & -15x_4 & +x_5 & & & & = & 10 \\
 & & x_2 & -4x_3 & +6x_4 & & +x_6 & & & = & 12 \\
 & -x_1 & & +x_3 & -2x_4 & & & +x_7 & & = & 4 \\
 & & x_2 & +x_3 & & & & & +x_8 & = & 16 \\
 & & & & & & & x_1, \ldots, x_8 & \geq & 0 &
\end{array}
$$

$$
A = \begin{bmatrix} 2 & -3 & 7 & -15 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 10 \\ 12 \\ 4 \\ 16 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Example Problem

$$A = \begin{bmatrix} 2 & -3 & 7 & -15 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 10 \\ 12 \\ 4 \\ 16 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B = \{5, 6, 7, 8\}$$

Constructing Objective Coefficient

$$A_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_I = \begin{bmatrix} 2 & -3 & 7 & -15 \\ 0 & 1 & -4 & 6 \\ -1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{c}_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{c}_I = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\pi = \mathbf{c}_B^\intercal A_B^{-1} = [0\ 0\ 0\ 0]$$

$$\hat{\mathbf{c}} = \mathbf{c}_I^\intercal - \pi A_I = [1\ \ -1\ 1\ \ -1]$$

Constructing dictionary

$$\boxed{B = \{5, 6, 7, 8\}}$$

- Choose x_3 as entering variable ($j = 3$).
- Leaving variable analysis requires $\hat{\mathbf{b}}$ and $\hat{\mathbf{a}}_j$

$$\hat{\mathbf{b}} = A_B^{-1}\mathbf{b} = \begin{bmatrix} 10 \\ 12 \\ 4 \\ 16 \end{bmatrix} \qquad \hat{\mathbf{a}}_j = -A_B^{-1}A_j = \begin{bmatrix} -7 \\ 4 \\ -1 \\ -1 \end{bmatrix}$$

Result

$$A = \begin{bmatrix} 2 & -3 & 7 & -15 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 10 \\ 12 \\ 4 \\ 16 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B = \{5, 6, 7, 8\}$$

Entering Variable is x_3 Leaving Variable is x_5

$$B = \{3, 6, 7, 8\}$$

Second Dictionary

$B = \{3, 6, 7, 8\}$

$$A = \begin{bmatrix} 2 & -3 & 7 & -15 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 10 \\ 12 \\ 4 \\ 16 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Calculations for Pivoting

Ab =

7	0	0	0
-4	1	0	0
1	0	1	0
1	0	0	1

pl = [0.2857 0 0 0]
chat = [0.4286 -0.1429 -0.2857 3.2857]

Entering Index = 4

Ai =

2	-3	1	-15
0	1	0	6
-1	0	0	-2
0	1	0	0

bhat =
[1.4286
17.7143
2.5714
14.5714]

Aj =
[2.1429
2.5714
-0.1429
-2.1429]

Leaving Index = 8

Third Dictionary

$$A = \begin{bmatrix} 2 & -3 & 7 & -15 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 10 \\ 12 \\ 4 \\ 16 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B = \{3, 6, 7, 8\}$$

Variable x_4 enters and x_8 leaves

$$B = \{3, 6, 7, {\color{red}4}\}$$

Calculations for Pivot

Ab =

7	0	0	-15
-4	1	0	6
1	0	1	-2
1	0	0	0

pI = [0.0667 0 0 1.5333]

cHat = [0.8667 -2.3333 -0.0667 -1.5333]

Entering Index: 1

Ai =

2	-3	1	0
0	1	0	0
-1	0	0	0
0	1	0	1

bHat =	Aj =
16.0000	0
35.2000	-0.8000
1.6000	1.2667
6.8000	0.1333

Leaving Index: 6

Third Dictionary

$$A = \begin{bmatrix} 2 & -3 & 7 & -15 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 10 \\ 12 \\ 4 \\ 16 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B = \{3, 6, 7, 4\}$$

Variable x_1 enters and x_6 leaves

$$B = \{3, \textcolor{red}{1}, 7, 4\}$$

Pivot Calculations

Ab =

7	2	0	-15
-4	0	0	6
1	-1	1	-2
1	0	0	0

Ai =

0	-3	1	0
1	1	0	0
0	0	0	0
0	1	0	1

pII =

0.5000	1.0833	0	2.8333

chat =

-1.0833	-3.4167	-0.5000	-2.8333

Final Dictionary. Objective Value: 47.333

DUALITY: BASIC IDEAS

Dual Problem

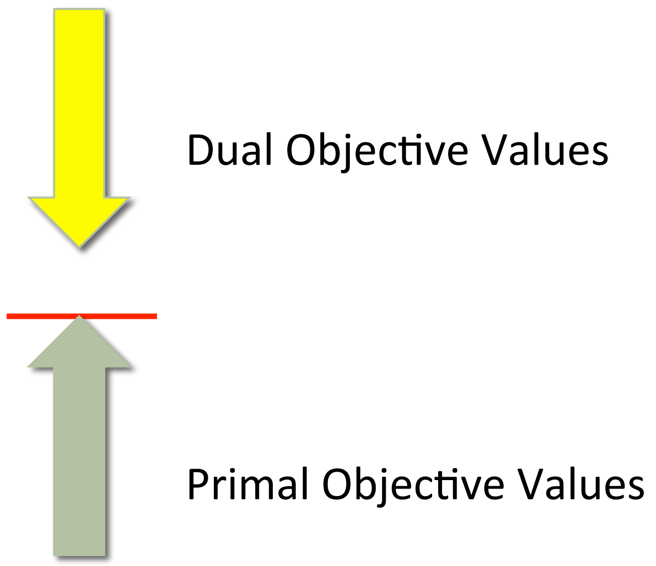
Powerful way of viewing optimization problem

$$\begin{array}{llll} \max & f(\vec{x}) & & \\ \text{s.t.} & C(\vec{x}) & \leq & \vec{d} \end{array} \xrightarrow{\text{Dualize}} \boxed{\begin{array}{llll} \min & g(\vec{y}) & & \\ \text{s.t.} & P(\vec{y}) & \leq & \vec{q} \end{array}}$$

General Opt. Problem
(Primal Problem)

Dual Problem

Dualization Motivation

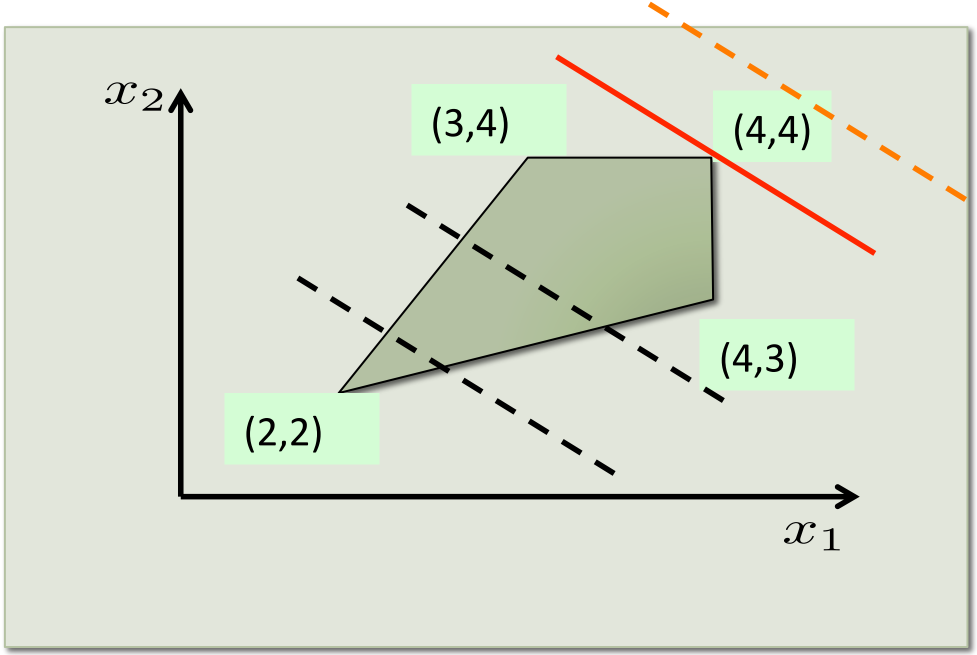


Get upper bounds to primal optimum and obtain best upper bound

Example

$$
\begin{array}{lrcr}
\text{max.} & x_1 + 2x_2 & & \\
\text{s.t.} & -2x_1 + x_2 & \leq & -2 \\
 & x_2 & \leq & 4 \\
 & x_1 - 2x_2 & \leq & -2 \\
 & x_1 & \leq & 4 \\
 & x_1, x_2 & \geq & 0
\end{array}
$$

Optimal Solution: z = 12



Example

$$
\begin{array}{rrcl}
\text{max.} & x_1 + 2x_2 & & \\
\text{s.t.} & -2x_1 + x_2 & \leq & -2 \\
& x_2 & \leq & 4 \\
& x_1 - 2x_2 & \leq & -2 \\
& x_1 & \leq & 4 \\
& x_1, x_2 & \geq & 0
\end{array}
$$

$$
\begin{array}{rcrrcr}
-2x_1 + x_2 \leq -2 & \Rightarrow & -4x_1 & +2x_2 & \leq & -4 \\
x_1 \leq 4 & \Rightarrow & 5x_1 & & \leq & 20 \\
\hline
& & x_1 & +2x_2 & \leq & 16
\end{array}
$$

