LINEAR PROGRAMMING PROBLEM

Definition and Examples

Linear Program

Decision Variables

$$x_1, x_2, x_3, x_4$$

Objective Function

$$\text{maximize} \quad 2x_1 + 3x_2 - x_3 + x_4$$

$$\begin{array}{lrrrrcr} \text{subject to} & x_1 & -x_2 & & & \leq & 10 \\ & 2x_1 & +x_2 & -x_3 & & \geq & -5 \\ & & -x_2 & & +x_4 & = & 4 \end{array}$$

Constraints

Linear Program (General Form)

Objective Function

$$
\begin{array}{lllllll}
\max & c_1x_1 & +c_2x_2 & +\cdots+ & c_nx_n & & \\
\text{s.t.} & a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 \\
 & & & \ddots & & \vdots & \\
 & a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m
\end{array}
$$

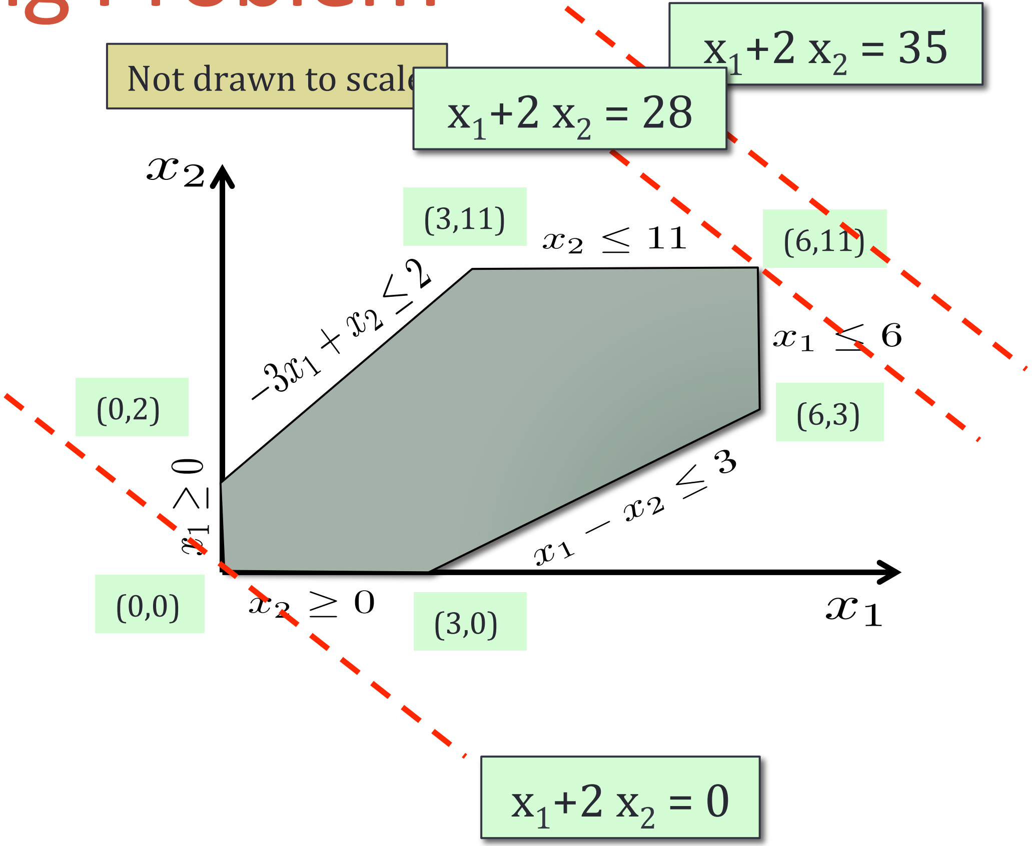
$\{\leq, \geq, =\}$

Constraints

Linear Programming Problem

$$
\begin{array}{llllll}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$

Solution: $x_1 = 6$, $x_2 = 11$
Optimal Objective Value: 28



Overview

- Solving a Linear Program.
  - Visualizing Linear Programs.
  - What does solving a Linear Program mean?

