1. (a) $P(MM) = P(MM \mid \text{it's identical}) + P(MM \mid \text{it's not identical})$

$$= \frac{1}{2} \times a + \frac{1}{4} \times (1-a) = \frac{1}{2}a + \frac{1}{4} - \frac{1}{4}a$$

$$= \frac{1}{4} + \frac{1}{4}a = \boxed{\frac{1+a}{4}}$$

$P(FF) = P(FF \mid \text{it's identical}) + P(FF \mid \text{it's not identical})$

$$= \frac{1}{2} \times a + \frac{1}{4} \times (1-a) = \frac{1}{2}a + \frac{1}{4} - \frac{1}{4}a$$

$$= \frac{1}{4} + \frac{1}{4}a = \boxed{\frac{1+a}{4}}$$

$P(MF) = P(MF \mid \text{it's not identical})$

$$= \frac{1}{2} \times (1-a)$$

$$= \boxed{\frac{1-a}{2}}$$

(b) $L(a) = f(x_1) \times f(x_2) \times f(x_3)$

$$= \frac{1+a}{4} \times \frac{1+a}{4} \times \frac{1-a}{2} = \frac{(1+2a+a^2)(1-a)}{32}$$

$$= \frac{1-a+2a-2a^2+a^2-a^3}{32} = \frac{1+a-a^2-a^3}{32}$$

$$\frac{d}{da}L(a) = \frac{1-2a-3a^2}{32} = 0$$

$$3a^2+2a-1=0$$

$$a = \frac{-2 \pm \sqrt{4+12}}{6} = \frac{-2 \pm 4}{6} = -1 \text{ or } \frac{1}{3}$$

$$L(a) = \frac{n!}{x_1!x_2!x_3!} \left(\frac{a+1}{4}\right)^{x_1} \left(\frac{a+1}{4}\right)^{x_2} \left(\frac{1-a}{2}\right)^{x_3}$$

$$\ln L(a) = \ln\left(\frac{n!}{x_1!x_2!x_3!}\right) + x_1 \ln\left(\frac{a+1}{4}\right) + x_2 \ln\left(\frac{a+1}{4}\right) + x_3 \ln\left(\frac{1-a}{2}\right)$$

$$\frac{d}{da}\ln L(a) = x_1 \frac{1}{a+1} + x_2 \frac{1}{a+1} + x_3 \frac{1}{a-1} = 0$$

$$= \frac{x_1+x_2}{a+1} + \frac{x_3}{a-1} = 0$$

$$\frac{x_1+x_2}{a+1} = -\frac{x_3}{a-1}$$

$$(x_1+x_2)(1-a) = x_3(a+1)$$

$$x_1 - ax_1 + x_2 - ax_2 = ax_3 + x_3$$

$$-ax_3 - ax_1 - ax_2 = x_3 - x_1 - x_2$$

$$-an = x_3 - x_1 - x_2$$

$$\boxed{a = \frac{x_1+x_2-x_3}{n}}$$

2. $L(\mu) = \dfrac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} e^{\left(-\frac{1}{2}\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sigma^2}\right)}$

$$l(\mu) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2}\frac{\Sigma(x_i-\mu)^2}{\sigma^2}$$

$$l(\hat{\mu}) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2}\frac{\Sigma(x_i-\hat{\mu})^2}{\sigma^2}$$

$$\frac{d}{d\mu} l(\hat{\mu}) = -\frac{1}{2\sigma^2}(-2)\,\Sigma(x_i-\hat{\mu}) = \frac{\Sigma(x_i-\hat{\mu})}{\sigma^2} = \frac{\Sigma x_i - n\hat{\mu}}{\sigma^2}$$

$$\frac{d^2}{d\mu^2} l(\hat{\mu}) = -\frac{n}{\sigma}$$

$$I(\hat{\mu}) = \frac{n}{\sigma^2}$$

$$l(\mu) - l(\hat{\mu}) = -\frac{1}{2}\frac{\Sigma(x_i-\mu)^2}{\sigma^2} + \frac{1}{2}\frac{\Sigma(x_i-\hat{\mu})^2}{\sigma^2}$$

$$= \frac{1}{2} \times \frac{\Sigma(x_i-\hat{\mu})^2 - \Sigma(x_i-\mu)^2}{\sigma^2}$$

$$= \frac{1}{2} \times \frac{\Sigma[(x_i-\hat{\mu}+x_i-\mu)(\cancel{x_i}-\hat{\mu}-\cancel{x_i}+\mu)]}{\sigma^2}$$

$$= \frac{1}{2} \times \frac{(\mu-\hat{\mu})\,\Sigma(2x_i-\mu-\hat{\mu})}{\sigma^2}$$

