AMATH 301 - Spring 2020

Homework #8

Due on Friday, May 29, 2020

Instructions for submitting:

- Problems 1-3 should be submitted to MATLAB Grader. You have 3 attempts for each problem.
- Problems 4-5 should be submitted to Gradescope. The solutions and the code used to get those solutions should be submitted as a single pdf. All code should be at the end of the file. You **must** select which page each problem is on when you submit to Gradescope.

Note: All three MATLAB Grader problems in the homework have corresponding data held in a `.mat` file. You should download them to the same directory as your script file and access them by using the `load` command. MATLAB Grader has its own copy of the data files.

(25 points) Problem 1: MATLAB Grader

In order to determine the half-life of Plutonium-239, scientists start with a sample of approximately 50 kg of Plutonium-239 and measure the remaining amount each year for 40 years. The data is contained in the file `Plutonium.mat` which is included with the homework. `Plutonium.mat` contains two vectors, `t` and `P`. The vector `t` contains the number of years since the beginning of the experiment, and the vector `P` contains the corresponding amounts of Plutonium-239 remaining measured in kg. Let the function $P(t)$ denote the amount of remaining Plutonium as a function of time.

(a) Use a central difference to approximate the derivative $\frac{dP}{dt}$ at time $t = 10$. Store the result in the variable `ans1`.

(b) Use the second order difference formula that was derived in Activity 8 to approximate the derivative $\frac{dP}{dt}$ at time $t = 0$. Store the result in the variable `ans2`.

(c) Use a second order accurate difference scheme to approximate the derivative $\frac{dP}{dt}$ at time $t = 40$. Store the result in the variable `ans3`.

(d) The decay rate of Plutonium-239 at a time t is given by $-\frac{1}{P}\frac{dP}{dt}$. Use a second order difference scheme to approximate $\frac{dP}{dt}$ at all 41 times in `t`. You should use your answers to (b) and (c) for times $t = 0$ and $t = 40$ and a central difference for every other time. Then use these approximations to estimate the decay rate at all 41 times in `t`. Create a 1×41 row vector named `ans4` with the estimates at each time in chronological order.

(e) If λ is the average of the decay rates that you found in part (d), then the half-life of Plutonium-239, denoted by $t_{1/2}$, is given by the formula

$$t_{1/2} = \frac{\ln(2)}{\lambda}.$$

Calculate the half-life, and store it in the variable `ans5`.

(15 points) Problem 2: MATLAB Grader

Consider blood flow through an artery or vein. For laminar flow (i.e. flow in which the fluid moves in parallel layers), the velocity of the blood is given by the equation

$$v(r) = \frac{\Delta p}{4\mu L}(R^2 - r^2),$$

where r is the distance from the center of the blood vessel, R is the radius of the blood vessel (distance from the center to the wall), Δp is the change in pressure from the beginning of the blood vessel to the end, μ is the viscosity of the blood, and L is the length of the blood vessel. By examining this equation, we can see that the blood moves fastest in the center of the blood vessel ($r = 0$) and slowest near the walls (r is close to R). The volumetric flow rate Q (the volume of fluid that passes through a cross section per unit time) is given by

$$Q = \int_0^R 2\pi r v(r) dr.$$

If all of the parameters in the function $v(r)$ are known, this is a very easy integral to evaluate by hand. However, the parameters are often not known. Instead, measurements of the velocity can be taken at different values of r.

(a) The file `BloodFlow.mat` contains two vectors, `r` and `v`. The vector `r` contains the values of r at which measurements were taken, and the vector `v` contains the corresponding velocities in m/s. Using this data, use the trapezoidal rule to approximate the volumetric flow rate Q and store the result in the variable `ans6`.

(b) The cross-sectional area A of the blood vessel can be calculated with the integral

$$A = \int_0^R 2\pi r dr.$$

Approximate this integral using the vector `r` and the right-sided rectangle rule. Store the result in the variable `ans7`.

(c) The mean velocity is the volumetric flow rate Q divided by the cross-sectional area A. Calculate the mean velocity using your answers to part (a) and (b) and store the result in the variable `ans8`.

