1. H_0: Pvalues come from a uniform distribution

H_1: Pvalues don't all come from a uniform distribution.

$L(\theta_0) = 27! \prod_{i=1}^{5} \frac{(\frac{1}{5})^{x_i}}{x_i!}$

$L(\theta) = 27! \prod_{i=1}^{5} \frac{\pi_i^{x_i}}{x_i!}$

$\log L(\theta) = \log\left(\frac{27!}{\prod_{i=1}^{5} x_i!}\right) + \sum_{i=1}^{5} x_i \log(\pi_i)$

$= \log L(\theta) + \lambda(1 - \sum_{i=1}^{5} \pi_i)$

$\frac{\partial}{\partial \pi_i} \log L(\theta) = \frac{\partial}{\partial \pi_i} \log L(\theta) - \lambda \frac{\partial}{\partial \pi_i} \Sigma \pi_i = 0$

$\Rightarrow \pi_i = \frac{x_i}{\lambda}$

$\Rightarrow \lambda = n$

$MLE = \frac{x_i}{n}$

$\lambda = \frac{27! \prod_{i=1}^{5} \frac{(\frac{1}{5})^{x_i}}{x_i!}}{27! \prod_{i=1}^{5} \frac{(\frac{x_i}{27})^{x_i}}{x_i!}} = \frac{\prod_{i=1}^{5} (\frac{1}{5})^{x_i}}{\prod_{i=1}^{5} (\frac{x_i}{27})^{x_i}}$

$-2\log(\Lambda) \sim Chisq(df=4)$

$p(\pi) = P(chisq(df=4) \geq -2\log(\lambda))$

[0,0.2)	[0.2, 0.4)	[0.4, 0.6)	[0.6, 0.8)	[0.8, 1.0]
10	4	5	3	5

$\lambda = \frac{(\frac{1}{5})^{27}}{(\frac{10}{27})^{10}(\frac{4}{27})^{4}(\frac{5}{27})^{5}(\frac{3}{27})^{3}(\frac{5}{27})^{5}} = 0.088$

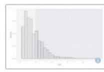
$pvalue = P(Chisq(df=4) \geq -2\log(0.088))$

$= 1 - P(Chisq(df=4) < -2\log(0.088))$

$= 0.3$

```{r}
#1
pvalues=c(0.01, 0.01, 0.02, 0.04, 0.04, 0.05,0.07, 0.07, 0.10, 0.19, 0.24, 0.27,0.34, 0.37, 0.44, 0.50,
0.53, 0.54,0.55, 0.61, 0.70,0.77,0.80,0.80,0.82,0.94,0.99)

p <- c(10,4,5,3,5)
rp <- rmultinom(n=10000,size=27,prob=c(0.2,0.2,0.2,0.2,0.2))
stat <- function(data) {
  -2*(dmultinom(x=data,prob=c(0.2,0.2,0.2,0.2,0.2),log=T)-dmultinom(x=data,prob=data/sum(data),log=T))
}
statTally(sample=p,rdata=rp,fun=stat,alternative="greater")
```

R Console



```
Test statistic applied to sample data =  4.857

Quantiles of test statistic applied to random data:
      50%       90%       95%       99%
 3.453128  8.068054  9.848351 14.417659

Of the  10001  samples (1 original +  10000  random),
         127 ( 1.27 % ) had test stats = 4.857
         3168 ( 31.68 % ) had test stats >= 4.857
```

I failed to reject the null hypothesis at significant level of 5%.

2\. a. $L(\theta) = \frac{3839!}{1997!\,906!\,904!\,32!}(0.25(2+\theta))^{1997}(0.25(1-\theta))^{1810}(0.25\theta)^{32}$

$\log L(\theta) = \log\left(\frac{3839!}{1997!\,906!\,904!\,32!}\right) + 1997\log(0.25(2+\theta)) + 1810\log(0.25(1-\theta)) + 32\log(0.25\theta)$

$\frac{\partial}{\partial\theta}\log L(\theta) = \frac{1997}{2+\theta} - \frac{1810}{1-\theta} + \frac{32}{\theta} = 0$

We get the MLE $= 0.0357033$ from R.

By doing the second derivative test, we see

$\frac{\partial^2}{\partial\theta^2}\log L(\theta) = -\frac{1997}{(2+\theta)^2} - \frac{1810}{(1-\theta)^2} - \frac{32}{\theta} < 0$. It's satisfied.