$$= \frac{1}{2}\frac{\mu-\hat{\mu}}{\sigma^2}(2n\bar{x} - n\mu - n\hat{\mu})$$

$$= \frac{1}{2}\frac{\mu-\bar{x}}{\sigma^2}(2n\bar{x} - n\mu - n\hat{\mu})$$

$$= \frac{1}{2}\frac{\mu-\bar{x}}{\sigma^2}(n\bar{x} - n\mu)$$

$$= \frac{1}{2}\frac{-n(\mu-\bar{x})^2}{\sigma^2}$$

$$= -\frac{1}{2}(\mu-\hat{\mu})^2 I(\hat{\mu})$$

Thus, we get $\boxed{l(\mu) = l(\hat{\mu}) - \frac{1}{2}(\mu-\hat{\mu})^2 I(\hat{\mu})}$

3. (a) $L(\theta) = f_\theta(x_1) \times f_\theta(x_2) \times \cdots \times f_\theta(x_n)$

$$= (\theta+1)x_1^\theta \times (\theta+1)x_2^\theta \times \cdots \times (\theta+1)x_n^\theta$$

$$= (\theta+1)^n \prod_{i=1}^{n} x_i^\theta$$

$$\ln L(\theta) = n\ln(\theta+1) + \theta \ln \prod_{i=1}^{n} x_i$$

$$= n\ln(\theta+1) + \theta \sum_{i=1}^{n} \ln(x_i)$$

$$\frac{d}{d\theta} \ln L(\theta) = \frac{n}{\theta+1} + \sum_{i=1}^{n} \ln(x_i) = 0$$

$$\frac{n}{\theta+1} = -\sum_{i=1}^{n} \ln(x_i)$$

$$\theta+1 = \frac{n}{-\sum_{i=1}^{n} \ln(x_i)}$$

$$\hat{\theta} = \frac{n}{-\sum_{i=1}^{n} \ln(x_i)} - 1$$

$$\lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{1}{\left(\frac{n}{-\sum_{i=1}^{n} \ln(x_i)} - 1 + 1\right)^n \prod_{i=1}^{n} x_i^{\frac{n}{-\sum_{i=1}^{n} \ln(x_i)} - 1}}$$

(b)

````
```{r}
#3b
x<-c(0.64,0.92,0.73,0.96,0.98,0.33,0.80,0.96,0.81,0.76,0.98,0.75,
    0.87,0.82,0.44,0.96,0.61,0.32,0.67,0.98,0.96,0.88,0.85,1.00,0.86,0.88,0.80,0.83,0.64,0.5)
n <- length(x)

lambda <- 1/(((-1-n/sum(log(x)))+1)^n*((prod(x))^(-1-n/sum(log(x)))))
lambda
```
````

```
[1] 7.733484e-08
```

(c)

````
```{r}
#3c
1-pchisq(q=-2*log(lambda),df=1)
```
````

```
[1] 1.047932e-08
```

(d) To use the chi-squared distribution, we should satisfy the three conditions in Theorem 2.1 in class. And the null value should be a true value. However, we can see that λ is very close to 0. So chisquare distribution is not a good approximation in this case.

4. $f(\theta) = \frac{1}{\theta} e^{-\frac{1}{\theta} x}$

$$L(\theta_1) = \frac{1}{\theta} e^{-\frac{1}{\theta} x_1} \times \frac{1}{\theta} e^{-\frac{1}{\theta} x_2} \times \cdots \times \frac{1}{\theta} e^{-\frac{1}{\theta} x_{n_1}}$$

$$= \frac{1}{\theta^{n_1}} e^{-\frac{1}{\theta} \Sigma x_i}$$

$$\ln L(\theta_1) = \ln \frac{1}{\theta^{n_1}} + \left(-\frac{1}{\theta} \sum_{i=1}^{n_1} x_i\right) \ln e$$

$$\frac{d}{d\theta_1} = -\frac{n_1}{\theta} + \frac{\sum_{i=1}^{n_1} x_i}{\theta^2} = 0$$

$$\frac{\sum_{i=1}^{n_1} x_i - n_1\theta}{\theta^2} = 0$$

$$n_1\theta = \sum_{i=1}^{n_1} x_i$$

$$\boxed{\theta_1 = \bar{x}}$$

$$L(\theta_2) = \frac{1}{k\theta} e^{-\frac{1}{k\theta} y_1} \times \frac{1}{k\theta} e^{-\frac{1}{k\theta} y_2} \times \cdots \times \frac{1}{k\theta} e^{-\frac{1}{k\theta} y_{n_2}}$$