Example

$$
\begin{array}{rrcl}
\text{max.} & x_1 + 2x_2 & & \\
\text{s.t.} & -2x_1 + x_2 & \leq & -2 \\
& x_2 & \leq & 4 \\
& x_1 - 2x_2 & \leq & -2 \\
& x_1 & \leq & 4 \\
& x_1, x_2 & \geq & 0
\end{array}
$$

$$
\begin{array}{lcllcl}
x_1 \leq 4 & \Rightarrow & x_1 & & \leq & 4 \\
x_2 \leq 4 & \Rightarrow & & 2x_2 & \leq & 8 \\
\hline
& & x_1 & +2x_2 & \leq & 12
\end{array}
$$

General Principle (Attempt 1 of 2)

$$
\begin{array}{llllllll}
\max\ z = & c_1x_1 & +c_2x_2 & +\cdots+ & c_nx_n & & & \\
 & a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 & \leftarrow {\color{red} e_1 \leq b_1} \\
 & a_{21}x_1 & +a_{22}x_2 & +\cdots+ & a_{2n}x_n & \leq & b_2 & \leftarrow {\color{red} e_2 \leq b_2} \\
 & \vdots & & \ddots & & \vdots & & \\
 & a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m & \leftarrow {\color{red} e_m \leq b_m} \\
 & & & & x_1, \ldots, x_n & \geq & 0 &
\end{array}
$$

$$c_1x_1 + \ldots + c_mx_m \equiv y_1 \times e_1 + y_2 \times e_2 + \cdots + y_m \times e_m$$

$$z^* \leq y_1b_1 + y_2b_2 + \cdots + y_mb_m$$

$$y_1, y_2, \ldots, y_m \geq 0$$

Example

$$
\begin{array}{lrrrcr}
\max & -2x_1 & +3x_2 & +x_3 & & \\
 & x_1 & -x_2 & +x_3 & \leq & 10 \\
 & 2x_1 & -3x_2 & -x_3 & \leq & 10 \\
 & & +6x_2 & +2x_3 & \leq & 10 \\
 & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

General Principle (Attempt 1 of 2)

$$
\begin{array}{lccccccl}
\max\ z = & c_1x_1 & +c_2x_2 & +\cdots+ & c_nx_n & & & \\
& a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 & \leftarrow \color{red}{e_1 \leq b_1} \\
& a_{21}x_1 & +a_{22}x_2 & +\cdots+ & a_{2n}x_n & \leq & b_2 & \leftarrow \color{red}{e_2 \leq b_2} \\
& \vdots & & \ddots & & \vdots & & \\
& a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m & \leftarrow \color{red}{e_m \leq b_m} \\
& -x_1 & & & & \leq & 0 & \\
& & -x_2 & & & \leq & 0 & \\
& & & \ddots & & & & \\
& & & & -x_n & \leq & 0 &
\end{array}
$$

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n \ \equiv\ \sum_{i=1}^{m} y_i(e_i) + y_{m+1}(-x_1) + \cdots + y_{n+m}(-x_n)$$

Example

$$\begin{array}{lcccrr}
\max & -2x_1 & +3x_2 & +x_3 & & \\
 & x_1 & -x_2 & +x_3 & \leq & 10 \\
 & 2x_1 & -3x_2 & -x_3 & \leq & 10 \\
 & & +6x_2 & +2x_3 & \leq & 10 \\
 & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$\begin{array}{lcccrr}
\max & -2x_1 & +3x_2 & +x_3 & & \\
 & x_1 & -x_2 & +x_3 & \leq & 10 \\
 & 2x_1 & -3x_2 & -x_3 & \leq & 10 \\
 & & +6x_2 & +2x_3 & \leq & 10 \\
 & -x_1 & & & \leq & 0 \\
 & & -x_2 & & \leq & 0 \\
 & & & -x_3 & \leq & 0
\end{array}$$

General Principle

$$
\begin{array}{lccccccl}
\max\ z = & c_1x_1 & +c_2x_2 & +\cdots+ & c_nx_n & & & \\
\\
& a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 & \leftarrow {\color{red}\times\ y_1} \\
& a_{21}x_1 & +a_{22}x_2 & +\cdots+ & a_{2n}x_n & \leq & b_2 & \leftarrow {\color{red}\times\ y_2} \\
& \vdots & & \ddots & & \vdots & & \\
& a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m & \leftarrow {\color{red}\times\ y_m} \\
& -x_1 & & & & \leq & 0 & \leftarrow \times\ y_{m+1} \\
& & -x_2 & & & \leq & 0 & \leftarrow \times\ y_{m+2} \\
& & & \ddots & & & & \\
& & & & -x_n & \leq & 0 & \leftarrow \times\ y_{m+n}
\end{array}
$$

Valid Upper Bounds

$$
\begin{array}{cccccccccc}
a_{11}y_1 & +a_{21}y_2 & +\cdots & +a_{m1}y_m & -y_{m+1} & & & & = & c_1 \\
a_{12}y_2 & +a_{22}y_2 & +\cdots & +a_{m2}y_m & & -y_{m+2} & & & = & c_2 \\
\vdots & & \ddots & & & & \ddots & & & \\
a_{1n}y_1 & +a_{2n}y_2 & +\cdots & +a_{mn}y_m & & & & -y_{m+n} & = & c_n \\
y_1, & y_2, & \cdots & y_m, & y_{m+1}, & y_{m+2}, & \cdots & y_{m+n} & \geq & 0
\end{array}
$$

$$\boxed{b_1y_1 + b_2y_2 + \cdots + b_my_m}$$

Matrix Form.

$$\begin{array}{rcl} \max & \mathbf{c}^\intercal \mathbf{x} & \\ & A\,\mathbf{x} \leq & \mathbf{b} \\ & \mathbf{x} \geq & 0 \end{array}$$

Original Problem

$$\begin{array}{rcl} A^\intercal \mathbf{y} \;\; -\mathbf{y_s} & = & c \\ \mathbf{y} & \geq & 0 \\ \mathbf{y_s} & \geq & 0 \end{array}$$

Yields Upper Bound: $\mathbf{b}^\intercal \mathbf{y}$

Most Stringent Upper Bound

$$\boxed{\begin{array}{llll} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ & A\,\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}} \Rightarrow \boxed{\begin{array}{llll} A^\intercal \mathbf{y} & -\mathbf{y_s} & = & c \\ \mathbf{y} & & \geq & 0 \\ & \mathbf{y_s} & \geq & 0 \end{array}} \Rightarrow \boxed{\mathbf{b}^\intercal \mathbf{y}}$$

$$\begin{array}{lllll} \min & \mathbf{b}^\intercal \mathbf{y} & & & \\ & A^\intercal\,\mathbf{y} & -\mathbf{y_s} & = & \mathbf{c} \\ & \mathbf{y} & & \geq 0 & \\ & & \mathbf{y_s} & \geq & 0 \end{array}$$

Dual Problem

Dual Problem (cont)

$$\boxed{\begin{array}{llll} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ & A\,\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}}$$

$$\begin{array}{lllll} \min & \mathbf{b}^\intercal \mathbf{y} & & & \\ & A^\intercal\,\mathbf{y} & -\mathbf{y_s} & = & \mathbf{c} \\ & \mathbf{y} & & \geq 0 & \\ & & \mathbf{y_s} & \geq & 0 \end{array}$$

$$\begin{array}{llll} \min & \mathbf{b}^\intercal \mathbf{y} & & \\ & A^\intercal\,\mathbf{y} & \geq & \mathbf{c} \\ & \mathbf{y} & \geq 0 & \end{array}$$

Dual Problem (Example)

$$\begin{array}{lrrrcl}
\max & -2x_1 & +3x_2 & +x_3 & & \\
& x_1 & -x_2 & +x_3 & \leq & 10 \\
& 2x_1 & -3x_2 & -x_3 & \leq & 10 \\
& & +6x_2 & +2x_3 & \leq & 10 \\
& & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

Dual Problem (Example)

$$
\begin{array}{lrrrcr}
\max & -2x_1 & +3x_2 & +x_3 & & \\
 & x_1 & -x_2 & +x_3 & \leq & 10 \\
 & 2x_1 & -3x_2 & -x_3 & \leq & 10 \\
 & & +6x_2 & +2x_3 & \leq & 10 \\
 & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Example

- Dual Decision Variables $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$\begin{array}{lllllll} \max & -2x_1 & +3x_2 & +x_3 & & \\ & x_1 & -x_2 & +x_3 & \leq & 10 \\ & 2x_1 & -3x_2 & -x_3 & \leq & 10 \\ & & +6x_2 & +2x_3 & \leq & 10 \\ & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

- Dual Problem:

$$\begin{array}{llllll} \min & 10y_1 & +10y_2 & +10y_3 & & \\ & y_1 & +2y_2 & & \geq & -2 \\ & -y_1 & -3y_2 & +6y_3 & \geq & 3 \\ & y_1 & -y_2 & +2y_3 & \geq & 1 \\ & y_1, & y_2, & y_3 & \geq & 0 \end{array}$$

Example-2

$$\begin{array}{rrcr} \text{max.} & x_1 + 2x_2 & & \\ \text{s.t.} & -2x_1 + x_2 & \leq & -2 \\ & x_2 & \leq & 4 \\ & x_1 - 2x_2 & \leq & -2 \\ & x_1 & \leq & 4 \\ & x_1, x_2 & \geq & 0 \end{array}$$

Example-3

$$
\begin{array}{rrcl}
\text{maximize} & 2x_1 + 3x_2 - 5x_3 & & \\
\text{s.t.} & x_1 - x_2 & \leq & 5 \\
& -x_1 + \qquad x_3 & \leq & 6 \\
& -2x_1 + \qquad x_3 & \leq & 2 \\
& -x_1 + x_2 & \leq & 4 \\
& x_1, x_2, x_3 & \geq & 0
\end{array}
$$

DUAL LINEAR PROGRAM: WEAK/STRONG DUALITY

Linear Program (Dual)

$$\begin{array}{llll} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ & A\,\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}$$

