1. Suppose an $i.i.d$ sample of size $n = 9$ is taken from the population distribution

$$f_\theta(x) = \begin{cases} \frac{1}{3} - \theta & x = 0 \\ \frac{1}{3} & x = 1 \\ \frac{1}{3} + \theta & x = 2 \end{cases}$$

where $-\frac{1}{3} \le \theta \le \frac{1}{3}$. We observe the sample 0 0 1 0 1 2 1 0 0.

a. Write the likelihood function $L(\theta)$ for the observed sample.

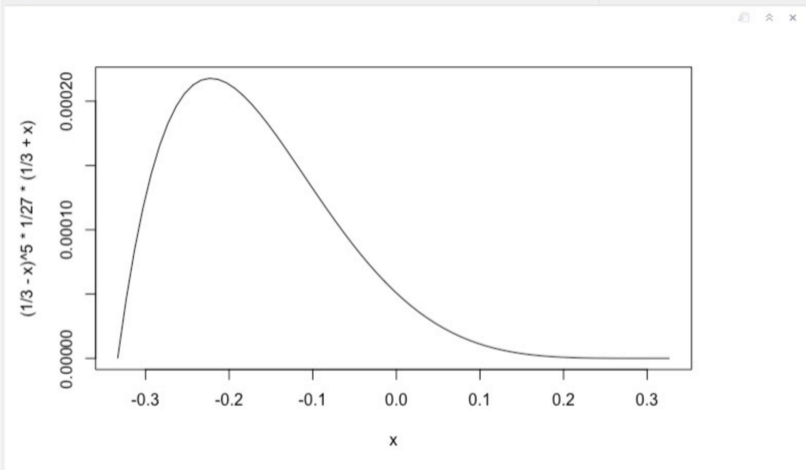
b. Draw a graph showing the likelihood function as a function of θ.

c. Calculate the MLE of θ showing your work.

(a) $L(\theta) = (\frac{1}{3} - \theta)^5 \times (\frac{1}{3})^3 \times (\frac{1}{3} + \theta)$ $\quad (-\frac{1}{3} \le \theta \le \frac{1}{3})$

(b)

```{r}
x <- seq(-1/3,1/3,0.01)
plot(x=x,y=(1/3-x)^5*1/27*(1/3+x),type="l")
```



(c) $\ln L(\theta) = \ln[(\frac{1}{3} - \theta)^5 \times \frac{1}{27} \times (\frac{1}{3} + \theta)]$

$\frac{d}{d\theta} \ln L(\theta) = \frac{d}{d\theta} (\ln(\frac{1}{3} - \theta)^5 + \frac{1}{27} + \ln(\frac{1}{3} + \theta))$

$= \frac{15}{3\theta - 1} + \frac{1}{\theta + \frac{1}{3}}$

$= \frac{54\theta + 12}{9\theta^2 - 1} \Rightarrow 0 \qquad 54\theta + 12 = 0 \qquad 54\theta = -12 \qquad \hat{\theta} = -\frac{2}{9}$

$\frac{d^2}{d\theta^2} = \frac{54}{9\theta^2 - 1} - \frac{18\theta(54\theta + 12)}{(9\theta^2 - 1)^2} < 0$

2. (1 point) Let X have PMF

$$P(X = x) = (1-\pi)^{x-1}\pi, \quad x = 1, 2, \ldots$$

and let $Y \sim Binom(n, \pi)$.

a. Find the MLE of π based on X.

b. Find the MLE of π based on Y.

c. Find the bias of the two estimators.

a. $L(\pi) = (1-\pi)^{x_1-1}\pi \times (1-\pi)^{x_2-1}\pi \times \cdots \times (1-\pi)^{x_n-1}\pi = (1-\pi)^{(\sum_{i=1}^{n} x_i)-n} \cdot \pi^n$