- Algorithms for Linear Programming.
  - Simplex.
  - Ellipsoidal Methods.
  - Interior Point Methods.

VISUALIZING LINEAR PROGRAMS

Linear Program (General Form)

Objective Function

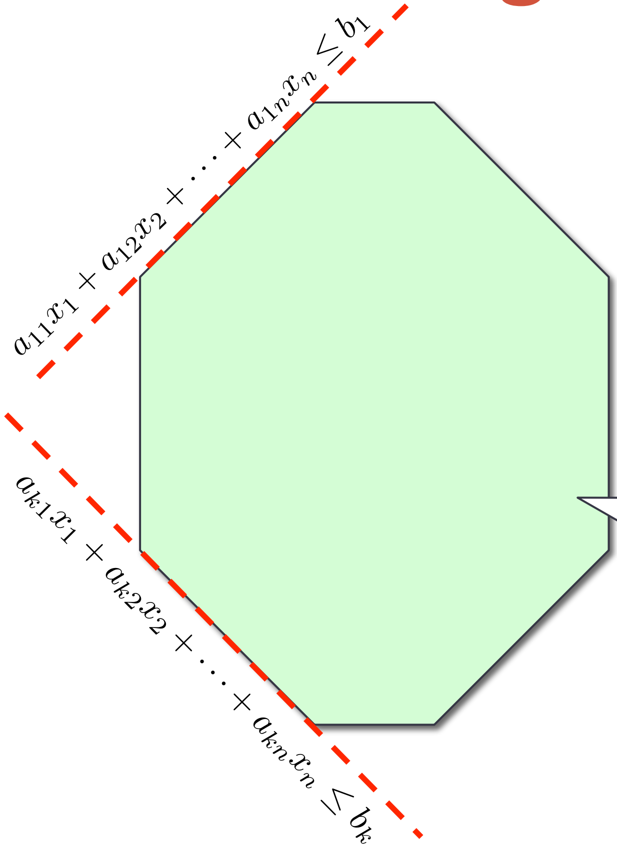
$$\begin{array}{lllllll} \max & \boxed{c_1x_1 & +c_2x_2 & +\cdots+ & c_nx_n} & & \\ \text{s.t.} & a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \le & b_1 \\ & & & \ddots & & \vdots & \\ & a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \le & b_m \end{array}$$