Dual Problem

$$\begin{array}{llll} \min & \mathbf{b}^\intercal \mathbf{y} & & \\ & A^\intercal\,\mathbf{y} & \geq & \mathbf{c} \\ & \mathbf{y} & \geq & 0 \end{array}$$

Standard Form Converted Dual

$$\begin{array}{llll} \max & -\mathbf{b}^\intercal \mathbf{y} & & \\ & -A^\intercal\,\mathbf{y} & \leq & -\mathbf{c} \\ & \mathbf{y} & \geq & 0 \end{array}$$

Primal vs. Dual

$$\boxed{\begin{array}{llll} \max & \mathbf{c}^{\intercal}\mathbf{x} & & \\ & A\,\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}}$$

- n variables
- m constraints
- Problem Matrix: A
- Objective vector: c (max)
- RHS of inequality: b (<=)

$$\boxed{\begin{array}{llll} \min & \mathbf{b}^{\intercal}\mathbf{y} & & \\ & A^{\intercal}\,\mathbf{y} & \geq & \mathbf{c} \\ & \mathbf{y} & \geq & 0 \end{array}}$$

- m variables
- n constraints
- Problem Matrix: A^T
- Objective Vector: b (min)
- RHS of inequality: c (>=)

Weak Duality Theorem

$$\boxed{\begin{aligned} \max \quad & \mathbf{c}^\intercal \mathbf{x} \\ & A\,\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}}$$

$$\boxed{\begin{aligned} \min \quad & \mathbf{b}^\intercal \mathbf{y} \\ & A^\intercal\,\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{aligned}}$$

Let **y** be any dual feasible solution and let **x** be any primal feasible solution.

$$\mathbf{b}^\intercal\,\mathbf{y} \;\geq\; \mathbf{c}^\intercal\mathbf{x}$$

Weak Duality Theorem

Let **y*** be an optimal solution for the dual and **x*** be an optimal solution to the primal.

$$\mathbf{b}^\intercal\ \mathbf{y}^* \ \geq\ \mathbf{c}^\intercal \mathbf{x}^*$$

Another Useful Result

- Let **x** be any primal feasible solution and **y** be any dual feasible solution such that

$$\mathbf{b}^\intercal\, \mathbf{y} = \ \mathbf{c}^\intercal \mathbf{x}$$

  - **x** is primal optimal, and
  - **y** is dual optimal.

Strong Duality

- Let $\mathbf{x}^*$ be a primal optimal solution for an LP.
  - The dual problem has an optimal solution, and
  - Any dual optimal solution $\mathbf{y}^*$ satisfies

$$\mathbf{b}^\intercal \mathbf{y}^* = \mathbf{c}^\intercal \mathbf{x}^*$$

Unboundedness of Primal/Dual

- If Primal unbounded then dual is infeasible.
- If Dual unbounded then primal is infeasible.

Relation between Primal and Dual Problems

	Infeasible	Unbounded	Optimal
Infeasible	Possible	Possible	
Unbounded	Possible		
Optimal			Possible

If Primal is unbounded then dual is infeasible.
If Dual is unbounded then primal is infeasible.

Both problems can be infeasible

Both Primal and Dual Infeasible.

$$\begin{array}{lrrcr} \max & & x_2 & & \\ & x_1 & & \leq & -1 \\ & & -x_2 & \leq & -1 \\ & x_1 \; , & x_2 & \geq & 0 \end{array}$$

$$\begin{array}{lrrcr} \min & -y_1 & -y_2 & & \\ & y_1 & & \geq & 0 \\ & & -y_2 & \geq & 1 \\ & y_1 \; , & y_2 & \geq & 0 \end{array}$$

Dual Certificate

$$\boxed{\begin{array}{llll} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ & A\,\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}}$$

Someone tells us $\mathbf{x}^*$ is an optimal solution.

Do we trust them?

- Check if x^* is feasible?
- **How to check if it is optimal?**

Dual Certificates

- Given x (Claimed to be primal optimal solution) and
- Given y (Claimed to be dual optimal solution).

We can use both to convince ourselves.

1. Check feasibility of x using primal problem
2. Form dual problem and check feasibility of y
3. Check that primal objective value is equal to dual objective value.

Example

$$\boxed{\begin{array}{lccc} \text{max.} & x_1 + 2x_2 & & \\ \text{s.t.} & -2x_1 + x_2 & \leq & -2 \\ & x_2 & \leq & 4 \\ & x_1 - 2x_2 & \leq & -2 \\ & x_1 & \leq & 4 \\ & x_1, x_2 & \geq & 0 \end{array}}$$

$$\begin{array}{lcccccc} \min & -2y_1 & +4y_2 & -2y_3 & +4y_4 & & \\ & -2y_1 & & +y_3 & +y_4 & \geq & 1 \\ & y_1 & +y_2 & -2y_3 & & \geq 2 & \\ & y_1, & y_2, & y_3, & y_4 & \geq & 0 \end{array}$$

$$\boxed{x_1 = 4,\ x_2 = 4,\ z = 12}$$

$$\boxed{y_1 = 0, y_2 = 2, y_3 = 0, y_4 = 1}$$

PRIMAL/DUAL CORRESPONDENCE

Complementary Pairs

Primal vs. Dual

$$\boxed{\begin{array}{llll} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ & A\,\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}}$$

$$\boxed{\begin{array}{llll} \min & \mathbf{b}^\intercal \mathbf{y} & & \\ & A^\intercal\,\mathbf{y} & \geq & \mathbf{c} \\ & \mathbf{y} & \geq & 0 \end{array}}$$

$$\begin{array}{lllll} \max & \mathbf{c}^\intercal \mathbf{x} & & & \\ & A\,\mathbf{x} & +\mathbf{x_s} & = & \mathbf{b} \\ & \mathbf{x} & & \geq & 0 \\ & & \mathbf{x_s} & \geq & 0 \end{array}$$

$$\begin{array}{lllll} \min & \mathbf{b}^\intercal \mathbf{y} & & & \\ & A^\intercal\,\mathbf{y} & -\mathbf{y_s} & = & \mathbf{c} \\ & \mathbf{y} & & \geq & 0 \\ & & \mathbf{y_s} & \geq & 0 \end{array}$$

Dual of Dual is the Primal

Complementary Variable Pairs

Primal Decision Variables	x_1	$\longleftrightarrow$	y_{m+1}	Dual Slack Variables
	x_2	$\longleftrightarrow$	y_{m+2}	
	$\vdots$		$\vdots$	
	x_n	$\longleftrightarrow$	y_{m+n}	
Primal Slack Variables	x_{n+1}	$\longleftrightarrow$	y_1	Dual Decision Variables
	$\vdots$		$\vdots$	
	x_{n+m}	$\longleftrightarrow$	y_m	

Complementary Pairs (Example)

$$
\begin{array}{lccccccc}
\max & 2x_1 & +3x_2 & -x_3 & & & & \\
 & x_1 & -x_2 & & \leq & 5 & \leftarrow & y_1 \\
 & 2x_2 & +x_2 & -x_3 & \leq & -1 & \leftarrow & y_2 \\
 & x_1 & -x_2 & +x_3 & \leq & 2 & \leftarrow & y_3 \\
 & x_1 & +x_2 & -x_3 & \leq & 1 & \leftarrow & y_4 \\
 & -x_1 & & & \leq & 0 & \leftarrow & \color{red}{y_5} \\
 & & -x_2 & & \leq & 0 & \leftarrow & \color{red}{y_6} \\
 & & & -x_3 & \leq & 0 & \leftarrow & \color{red}{y_7}
\end{array}
$$

DUAL DICTIONARIES

Primal vs. Dual

$$\begin{array}{llll} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ & A\,\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}$$

$$\begin{array}{llll} \min & \mathbf{b}^\intercal \mathbf{y} & & \\ & A^\intercal\,\mathbf{y} & \geq & \mathbf{c} \\ & \mathbf{y} & \geq & 0 \end{array}$$

$$\begin{array}{llll} \max & -\mathbf{b}^\intercal \mathbf{y} & & \\ \text{s.t.} & -A^\intercal \mathbf{y} & \leq & -\mathbf{c} \\ & \mathbf{y} & \geq & 0 \end{array}$$

Complementary Variable Pairs

Primal Decision Variables

Dual Slack Variables

$$\begin{array}{ccc} x_1 & \longleftrightarrow & y_{m+1} \\ x_2 & \longleftrightarrow & y_{m+2} \\ \vdots & & \vdots \\ x_n & \longleftrightarrow & y_{m+n} \\ x_{n+1} & \longleftrightarrow & y_1 \\ \vdots & & \vdots \\ x_{n+m} & \longleftrightarrow & y_m \end{array}$$

Primal Slack Variables

Dual Decision Variables

Example

$$\begin{array}{lrrcr}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
& & +x_2 & \leq & 11 \\
& x_1 & -x_2 & \leq & 3 \\
& x_1 & & \leq & 6 \\
& x_1, & x_2 & \geq & 0
\end{array}$$

$$\begin{array}{lrrrrcr}
\max & -2y_1 & -11y_2 & -3y_3 & -6y_4 & & \\
& 3y_1 & & -y_3 & -y_4 & \leq & -1 \\
& -y_1 & -y_2 & +y_3 & & \leq & -2 \\
& y_1, & y_2, & y_3, & y_4, & \geq & 0
\end{array}$$

x_1	x_2	x_3	x_4	x_5	x_6
y_5	y_6	y_1	y_2	y_3	y_4

Primal vs. Dual Dictionaries

$$
\begin{array}{rclll}
x_3 & = & 2 & +3x_1 & -x_2 \\
x_4 & = & 11 & +0x_1 & -x_2 \\
x_5 & = & 3 & -x_1 & +x_2 \\
x_6 & = & 6 & -x_1 & +0x_2 \\
\hline
z & = & 0 & +x_1 & +2x_2
\end{array}
$$

x_1	x_2	x_3	x_4	x_5	x_6
y_5	y_6	y_1	y_2	y_3	y_4

Primal Problem Dictionary

Dual Problem Dictionary

$$
\begin{array}{c|cccccc}
y_5 & -1 & -3y_1 & & +y_3 & +y_4 \\
y_6 & -2 & +y_1 & +y_2 & -y_3 & \\
\hline
d & 0 & -2y_1 & -11y_2 & -3y_3 & -6y_4
\end{array}
$$