(10 points) Problem 3: MATLAB Grader

The cardiac output C_0 is the volume of blood pumped by the heart per unit time. One way to measure cardiac output is to inject dye into the right atrium and then measure the concentration of dye in the blood that is leaving the heart. If A is the amount of injected dye and $c(t)$ is the concentration of dye as a function of time, then the cardiac output is

$$C_0 = \frac{A}{\int_0^T c(t)dt}$$

where T is the final measurement time. The file `Dye.mat` contains two vectors. The vector `t` contains the times (in seconds) at which the dye concentration was measured, and the vector `c` contains the concentration of dye in the blood at those times. Use Simpson's rule to evaluate the integral in the denominator of the formula above. Store the answer in the variable `ans9`. Then use the value $A = 3$ ml to calculate the cardiac output in ml/s. Store the result in the variable `ans10`.

(45 points) Problem 4: Gradescope

Adult Alaskan Malamutes (i.e. Huskies) have weights that are normally distributed with a mean of 85 pounds and a standard deviation of 5 pounds. To compute the probability that a randomly selected Malamute has a weight between 76 and 86 pounds, you would compute the integral

$$P = \int_{76}^{86} \frac{1}{\sqrt{50\pi}} e^{-(x-85)^2/50} dx$$

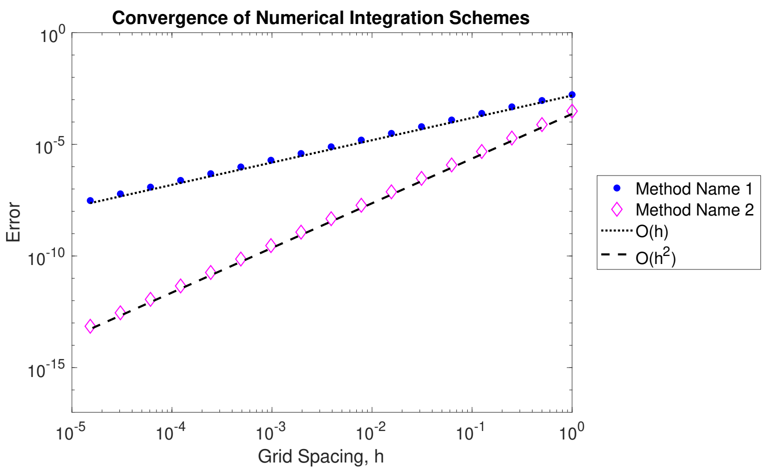
This integral cannot be evaluated exactly by using any of the methods you learned in Calculus class so we will evaluate it using numerical integration.

(a) Use the `integral` function to calculate the "exact" value of P.

(b) Use the left-sided rectangle rule to approximate P with step sizes of $h = 1, 2^{-1}, 2^{-2}, \ldots, 2^{-16}$. Calculate the error for each h by taking the absolute value of the exact solution minus the approximation. Store the errors in a vector.

(c) Do the same for the right-sided rectangle rule, the midpoint rule, the trapezoidal rule, and Simpson's rule.

(d) Plot the errors versus h on a log-log plot. Use a different color and marker type for each method. Plot a trend line that represents $\mathcal{O}(h)$ by plotting $c \cdot h$ versus h on the log-log plot. Choose the constant c so that the trend line falls near your error points. Also include trend lines for $\mathcal{O}(h^2)$ and any other orders that are represented by the numerical integration methods in your plot. Use different line styles for each trend line. Below is a sample of what the plot might look like for just two different integration schemes and two trend lines.



Also add a horizontal line at 10^{-16} which is (approximately) "machine precision". This is the lowest you could reasonably expect the error of one of the methods to be because of rounding error. Add appropriate labels to the x and y axes, a legend, and a title. You will be graded on how easy it is to see and interpret your plot and how well it illustrates the orders of each method.

(5 points) Problem 5: Gradescope

Which of the following finite difference approximations have error that is $\mathcal{O}(\Delta x^2)$? Choose all that apply.

(a) $f'(x) = \frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x}$

(b) $f'(x) = \frac{f(x)-f(x-\Delta x)}{\Delta x}$

(c) $f'(x) = \frac{f(x+\Delta x)-f(x)}{\Delta x}$

(d) $f'(x) = \frac{-3f(x)+4f(x+\Delta x)-f(x+2\Delta x)}{2\Delta x}$

(e) There is not enough information because the problem doesn't specify whether it is local error or global error.

Gradescope Deliverables Your Gradescope writeup should contain the following:

- **Problem 4:** The plot
- **Problem 5:** A letter or multiple letters corresponding to your answer choice
- **Code:** Code for problem 4