$\{\le, \ge, =\}$

Constraints

Feasible Region

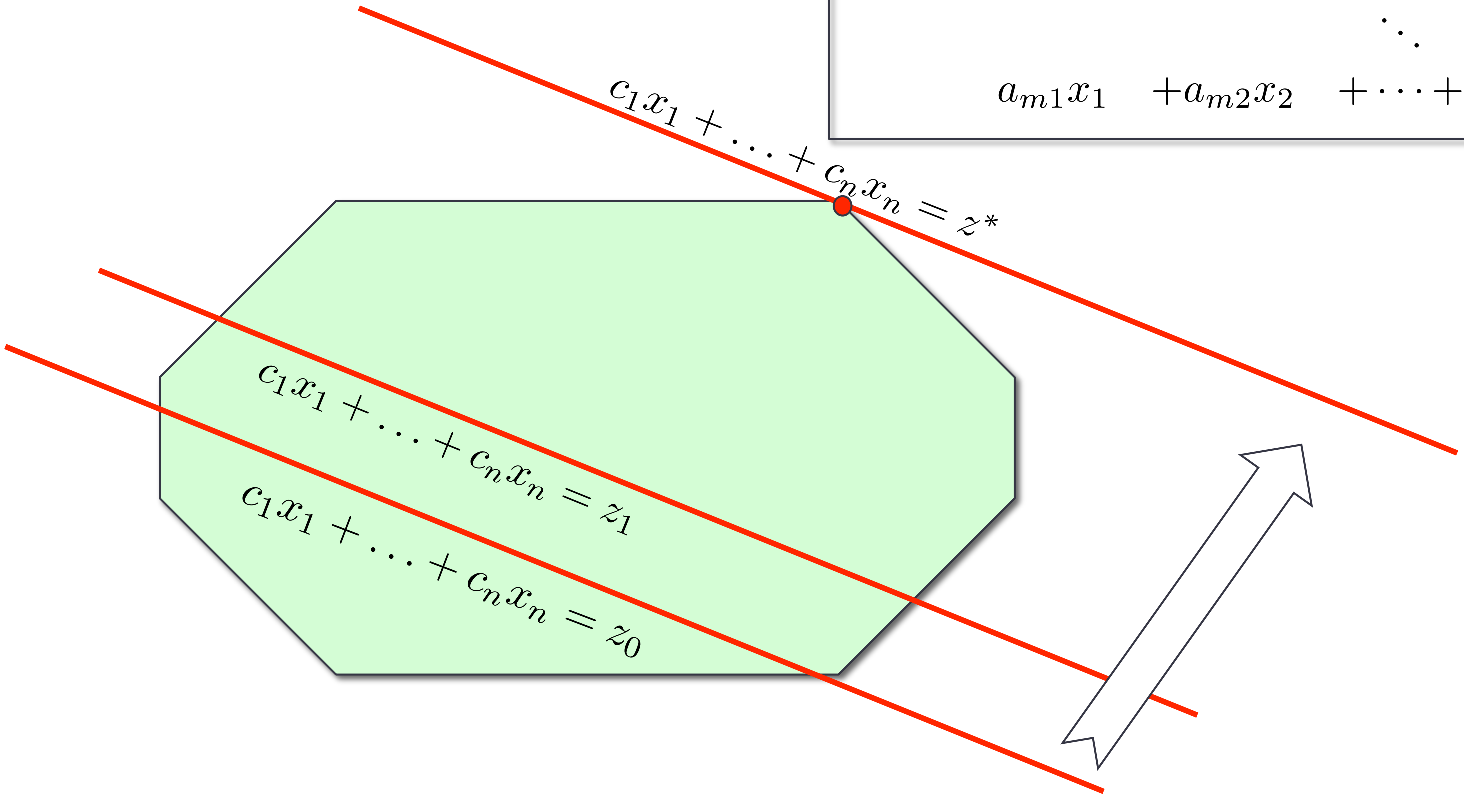
$$
\begin{array}{lccccccc}
\text{max} & c_1x_1 & +c_2x_2 & +\cdots+ & c_nx_n & & \\
\text{s.t.} & a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 \\
 & & & \ddots & & \vdots & \\
 & a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m
\end{array}
$$



Feasible Region: Polyhedron
(n dimensional)

Optimization

$$
\begin{array}{lccccccc}
\text{max} & c_1x_1 & +c_2x_2 & +\cdots+ & c_nx_n & & \\
\text{s.t.} & a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 \\
 & & & \ddots & & \vdots & \\
 & a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m
\end{array}
$$