$\ln L(\pi) = ((\sum_{i=1}^{n} x_i)-n)\ln(1-\pi) + n\ln(\pi)$

$\frac{d}{d\pi} \ln L(\pi) = \frac{(\sum_{i=1}^{n} x_i)-n}{\pi-1} + \frac{n}{\pi} = 0$

$\frac{(\sum_{i=1}^{n} x_i)-n}{\pi-1} = -\frac{n}{\pi}$

$n(\pi-1) = -\pi[(\sum_{i=1}^{n} x_i)-n]$

$n\pi - n = -\pi(\sum_{i=1}^{n} x_i) + n\pi$

$\pi(\sum_{i=1}^{n} x_i) = n$

$\hat{\pi} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}$

$\frac{d^2}{d\pi^2} = -\frac{(\sum_{i=1}^{n} x_i)-n}{(\pi-1)^2} - \frac{1}{\pi^2} < 0$

$\Rightarrow \hat{\pi}$ is a max

b. $L(\pi) = \binom{n}{y}\pi^y(1-\pi)^{n-y}$

$\ln L(\pi) = \ln\binom{n}{y} + y\ln(\pi) + (n-y)\ln(1-\pi)$

$\frac{d}{d\pi} \ln L(\pi) = \frac{y}{\pi} + \frac{n-y}{\pi-1} = 0$

$\frac{y}{\pi} = -\frac{n-y}{\pi-1}$

$\pi(y-n) = \pi y - y$

$\pi y - n\pi = \pi y - y$

$n\pi = y$

$\boxed{\hat{\pi} = \frac{y}{n}}$

$\frac{d^2}{d\pi^2} = -\frac{y}{\pi^2} - \frac{n-y}{(\pi-1)^2} < 0$

$\Rightarrow \hat{\pi}$ is a max

c. $\text{bias}(\hat{\pi}) = E(\frac{1}{x}) - \pi = \sum_{i=1}^{\infty}(\frac{1}{x}(1-\pi)^{x-1}\pi) - \pi$

$\text{bias}(\hat{\pi}) = E(\frac{y}{n}) - \pi = \frac{1}{n} \cdot n\pi - \pi = 0$

3. (1 point) A set of cheap light bulbs have a lifetime (in hours) which is exponentially distributed with the unknown mean θ. Choosing a random sample of ten light bulbs, they are turned on simultaneously and observed for 48 hours. During this period, six bulbs went out, at times $x_1, x_2, \ldots, x_6$. At the end of the experiment, four light bulbs were still working. How would you define the likelihood function in this situation? Find the MLE of θ.

({*Hint:* this is a survival analysis problem with censoring. The lifetimes of the four bulbs that are still working is said to be censored. All we observe is that the lifetime exceeds 48 hours.)

$$f(\theta) = \frac{1}{\theta} e^{-\frac{1}{\theta}x}$$

$$P(X>48) = e^{-\frac{48}{\theta}}$$

$$L(\theta) = f_1\theta \times f_2\theta \times \cdots \times f_6\theta \times f_7\theta \times \cdots f_{10}\theta$$

for last 4 f, we can use $e^{-\frac{48}{\theta}}$

$$L(\theta) = \frac{1}{\theta}e^{-\frac{1}{\theta}x_1} \times \frac{1}{\theta}e^{-\frac{1}{\theta}x_2} \times \cdots \times \frac{1}{\theta}e^{-\frac{1}{\theta}x_6} \times \left(e^{-\frac{48}{\theta}}\right)^4$$

$$L(\theta) = \frac{1}{\theta^6} \cdot e^{-\frac{1}{\theta}\sum_{i=1}^{6} x_i} \times e^{-\frac{192}{\theta}}$$

$$\ln L(\theta) = \ln\frac{1}{\theta^6} + \left(-\frac{1}{\theta}\sum_{i=1}^{6} x_i\right)\ln e - \frac{192}{\theta}\ln e$$