Dual Dictionary

$$\begin{array}{c|cc} \mathbf{x_B} & \mathbf{b} & +A\mathbf{x_I} \\ \hline z & z_0 & +\mathbf{c}^\intercal \mathbf{x_I} \end{array}$$

Primal Problem Dictionary

$\mathbf{x}$	$\mathbf{x_s}$
$\mathbf{y_s}$	$\mathbf{y}$

$$\begin{array}{c|cc} {\mathbf{x_I}}^c & -\mathbf{c} & -A^\intercal {\mathbf{x_B}}^c \\ \hline d & -z_0 & -\mathbf{b}^\intercal {\mathbf{x_B}}^c \end{array}$$

Dual Problem Dictionary

Example #2

$$
\begin{array}{rclll}
x_1 & = & 3 & -\frac{1}{3}x_4 & +\frac{1}{3}x_3 \\
x_2 & = & 11 & -x_4 & +0x_3 \\
x_5 & = & 11 & -\frac{2}{3}x_4 & -\frac{1}{3}x_3 \\
x_6 & = & 3 & +\frac{1}{3}x_4 & -\frac{1}{3}x_3 \\
\hline
z & = & 18 & -\frac{7}{3}x_4 & +\frac{1}{3}x_3
\end{array}
$$

x_1	x_2	x_3	x_4	x_5	x_6
y_5	y_6	y_1	y_2	y_3	y_4

Example #3

$$\begin{array}{rclll} x_3 & = & 9 & +x_4 & -3x_6 \\ x_1 & = & 6 & & -x_6 \\ x_2 & = & 11 & -x_4 & +0x_6 \\ x_5 & = & 8 & -x_4 & +x_6 \\ \hline z & = & 21 & -2x_4 & -x_6 \end{array}$$

x_1	x_2	x_3	x_4	x_5	x_6
y_5	y_6	y_1	y_2	y_3	y_4

Dual Dictionary

Primal Problem Dictionary

$$
\begin{array}{c|cc}
\mathbf{x_B} & \mathbf{b} & +A\mathbf{x_I} \\
\hline
z & z_0 & +\mathbf{c}^\intercal \mathbf{x_I}
\end{array}
$$

$\mathbf{x}$	$\mathbf{x_s}$
$\mathbf{y_s}$	$\mathbf{y}$

Dual Problem Dictionary

$$
\begin{array}{c|cc}
\mathbf{x_I}^c & -\mathbf{c} & -A^\intercal \mathbf{x_B}^c \\
\hline
d & -z_0 & -\mathbf{b}^\intercal \mathbf{x_B}^c
\end{array}
$$

Primal vs. Dual Dictionary

$$\begin{array}{c|cc} \mathbf{x_B} & \mathbf{b} & +A\mathbf{x_I} \\ \hline z & z_0 & +\mathbf{c}^\intercal \mathbf{x_I} \end{array} \qquad \begin{array}{c|cc} \mathbf{x_I}^c & -\mathbf{c} & -A^\intercal \mathbf{x_B}^c \\ \hline d & -z_0 & -\mathbf{b}^\intercal \mathbf{x_B}^c \end{array}$$

Non-Final $\Longleftrightarrow$ Infeasible

Feasible + Final $\Longleftrightarrow$ Feasible + Final

Pivoting the primal

$$D_i \xrightarrow[x_j \text{ leaves}]{x_i \text{ enters}} D_{i+1}$$

$$D_i^c \xrightarrow[x_j^c \text{ enters}]{x_i^c \text{ leaves}} D_{i+1}^c$$

Simplex Optimization Phase

$$
\begin{array}{lccccccccc}
\text{Primal Dict.} & D_1 & \rightarrow & D_2 & \rightarrow & \cdots & \rightarrow & \mathbf{D}^* \\
& \downarrow & & \downarrow & & & & \downarrow \\
\text{Dual Dict.} & D_1^c & \rightarrow & D_2^c & \rightarrow & \cdots & \rightarrow & \mathbf{D^{c*}}
\end{array}
$$

PROOFS OF THEOREMS

Dual Dictionary

$$\begin{array}{c|cc} \mathbf{x_B} & \mathbf{b} & +A\mathbf{x_I} \\ \hline z & z_0 & +\mathbf{c}^\intercal \mathbf{x_I} \end{array}$$

Primal Problem Dictionary

$\mathbf{x}$	$\mathbf{x_s}$
$\mathbf{y_s}$	$\mathbf{y}$

$$\begin{array}{c|cc} \mathbf{x_I}^c & -\mathbf{c} & -A^\intercal \mathbf{x_B}^c \\ \hline d & -z_0 & -\mathbf{b}^\intercal \mathbf{x_B}^c \end{array}$$

Dual Problem Dictionary

Theorem

Let **x** be a primal solution and **y** be a dual solution such that

$$\mathbf{c}^\intercal \mathbf{x} = \mathbf{b}^\intercal \mathbf{y}$$

It follows that **x** is primal optimal and **y** is dual optimal.

Fundamental Result of Simplex

The solution associated with any (feasible) final dictionary of the primal problem is optimal.

$$\begin{array}{c|cc} \mathbf{x_B} & \mathbf{b} & +A\mathbf{x_I} \\ \hline z & z_0 & +\mathbf{c}^\intercal \mathbf{x_I} \end{array}$$

Feasible + Final Primal Dict.

$$\begin{array}{c|cc} \mathbf{x_I}^c & -\mathbf{c} & -A^\intercal \mathbf{x_B}^c \\ \hline d & -z_0 & -\mathbf{b}^\intercal \mathbf{x_B}^c \end{array}$$

Feasible + Final Dual Dict.

Primal Objective = Dual Objective Value

Strong Duality

- Let $\mathbf{x}^*$ be a primal optimal solution for an LP.
  - The dual problem has an optimal solution, and
  - Any dual optimal solution $\mathbf{y}^*$ satisfies

$$\mathbf{b}^\intercal \mathbf{y}^* = \mathbf{c}^\intercal \mathbf{x}^*$$

Proof of Strong Duality Theorem

INITIALIZATION USING THE DUAL

Simplex Algorithm

Feasible Dictionary

$$\color{red}{D_0} \xrightarrow{\text{pivot}} \color{red}{D_1} \xrightarrow{\text{pivot}} \cdots \color{red}{D_{k-1}} \xrightarrow{\text{pivot}} D_k \xrightarrow{\text{pivot}} D_{k+1} \cdots \xrightarrow{\text{pivot}} \color{green}{\mathbf{D}^*}$$

Initialization Phase

Optimization Phase

Feasible Dictionaries

Initialization

- Goal: Find a feasible dictionary for the problem.

$$\begin{array}{llll} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ & A\,\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}$$

Observation:
Objective does not matter.

Initialization

$$\begin{aligned} \max \quad & \mathbf{c}^\intercal \mathbf{x} \\ & A\,\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

Original Problem

If all entries in **b ≥ 0**
- Initialization not needed.

If any entry in **b <0**
- initialization needed.

Linear Program (Dual)

$$\begin{array}{llll} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ & A\,\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}$$

Dual Problem

$$\begin{array}{llll} \min & \mathbf{b}^\intercal \mathbf{y} & & \\ & A^\intercal\,\mathbf{y} & \geq & \mathbf{c} \\ & \mathbf{y} & \geq & 0 \end{array}$$

Standard Form Converted Dual

$$\begin{array}{llll} \max & -\mathbf{b}^\intercal \mathbf{y} & & \\ & -A^\intercal\,\mathbf{y} & \leq & -\mathbf{c} \\ & \mathbf{y} & \geq & 0 \end{array}$$

Initialization Using Dual: Basic Idea

1. Change the objective function

$$\begin{aligned} \max \mathbf{d}^T \mathbf{x} & \\ A\,\mathbf{x} &\leq \mathbf{b} \\ \mathbf{x} &\geq 0 \end{aligned}$$

Dual Problem

$$\begin{aligned} \max \quad & -\mathbf{b}^\intercal \mathbf{y} \\ & -A^\intercal\,\mathbf{y} \leq -\mathbf{d} \\ & \mathbf{y} \geq 0 \end{aligned}$$

All
Positive

Idea: choose **d** to be all negative entries.