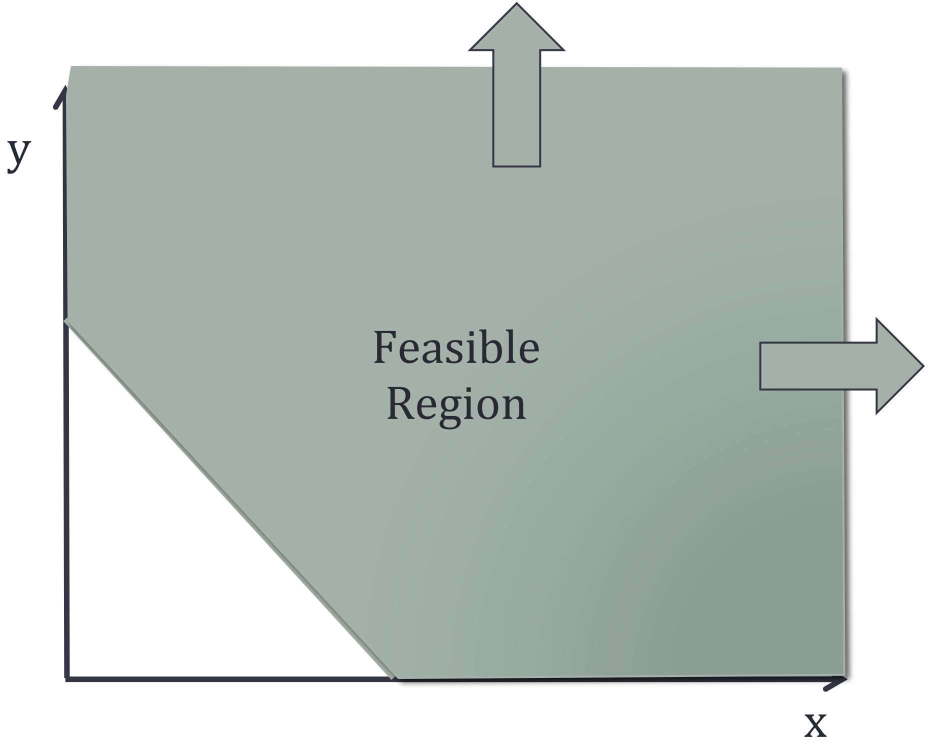


Solving Linear Programs

$$\begin{array}{llll} \max & \mathbf{c}^\mathsf{T}\,\mathbf{x} & & \\ & A\,\mathbf{x} & \leq & \mathbf{b} \end{array}$$

- Outcome #1: Optimal Solution(s) exists.
- Outcome #2: Objective Function is unbounded.
- Outcome #3: Feasible Region is empty.

Unbounded Problem (Example)



$$\begin{array}{llll} \max & x & & \\ \text{s.t.} & x & \geq & 0 \\ & x + y & \geq & 1 \\ & y & \geq & 0 \end{array}$$

Infeasible Problem

- Issue: Constraints contradict each other.

$$
\begin{aligned}
\max \quad & x \\
\text{s.t.} \quad & x \geq 0 \\
& \boxed{x + y \geq 1} \\
& y \geq 0 \\
& \boxed{x + y \leq \tfrac{1}{2}}
\end{aligned}
$$

Solving Linear Programs

1. Find which of the three cases are applicable.
   - Infeasible?
   - Unbounded?
   - Feasible + Bounded = Optimal?

2. If Optimal, find optimal solution.
   - Note multiple optimal solutions possible.

LINEAR PROGRAMMING ALGORITHMS

Linear Programming

- Solving systems of Linear Inequalities.
  - Early work by Fourier (Fourier-Motzkin Elimination Algorithm).
  - In symbolic logic, this is called "Linear Arithmetic".

- World War II: Optimal allocation of resources.
  - Advent of electronic/mechanical calculating machines.
  - L.V. Kantorovich in USSR (1940) and G.B. Dantzig et al. in the USA (1947).

SIMPLEX

- *Simplex:* algorithm for solving LPs.

- First Published by George B. Dantzig

G.B Dantzig: Maximization of a linear function of variables subject to linear inequalities, 1947.