$$\frac{d}{d\theta}\ln L(\theta) = -\frac{6}{\theta} + \frac{\sum_{i=1}^{6} x_i}{\theta^2} + \frac{192}{\theta^2} = 0$$

$$\frac{d}{d\theta}\ln L(\theta) = \frac{-6\theta + \sum_{i=1}^{6} x_i + 192}{\theta^2} = 0$$

$$6\theta = \sum_{i=1}^{6} x_i + 192$$

$$\boxed{\theta = \frac{\sum_{i=1}^{6} x_i + 192}{6}}$$

$$\frac{d^2}{d\theta^2} = \frac{6}{\theta^2} - \frac{\sum_{i=1}^{6} x_i}{\theta^3} - \frac{384}{\theta^3} < 0$$

So $\hat{\theta}$ is a max.

4. (2 points) A sample of size 10 drawn from $Gamma(\alpha, 2)$ distribution is shown below:

x	1.425	2.216	0.738	0.590	1.266	0.601	0.483	1.313	1.707	1.008

a. Use the method of moments to find an estimate of α.

b. Use this estimate as a starting value along with the sample and use `maxLik()` function in R, to compute the MLE of α.

c. Use the starting value obtained in part (a) and derive 3 iterations of Newton Raphson method by hand and report the values α obtained from them. Clearly state your g(.) function and g'(.) function that are used in Newton Raphson method. Comment on the convergence using values obtained in 3rd iteration here and value obtained in part (b).

(*Hint:* You may use `digamma` and `trigamma` function in R to calculate the first and second derivatives of the logarithm of the gamma function at a user supplied value of α.

a. $\text{mean}(x) = \dfrac{\alpha}{\beta}$

$\alpha = 2 \cdot \text{mean}(x)$

$\alpha = 2 \cdot \dfrac{1.425+2.216+0.738+0.59+1.266+0.601+0.483+1.313+1.707+1.008}{10}$

$\alpha = 2.2694$

b. $f(x; \alpha, \lambda) = \dfrac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$

$L(\alpha) = \prod_{i=1}^{n} \dfrac{2^{n\alpha} e^{-2x_i} x_i^{(\alpha-1)}}{\Gamma(\alpha)}$

$\ln L(\alpha) = n\alpha \ln(2) - 2\sum x_i + (\alpha-1)\sum \log(x_i) - n\ln(\Gamma(\alpha))$

```{r}
x <- c(1.425,2.216,0.738,0.590,1.266,0.601,0.483,1.313,1.707,1.008)
start <- mean(x)*2

loglik<-function(alpha,x,n){ #first argument must be theta
  if(alpha < 0)
    NA
  else
    n*alpha*log(2)-2*sum(x)+(alpha-1)*sum(log(x))-n*log(gamma(alpha))
}
maxLik(logLik=loglik,start=start,method="NR",x=x,n=10,tol=0.01)

```

```
Maximum Likelihood estimation
Newton-Raphson maximisation, 2 iterations
Return code 2: successive function values within tolerance limit (tol)
Log-Likelihood: -8.01342 (1 free parameter(s))
Estimate(s): 2.506546
```

c. $\frac{d}{d\alpha} = n \cdot \ln(\lambda) + \Sigma \ln(x_i) - \frac{n}{\Gamma(\alpha)} (\Gamma(\alpha))' = g(x)$

$\frac{d^2}{d\alpha^2} = n(\Gamma(\alpha)^{-2})(\Gamma(\alpha))' - \frac{n}{\Gamma(\alpha)} (\Gamma(\alpha))'' = g'(x)$

$\alpha_1 = 2.2694 - \frac{g(2.2694)}{g'(2.2694)} = 2.480423$

$\alpha_2 = 2.480423 - \frac{g(2.480423)}{g'(2.480423)} = 2.506561$

$\alpha_3 = 2.506561 - \frac{g(2.506561)}{g'(2.506561)} = 2.506609$

It's the same with what I get from part b. It converges to the root.