Initialization Using Dual

$$\begin{array}{llcl} \max \mathbf{d}^T & \mathbf{x} & & \\ & A\,\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}$$

d has all negative entries

Dualize

$$\begin{array}{llcl} \max & -\mathbf{b}^\intercal \mathbf{y} & & \\ & -A^\intercal\,\mathbf{y} & \leq & -\mathbf{d} \\ & \mathbf{y} & \geq & 0 \end{array}$$

Optimization
Phase Simplex

$$\begin{array}{l|ll} \mathbf{y}_b & \mathbf{q} & +P\ \mathbf{y}_n \\ \hline w & w_0 & +\mathbf{r}^T\ \mathbf{y}_n \end{array}$$

Final
Dual
Dictionary

Dualize

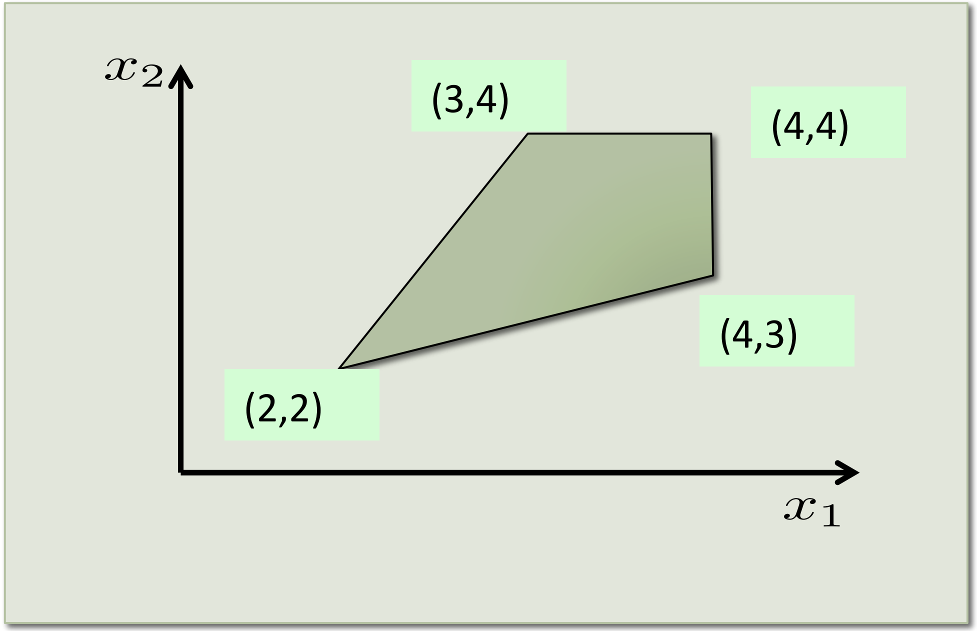
$$\begin{array}{l|ll} \mathbf{y}_n^c & -\mathbf{r} & -P^T\ \mathbf{y}_b^c \\ \hline \end{array}$$

Restore Original Obj.

Feasible
Primal Dictionary

Example

$$\begin{array}{rrcr} \text{max.} & x_1 + 2x_2 & & \\ \text{s.t.} & -2x_1 + x_2 & \leq & -2 \\ & x_2 & \leq & 4 \\ & x_1 - 2x_2 & \leq & -2 \\ & x_1 & \leq & 4 \\ & x_1, x_2 & \geq & 0 \end{array}$$



Example #1: Problem Transformation

$$\begin{array}{llcl}
\max & -x_1 - x_2 & & \\
\text{s.t.} & -2x_1 + x_2 & \leq & -2 \\
& x_2 & \leq & 4 \\
& x_1 - 2x_2 & \leq & -2 \\
& x_1 & \leq & 4 \\
& x_1, x_2 & \geq & 0
\end{array}$$

$$\begin{array}{llcl}
\max & -x_1 - x_2 & & \\
\text{s.t.} & -2x_1 + x_2 + x_3 & = & -2 \\
& x_2 + x_4 & = & 4 \\
& x_1 - 2x_2 + x_5 & = & -2 \\
& x_1 + x_6 & = & 4 \\
& x_1, x_2, x_3, \ldots, x_6 & \geq & 0
\end{array}$$

$$\begin{array}{lccccccc}
\max & 2y_1 & -4y_2 & +2y_3 & -4y_4 & & \\
\text{s.t.} & 2y_1 & & -y_3 & -y_4 & \leq & 1 \\
& -y_1 & -y_2 & +2y_3 & & \leq & 1 \\
& & & & y_1, \ldots, y_4 & \geq & 0
\end{array}$$

$$\begin{array}{lcccccccc}
\max & 2y_1 & -4y_2 & +2y_3 & -4y_4 & & & \\
\text{s.t.} & 2y_1 & & -y_3 & -y_4 & +y_5 & = & 1 \\
& -y_1 & -y_2 & +2y_3 & & +y_6 & = & 1 \\
& & & & & y_1, \ldots, y_4, y_5, y_6 & \geq & 0
\end{array}$$

Solving the modified dual

$$
\begin{array}{lrrrrrcl}
\text{max} & 2y_1 & -4y_2 & +2y_3 & -4y_4 & & & \\
\text{s.t.} & 2y_1 & & -y_3 & -y_4 & +y_5 & = & 1 \\
 & -y_1 & -y_2 & +2y_3 & & +y_6 & = & 1 \\
 & & & & & y_1, \ldots, y_4, y_5, y_6 & \geq & 0
\end{array}
$$

$$
\begin{array}{c|rrrrr}
y_5 & 1 & -2y_1 & & +y_3 & +y_4 \\
y_6 & 1 & +y_1 & +y_2 & -2y_3 & \\
\hline
w & 0 & +2y_1 & -4y_2 & +2y_3 & -4y_4
\end{array}
$$

y_1 enters and y_5 leaves

Solving the modified dual (step 2)

$$
\begin{array}{c|cccccc}
y_1 & 0.5 & -0.5y_5 & & +0.5y_3 & +0.5y_4 \\
y_6 & 1.5 & -0.5y_5 & +1y_2 & -1.5y_3 & +0.5y_4 \\
\hline
w & 1 & -1y_5 & -4y_2 & +3y_3 & -3y_4
\end{array}
$$

y_3 enters and y_6 leaves

Solving the modified dual (step 3)

$$\begin{array}{c|cl} y_1 & 1 & -\frac{2}{3}y_5 + \frac{1}{3}y_2 - \frac{1}{3}y_6 + \frac{2}{3}y_4 \\ y_3 & 1 & -\frac{1}{3}y_5 + \frac{2}{3}y_2 - \frac{2}{3}y_6 + \frac{1}{3}y_4 \\ \hline w & 4 & -2y_5 \; -2y_2 \; -2y_6 \; -2y_4 \end{array}$$

Final Dual Dictionary

Convert Dual back to Primal

$$\begin{array}{lrcl} \max & -x_1 - x_2 & & \\ \text{s.t.} & -2x_1 + x_2 + x_3 & = & -2 \\ & x_2 + x_4 & = & 4 \\ & x_1 - 2x_2 + x_5 & = & -2 \\ & x_1 + x_6 & = & 4 \\ & x_1, x_2, x_3, \ldots, x_6 & \geq & 0 \end{array}$$

$$\Downarrow$$

$$\begin{array}{lcccccccc} \max & 2y_1 & -4y_2 & +2y_3 & -4y_4 & & & \\ \text{s.t.} & 2y_1 & & -y_3 & -y_4 & +y_5 & = & 1 \\ & -y_1 & -y_2 & +2y_3 & & +y_6 & = & 1 \\ & & & & & y_1, \ldots, y_4, y_5, y_6 & \geq & 0 \end{array}$$

x_1	y_5
x_2	y_6
x_3	y_1
x_4	y_2
x_5	y_3
x_6	y_4

Conversion to Primal Dictionary

$$
\begin{array}{c|ccccc}
y_1 & 1 & -\frac{2}{3}y_5 & +\frac{1}{3}y_2 & -\frac{1}{3}y_6 & +\frac{2}{3}y_4 \\
y_3 & 1 & -\frac{1}{3}y_5 & +\frac{2}{3}y_2 & -\frac{2}{3}y_6 & +\frac{1}{3}y_4 \\
\hline
w & 4 & -2y_5 & -2y_2 & -2y_6 & -2y_4
\end{array}
$$

x_1	y_5
x_2	y_6
x_3	y_1
x_4	y_2
x_5	y_3
x_6	y_4

$$\Downarrow$$

$$
\begin{array}{c|ccc}
x_1 & 2 & +\frac{2}{3}x_3 & +\frac{1}{3}x_5 \\
x_4 & 2 & -\frac{1}{3}x_3 & -\frac{2}{3}x_5 \\
x_2 & 2 & +\frac{1}{3}x_3 & +\frac{2}{3}x_5 \\
x_6 & 2 & -\frac{2}{3}x_3 & -\frac{1}{3}x_5 \\
\hline
w & -4 & -x_3 & -x_5
\end{array}
$$

Initialization: Restoring Objective

$$
\begin{array}{c|llll}
x_1 & 2 & +\frac{2}{3}x_3 & +\frac{1}{3}x_5 \\
x_4 & 2 & -\frac{1}{3}x_3 & -\frac{2}{3}x_5 \\
x_2 & 2 & +\frac{1}{3}x_3 & +\frac{2}{3}x_5 \\
x_6 & 2 & -\frac{2}{3}x_3 & -\frac{1}{3}x_5 \\
\hline
\boxed{\text{z}} & \boxed{6 + \frac{4}{3}x_3 + \frac{5}{3}x_5} & &
\end{array}
$$