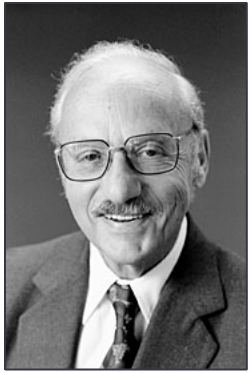
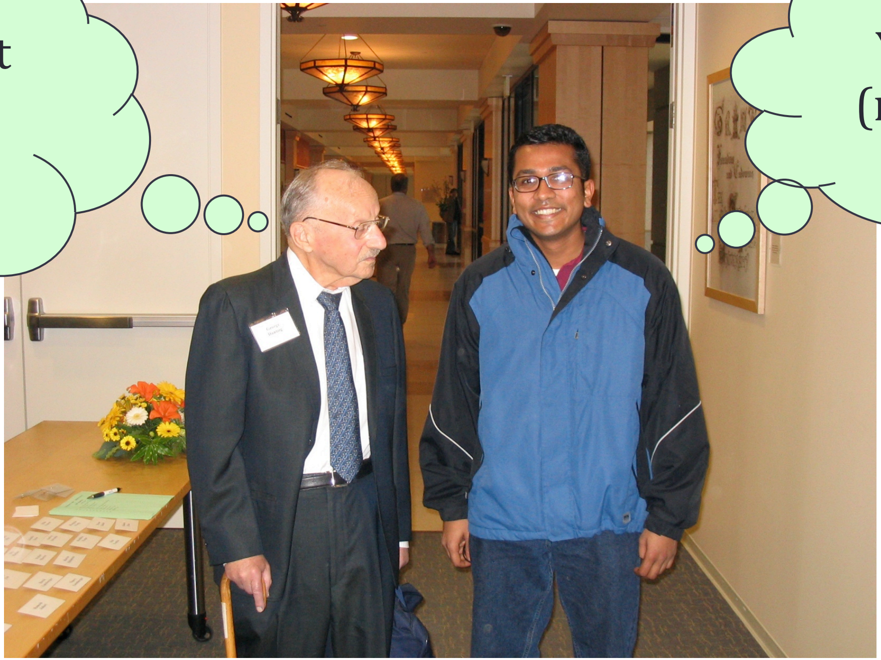


Photo credit: Stanford University

- Prof. Dantzig contributed numerous seminal ideas to this field.

Prof. Dantzig at his 90th birthday celebrations



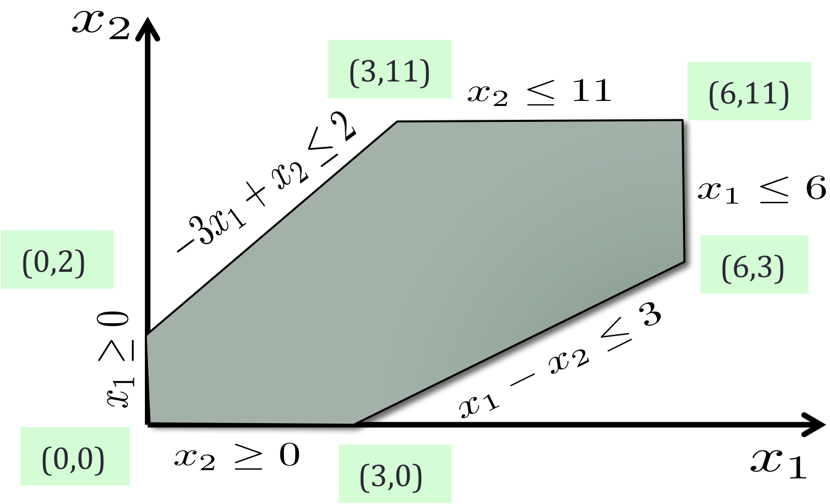
Your instructor (many years ago)

A very special picture for your instructor!!

Visualizing the Simplex Algorithm

Not drawn to scale

$$
\begin{array}{lrrcr}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$



Solution: $x_1 = 6$, $x_2 = 11$
Objective Value: 28

Linear Programming Theory

- Duality: John Von Neumann
  - Early work by Lagrange.
  - Connections to game theory.

- Generalized to Karush-Kuhn-Tucker Conditions.

- Complexity of Simplex:
  - Exponential time in the worst case (Klee + Minty).
  - Polynomial time in the “average case”.
  - Much remains to be understood.

