In order to test the goodness of fit, we apply the bootstrap to see how the $\theta = \frac{\pi_1}{\pi_1+\pi_2}$ fits $Beta(\alpha = 0.7000102, \beta = 1)$.

First Method

Our hypothesis is:

$$H_0 : \theta \sim Beta(\alpha = 0.7000102, \beta = 1) H_a : \theta \nsim Beta(\alpha = 0.7000102, \beta = 1)$$

The likelihood function is:

$$\begin{aligned} L(\theta) &= P(x_1, \theta) \\ &= P(x_1|\theta)P(\theta) \\ &= \binom{x_1 + x_2}{x_1} \theta^{x_1}(1-\theta)^{x_2} P(\theta) \end{aligned}$$

Let $\theta \sim Beta(\alpha, \beta)$. Then we get the function of $L(\theta)$:

$$L(\theta) = \binom{x_1 + x_2}{x_1} \theta^{x_1}(1-\theta)^{x_2} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$l(\theta) = ln(\binom{x_1 + x_2}{x_1}) + x_1 log(\theta) + x_2(1-\theta) + log(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}) + (\alpha-1)log(\theta) + (\beta-1)log(1-\theta)$$

$$\begin{aligned} \frac{\partial l(\theta)}{\partial \theta} &= \frac{x_1}{\theta} - \frac{x_2}{1-\theta} + \frac{\alpha-1}{\theta} - \frac{\beta-1}{1-\theta} \\ &= \frac{x_1}{\theta} + \frac{x_2}{\theta-1} + \frac{\alpha-1}{\theta} + \frac{\beta-1}{\theta-1} \\ &= \frac{\alpha+x_1-1}{\theta} + \frac{\beta+x_2-1}{\theta-1} \end{aligned}$$

$$\begin{aligned} \frac{\partial l(\theta)}{\partial \theta} &= 0 \\ \frac{\alpha+x_1-1}{\theta} + \frac{\beta+x_2-1}{\theta-1} &= 0 \\ (\alpha+x_1-1)\cdot(\theta-1) + (\beta+x_2-1)\cdot\theta &= 0 \\ (\alpha+\beta+x_1+x_2-2)\cdot\theta &= \alpha+x_1-1 \\ \theta &= \frac{\alpha+x_1-1}{\alpha+\beta+x_1+x_2-2} \end{aligned}$$

$\hat{\theta} = \frac{8}{8+162} = 0.04705882$

Second Method

Our hypothesis is:

$$H_0 : \theta \sim Beta(\alpha = 0.7000102, \beta = 1) H_a : \textit{at least one is not equal to the null value}$$

The likelihood function is:

$$L(\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$l(\alpha, \beta) = log(\Gamma(\alpha + \beta)) - \Gamma(\alpha) - \Gamma(\beta)) + (\alpha - 1)log(\theta) + (\beta - 1)log(1 - \theta)$$

$$\frac{\partial l(\alpha, \beta)}{\partial \alpha} = \frac{\Gamma'(\alpha + \beta)}{\Gamma(\alpha + \beta)}$$

Third Method

Note that we only have one sample, which is $\theta = \frac{\pi_1}{\pi_1 + \pi_2} = \frac{\frac{x_1}{n_1}}{\frac{x_1}{n_1} + \frac{x_2}{n_2}} = 0.04707534$

```
theta <- 8/21720/(8/21720+162/21728)
sample <- rbeta(10000, 0.7000102, 1)
length(sample[sample <= theta])/length(sample)
```

```
## [1] 0.1199
```

```
length(sample[sample >= theta])/length(sample)
```

```
## [1] 0.8801
```