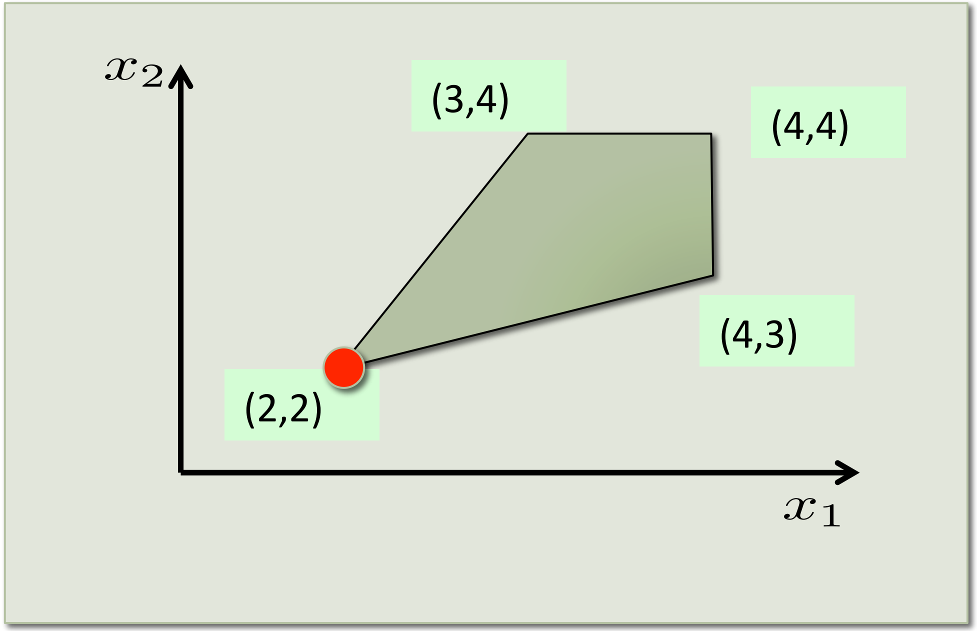
$$
\begin{aligned}
z &= x_1 + 2x_2 \\
&= 2 + \tfrac{2}{3}x_3 + \tfrac{1}{3}x_5 + 2(2 + \tfrac{1}{3}x_3 + \tfrac{2}{3}x_5) \\
&= 6 + \tfrac{4}{3}x_3 + \tfrac{5}{3}x_5
\end{aligned}
$$

Initial Feasible Dictionary Found.
We can now proceed to optimize!!

Example

$$
\begin{array}{rrcr}
\text{max.} & x_1 + 2x_2 & & \\
\text{s.t.} & -2x_1 + x_2 & \leq & -2 \\
 & x_2 & \leq & 4 \\
 & x_1 - 2x_2 & \leq & -2 \\
 & x_1 & \leq & 4 \\
 & x_1, x_2 & \geq & 0
\end{array}
$$

$$
\begin{array}{c|cll}
x_1 & 2 & +\frac{2}{3}x_3 & +\frac{1}{3}x_5 \\
x_4 & 2 & -\frac{1}{3}x_3 & -\frac{2}{3}x_5 \\
x_2 & 2 & +\frac{1}{3}x_3 & +\frac{2}{3}x_5 \\
x_6 & 2 & -\frac{2}{3}x_3 & -\frac{1}{3}x_5 \\
\hline
\boxed{z} & \boxed{6 + \frac{4}{3}x_3 + \frac{5}{3}x_5} & &
\end{array}
$$



Initialization Using Dual

$$\begin{aligned} \max\ & \mathbf{d}^T \mathbf{x} \\ & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

d has all negative entries

Dualize

$$\begin{aligned} \max\ & -\mathbf{b}^\intercal \mathbf{y} \\ & -A^\intercal \mathbf{y} \leq -\mathbf{d} \\ & \mathbf{y} \geq 0 \end{aligned}$$

Optimization Phase Simplex

$$\begin{array}{c|c} \mathbf{y}_b & \mathbf{q} \quad + P\ \mathbf{y}_n \\ \hline w & w_0 \quad + \mathbf{r}^T\ \mathbf{y}_n \end{array}$$

Final Dual Dictionary

Unbounded!

Original primal problem is infeasible.

INFEASIBLE PROBLEM EXAMPLE

Infeasible Problem Example

$$
\begin{array}{lrrrcr}
\max & -x_1 & -x_2 & -x_3 & & \\
\text{s.t.} & x_1 & -x_2 & & \leq & 5 \\
 & & x_2 & -x_3 & \leq & 4 \\
 & -x_1 & & +2x_3 & \leq & -10 \\
 & x_1, & x_2, & x_3 & \geq & 0
\end{array}
$$

Initial Primal Dictionary

$$\begin{array}{c|rrrr} x_4 & 5 & -x_1 & +x_2 & \\ x_5 & 4 & & -x_2 & +x_3 \\ x_6 & -10 & +x_1 & & -2x_3 \\ \hline z & & -x_1 & -x_2 & -x_3 \end{array}$$

Infeasible

Dualize

$$\begin{array}{c|rrrr} y_4 & 1 & +y_1 & & -y_3 \\ y_5 & 1 & -y_1 & +y_2 & \\ y_6 & 1 & & -y_2 & +2y_3 \\ \hline w & 0 & -5y_1 & -4y_2 & +10y_3 \end{array}$$

Feasible!

Dual Simplex Method

$$
\begin{array}{c|cccc}
y_4 & 1 & +y_1 & & -y_3 \\
y_5 & 1 & -y_1 & +y_2 & \\
y_6 & 1 & & -y_2 & +2y_3 \\
\hline
w & 0 & -5y_1 & -4y_2 & +10y_3
\end{array}
$$

y3 enters
y4 leaves

$$
\begin{array}{c|cccc}
y_3 & 1 & +y_1 & & -y_4 \\
y_5 & 1 & -y_1 & +y_2 & \\
y_6 & 3 & +2y_1 & -y_2 & -2y_4 \\
\hline
w & 10 & +5y_1 & -4y_2 & -10y_4
\end{array}
$$

y1 enters
y5 leaves

DUAL UNBOUNDED
= PRIMAL INFEASIBLE

$$
\begin{array}{c|cccc}
y_3 & 2 & -y_5 & \boxed{+y_2} & -y_4 \\
y_1 & 1 & -y_5 & \boxed{+y_2} & \\
y_6 & 5 & -2y_5 & \boxed{+y_2} & -2y_4 \\
\hline
w & 15 & -5y_5 & \boxed{+y_2} & -10y_4
\end{array}
$$

Initialization Using Dual (Summary)

1. Change problem objective to $\boxed{\sum_{j=1}^{n} -x_j}$
2. Construct initial primal dictionary D_0
3. Convert to dual dictionary.
4. Perform optimization phase simplex on dual.
5. If UNBOUNDED, original primal is INFEASIBLE.
6. If Optimal Solution found,
   a) Convert back to primal
   b) Restore original objective function.

SIMPLEX: REVISED SIMPLEX METHOD

Revised Simplex Method

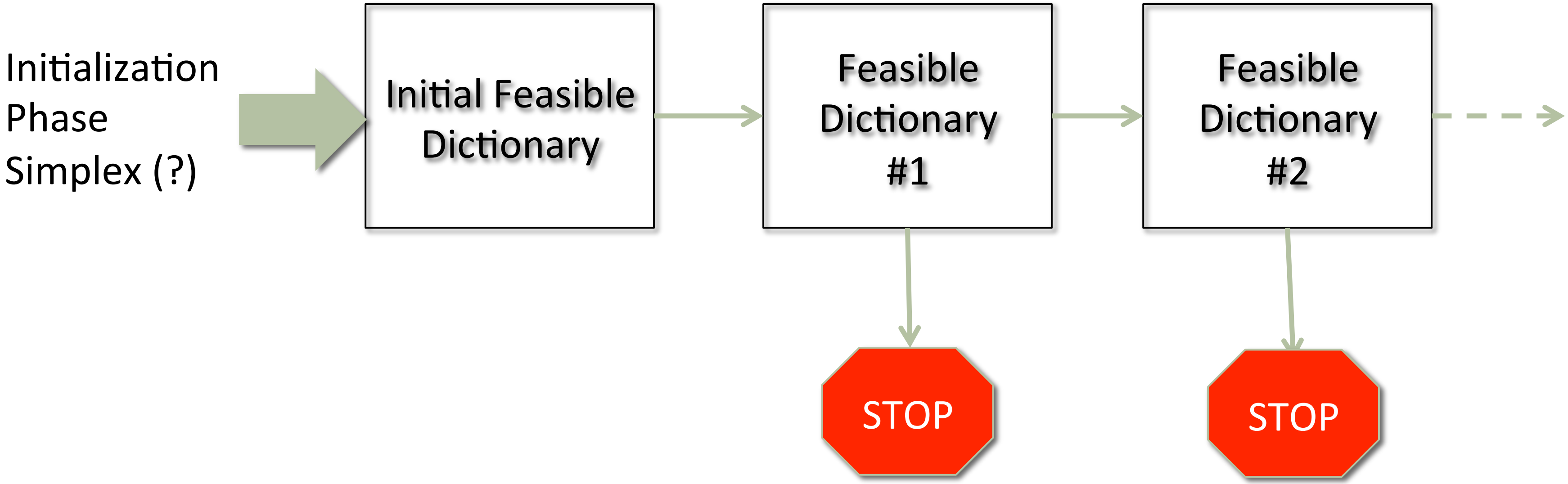
Original LP

n decision variables
m rows in A.

$$\begin{array}{lrcl} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \max & \mathbf{c}^\intercal \mathbf{x} & & \\ & A\mathbf{x} + \mathbf{x_s} & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x_s} & \geq & 0 \end{array}$$

Standard Simplex Method



Complexity of Each Pivoting Operation

$$
\begin{array}{ccccccc}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +{\color{red}a_{1j}}x_{Ij} & +\cdots & +a_{1n}x_{In} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +{\color{red}a_{2j}}x_{Ij} & +\cdots & +a_{2n}x_{In} \\
 & \vdots & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +{\color{red}a_{mj}}x_{Ij} & +\cdots & +a_{mn}x_{In} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +{\color{red}c_j}x_{Ij} & +\cdots & +c_nx_{In}
\end{array}
$$

Choice of Entering Variable: O(n)
Choice of Leaving Variable: O(m)
Updating Dictionary: O(m * n)

Storage: O(m*n) floating point numbers

Problems with Standard Simplex Algorithm

- Storage Cost is High.

$$\begin{array}{ccccccc}
x_{B1} & = & b_1 & +a_{11}x_{I1} & +\cdots & +a_{1j}x_{Ij} & +\cdots & +a_{1n}x_{In} \\
x_{B2} & = & b_2 & +a_{21}x_{I1} & +\cdots & +a_{2j}x_{Ij} & +\cdots & +a_{2n}x_{In} \\
 & \vdots & & & & & & \\
x_{Bm} & = & b_m & +a_{m1}x_{I1} & +\cdots & +a_{mj}x_{Ij} & +\cdots & +a_{mn}x_{In} \\
\hline
z & = & c_0 & +c_1x_{I1} & +\cdots & +c_jx_{Ij} & +\cdots & +c_nx_{In}
\end{array}$$

- Sparsity of Original Problem is Lost.