Thus MLE $= 0.0357033$

b. $H_0: \theta_0 = 0.05$ $\quad H_1: \theta_0 \neq 0.05$

$L(\theta_0) = \frac{3839!}{1997!\,906!\,904!\,32!}(0.25(2+0.05))^{1997}(0.25(1-0.05))^{1810}(0.25\times 0.05)^{32}$

$L(\hat{\theta}) = \frac{3839!}{1997!\,906!\,904!\,32!}(0.25(2+0.0357033))^{1997}(0.25(1-0.0357033))^{1810}(0.25\times 0.0357033)^{32}$

According to R, we get p-value $= 0.03261$.

e. $\lambda = \frac{L(\hat{p_0})}{L(MLE)} = \frac{(0.25(2+0.0357033))^{1997}(0.25(1-0.0357033))^{1810}(0.25\times 0.0357033)^{32}}{\left(\frac{1997}{3839}\right)^{1997}\left(\frac{906}{3839}\right)^{906}\left(\frac{904}{3839}\right)^{904}\left(\frac{32}{3839}\right)^{32}} = 0.3644511$

According to R, we get p-value $= 0.3644511$

Since $0.3644511 > 5\%$, so we believe the data fit the model very well.

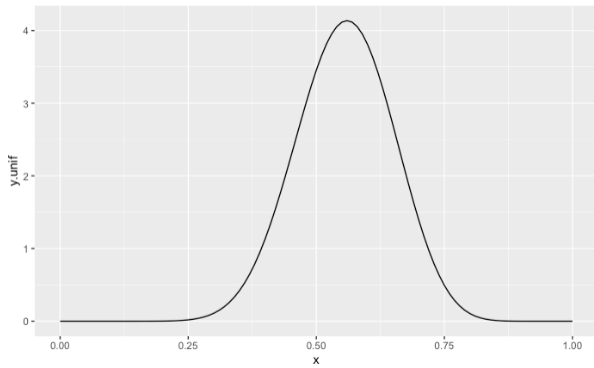
f. From R, we get 95% CI is $[0.025, 0.04855]$.

3.a.

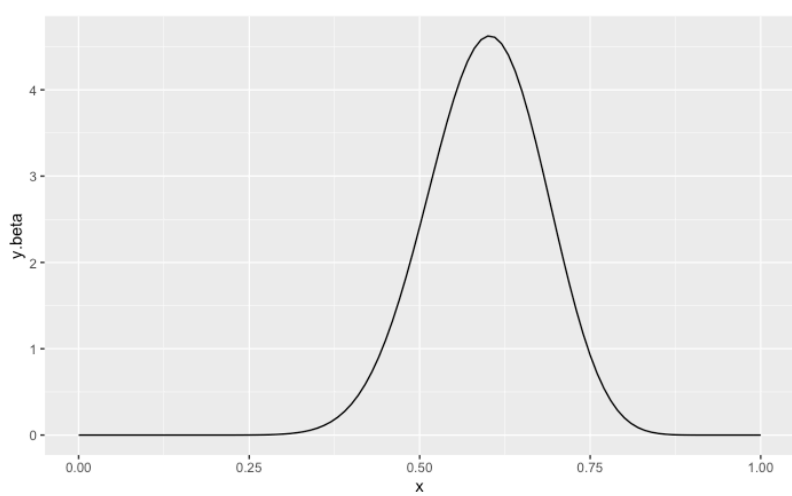
```{r}
library(LearnBayes)
set.seed(1234)
pos <- rgeo(25)
leaflet_map(position = pos, mark = TRUE)

x <- seq(0, 1, 0.01)
y.unif <- dbeta(x, 15, 12)
ggplot(tibble(x, y.unif), aes(x = x, y = y.unif)) + geom_line()
beta.select(list(p = 0.1, x = 0.5), list(p = 0.9, x = 0.9))
```

leaflet
htmlwidget

R Console




```{r}
y.beta <- dbeta(x, 5.51+14, 25-14+2.21)
ggplot(tibble(x, y.beta), aes(x = x, y = y.beta)) + geom_line()
```



b. For uniform : we know beta (15, 12), with a sample of 25,

we get beta (33, 19), which equals to beta (1+32, 50-32+1)

For beta : we get

beta (20.51 + 19, 25-19+12.21) = beta (5.51+32, 50-32+2.21)

which equals to beta (38.51, 18.21)

4. $x_1 \sim \text{Poisson}(\lambda)$

$\lambda \sim \text{Gamma}(a, \theta)$

$P(\lambda | x) = P(\lambda) \times P(x | \lambda)$

$= \frac{\theta^a}{\Gamma(a)} \lambda^{a-1} e^{-\theta\lambda} \frac{e^{-n\lambda} \lambda^{\Sigma x_i}}{\prod_{i=1} (x_i!)}$, which is a gamma distribution.

$\text{Gamma}(2+3+4+3+4,\ 1+10) = \text{Gamma}(16, 11)$