$$= \frac{1}{k^{n_2}\theta^{n_2}} e^{-\frac{1}{k\theta} \Sigma y_i}$$

$$\ln L(\theta_2) = -n_2 \ln(k\theta) + \left(-\frac{1}{k\theta} \sum_{i=1}^{n_2} y_i\right) \ln e$$

$$\frac{d}{d\theta_2} = -\frac{n_2}{\theta} + \frac{1}{k\theta^2} \Sigma y_i = 0$$

$$\frac{\sum_{i=1}^{n_2} y_i - k\theta n_2}{k\theta^2} = 0$$

$$k\theta n_2 = \sum_{i=1}^{n_2} y_i$$

$$\boxed{\theta_2 = \frac{\bar{y}}{k}}$$

$$var(\bar{x}) = var\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) = \frac{1}{n^2} (\theta^2 + \theta^2 + \cdots + \theta^2) = \frac{n\theta^2}{n_1^2} = \frac{\theta^2}{n_1}$$

$$var(\bar{y}) = var\left(\frac{1}{kn} \sum_{i=1}^{n} y_i\right) = \frac{1}{k^2 n^2} (k^2\theta^2 + k^2\theta^2 + \cdots k^2\theta^2) = \frac{nk^2\theta^2}{k^2 n_2^2} = \frac{\theta^2}{n_2}$$

$$var(a\bar{x} + (1-a)\bar{y}) = a^2 var(\bar{x}) + (1-a)^2 var(\bar{y}) = a^2 \frac{\theta^2}{n_1} + (1-a)^2 \frac{\theta^2}{n_2}$$

$$\frac{d}{da} var = 2a \cdot \frac{\theta^2}{n_1} - 2(1-a) \frac{\theta^2}{n_2} = 0.$$

$$2a \frac{\theta^2}{n_1} - 2 \frac{\theta^2}{n_2} + 2a \frac{\theta^2}{n_2} = 0$$

$$2a\left(\frac{\theta^2}{n_1} + \frac{\theta^2}{n_2}\right) = 2 \frac{\theta^2}{n_2}$$

$$a = \frac{\frac{\theta^2}{n_2}}{\frac{n_2\theta^2 + n_1\theta^2}{n_1 n_2}} = \frac{\cancel{\theta^2}}{\cancel{n_2}} \times \frac{n_1 \cancel{n_2}}{n_2\cancel{\theta^2} + n_1\cancel{\theta^2}} = \frac{n_1}{n_2 + n_1}$$

$$a\hat{\theta}_1 + (1-a)\hat{\theta}_2 = \frac{n_1}{n_2 + n_1} \bar{x} + \frac{n_2}{n_2 + n_1} \frac{\bar{y}}{k} = \frac{n_1\bar{x} + n_2\bar{y}}{k(n_1 + n_2)} = \boxed{\frac{k\sum_{i=1}^{n_1} x_i + \sum_{i=1}^{n_2} y_i}{k(n_1 + n_2)}}$$

$$L(\theta) = \frac{1}{\theta} e^{-\frac{1}{\theta}x_1} \times \frac{1}{\theta} e^{-\frac{1}{\theta}x_2} \times \cdots \times \frac{1}{\theta} e^{-\frac{1}{\theta}x_{n_1}} \times \frac{1}{k\theta} e^{-\frac{1}{k\theta}y_1} \times \frac{1}{k\theta} e^{-\frac{1}{k\theta}y_2} \times \cdots \times \frac{1}{k\theta} e^{-\frac{1}{k\theta}y_{n_2}}$$

$$= \frac{1}{\theta^{n_1}} \times \frac{1}{k^{n_2}\theta^{n_2}} \times e^{-\frac{1}{\theta}\sum x_i - \frac{1}{k\theta}\sum y_i}$$

$$\ln L(\theta) = -n_1 \ln(\theta) - n_2 \ln(k\theta) - \frac{1}{\theta}\sum x_i - \frac{1}{k\theta}\sum y_i$$

$$\frac{d}{d\theta} \ln L(\theta) = -\frac{n_1}{\theta} - \frac{n_2}{\theta} + \frac{1}{\theta^2}\sum x_i + \frac{1}{k\theta^2}\sum y_i = 0$$

$$\frac{n_1}{\theta} + \frac{n_2}{\theta} = \frac{1}{\theta^2}\sum x_i + \frac{1}{k\theta^2}\sum y_i$$

$$\frac{n_1+n_2}{\theta} = \frac{k\sum x_i + \sum y_i}{k\theta^2}$$

$$\theta = \boxed{\frac{k\sum_{i=1}^{n_1} x_i + \sum_{i=1}^{n_2} y_i}{k(n_1+n_2)}}$$