- Accumulation of Floating Point Errors over number of iterations.

Problem # 1 : Storage Cost

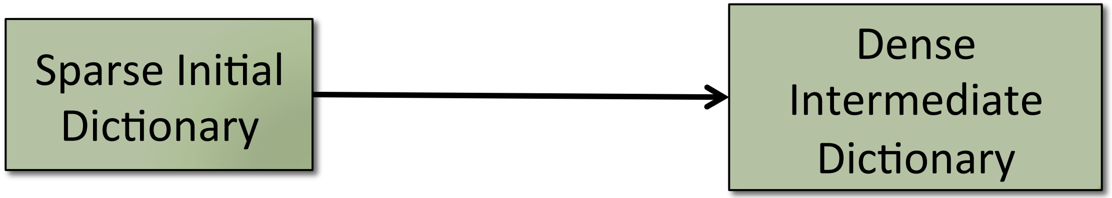
- Typical problem size.
  - m = 500 and n = 10000

- Cost of storing dictionary: $5x10^6$ floating point numbers.
  - Approx. 20 MB

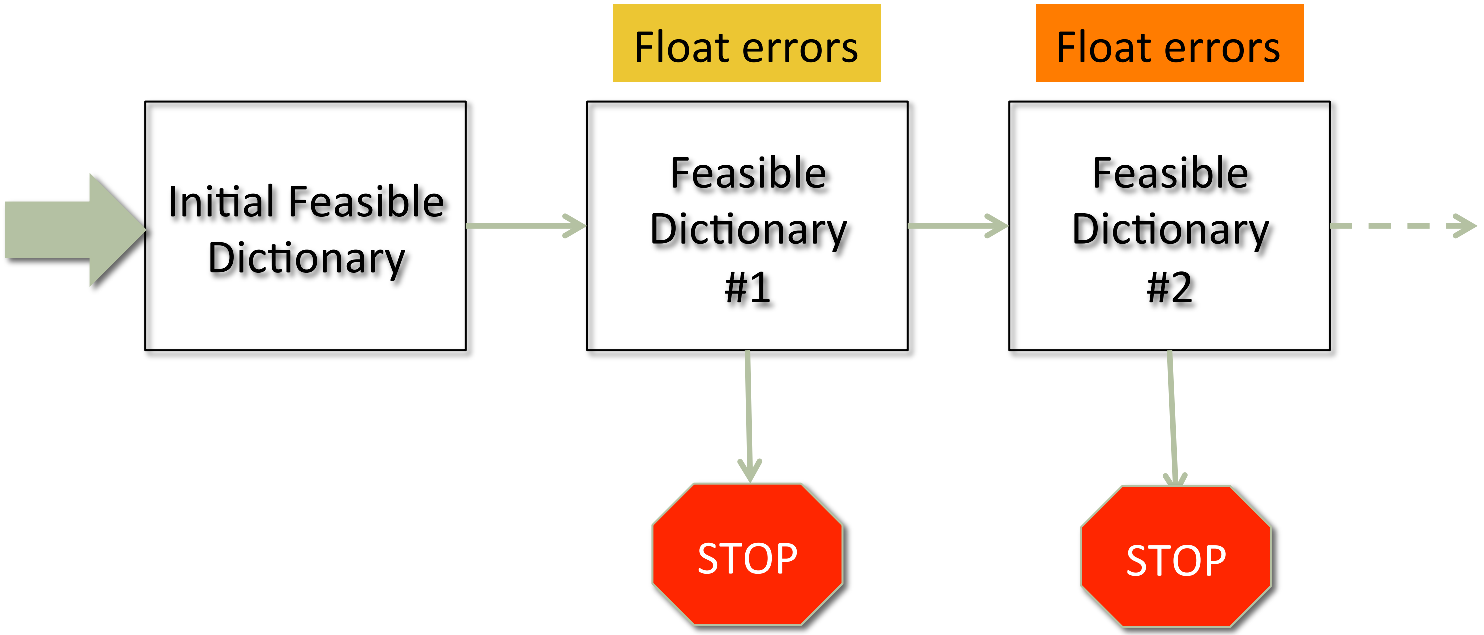
Problem #2: Loss in Sparsity

- Most practical LP instances are sparse.
- Lots of variables in the problem.
- But each inequality involves few variables.

- Fill In Problem:



Problem #3: Floating Point Error Accumulation



Revised Simplex Method

- **Basic Idea:** Do not store the intermediate dictionary.
- Store the set of basic and non-basic variables.
- At each step, **reconstruct** dictionary from data:
  - Original problem data: A,b,c
  - Set of basic (and non-basic) variables: B

Storage Cost:
  Original problem data (sparse)
  Basis set $O(m + n)$

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- **Basic Idea:** Do not store the intermediate dictionary.
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Storage Cost:
Original problem data (sparse)
Basis set O(m + n)

Recap: Dictionary Reconstruction

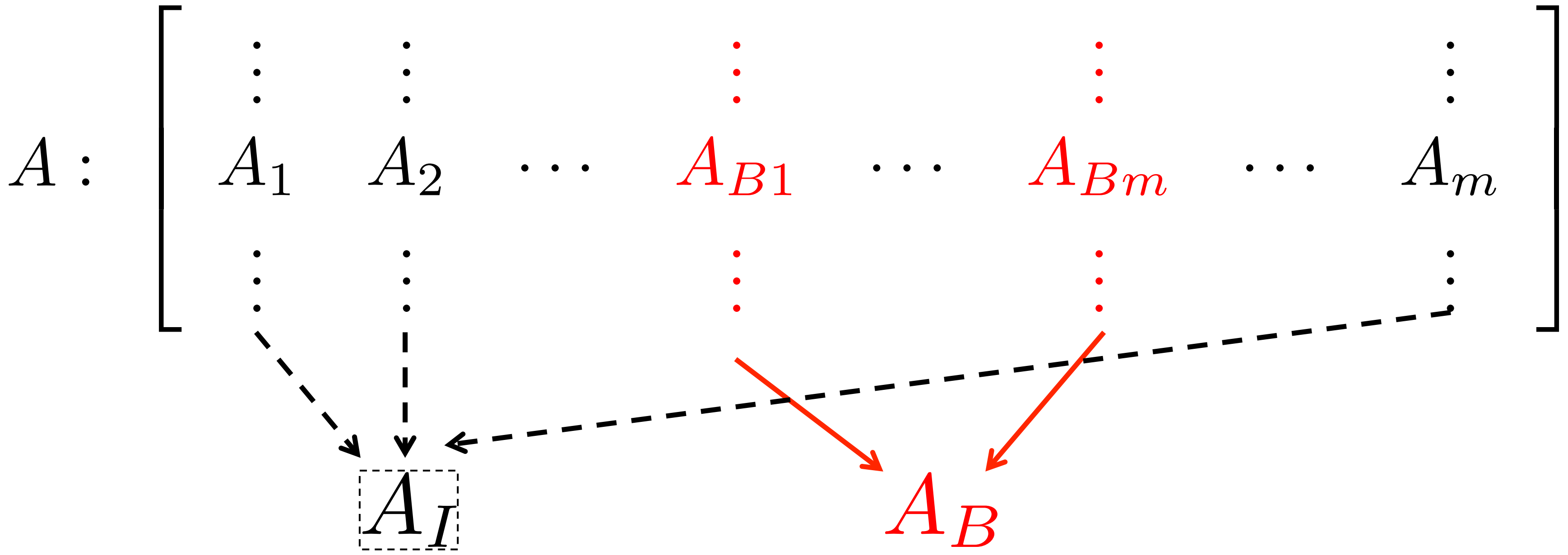
$$\begin{array}{lccc} \max & \mathbf{c}^\intercal \ \mathbf{x} & & \\ & A\mathbf{x} + \mathbf{x_s} & = & \mathbf{b} \\ & \mathbf{x}, \mathbf{x_s} & \geq & 0 \end{array}$$

$$\begin{array}{cccc} \mathbf{x}_B & = & \hat{\mathbf{b}} & + \hat{A}\mathbf{x}_I \\ \hline z & = & z_0 & + \hat{c}\ \mathbf{x}_I \end{array}$$

$$B = \{x_{B1}, \ldots, x_{Bm}\}$$

Basic Variables

Recap: Splitting the Matrix



Recap: Rewriting the Equation

$$A\mathbf{x} + \mathbf{x_s} = \mathbf{b}$$

Step 1

$$A_B\mathbf{x_B} + A_I\mathbf{x_I} = b$$

Step 2

$$A_B\ \mathbf{x_B} = \mathbf{b} - A_I\mathbf{x_I}$$

Is A_B always invertible?

Recap: Dictionary Reconstruction

$$A_B \ \mathbf{x_B} = \mathbf{b} - A_I \mathbf{x_I}$$

$$\mathbf{x_B} = A_B^{-1}\mathbf{b} - A_B^{-1} A_I \mathbf{x_I}$$

Is A_B always invertible?