Polynomial Time Algorithms

- Leonid Khachiyan’s ellipsoidal algorithm [Kachiyan’1980]
  - First polynomial time algorithm.

- Interior Point Methods
  - Ideas go back to Isaac Newton (Newton-Raphson).
  - First algorithms for Linear Programs by Narendra Karmarkar [Karmarkar’1984]
    - Interior point methods are useful for non-linear programming (Cf. Nocedal + Wright textbook).

Applications of Linear Programming Theory

- Too numerous to list exhaustively…

- Major application areas:
  - Operations Research.
    - Optimal allocation of resources.
    - Decision making.
  - Computer Science
    - Algorithms, Machine Learning, Automated Reasoning, Robotics.
  - Engineering
    - Control Theory

In this course…

- We will first study Simplex algorithm.
  - Understand duality of Linear Programs.

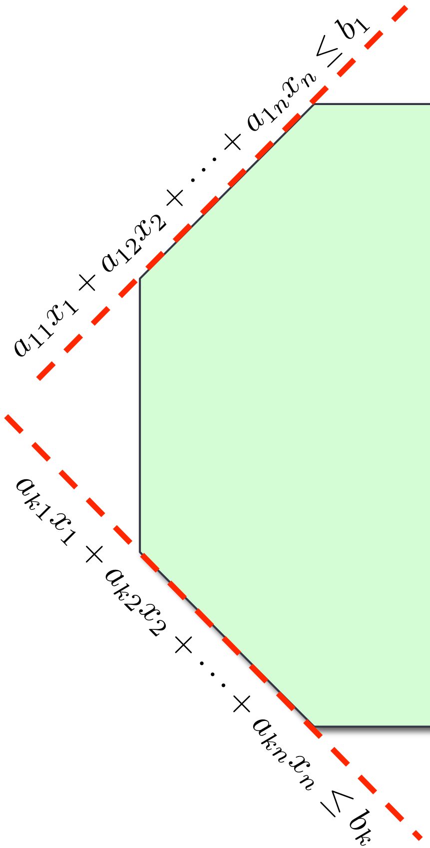
- Finally, study interior point methods.

INTEGER LINEAR PROGRAMMING

Real vs. Integer Variables

Feasible Region

$$
\begin{array}{lccccccc}
\text{max} & c_1x_1 & +c_2x_2 & +\cdots+ & c_nx_n & & \\
\text{s.t.} & a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 \\
 & & & \ddots & & \vdots & \\
 & a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m
\end{array}
$$



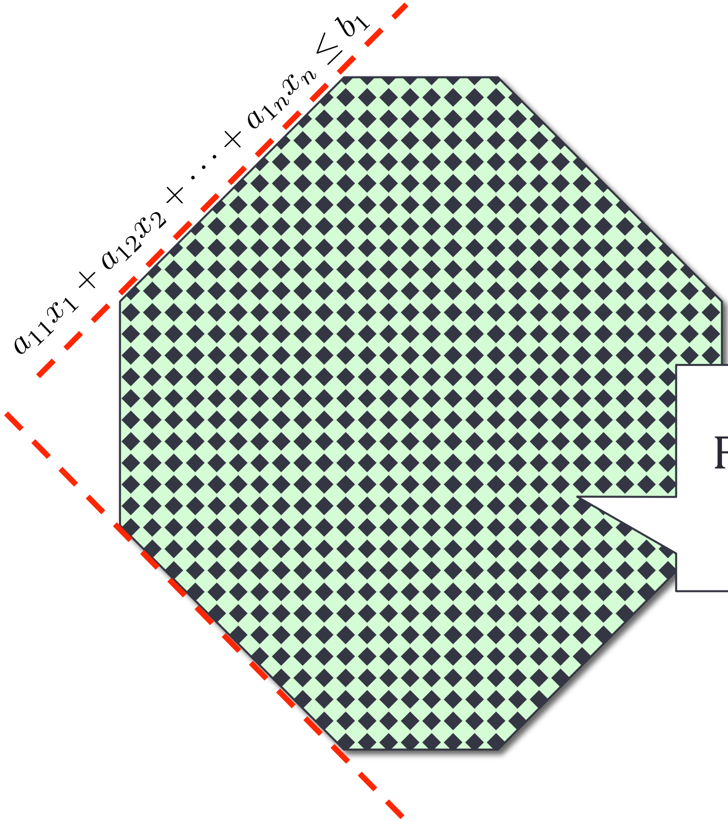
Feasible Region: Polyhedron
(n dimensional)

Linear vs. Integer Linear Programs

$$
\begin{array}{lccccc}
\text{max} & c_1x_1 & +c_2x_2 & +\cdots+ & c_nx_n & & \\
\text{s.t.} & a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & \leq & b_1 \\
 & & & \ddots & & \vdots & \\
 & a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \leq & b_m
\end{array}
$$

Integer Linear Programming

$$
\begin{array}{lcccccc}
\text{max} & c_1 x_1 & +c_2 x_2 & + \cdots + & c_n x_n & & \\
\text{s.t.} & a_{11} x_1 & +a_{12} x_2 & + \cdots + & a_{1n} x_n & \leq & b_1 \\
 & & & \ddots & & \vdots & \\
 & a_{m1} x_1 & +a_{m2} x_2 & + \cdots + & a_{mn} x_n & \leq & b_m
\end{array}
$$



Feasible Region: Z-Polyhedron
(n dimensional)

Linear vs. Integer Linear Programs (Complexity)

Linear Programming (Integers)

Nondeterministic Polynomial Time

Million Dollar Question: Can Integer Linear Programs be solved in polynomial time?

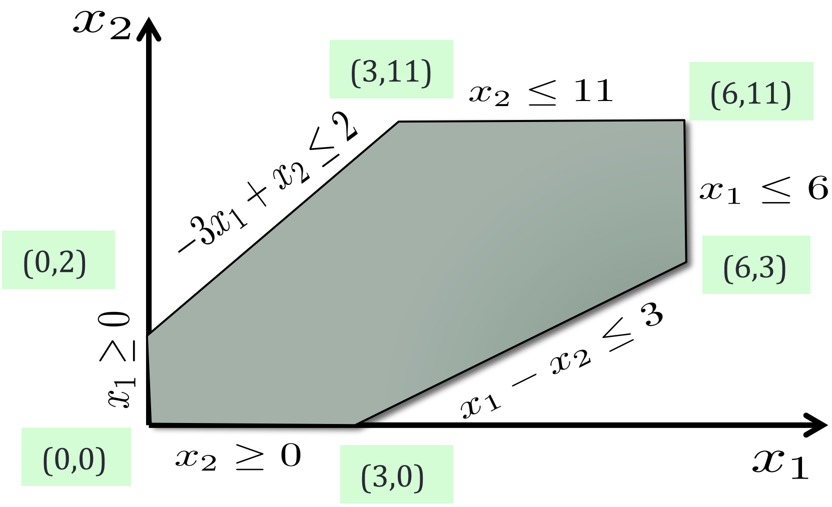
$(P =^{?}= NP)$

Example #1

$$
\begin{array}{llllll}
\text{max.} & x_1 & +2x_2 & & \\
\text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\
 & & +x_2 & \leq & 11 \\
 & x_1 & -x_2 & \leq & 3 \\
 & x_1 & & \leq & 6 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$

Solution: $x_1 = 6$, $x_2 = 11$0pt.
Objective Value: 28

Not drawn to scale



Example #2

$$
\begin{array}{llllll}
\max & & x_2 & & \\
\text{s.t.} & 3x_1 & +2x_2 & \leq & 6 \\
 & -3x_1 & +2x_2 & \leq & 0 \\
 & x_1, & x_2 & \geq & 0 \\
 & x_1, & x_2 & \in & Z
\end{array}
$$