Recap: Result Dictionary

$$\mathbf{c}^\intercal \ \mathbf{x} = \mathbf{c_B}^\intercal \mathbf{x_B} + \mathbf{c_I}^\intercal \mathbf{x_I}$$

$$\begin{array}{ccccc} \mathbf{x_B} & = & A_B^{-1}\mathbf{b} & & -A_B^{-1}A_I\mathbf{x_I} \\ \hline \mathbf{c} & = & \mathbf{c_B}^\intercal \ A_B^{-1}\mathbf{b} & & +(-\mathbf{c_B}^\intercal A_B^{-1}A_I + \mathbf{c_I}^\intercal) \ \mathbf{x_I} \end{array}$$

Invert matrix A_B: $O(m^3)$ (Gauss-Jordan).

Compute $A_B^{-1} \times A_I$ takes $O(m^2 n)$.

Compute $A_B^{-1}\mathbf{b}$ takes $O(m^2)$

Overall complexity: $O(m^2 * (m+n))$.

REVISED SIMPLEX: BASIS FACTORIZATION

Reducing the complexity of revised Simplex method.

$$\pi A_B = \mathbf{c}_B^{\mathsf{T}}$$

$$\hat{\mathbf{c}} = \mathbf{c}_I^{\mathsf{T}} - \pi A_I$$

Entering Variable Analysis

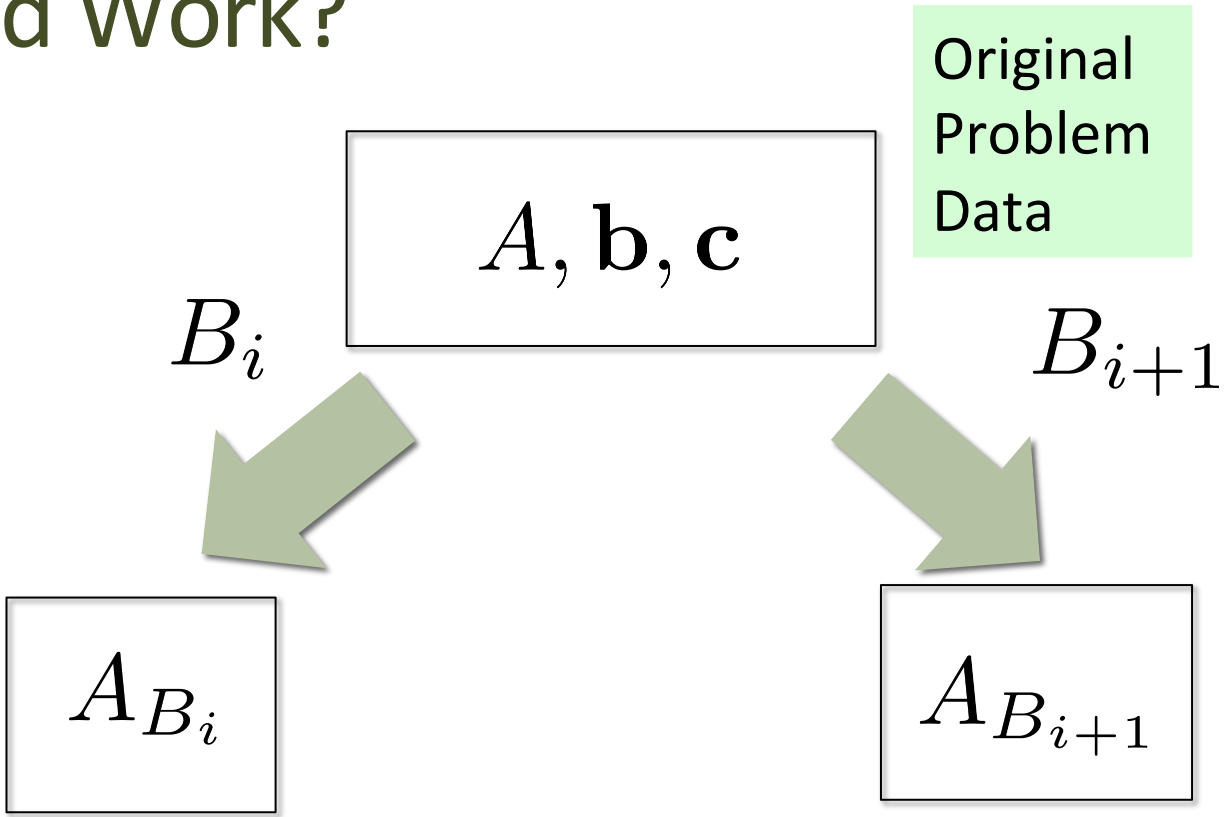
$$A_B \hat{\mathbf{b}} = \mathbf{b}$$

$$A_B \hat{\mathbf{a}}_j = -A_j$$

Leaving Variable Analysis

Update New Basis

Wasted Work?



Ideas

1. How does basis matrix A_B ...?

2. Can we reuse ...

- Using ... (SMW) formula.
- Eta Matr...
- Updating ... (Forrest-Tomlin Method).

Practical Simplex implementations use these ideas + a lot more!

Understanding how the basis changes.

$$A = \begin{bmatrix} 2 & -3 & 7 & -15 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 10 \\ 12 \\ 4 \\ 16 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B = \{3, 6, 7, 8\}$$

x_4 enters and x_8 leaves

How is the basis updated?

$$A_B : \begin{bmatrix} \vdots & \vdots & & \color{red}{\vdots} & & \vdots \\ A_1 & A_2 & \cdots & \color{red}{A_i} & \cdots & A_m \\ \vdots & \vdots & & \color{red}{\vdots} & & \vdots \end{bmatrix}$$

OLD

$$\widetilde{A_B} : \begin{bmatrix} \vdots & \vdots & & \color{blue}{\vdots} & & \vdots \\ A_1 & A_2 & \cdots & \color{blue}{A_k} & \cdots & A_m \\ \vdots & \vdots & & \color{blue}{\vdots} & & \vdots \end{bmatrix}$$

NEW

USING SHERMAN-MORRISON-WOODBURY FORMULA

Understanding the basis update

$$\widetilde{A_B} = A_B + \left[\ \mathbf{0} \quad \mathbf{0} \quad \cdots \quad {\color{red}(A_k - A_i)} \quad \cdots \quad \mathbf{0}\ \right]$$

$$\tilde{A_B} : \begin{bmatrix} 7 & 0 & 0 & {\color{red}15} \\ -4 & 1 & 0 & {\color{red}6} \\ 1 & 0 & 1 & {\color{red}-2} \\ 1 & 0 & 0 & {\color{red}0} \end{bmatrix} \qquad A_B : \begin{bmatrix} 7 & 0 & 0 & {\color{red}0} \\ -4 & 1 & 0 & {\color{red}0} \\ 1 & 0 & 1 & {\color{red}0} \\ 1 & 0 & 0 & {\color{red}1} \end{bmatrix}$$

Understanding the basis update

$$\widetilde{A_B} = A_B + \boxed{\begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & (A_k - A_i) & \cdots & \mathbf{0} \end{bmatrix}}$$

$$(A_k - A_i) \times (0\ 0 \cdots 1\ 0 \cdots 0)$$

$$\begin{bmatrix} 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{pmatrix} 15 \\ 6 \\ -2 \\ -1 \end{pmatrix} \times (0\ 0\ 0\ 1)$$

Sherman-Morrison-Woodbury (SMW) Formula

$$(A + \mathbf{u}\mathbf{v}^\intercal)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^\intercal A^{-1}}{1+\mathbf{v}^\intercal A^{-1}\mathbf{u}}$$

$$\widetilde{A_B} = A_B + \underbrace{(A_k - A_i)}_{\mathbf{u}} \times \underbrace{\mathbf{e_i}^\intercal}_{\mathbf{v}^\intercal}$$

SMW formula for inverse update

$$\widetilde{A_B}^{-1} = \left(I - \frac{(\hat{a}_k - \mathbf{e_i}) \times \mathbf{e_i}^\intercal}{\hat{a}_k(i)}\right) A_B^{-1}$$

$$\text{where, } \hat{a_k} = A_B^{-1} A_k$$

Example

$$A_B : \begin{bmatrix} 7 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{A_B} = A_B + \begin{pmatrix} 15 \\ 6 \\ -2 \\ -1 \end{pmatrix} \times (0\ 0\ 0\ 1)$$

$$\tilde{A_B} : \begin{bmatrix} 7 & 0 & 0 & 15 \\ -4 & 1 & 0 & 6 \\ 1 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\widetilde{A_B}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 6.8 \\ 0 & 0 & 1 & -1.9333 \\ 0 & 0 & 0 & -0.4666 \end{bmatrix} \times A_B^{-1}$$

Summary

$$A_{B_i}^{-1} \xrightarrow{\text{O(m}^2\text{) steps}} A_{B_{i+1}}^{-1}$$

Matrix $A_{B_{i+1}}^{-1}$ is no longer sparse.

Extra storage cost: O(m^2)

ETA FACTORIZATION

SMW formula for inverse update

$$\widetilde{A_B}^{-1} = \left(I - \frac{(\hat{a}_k - \mathbf{e_i}) \times \mathbf{e_i}^{\mathsf{T}}}{\hat{a}_k(i)} \right) A_B^{-1}$$

$$\text{where, } \hat{a_k} = A_B^{-1} A_k$$

Eta Matrix

Basis Update

$$A_{B_i}^{-1} \xrightarrow{\text{O(m}^2\text{) steps}} A_{B_{i+1}}^{-1}$$

Matrix $A_{B_{i+1}}^{-1}$ is no longer sparse.

Extra storage cost: O(m^2)

Eta File: Basic Idea

$$A_{B_i}^{-1} = E_i^{-1} E_{i-1}^{-1} \cdots E_1^{-1} A_1^{-1}$$

Each E_j is sparse: requires O(m) storage.